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Homotopy type of gauge groups of quaternionic line bundles over spheres

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ABSTRACT

Let P be a principal S^3-bundle over a sphere S^n, with $n \geqslant 4$. Let $\mathcal{G}_P$ be the gauge group of P. The homotopy type of $\mathcal{G}_P$ when $n = 4$ was studied by A. Kono in [A. Kono, A note on the homotopy type of certain gauge groups, Proc. Roy. Soc. Edinburgh Sect. A 117 (1991) 295–297]. In this paper we extend his result and we study the homotopy type of the gauge group of these bundles for all $n \leqslant 25$.

1. Introduction

Let $P \to B$ be a principal G-bundle over a finite complex B, where G is a compact Lie group. The gauge group $\mathcal{G}_P$ of P is the group of the G-equivariant maps of P covering the identity. When B and G are connected, M.C. Crabb and W. Sutherland proved in [2] that the number of homotopy types of $\mathcal{G}_P$ is finite. Moreover, explicit results have been given by A. Kono, and A. Kono and H. Hamanaka, that studied the case of $B = S^4$ and $G = SU(n)$. These bundles $P_{n,k}$ are classified by the Chern class $c_2(P_{n,k}) = k$, and it was proved in [6] that $\mathcal{G}_{P_{2,k}} \sim \mathcal{G}_{P_{2,k'}}$ if and only if $(12, k) = (12, k')$ (here (m, n) denotes the GCD of n and m), and there are six homotopy types. It was proved in [4] that $\mathcal{G}_{P_{3,k}} \sim \mathcal{G}_{P_{3,k'}}$ if and only if $(24, k) = (24, k')$. Recently, the same authors also studied the case of $B = S^6$ and $G = SU(3)$. These bundles are classified by the Chern class $c_3(P_k) = k$, and they proved in [5] that $\mathcal{G}_{P_k} \sim \mathcal{G}_{P_{k'}}$ if and only if $(120, k) = (120, k')$. Continuing along this line of investigation, we study the case of principal $SU(2)$-bundles over a sphere S^n. Our approach consists in generalizing the method introduced in [6] for $n = 4$ to higher n. In particular, we give explicit formulas for the boundary operator in the homotopy exact sequence associated with the evaluation fibration $ev : m(S^n, BS^3) \to BS^3$. This reduces the problem in calculations involving homotopy groups of the spheres, and can be solved in principle as far as information on these groups is available. We study all the classical cases, namely those covered by the results contained in the book of H. Toda [11]. However, this method fails whenever all the homotopy groups involved are elementary 2-groups. In such a case, a direct approach is necessary, in order to realize the appropriate homotopy equivalence (see Section 5).

Let $P \to S^n$ be a principal S^3-bundle. Such bundles are classified by elements in $\pi_{n-1}(S^3)$, and we will use this identification without further comment. By classical results [1,9], the classifying space $B\mathcal{G}_P$ of the gauge group of P is homotopy equivalent to the mapping space $m(S^n, BS^3; f)$, where $f : S^n \to BS^3$ is the classifying map of P. We will also use the notation $\mathcal{G}_f$ for the gauge group of the bundle P classified by f. We have the fiber sequence

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Table 1

n	$\pi_{n-1}(S^3)$	$\pi_{n+2}(S^3)$
5	$\mathbb{Z}/2$	$\mathbb{Z}/2$
6	$\mathbb{Z}/2$	$\mathbb{Z}/2$
7	$\mathbb{Z}/12$	$\mathbb{Z}/3$
8	$\mathbb{Z}/2$	$\mathbb{Z}/15$
9	$\mathbb{Z}/2$	$\mathbb{Z}/2$
10	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$
11	$\mathbb{Z}/15$	$\mathbb{Z}/2 \oplus \mathbb{Z}/12$
12	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/84$
13	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
14	$\mathbb{Z}/2 \oplus \mathbb{Z}/12$	$\mathbb{Z}/6$
15	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/84$	$\mathbb{Z}/30$
16	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/30$
17	$\mathbb{Z}/6$	$\mathbb{Z}/2 \oplus \mathbb{Z}/6$
18	$\mathbb{Z}/30$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/12$
19	$\mathbb{Z}/30$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/12$
20	$\mathbb{Z}/2 \oplus \mathbb{Z}/6$	$\mathbb{Z}/2 \oplus \mathbb{Z}/132$
21	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/12$	$(\mathbb{Z}/2)^2$
22	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/12$	$\mathbb{Z}/2$
23	$\mathbb{Z}/2 \oplus \mathbb{Z}/132$	$\mathbb{Z}/210$
24	$(\mathbb{Z}/2)^2$	
25	$\mathbb{Z}/2$	

$$\cdots \longrightarrow \mathcal{G}_f \longrightarrow S^3 \xrightarrow[\hat{f}]{} \Omega_f^{n-1}(S^3) \longrightarrow m(S^{n-1}, S^3; f) \xrightarrow[ev]{} BS^3,$$

and the class of $\hat{f}$ is given by some Samelson product (see Proposition 1). Since the different components of the loop space have the same type, the types of the gauge groups for different f are a subset of the set of the types of the fibres of the maps $h : S^3 \to \Omega_0^{n-1} S^3$, i.e. are a subset of $\pi_{n+2}(S^3)$. We collect this information in Table 1. By [7], we also know $\pi_{26}(S^3; 2) = \mathbb{Z}/4$ and $\pi_{27}(S^3; 2) = (\mathbb{Z}/2)^3$.

In order to determine the type of the gauge groups, we give an explicit formula for the boundary operator in the homotopy exact sequence associated to the evaluation fibration in Section 2. This is equivalent to fixing the class of the map $\hat{f}$, and therefore determining all the homotopy information of its fiber, namely of $\mathcal{G}_f$. This solves the problem in the majority of cases, $n \neq 13, 14, 19, 20, 21$, and 22, see Section 3. In order to deal with the cases $n = 14, 19, 20$ and 21, we need more information about the possible types of the fiber of the maps h, this is in Section 4. Eventually, in Section 5, we deal with the cases $n = 13$ and 21, that are harder.

In the following we will use the notation of [11] for the elements in the homotopy groups of the spheres, and we will identify maps and classes when possible.

2. Boundary operator and the class of $\hat{f}$

Consider the homotopy exact sequence associated to the fibration ev,

$$\longrightarrow \pi_{k+n-1}(S^3) \longrightarrow \pi_{k-1}(\mathcal{G}_f) \xrightarrow[ev_{k*}]{} \pi_{k-1}(S^3) \xrightarrow[\partial_k]{} \pi_{k+n-2}(S^3) \longrightarrow \tag{1}$$

then we can prove the following result, where we denote by $\alpha(p)$ the p component of an element α in some homotopy group.

