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On the Barnes double zeta and Gamma functions

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ABSTRACT

We present a complete description of the analytic properties of the Barnes double zeta and Gamma functions.

Contents

1. Introduction

Let a and b be real positive numbers and x a real number such that $am+bn+x>0$ for all natural numbers n and m. Let s be a complex number. For $\mathrm{Re}(s)>2$, the Barnes double zeta function is defined by the double series [3,8]

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$$\zeta_2(s; a, b, x) = \sum_{m,n=0} (am + bn + x)^{-s},$$

while the Barnes double Gamma function is defined as

$$\log \Gamma_2(x; a, b) = \zeta_2'(0; a, b, x) + \log \rho_2(a, b),$$

where

$$\log \rho_2(a, b) = -\lim_{x \to 0}\big(\zeta_2'(0; a, b, x) + \log x\big).$$

Note that the range for the parameters has been chosen for convenience. Larger ranges could be chosen (for example, a, b and x to be real numbers such that $am+bn+x > 0$ for all natural numbers n and m). Furthermore, all the parameters can be analytically extended to suitable complex domains (cpr. Sections 7 and 9).

The double zeta function was introduced and studied by Barnes [3] in order to study the double Gamma function [3–5]. Nowadays, double zeta and Gamma functions are quite important in number theory in particular by the works of Shintani [20–22] and Zagier [33,34], and different properties of these functions have been investigated by various authors, and are strictly related, by methods and applications, to many works appeared in the recent literature, where different analytic properties of various type of zeta functions are investigated (see for example [28] for a list of reference). In particular, the asymptotic expansion of double and multiple Gamma functions as functions of x in the case when $a = b = 1$, has been investigated by Shuster [23] with applications to the study of topological zeros of the Selberg zeta function on forms for compact hyperbolic space forms, and a first attempt to the study of the case with a non-trivial parameter (namely $a \neq 1$) has been done by Actor [1], where however the formula for the derivative of the double zeta function at zero seems to have some problems. More recently, in a series of works, Matsumoto has studied the asymptotic series for both the double zeta and Gamma function, as functions of one of the parameters a or b, with applications to asymptotic series of Hecke L-functions of real quadratic fields, while a formula for $\zeta_2'(0; a, 1, a)$ has been given in [28] as a particular case of a more general Kronecker limit formula for double quadratic zeta series. Eventually, note that multiple and in particular double zeta and Gamma functions appear very frequently in mathematical physics in zeta regularization methods (see works of Sarnak [19], Vardi [30] and Voros [31] or more recently [17] or [9] for a list of formulas and applications in theoretical physics).

Motivated by these works, we present here a complete investigation of the main analytic properties of these functions. A few comments on our results. About the zeta function, we have three remarks. First, applying standard heat kernels methods [10] we can obtain all information on poles, residues and values at non-positive integers. Second, using classical techniques (in particular the Plana theorem [18]), we can easily provide an integral representation of the zeta function that determines its analytic extension. This gives, on one side another way to reobtain poles, residues and particular values, and on the other side also an integral representation for the derivative at zero; from this equation, an integral representation for the Gamma function can also be obtained. Third, using more recent techniques introduced in [27–29], we obtain a series representation for the zeta function. This representation can be differentiated and therefore provides a series representation for the derivative at zero of the zeta function and hence also for the Gamma function. About the Gamma function, first we provide the integral and series representation just mentioned. Second, applying the approach of [27], we perform a detailed study of the analytic properties of $\Gamma_2(x; a, b)$ as a function of x, reproducing some of the basic properties of the classical Euler function (compare with [23] or [25]). We conclude our analysis by presenting a very simple proof of the asymptotic formulas for large and small a given by Matsumoto [14,15].

We conclude this section introducing some notation and some elementary equations. First, mimic the duality Riemann/Hurwitz zeta function, it is natural to introduce the corresponding Riemann Barnes double zeta function

$$\chi(s;a,b)=\sum_{(m,n)\in\mathbb{N}_0^2}(am+bn)^{-s},$$

defined for similar range of the parameters and $\mathrm{Re}(s)>2$, and where the notation is $\mathbb{N}_0^2=\mathbb{N}\times\mathbb{N}-\{(0,0)\}$, with $\mathbb{N}=\{0,1,2,\dots\}$.