Proposition 1. *Fix $f \in m(S^n, BS^3; f)$, $n > 4$, and let ∂f be the class that represents f in $\pi_{n-1}(S^3)$. Then, for all $k \geqslant 4$, $\zeta \in \pi_{k-1}(S^3)$,*

$$\partial_k(f)(\zeta) = \nu' \circ \Sigma^3 \partial f(2) \circ \Sigma^{n-1} \zeta(2),$$

where ν' is the element of order 4 of $\pi_6(S^3)$.

Proof. It is convenient to work with homotopy groups of S^4. This can be done recalling that each map $f : S^n \to BS^3$ ($n \geqslant 4$) factors through the inclusion j of S^4 into BS^3, as $f = jf'$ (see for example [3, Lemma 1]). The induced homomorphism j_* in homotopy, has a right inverse $\Sigma\partial$, where ∂ is the boundary homomorphism $\partial : \pi_n(BS^3) \to \pi_{n-1}(S^3)$, namely $f' = \Sigma\partial f$.

Using the construction of [6], we find that the boundary operator ∂_k sends the class of a map $u : S^k \to BS^3$ to the class of the Whitehead product $[u, f]$ in $\pi_{n+k-1}(BS^3)$. Using the commutative diagram

$$\begin{array}{ccc} \pi_k(S^4) & \xrightarrow{[\_, f']} & \pi_{n+k-1}(S^4) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ \pi_k(BS^3) & \xrightarrow[\partial_k]{} & \pi_{n+k-1}(BS^3) \end{array}$$

we obtain

$$[u, f] = \big[j_*(\Sigma\partial u), j_*(\Sigma\partial f)\big] = j_*\big([\iota_4 \circ \Sigma\partial u, \iota_4 \circ \Sigma\partial f]\big) = j_*\big([\iota_4, \iota_4] \circ \Sigma(\partial u \wedge \partial f)\big) = j_*\big([\iota_4, \iota_4] \circ \Sigma^4\partial f \circ \Sigma^n\partial u\big),$$

in $\pi_{n+k-1}(BS^3) = j_*(\pi_{n+k-1}(S^4))$.

Next observe that, for all $\psi \in \pi_n(S^6)$,

$$j_*\big([\iota_4, \iota_4] \circ \Sigma\psi\big) = j_*\Sigma(\xi \circ \psi),$$

in $\pi_{n+1}(BS^3) = j_*(\pi_{n+1}(S^4))$, where ξ generates $\pi_6(S^3)$. For, $[\iota_4, \iota_4] = 2\nu_4 - \Sigma\xi$, and ν_4 is in the kernel of j_* by definition, using the isomorphism

$$\pi_6\big(S^3\big) \oplus \pi_7\big(S^7\big) \to \pi_7\big(S^4\big), \qquad (\alpha, \beta) \mapsto \Sigma\alpha + \nu_4 \circ \beta,$$

of the Hopf fibering [10]. Also, by the mod p Serre isomorphism (see for example [11, (13.1)]), with p odd, we have that $j_*([\iota_4, \iota_4] \circ \Sigma\psi(p)) = 0$. This gives the formula stated in the proposition. □

The formula of the boundary operator allows in theory to compute $\pi_k(\mathcal{G}_f)$. In particular, Kono used π_2 in order to solve the problem when $n = 4$ in [6].

We have the following immediate consequences of Proposition 1, where

$$ad : m_0\big(S^3, \Omega_0^{n-1}S^3\big) \to \Omega_0^{n+2}S^3,$$

denotes the adjoint map.

Corollary 1. *For all $n > 4$, $ad(\hat{f})$ is in the class of $\nu' \circ \Sigma^3\partial f(2)$ in $\pi_{n+2}(S^3)$.*

Corollary 2. *For all $n > 4$, if $ad(\hat{f})$ and $ad(\hat{f}')$ are in the same class of $\pi_{n+2}(S^3)$, then $\mathcal{G}_f \sim \mathcal{G}_{f'}$.*

Note that the converse is not true. In fact a homotopy self equivalence s of $\Omega_0^{n-1}S^3$ such that $ad(\hat{f}') \sim s_*ad(\hat{f})$ would imply $\mathcal{G}_f \sim \mathcal{G}_{f'}$, even if $ad(\hat{f}) \nsim ad(\hat{f}')$. This happens for example when $n = 4$ (see [6]) or $n = 13$ (see Section 4).

Corollary 3. *For all $n > 4$, $k > 4$,*

$$\operatorname{Im} ev_{k*}(f) = \big\{\zeta \in \pi_{k-1}\big(S^3\big) \;\big|\; \nu' \circ \Sigma^3\partial f(2) \circ \Sigma^{n-1}\zeta = 0 \in \pi_{n+k-2}\big(S^3; 2\big)\big\};$$
$$\operatorname{Im} ev_{4*}(f) = \big\{m \in \mathbb{Z} \;\big|\; m\nu' \circ \Sigma^3\partial f(2) = 0 \in \pi_{n+2}\big(S^3; 2\big)\big\}.$$

3. The cases $5 \leqslant n \leqslant 12$, $n = 16, 17, 18, 23, 24, 25$

Proposition 2. *For each fixed n with $n = 7, 8, 9, 10, 11, 15, 16, 17, 18, 23, 24, 25$, the homotopy type of the gauge group of all the principal S^3-bundles over S^n is the same, and it is the one of the trivial bundle, namely $m(S^n, S^3) \sim \Omega_0^n S^3 \times S^3$.*

Proof. We use Corollary 1 to show that, for each of these values of n, $ad(\hat{f})$ is the trivial element of $\pi_{n+2}(S^3)$. Therefore, the thesis follows from Corollary 2. The notation is that of Proposition 1.

Case $n = 7$. Since $\partial f \in \pi_6(S^3) = \mathbb{Z}/12[\xi]$, we have 12 bundles $\partial f_m = m\xi$. Since

$$m\nu' \circ \Sigma^3\nu' = m\nu' \circ 2\nu_6 = 0, \tag{2}$$

by [11, (5.4)], and $\pi_9(S^3)$ that has no elements of even order, $ad(\hat{f}) = 0$, for all f_m.

Case $n = 8, 9$, and 25. $\partial f \in \pi_7(S^3) = \mathbb{Z}/2[\nu' \circ \eta_6]$, $\partial f \in \pi_8(S^3) = \mathbb{Z}/2[\nu' \circ \eta_6^2]$, and $\partial f \in \pi_{24}(S^3) = \mathbb{Z}/2[\nu' \circ \eta_6 \circ \bar{\mu}_7]$, respectively for these values of n, so we have only one non-trivial bundle. By (2), $\nu' \circ \Sigma^3\partial f = 0$ in all cases, so $ad(\hat{f}) = 0$ for all f.

Cases $n = 10$, *and* 11. Since $\partial f \in \pi_9(S^3; 2) = \pi_{10}(S^3; 2) = 0$, it is clear that $ad(\hat{f}) = 0$ for all f.

Case $n = 15$. $\partial f \in \pi_{14}(S^3) = \mathbb{Z}/2[\nu' \circ \epsilon_6] \oplus \mathbb{Z}/2[\epsilon_3 \circ \nu_{11}] \oplus \mathbb{Z}/4[\mu'] \oplus \mathbb{Z}/21[\alpha_{14}]$, so we have 336 bundles, $\partial f_{m,n,k,l} = m\nu' \circ \epsilon_6 + n\epsilon_3 \circ \nu_{11} + k\mu' + l\alpha_{14}$. Using (2), $\nu' \circ \Sigma^3(\nu' \circ \epsilon_6) = 0$. By [11, (7.14)], and since $\nu' \circ \zeta_6 \in \pi_{17}(S^3; 2) = \mathbb{Z}/2$,

$$\nu' \circ \Sigma^3 \mu' = \nu' \circ 2\zeta_6 = 2\nu' \circ \zeta_6 = 0,$$

and since $\nu' \circ \nu_6 \in \pi_9(S^3; 2) = 0$, we have by [11, Proposition 3.1],

$$\nu' \circ \Sigma^3(\epsilon_3 \circ \nu_4) = \nu' \circ \epsilon_6 \circ \nu_7 = \nu' \circ \nu_6 \circ \epsilon_9 = 0.$$

Therefore, $ad(\hat{f}_{m,n,k,l}) = 0$.

Case $n = 16$, *and* 17. ∂f belongs to $\pi_{15}(S^3) = \mathbb{Z}/2[\nu' \circ \mu_6] \oplus \mathbb{Z}/2[\nu' \circ \eta_6 \circ \epsilon_7]$, and ∂f belongs to $\pi_{16}(S^3) = \mathbb{Z}/2[\nu' \circ \eta_6 \circ \mu_7] \oplus \mathbb{Z}/3[\alpha_{16}]$, respectively for these two values of n, so we have 4 and 6 bundles. If ζ is a generator in the even component, we see that $\nu' \circ \Sigma^3 \zeta = 0$ by (2).

Case $n = 18$. $\partial f \in \pi_{17}(S^3) = \mathbb{Z}/2[\epsilon_3 \circ \nu_{11}^2] \oplus \mathbb{Z}/15[\alpha_{17}]$, so we have 30 bundles, $f_{m,n} = m\epsilon_3 \circ \nu_{11}^2 + n\alpha_{17}$. By [11, (7.12)], and (2),

$$\nu' \circ \Sigma^3\left(\epsilon_3 \circ \nu_{11}^2\right) = \nu' \circ \Sigma^3(\nu' \circ \bar{\nu}_6 \circ \nu_{14}) = 0,$$

and hence $ad(\hat{f}_{m,n}) = 0$.

Case $n = 23$. $\partial f \in \pi_{22}(S^3) = \mathbb{Z}/2[x = \nu' \circ \mu_6 \circ \sigma_{15}] \oplus \mathbb{Z}/4[\bar{\mu}'] \oplus \mathbb{Z}/33[\alpha_{22}]$, so we have 264 bundles, $\partial f_{m,n,k} = mx + n\bar{\mu}' + k\alpha_{22}$. By (2), $\nu' \circ \Sigma x = 0$. Next, by [11, Lemma 12.4], and since $\nu' \circ \bar{\zeta}_6 \in \pi_{25}(S^3; 2) = \mathbb{Z}/2$,

$$\nu' \circ \Sigma^3 \bar{\mu}' = \nu' \circ 2\bar{\zeta}_6 = 2\nu' \circ \bar{\zeta}_6 = 0;$$

thus, $ad(\hat{f}_{m,n,k}) = 0$.

Case $n = 24$. $\partial f \in \pi_{23}(S^3) = \mathbb{Z}/2[x = \nu' \circ \bar{\mu}_6] \oplus \mathbb{Z}/2[y = \nu' \circ \eta_6 \circ \mu_7 \circ \sigma_{16}]$, so we have 4 bundles, $\partial f_{m,n} = mx + ny$. By (2) both $\nu' \circ \Sigma^3 x = \nu' \circ \Sigma^3 y = 0$, therefore $ad(\hat{f}_{m,n}) = 0$. □

Proposition 3. *For $n = 5, 6$ and 12, the two principal S^3-bundles over S^n have gauge groups with different homotopy type.*

Proof. For all these values of n, we have only one non-trivial bundle f. Using Corollary 1, we show that $ad(\hat{f}) \neq 0$. This is not enough, but using the exact sequence (1) and Corollary 3, we compute $\pi_2(\mathcal{G}_f) = \pi_{n+2}(S^3)/\operatorname{Im}\partial_4$, $\operatorname{Im}\partial_4 = \mathbb{Z}/\operatorname{Im} ev_{4*}$, and we show that $\pi_2(\mathcal{G}_f) \neq \pi_{n+2}(S^3)$.

Case $n = 5$. Since $\partial f \in \pi_4(S^3) = \mathbb{Z}/2[\eta_3]$, $\partial f = \eta_3$. Since $\nu' \circ \Sigma^3 \eta_3 = \nu' \circ \eta_6$ is the generator of $\pi_7(S^3) = \mathbb{Z}/2[\nu' \circ \eta_6]$, then $ad(\hat{f})$ is not trivial; hence, $\operatorname{Im} ev_{4*}(f) = 2\mathbb{Z}$, and $\pi_2(\mathcal{G}_f) = 0 \neq \pi_7(S^3)$.

Case $n = 6$. $\partial f = \eta_3^2 \in \pi_5(S^3) = \mathbb{Z}/2[\eta_3^2]$, and $\nu' \circ \Sigma^3 \eta_3^2 = \nu' \circ \eta_6 \circ \eta_7$ is the generator of $\pi_8(S^3) = \mathbb{Z}/2[\nu' \circ \eta_6^2]$. Then, $\operatorname{Im} ev_{4*}(f) = 2\mathbb{Z}$, and $\pi_2(\mathcal{G}_f) = 0 \neq \pi_8(S^3)$.