The following equations are clear, and will be used to obtain all the results concerning the function $\chi(s;a,b)$, that will be stated in form of corollaries of the related results for the function $\zeta_2(s;a,b,x)$.

$$\zeta_2(s;a,b,a)=\chi(s;a,b)-b^{-s}\zeta_R(s), \tag{1}$$

$$\zeta_2(s;a,b,b)=\chi(s;a,b)-a^{-s}\zeta_R(s). \tag{2}$$

Also, note that all these functions have the symmetry $\zeta_2(s;a,b,x)=\zeta_2(s;b,a,x)$, $\chi(s;a,b)=\chi(s;b,a)$.

In the following, we will use classical methods and some new techniques defined in [27,29]. Using the approach of those works, we obtain the analytic properties of the zeta and of the Gamma function from the defining sequences, namely the sequence: $S_0=\{am+bn\}_{(m,n)\in\mathbb{N}_0^2}$ and the shifted sequence $S_x=\{am+bn+x\}_{(m,n)\in\mathbb{N}_0^2}$, and the sequence $S=\{am+bn+x\}_{(m,n)\in\mathbb{N}^2}$, with the previous restrictions on the coefficients. All these sequences have a unique accumulation point at infinity, exponent of convergence 2 and genus 2. Moreover they are regular sequences of spectral type as defined in [27], and totally regular as defined in [29]. This can be immediately proved using heat kernel techniques and Lemma 2.2 of [27]. Following [27], we introduce the associated spectral functions: the heat function $(t>0)$

$$f(t,S_x)=\sum_{(m,n)\in\mathbb{N}^2}\mathrm{e}^{-(am+bn+x)t},$$

the logarithmic Fredholm determinant or logarithmic Gamma function (see also [29]), where the variable λ belongs to the domain $\mathbb{C}-\Sigma_{\theta,c}$, where $\Sigma_{\theta,c}$ is the sector $\Sigma_{\theta,c}=\{z\in\mathbb{C}\mid |\arg(z-c)|\leqslant\frac{\theta}{2}\}$, with $0<c<\min\{a,b,x\}$, and $0<\theta<\pi$,

$$\log\Gamma(-\lambda,S_0)=-\log\prod_{(m,n)\in\mathbb{N}_0^2}\left(1-\frac{\lambda}{am+bn}\right)\mathrm{e}^{\frac{\lambda}{am+bn}+\frac{\lambda^2}{2(am+bn)^2}},$$

and the zeta functions:

$$\zeta(s,S_0)=\sum_{(m,n)\in\mathbb{N}_0^2}(am+bn)^{-s}=\chi(s;a,b),$$

$$\zeta(s,S_x)=\sum_{(m,n)\in\mathbb{N}_0^2}(am+bn+x)^{-s},$$

$$\zeta(s,S)=\sum_{(m,n)\in\mathbb{N}^2}(am+bn+x)^{-s}=\zeta_2(s;a,b,x)=\zeta(s,S_x)+x^{-s}.$$

The further zeta function $\zeta(s,S_x)$ has been introduced for technical reasons, in order to apply directly the results of [27]. It is clear that it could be taken as the main zeta function instead of the function $\zeta_2(s;a,b,x)$. In particular, note that we can take the limit $x\to 0$ and obtain the function $\chi(s;a,b)$.

As a first application, we have the following generalization of the classical Lerch formula (see for example [12,27] for other generalizations).

Proposition 1.1 *(Lerch formula).*

$$\zeta_2'(0; a, b, x) = \log \Gamma_2(x; a, b) + \chi'(0; a, b). \tag{3}$$

We will give in Section 8 the relation between the Barnes Gamma function and $\Gamma(-\lambda, S_0)$, that shows that they are the same function up to some normalization.