Case $n = 12$. $\partial f = \epsilon_3 \in \pi_{11}(S^3) = \mathbb{Z}/2[\epsilon_3]$, and $\nu' \circ \Sigma^3 \epsilon_3$, is one of the generators of $\pi_{14}(S^3; 2) = \mathbb{Z}/4[\mu'] \oplus \mathbb{Z}/2[\epsilon_3 \circ \nu_{11}] \oplus \mathbb{Z}/2[\nu' \circ \epsilon_6]$, $\operatorname{Im} ev_{4*}(f) = 2\mathbb{Z}$, and $\pi_2(\mathcal{G}_f)$ is $\pi_{14}(S^3)/(\mathbb{Z}/2) \neq \pi_{14}(S^3)$. □

4. The cases $n = 14, 19, 20, 22$

The proof is the same in the four cases, so we give it in details for $n = 14$, and we just sketch the other ones.

Proposition 4. *On S^{14} there are 24 principal S^3-bundles, classified by the elements $\partial f_{m,n,k}$ of $\pi_{13}(S^3) = \mathbb{Z}/2[\eta_3 \circ \mu_4] \oplus \mathbb{Z}/4[\epsilon'] \oplus \mathbb{Z}/3[\alpha_{13}]$. The gauge groups of $\partial f_{m,n,k} = m\eta_3 \circ \mu_4 + n\epsilon' + k\alpha_{13}$ and $\partial f'_{m',n',k'} = m'\eta_3 \circ \mu_4 + n'\epsilon' + k'\alpha_{13}$ have the same type if and only if $m = m' \bmod 2$ $(m, n, k, m', n', k' \in \mathbb{Z})$.*

Proof. First, we compute $ad(\hat{f}_{m,n,k})$, with $\partial f_{m,n,k} = m\eta_3 \circ \mu_4 + n\epsilon' + k\alpha_{13}$, using Corollary 1. By [11, (7.10)], and (2),

$$\nu' \circ \Sigma^3 \epsilon' = \nu' \circ 2(\nu_6 \circ \sigma_9) = 0,$$

while by [11, Theorem 7.7],

$$\nu' \circ \Sigma^3(\eta_3 \circ \mu_4) = \nu' \circ \eta_6 \circ \mu_7,$$

and this is the generator of the 2 component of $\pi_{16}(S^3)$. By Corollary 2, we have at most two types for $\mathcal{G}_f$. Next, in order to show that they are not the same type we compute $\pi_2(\mathcal{G}_f)$. Using Corollary 3, $\operatorname{Im} ev_{4*} = \mathbb{Z}$, for all the bundles $f_{0,n,k}$, and $\operatorname{Im} ev_{4*} = 2\mathbb{Z}$, for all the bundles $f_{1,n,k}$. Therefore, $\pi_2(\mathcal{G}_{f_{0,n,k}}) = \pi_{16}(S^3) = \pi_2(\mathcal{G}_{c_0})$, while $\pi_2(\mathcal{G}_{f_{1,n,k}}) = \pi_{16}(S^3)/\mathbb{Z}/2 = \mathbb{Z}/3$, and this completes the proof. □

Remark 1. It is interesting to find out the possible homotopy types of the fibres of the different maps $h : S^3 \to \Omega_0^{n-1} S^3$. We have the exact sequence

$$\xrightarrow[\partial_{k+1}]{} \pi_k(F_h) \xrightarrow[i_k]{} \pi_k(S^3) \xrightarrow[h_{k*}]{} \pi_{k+n-1}(S^3) \xrightarrow[\partial_k]{}$$

where $h_{k*}(\zeta) = ad(h) \circ \Sigma^{n-1}\zeta$. Taking $k = 2$, we get $\pi_2(F_h) = \pi_{n+2}(S^3)/\operatorname{Im} h_{3*}$, and $h_{3*} \in Hom(\mathbb{Z}, \pi_{n+2}(S^3))$. With $n = 14$,

$$ad(h) \in \pi_{16}\big(S^3\big) = \mathbb{Z}/2[a = \nu' \circ \eta_6 \circ \mu_7] \oplus \mathbb{Z}/3[\beta],$$

and we compute $\pi_2(F_h) = \pi_{16}(S^3)/\operatorname{Im} h_{3*}$. Since h_{3*} sends $m\iota_3$ to $mad(h)$, we obtain

$ad(h)$	$\operatorname{Im} h_{3*}$
0	0
a	$\mathbb{Z}/2[a]$
β	$\mathbb{Z}/3[\beta]$
$2\beta = -\beta$	$\mathbb{Z}/3[\beta]$
$a + \beta$	$\mathbb{Z}/6$
$a + 2\beta = a - \beta$	$\mathbb{Z}/6.$

Thus, there are at most 6 types. Actually, the two fibres $F_{\pm\beta}$ and $F_{a\pm\beta}$ also have the same type, respectively. For we can use the homotopy equivalence determined on $\Omega^{13}(S^3)$ by the loop inverse, and observe that this can be identified with the inversion of the prime 3. So there are exactly 4 different types.

Proposition 5. *On S^{19} there are 30 principal S^3-bundles, classified by the elements $\partial f_{m,n}$ of $\pi_{18}(S^3) = \mathbb{Z}/2[\bar\epsilon_3] \oplus \mathbb{Z}/15[\alpha_{18}]$. The gauge groups of $\partial f_{m,n} = m\bar\epsilon_3 + n\alpha_{18}$ and $\partial f'_{m',n'} = m'\bar\epsilon_3 + n'\alpha_{18}$ have the same type if and only if $m = m' \bmod 2$ $(m, n, m', n' \in \mathbb{Z})$.*

Proof. We only need to compute $\nu' \circ \Sigma^3\bar\epsilon_3 = \nu' \circ \bar\epsilon_6$, and this the generator of the 2 component of $\pi_{21}(S^3)$ by [11, Proposition 12.8]. □

Proposition 6. *On S^{20} there are 12 principal S^3-bundles, classified by the elements $\partial f_{m,n,k}$ of $\pi_{19}(S^3) = \mathbb{Z}/2[x = \eta_3 \circ \bar\epsilon_4] \oplus \mathbb{Z}/2[y = \mu_3 \circ \sigma_{12}] \oplus \mathbb{Z}/3[\alpha_{19}]$. The gauge groups of $f_{m,n,k} = mx + ny + k\alpha_{19}$ and $f'_{m',n',k'} = m'x + n'y + k'\alpha_{19}$ have the same type if and only if $n = n' \bmod 2$ $(m, n, k, m', n', k' \in \mathbb{Z})$.*

Proof. We have that

$$\nu' \circ \Sigma^3(\eta_3 \circ \bar\epsilon_4) = \nu' \circ \eta_6 \circ \bar\epsilon_7 = \nu' \circ \nu_6 \circ \sigma_9 \circ \nu_{16}^2 = 0,$$

by [11, Lemma 12.10], and since $\nu' \circ \nu_6 \in \pi_9(S^3; 2) = 0$, while

$$\nu' \circ \Sigma^3(\mu_3 \circ \sigma_{12}) = \nu' \circ \mu_6 \circ \sigma_{15},$$

that is one of the generators of the 2 component of $\pi_{22}(S^3)$. □

Proposition 7. *On S^{22} there are 48 principal S^3-bundles, classified by the elements $\partial f_{m,n,l,k}$ of $\pi_{21}(S^3) = \mathbb{Z}/2[x] \oplus \mathbb{Z}/2[y] \oplus \mathbb{Z}/4[z] \oplus \mathbb{Z}/3[\alpha_{21}]$, where $x = \nu' \circ \bar\epsilon_6$, $y = \eta_3 \circ \bar\mu_4$, and $z = \mu' \circ \sigma_{14}$. The gauge groups of $\partial f_{m,n,l,k} = mx + ny + lz + k\alpha_{21}$ and $\partial f'_{m',n',l',k'} = m'x + n'y + l'z + k'\alpha_{21}$ have the same type if and only if $n = n' \bmod 2$ $(m, n, l, k, m', n', l', k' \in \mathbb{Z})$.*

Proof. By (2), $\nu' \circ \Sigma^3(\nu' \circ \bar\epsilon_6) = 0$. By [11, (7.14)], since $\nu' \circ \zeta_6 \circ \sigma_{17}$ belongs to $\pi_{24}(S^3; 2) = \mathbb{Z}/2$, we have that

$$\nu' \circ \Sigma^3(\mu' \circ \sigma_{14}) = \nu' \circ 2\zeta_6 \circ \sigma_{17} = 2\nu' \circ \zeta_6 \circ \sigma_{17} = 0,$$

while

$$\nu' \circ \Sigma^3(\eta_3 \circ \bar\mu_4) = \nu' \circ \eta_6 \circ \bar\mu_7$$

is a generator of the 2 component of $\pi_{24}(S^3)$. □

5. The cases $n = 13$ and 21

Proposition 8. *On S^{13} there are four principal S^3-bundles. The non-trivial ones are classified by the following elements of $\pi_{12}(S^3)$: μ_3, $\eta_3 \circ \epsilon_4$ and $\mu_3 + \eta_3 \circ \epsilon_4$. The gauge groups of these bundles have three different homotopy types as follows: $\mathcal{G}_{\mu_3} \sim \mathcal{G}_{\mu_3+\eta_3\circ\epsilon_4}$, $\mathcal{G}_{\mu_3} \nsim \mathcal{G}_0$, $\mathcal{G}_{\eta_3\circ\epsilon_4} \nsim \mathcal{G}_0$, $\mathcal{G}_{\mu_3} \nsim \mathcal{G}_{\eta_3\circ\epsilon_4}$, where $\mathcal{G}_0$ denotes the gauge group of the trivial bundle.*

Proof. Following Corollary 1, we compute $ad(\hat{f}_{m,l})$, where $\partial f_{m,l} = m\mu_3 + l\eta_3 \circ \epsilon_4$ belongs to $\pi_{12}(S^3) = \mathbb{Z}/2[\mu_3, \eta_3 \circ \epsilon_4]$. By [11, Theorem 7.6],

$$\nu' \circ \Sigma^3 \mu_3 = \nu' \circ \mu_6$$

and

$$\nu' \circ \Sigma^3(\eta_3 \circ \epsilon_4) = \nu' \circ \eta_6 \circ \epsilon_7$$

are the two generators of $\pi_{15}(S^3; 2) = \mathbb{Z}/2[\nu' \circ \mu_6, \nu' \circ \eta_6 \circ \epsilon_7]$. Since, by Corollary 3, $\operatorname{Im} ev_{4*} = 2\mathbb{Z}$, for all $f_{m,l} \neq f_{0,0}$ it follows that $\pi_2(\mathcal{G}_{f_{(m,l)\neq(0,0)}}) = \mathbb{Z}/2$ while $\pi_2(\mathcal{G}_{f_{(0,0)}}) = \pi_{15}(S^3) \neq \mathbb{Z}/2$. This shows that the non-trivial bundles have gauge group of different type from that of the trivial one.

At this point it is worth observing that a technique like the one followed in Remark 1 or in [6], would not help in this case, since everything reduces modulo 2. Therefore, we try to compute higher homotopy groups, in order to distinguish the types of the non-trivial bundles. Take the sequence (1) with $k = 4$ and $n = 13$,

$$\xrightarrow{ev_{5*}} \mathbb{Z}/2 \xrightarrow{\partial_5} \mathbb{Z}/6 \xrightarrow{\phi} \pi_3(\mathcal{G}_f) \xrightarrow{ev_{4*}} \mathbb{Z} \xrightarrow{\partial_4} \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

We compute $\operatorname{Im} ev_{5*}$ for the various f. By Proposition 1 (with $k = 5$), we need to solve the equation

$$\nu' \circ \Sigma^3 \partial f(2) \circ \Sigma^{12} \zeta(2) = 0$$

in $\pi_{16}(S^3; 2) = \mathbb{Z}/2[\nu' \circ \eta_6 \circ \mu_7]$, where $\zeta \in \pi_4(S^3) = \mathbb{Z}/2[\eta_3]$. When $f = f_{1,0}$, by [11, Proposition 3.1],

$$\nu' \circ \Sigma^3(\mu_3) \circ \eta_{15} = \nu' \circ \mu_6 \circ \eta_{15} = \nu' \circ \eta_6 \circ \mu_7,$$