Eventually, we recall some basic formulas concerning the Riemann zeta function ζ_R and Hurwitz zeta function ζ_H (see for example [11, 9.5] for the definitions) that will be implicitly used in the following without further comment.

$$\begin{gathered}
\operatorname*{Res_1}_{s=1} \zeta_H(s, q) = 1, \\
\operatorname*{Res_0}_{s=1} \zeta_H(s, q) = -\psi(q), \\
\zeta_H(-1, q) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{12}, \\
\zeta_H(0, q) = \frac{1}{2} - q, \\
\zeta_H'(0, q) = \log \Gamma(q) - \frac{1}{2}\log 2\pi.
\end{gathered}$$

2. Heat kernel asymptotics, poles residues and particular values of $\zeta_2(s; a, b, x)$ and $\chi_2(s; a, b)$

The asymptotic expansion of the heat function can be immediately obtained by using classical expansions of elementary functions (here the B_k are the Bernoulli numbers in the notation of [11]).

Lemma 2.1. *For small t,*

$$f(t, S_x) \sim \sum_{k=-2}^{\infty} P_k(x) t^k,$$

where

$$P_k(x) = \sum_{j=0}^{k+2} \frac{(-1)^j}{j!} e_{k-j} x^j,$$

and

$$\begin{gathered}
e_{-2} = \frac{1}{ab}, \\
e_{-1} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right), \\
e_0 = \frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a}\right), \\
e_1 = \frac{1}{24}(a + b) + \frac{B_3}{6}\frac{1}{ab}\left(a^3 + b^3\right),
\end{gathered}$$

$$e_k = \frac{1}{2}\frac{B_{k+1}}{(k+1)!}\left(a^k + b^k\right) + \frac{B_{k+2}}{(k+2)!}\frac{a^{k+2}+b^{k+2}}{ab} + \sum_{j=0}^{k-2}\frac{B_{k-j}B_{j+2}}{(k-j)!(j+2)!}a^{k-j-1}b^{j+1}, \quad k>1.$$

Corollary 2.2. *For small t,*

$$f(t,S_0) \sim \sum_{k=-2}^{\infty} e_k t^k.$$

From this information, we can easily localize the poles of the zeta functions, and obtain the values of the residues and the values at negative integers using standard heat kernel methods (see [10]).

Proposition 2.3. *The function $\zeta_2(s;a,b,x)$ has an analytic extension to the complex s-plane with simple poles at $s=1$ and 2. The residues at the poles are*

$$\operatorname*{Res_1}_{s=2} \zeta_2(s;a,b,x) = \frac{1}{ab},$$

$$\operatorname*{Res_1}_{s=1} \zeta_2(s;a,b,x) = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) - \frac{x}{ab}.$$

The values at zero and at negative integers $-k=-1,-2,-3,\ldots$ are

$$\zeta_2(0;a,b,x) = \frac{1}{4} + \frac{1}{12}\left(\frac{a}{b}+\frac{b}{a}\right) - \frac{x}{2}\left(\frac{1}{a}+\frac{1}{b}\right) + \frac{x^2}{2ab},$$

$$\zeta_2(-k;a,b,x) = (-1)^k k! \sum_{j=0}^{k+2}\frac{(-1)^j}{j!}e_{k-j}x^j.$$

Proof. Consider the Mellin transform of the zeta function

$$\zeta_2(s;a,b,x) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} f(t,S_x)\,dt.$$

Then, we can split the integral at $t=1$: the part with large t gives an integral function of s, in the part with small t we can use the expansion given in Lemma 2.1 and integrate. This gives the result. □

Corollary 2.4. *The function $\chi(s;a,b)$ has a regular analytic extension to the complex s-plane up to two simple poles at $s=1$ and $s=2$, with residues:*

$$\operatorname*{Res_1}_{s=2} \chi(s;a,b) = \frac{1}{ab},$$

$$\operatorname*{Res_1}_{s=1} \chi(s;a,b) = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right).$$

The values at zero and at negative integers $-k=-1,-2,-3,\dots$ *are*

$$\chi(0;a,b) = -\frac{3}{4} + \frac{1}{12}\left(\frac{b}{a}+\frac{a}{b}\right),$$
$$\chi(-k;a,b) = (-1)^k k! e_k.$$

3. Analytic extensions: Integral representation of $\zeta_2(s;a,b,x)$ and $\chi_2(s;a,b)$

We give in this section some integral representations that can be used to obtain the analytic continuation of the function $\zeta_2(s;a,x)$, and $\chi(s;a,b)$ (see also [16], in particular Section 4). These results are obtained by application of the theorem of Plana [18], as for example in [24], and consequently they are particularly interesting because of the simplicity of the proofs. The importance of these representations, beside the analytic continuation, is double. From one side, they will be used in the next section when computing the finite part of the zeta functions at the poles, from the other, in Section 5 in order to analyze the behavior near $s=0$, and consequently to obtain formulas for the derivative at zero of the zeta functions. Observe that similar equations hold when exchanging the parameters a and b.