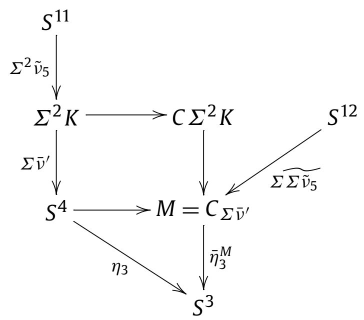
that is precisely the generator. When $f = f_{0,1}$, by [11, (7.5) and (7.10)],

$$\nu' \circ \Sigma^3(\eta_3 \circ \epsilon_4) \circ \eta_{15} = \nu' \circ \eta_6 \circ \epsilon_7 \circ \eta_{15} = \nu' \circ \eta_6 \circ \eta_7 \circ \epsilon_8 = \nu' \circ 4(\nu_6 \circ \sigma_9) = 4\nu' \circ \nu_6 \circ \sigma_9 = 0.$$

Therefore, $\operatorname{Im} ev_{5*} = 0$ for $f_{0,1}$, while $\operatorname{Im} ev_{5*} = \pi_4(S^3; 2) = \mathbb{Z}/2$ for $f_{1,l}$. This gives $\pi_3(\mathcal{G}_{f_{1,0}}) = \pi_3(\mathcal{G}_{f_{1,1}}) = \mathbb{Z} \oplus \mathbb{Z}/3$, while $\pi_3(\mathcal{G}_{f_{0,1}}) = \mathbb{Z} \oplus \mathbb{Z}/6$.

We still have ambiguity for $\mathcal{G}_{f_{1,0}}$ and $\mathcal{G}_{f_{1,1}}$. We computed the homotopy groups π_k of these two spaces using the approach above and results of [11] and [7], for $4 \leqslant k \leqslant 9$. We find that $\pi_k(\mathcal{G}_{f_{1,0}}) = \pi_k(\mathcal{G}_{f_{1,1}})$, for all these k. This suggests that the two spaces have the same type. We prove that this is the case, constructing a homotopy self equivalence s of $\Omega^{12} S^3$, such that $f_{1,1} \sim s f_{1,0}$. This will be done in 4 steps.

Step 1. We recall the construction of the elements μ_3 and ϵ_3. The element ϵ_3 is the unique element of the secondary composition $\{\epsilon_3\} = \{\eta_3, \Sigma\nu', \Sigma\nu_6\}_1 \subseteq \pi_{11}(S^3; 2)$, that is well defined since $\eta_3 \circ \Sigma\nu' = \nu' \circ \nu_6 = 0$ [11, Lemma 5.7 and Proposition 5.11]. Thus, ϵ_3 is the composition $\bar{\eta}_3^L \circ \Sigma\tilde{\nu}_6$, of an extension of η_3 and the suspension of a coextension of ν_6, and where $L = C_{\Sigma\nu'}$. Next, consider the following diagram



where $K = C_{8\iota_5}$. Since $\nu' \circ 8\iota_6 = 8\nu' = 0$, we have an extension $\bar{\nu}'$ of $\nu' : S^6 \to S^3$. Since $\nu' \circ \nu_6 = 0$, we have a coextension $\tilde{\nu}_6$ of $\nu_6 : S^9 \to S^6$. We also have a coextension $\tilde{\nu}_5$ of ν_5, and since $\eta_3 \circ \Sigma\nu' = \bar{\nu}' \circ \Sigma\tilde{\nu}_5 = 0$ [11, p. 56], the secondary composition $\{\eta_3, \Sigma\bar{\nu}', \Sigma^2\tilde{\nu}_5\}_1 \subseteq \pi_{12}(S^3; 2)$, is well defined. We have that μ_3 belongs to $\{\eta_3, \Sigma\bar{\nu}', \Sigma^2\tilde{\nu}_5\}_1 \subseteq \pi_{12}(S^3; 2)$, and therefore it is a composition $\mu_3 = \bar{\eta}_3^M \circ \Sigma\tilde{\gamma}_1$, where we denote by $\Sigma\tilde{\gamma}_1 = \Sigma\widetilde{\Sigma\tilde{\nu}_5}$. Observe that L is a subcomplex of M, and denote the inclusion by $j : L \to M$.

Step 2. Consider the two maps $\hat{f}_{1,l} : S^3 \to \Omega^{12} S^3$. Using the explicit description given in Corollary 1 for $ad(\hat{f})$, and the description of the classes involved given in the previous step, it is possible to see that the maps $\hat{f}_{1,l}$ factors (up to homotopy) through the space $\Omega^{12} M$. More precisely,

$$\hat{f}_{1,0} : x \mapsto \nu' \circ (\iota_3 \wedge \bar{\eta}_3^M) \circ (\iota_3 \wedge \Sigma\tilde{\gamma}_1)(x \wedge \_),$$
$$\hat{f}_{1,1} : x \mapsto \nu' \circ (\iota_3 \wedge \bar{\eta}_3^M) \circ (\iota_3 \wedge (\Sigma\tilde{\gamma}_1 + \Sigma\tilde{\gamma}_2))(x \wedge \_),$$

where $\Sigma\tilde{\gamma}_2 = j_* \Sigma\tilde{\nu}_6 \circ \eta_{11}$ in $\Omega^{12} M$. Therefore, if s is a homotopy self equivalence of M, left composition with $\bar{\eta}_3^M$ gives a homotopy self equivalence of $\Omega^{12} S^3$, that we denote with the same symbol s, and sends $\bar{\eta}_3^M \circ u$ to $s_*(\bar{\eta}_3^M \circ u) = \bar{\eta}_3^M \circ s_* u$.

Step 3. We construct a homotopy self equivalence of M. We do it using results of [8]. To fix notation, recall the construction of the space M in the following diagram

$$\begin{array}{ccccccc}
\Sigma^2 K & \longrightarrow & C\Sigma^2 K & & & & \\
\downarrow{\scriptstyle \Sigma\bar{\nu}'} & & \downarrow & & & & \\
S^4 & \xrightarrow{i} & M = S^4 \cup_{\Sigma\bar{\nu}'} C\Sigma^2 K & \xrightarrow{\kappa} & \Sigma^3 K & \xrightarrow{\Sigma^2\bar{\nu}'} & S^5
\end{array}$$

Consider the following portions of the exact sequences associated to $C_{\Sigma\bar{\nu}'}$

$$\begin{array}{ccccccccccc}
\longrightarrow & \pi_5(M) & \xrightarrow{(\Sigma^2\bar{\nu}')^*} & [\Sigma^3 K, M]_0 & \xrightarrow{\kappa^*} & [M, M]_0 & \xrightarrow{i^*} & \pi_4(M) & \xrightarrow{(\Sigma\bar{\nu}')^*} & \\
& \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \kappa_*} & & \\
\longrightarrow & \pi_5(\Sigma^3 K) & \xrightarrow{(\Sigma^2\bar{\nu}')^*} & [\Sigma^3 K, \Sigma^3 K]_0 & \xrightarrow{\kappa^*} & [M, \Sigma^3 K]_0 & \xrightarrow{i^*} & \pi_4(\Sigma^3 K) & \xrightarrow{(\Sigma\bar{\nu}')^*} &
\end{array}$$