Proposition 3.1 *(Hermite formula).*

$$\zeta_2(s;a,b,x) = \frac{1}{2}a^{-s}\zeta_H\left(s,\frac{x}{a}\right) + \frac{a^{1-s}}{b}\frac{1}{s-1}\zeta_H\left(s-1,\frac{x}{a}\right)$$
$$+ ia^{-s}\int_0^\infty \frac{\zeta_H(s,\frac{x+iby}{a}) - \zeta_H(s,\frac{x-iby}{a})}{e^{2\pi y}-1}\,dy.$$

Proof. Assume $\mathrm{Re}(s) > 2$, and apply the Plana theorem to the sum over n. Uniform convergence of series and integrals allows to do the following computations.

$$\zeta_2(s;a,b,x) = \frac{1}{2}\sum_{m=0}^\infty (am+x)^{-s} + \sum_{m=0}^\infty \int_0^\infty (am+bt+x)^{-s}\,dt$$
$$+ i\sum_{m=0}^\infty \int_0^\infty \frac{(am+x+iby)^{-s} - (am+x-iby)^{-s}}{e^{2\pi y}-1}\,dy$$
$$= \frac{1}{2}a^{-s}\sum_{m=0}^\infty \left(m+\frac{x}{a}\right)^{-s} + \frac{1}{b}\frac{1}{s-1}\sum_{m=0}^\infty (am+x)^{1-s}$$
$$+ ia^{-s}\sum_{m=0}^\infty \int_0^\infty \frac{(m+\frac{x+iby}{a})^{-s} - (m+\frac{x-iby}{a})^{-s}}{e^{2\pi y}-1}\,dy,$$

and after some simplification this gives the thesis. $\square$

A similar equation holds for the function $\zeta(s,S_x)$, and in particular for $\chi(s;a,b)$.

Corollary 3.2.

$$\chi(s;a,b) = b^{-s}\zeta_R(s) + \frac{1}{2}a^{-s}\zeta_R(s) + \frac{a^{1-s}}{b}\frac{1}{s-1}\zeta_R(s-1)$$
$$+ ia^{-s}\int_0^\infty \frac{\zeta_H(s, i\frac{b}{a}y+1) - \zeta_H(s, -i\frac{b}{a}y+1)}{e^{2\pi y}-1}\,dy.$$

Note that we cannot simplify further the integrals using the relation

$$\zeta_H(s, x+1) = \zeta_H(s,x) - x^{-s},$$

because of convergence's problems.

4. Finite part at the poles of $\zeta_2(s;a,b,x)$ and $\chi_2(s;a,b)$

The equations given in the previous section for the analytic extensions allow easy detection of the poles and calculation not only of the residues but also of the finite part. In all the equations, in fact, the integral is a regular function of s for all s, and therefore can be evaluated by simple substitution.

Proposition 4.1.

$$\operatorname*{Res_0}_{s=1}\zeta_2(s;a,b,x) = \left(\frac{x}{ab} - \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\right)\log b - \frac{1}{2b}\psi\left(\frac{x}{b}\right) - \frac{1}{2a}\log 2\pi + \frac{1}{a}\log\Gamma\left(\frac{x}{b}\right)$$
$$- \frac{i}{b}\int_0^\infty \frac{\psi(\frac{x+iay}{b}) - \psi(\frac{x-iay}{b})}{e^{2\pi y}-1}\,dy,$$

$$\operatorname*{Res_0}_{s=2}\zeta_2(s;a,b,x) = \frac{1}{2b^2}\zeta_H\left(2,\frac{x}{b}\right) - \frac{1}{ab}\left(\psi\left(\frac{x}{b}\right) + 1 + \log b\right)$$
$$+ \frac{i}{b^2}\int_0^\infty \frac{\zeta_H(2,\frac{x+iay}{b}) - \zeta_H(2,\frac{x-iay}{b})}{e^{2\pi y}-1}\,dy.$$