Explicit calculations and the fact that the attaching map is a suspension give

$$\begin{array}{ccccccccccc}
\longrightarrow & \mathbb{Z}/2 & \xrightarrow{(\Sigma^2\bar{\nu}')^*} & [\Sigma^3 K, M]_0 & \xrightarrow{\kappa^*} & [M, M]_0 & \xrightarrow{i^*} & \mathbb{Z} & \longrightarrow & 0 \\
& \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \kappa_*} & & \\
\longrightarrow & 0 & \longrightarrow & [\Sigma^3 K, \Sigma^3 K]_0 & \xrightarrow{=} & [M, \Sigma^3 K]_0 & \longrightarrow & 0 & &
\end{array}$$

By [8], we have homomorphisms π and λ that make the following square commutes

$$\begin{array}{ccc}
[\Sigma^3 K, M]_0 & \xrightarrow{\lambda} & [M, M]_0 \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \kappa_*} \\
[\Sigma^3 K, \Sigma^3 K]_0 & \xrightarrow{=} & [M, \Sigma^3 K]_0
\end{array}$$

and where $\pi(x) = 1 + k_*(x)$, and $\lambda(x) = 1^x = \nabla(1 \vee x)\theta$ is the map defined by the Hilton coaction. We know from [8, (1.8)], that $\lambda\pi^{-1}(1)$ is a subgroup of the group of self equivalences of M. We construct a map $x : \Sigma^3 K \to M$, such that $\pi(x) = 1$.

Since $\eta_4 \circ \nu_5 \circ 8\iota_8 = 0$, we have extensions $\overline{\eta_4 \circ \nu_5} : C_{8\iota_8} = \Sigma^3 K \to S^5$ of $\eta_4 \circ \nu_5$. Let $x = i_*(\overline{\eta_4 \circ \nu_5})$. Since $k_*(x) = 0$, $\pi(x) = 1$, and therefore $s = \lambda(x)$ is a homotopy self equivalence of M.

Step 4. We show that $s_*(\mu_3) = \mu_3 + \epsilon_3 \circ \eta_4$. From Step 2, $s_*(\mu_3) = \bar{\eta}_3^M \circ s_*(\Sigma\tilde{\gamma}_1)$, where $\tilde{\gamma}_1 = \widetilde{\Sigma\tilde{\nu}_5}$. By the definition of s in Step 3, $s_*(\Sigma\tilde{\gamma}_1) = \nabla(1 \vee x)\theta\Sigma\tilde{\gamma}_1$. It is easy to see, from the definition of coextension, that $\theta\Sigma\tilde{\gamma}_1 = \Sigma\tilde{\gamma}_1 + \Sigma^3\tilde{\nu}_5$. Therefore

$$s_*(\Sigma\tilde{\gamma}_1) = \nabla(1 \vee x)(\Sigma\tilde{\gamma}_1 \vee \Sigma^3\tilde{\nu}_5)\nu = \nabla(\Sigma\tilde{\gamma}_1 \vee x \circ \Sigma^3\tilde{\nu}_5)\nu = \Sigma\tilde{\gamma}_1 + x \circ \Sigma^3\tilde{\nu}_5,$$

in $\pi_{12}(M)$. This means that

$$s_*(\mu_3) = \bar{\eta}_3^M \circ (\Sigma\tilde{\gamma}_1 + x \circ \Sigma^3\tilde{\nu}_5) = \mu_3 + \bar{\eta}_3^M \circ x \circ \Sigma^3\tilde{\nu}_5.$$

We show that $\bar{\eta}_3^M \circ x \circ \Sigma^3\tilde{\nu}_5 = \eta_3 \circ \epsilon_4$. Recalling the definition of x,

$$\bar{\eta}_3^M \circ x \circ \Sigma^3\tilde{\nu}_5 = \bar{\eta}_3^M \circ i_*(\overline{\eta_4 \circ \nu_5}) \circ \Sigma^3\tilde{\nu}_5 = \eta_3 \circ \overline{\eta_4 \circ \nu_5} \circ \Sigma^3\tilde{\nu}_5.$$

We show that $\overline{\eta_4 \circ \nu_5} \circ \Sigma^3 \tilde{\nu}_5 = \epsilon_4$. By definition and [11, Proposition 1.3],

$$\Sigma\epsilon_3 = \Sigma\{\eta_3, \Sigma\nu', \nu_7\}_1 \subseteq -\{\eta_4, \Sigma^2\nu', \nu_8\}_2.$$

Now, $\{\eta_4, \Sigma^2\nu', \nu_8\}_2$ is a coset of $\pi_9(S^4) \circ \nu_9 + \eta_4 \circ \Sigma^2\pi_{10}(S^3)$ in $\pi_{12}(S^4)$. Direct calculation shows that $\pi_9(S^4) \circ \nu_9 + \eta_4 \circ \Sigma^2\pi_{10}(S^3)$ is 2-trivial, and therefore the 2 component of this secondary composition is a single element. So

$$\Sigma\epsilon_3 = -\{\eta_4, \Sigma^2\nu', \nu_8\}_2 = -\{\eta_4, 2\nu_5, \nu_8\}_2,$$

by [11, Proposition 1.3]. Next, by [11, Proposition 1.2(iii) and (iv)]

$$\{\eta_4, 2\nu_5, \nu_8\}_2 \supseteq \{\eta_4 \circ \nu_5, 2\iota_8, \nu_8\}_2 \supseteq \{\eta_4 \circ \nu_5, 2\iota_8, \nu_8\}_3,$$

and by definition $\{\eta_4 \circ \nu_5, 2\iota_8, \nu_8\}_3$ is the set of the compositions $\overline{\eta_4 \circ \nu_5} \circ \Sigma^3 \tilde{\nu}_5$. Since this set contains a single element, it follows that $\overline{\eta_4 \circ \nu_5} \circ \Sigma^3 \tilde{\nu}_5 = \epsilon_4$. □

Remark 2. It is easy to see that the problem of proving that the gauge groups $\mathcal{G}_{f_{1,0}}$ and $\mathcal{G}_{f_{1,1}}$ have all isomorphic homotopy groups reduces to show that for all $m \geqslant 4$, $\nu' \circ \eta_6 \circ \epsilon_7 \circ \Sigma\zeta(2) = 0 \in \pi_{m+12}(S^3)$, for all $\zeta \in \pi_m(S^3)$. We were not able to prove this fact.

We conclude with the case $n = 21$. We are not able to solve this case, but we can state a conjecture. On S^{21} we have 48 principal S^3-bundles, classified by the elements $\partial f_{m,n,k,l}$ of $\pi_{20}(S^3) = \mathbb{Z}/2[\bar{\mu}_3] \oplus \mathbb{Z}/2[\eta_3 \circ \mu_4 \circ \sigma_{13}] \oplus \mathbb{Z}/4[\bar{\epsilon}'] \oplus \mathbb{Z}/3[\alpha_{20}]$.

First, we compute $ad(\hat{f}_{m,n,k,l})$ using Corollary 1. Since $\nu' \circ \nu_6 \in \pi_9(S^3; 2) = 0$, using [11, Lemma 12.3 and (5.5)], we obtain

$$\nu' \circ \Sigma^3(\bar{\epsilon}') = \nu' \circ \Sigma^2(\Sigma\nu' \circ \kappa_7) = \nu' \circ 2\nu_6 \circ \kappa_9 = 0,$$

while

$$\nu' \circ \Sigma^3(\bar{\mu}_3) = \nu' \circ \bar{\mu}_6,$$

and

$$\nu' \circ \Sigma^3(\eta_3 \circ \mu_4 \circ \sigma_{13}) = \nu' \circ \eta_6 \circ \mu_7 \circ \sigma_{16},$$

and these are generators of the 2 component of $\pi_{23}(S^3)$. Using Corollary 3, we calculate $\operatorname{Im} ev_{4*} = 2\mathbb{Z}$, for all the bundles $f_{1,n,k,l}$, $f_{m,1,k,l}$, $f_{1,1,k,l}$ and $\operatorname{Im} ev_{4*} = \mathbb{Z}$, for all the bundles $f_{0,0,k,l}$. Therefore, $\pi_2(\mathcal{G}_{f_{0,0,k,l}}) = \pi_{23}(S^3) = \pi_2(\mathcal{G}_{c_0})$, while $\pi_2(\mathcal{G}_{f_{1,n,k,l}}) = \pi_2(\mathcal{G}_{f_{m,1,k,l}}) = \pi_2(\mathcal{G}_{f_{1,1,k,l}}) = \pi_{23}(S^3)/\mathbb{Z}/2$.

Second, in order to distinguish the type of $\mathcal{G}_{f_{m,n,k,l}}$ with $(m,n) \neq (0,0)$, we compute π_3. From sequence (1) with $k = 4$ and $n = 21$, we obtain

$$\xrightarrow[ev_{5*}]{} \mathbb{Z}/2 \xrightarrow[\partial_5]{} \mathbb{Z}/2 \xrightarrow[\phi]{} \pi_3(\mathcal{G}_f) \xrightarrow[ev_{4*}]{} \mathbb{Z} \xrightarrow[\partial_4]{} \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

We compute $\operatorname{Im} ev_{5*}$ for the various f. By Proposition 1 (with $k = 5$), we need to solve the equation

$$\nu' \circ \Sigma^3 \partial\beta_f(2) \circ \Sigma^{20}\zeta(2) = 0,$$

in $\pi_{24}(S^3; 2) = \mathbb{Z}/2[\nu' \circ \eta_6 \circ \bar{\mu}_7]$, where $\zeta(2) \in \pi_4(S^3; 2) = \mathbb{Z}/2[\eta_3]$. Since $\nu' \circ \nu_6$ belongs to $\pi_9(S^3; 2) = 0$, using [11, Lemma 12.3], and (5.5), we obtain that

$$\nu' \circ \Sigma^3\bar{\epsilon}' = \nu' \circ \Sigma^2(\Sigma\nu' \circ \kappa_7) = \nu' \circ 2\nu_6 \circ \kappa_9 = 0,$$

so $\nu' \circ \Sigma^3\bar{\epsilon}' \circ \eta_{23} = 0$. By [11, Proposition 3.1],

$$\nu' \circ \bar{\mu}_6 \circ \eta_{23} = \nu' \circ \eta_6 \circ \bar{\mu}_7,$$

that is the generator. By [11, Proposition 3.1, (7.7), and (7.14)],

$$\begin{aligned}\nu' \circ \eta_6 \circ \mu_7 \circ \sigma_{16} \circ \eta_{23} &= \nu' \circ \eta_6 \circ \mu_7 \circ \eta_{16} \circ \sigma_{17} = \nu' \circ \eta_6 \circ \eta_7 \circ \mu_8 \circ \sigma_{17} \\ &= \nu' \circ \Sigma^3(2\mu') \circ \sigma_{17} = \nu' \circ 2\Sigma(2\zeta_5) \circ \sigma_{17} = 4\nu' \circ \zeta_6 \circ \sigma_{17} = 0.\end{aligned}$$

Therefore, $\operatorname{Im} ev_{5*} = 0$ for $f_{1,0,k,l}$ and $f_{1,1,k,l}$, while $\operatorname{Im} ev_{5*} = \mathbb{Z}/2$ for $f_{0,1,k,l}$. This gives $\pi_3(\mathcal{G}_{f_{(1,1,k,l)}}) = \pi_3(\mathcal{G}_{f_{(1,0,k,l)}}) = \mathbb{Z}$, and $\pi_3(\mathcal{G}_{f_{(0,1,k,l)}}) = \mathbb{Z} \oplus \mathbb{Z}/2$.

We have the same problem that we had in the case of $n = 13$. Therefore, an explicit analysis, similar to the one performed in the proof of Proposition 8, is necessary. Unfortunately, we are not able to find all the necessary information about the generators of the groups involved, but we formulate the following conjecture.

Conjecture 1. *On S^{21} there are 48 principal S^3-bundles, classified by the elements $\partial f_{m,n,l,k} = m\bar{\mu}_3 + n\eta_3 \circ \mu_4 \circ \sigma_{13} + l\bar{\epsilon}' + k\alpha_{20}$ of $\pi_{20}(S^3) = \mathbb{Z}/2[\bar{\mu}_3, \eta_3 \circ \mu_4 \circ \sigma_{13}] \oplus \mathbb{Z}/4[\bar{\epsilon}'] \oplus \mathbb{Z}/3[\alpha_{20}]$. The gauge groups are of three homotopy types: the one of $f_{0,0,l,k}$, the one of $f_{1,n,l,k}$, and the one of $f_{0,1,l,k}$ $(m,n,l,k \in \mathbb{Z})$.*

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