Proof. We must compute

$$\lim_{s\to 1}\left(\zeta_2(s;a,b,x) - \left(\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) - \frac{x}{ab}\right)\frac{1}{s-1}\right).$$

Using the equation given in Proposition 3.1, we compute

$$\lim_{s\to 1}\left(\frac{1}{2}b^{-s}\zeta_H\left(s,\frac{x}{b}\right) + \frac{b^{1-s}}{a}\frac{1}{s-1}\zeta_H\left(s-1,\frac{x}{b}\right) + ib^{-s}\int_0^\infty \frac{\zeta_H(s,\frac{x+iay}{b}) - \zeta_H(s,\frac{x-iay}{b})}{e^{2\pi y}-1}\,dy\right.$$
$$\left. - \left(\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) - \frac{x}{ab}\right)\frac{1}{s-1}\right).$$

The non-regular term is the one in the first line; we expand it near $s=1$

$$\frac{1}{2b}\frac{1}{s-1}-\frac{1}{2b}\log b-\frac{1}{2b}\psi\left(\frac{x}{b}\right)+\frac{1}{a}\frac{1}{s-1}\zeta_H\left(0,\frac{x}{b}\right)-\frac{1}{a}\log b\zeta_H\left(0,\frac{x}{b}\right)$$
$$+\frac{1}{a}\left(\log\Gamma\left(\frac{x}{b}\right)-\frac{1}{2}\log 2\pi\right)+O(s-1).$$

The singular part cancels out, and substitution of the value of $s=1$ in the other regular terms gives the thesis. Similar computations give the second equation. □

Corollary 4.2.

$$\operatorname*{Res_0}_{s=1}\chi(s;a,b)=-\frac{1}{a}\log a+\frac{1}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\log b+\frac{\gamma}{a}+\frac{\gamma}{2b}-\frac{1}{2a}\log 2\pi$$
$$-\frac{i}{b}\int_0^\infty\frac{\psi(i\frac{a}{b}y+1)-\psi(-i\frac{a}{b}y+1)}{e^{2\pi y}-1}dy,$$
$$\operatorname*{Res_0}_{s=2}\chi(s;a,b)=\frac{1}{a^2}\zeta_R(2)+\frac{1}{2b^2}\zeta_R(2)+\frac{1}{ab}(\gamma-1-\log b)$$
$$+\frac{i}{b^2}\int_0^\infty\frac{\zeta_H(2,i\frac{a}{b}y+1)-\zeta_H(2,-i\frac{a}{b}y+1)}{e^{2\pi y}-1}dy,$$

where recall that $\zeta_R(2)=\frac{\pi^2}{6}$.

Next, we give some series representation for the finite parts.

Proposition 4.3.

$$\operatorname*{Res_0}_{s=1}\zeta_2(s;a,b,x)=-\left(\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)-\frac{x}{ab}\right)\log b-\frac{1}{2b}\psi\left(\frac{x}{b}\right)-\frac{1}{2a}\log 2\pi+\frac{1}{a}\log\Gamma\left(\frac{x}{b}\right)$$
$$+\begin{cases}\dfrac{1}{a}\displaystyle\sum_{n=0}^\infty\left(\sum_{j=2}^\infty\frac{(-1)^j}{j}\zeta_H\left(j,\frac{bn+x}{a}\right)-\frac{a}{2(bn+x)}\right),\\[2ex] \dfrac{1}{2a}\displaystyle\sum_{n=0}^\infty\left(\zeta_H\left(2,\frac{bn+x}{a}\right)-\frac{a}{bn+x}\right)+\frac{1}{a}\sum_{j=3}^\infty\frac{(-1)^j\zeta_2(j;a,b,x)a^j}{j},\quad |a|<1;\end{cases}$$
$$\operatorname*{Res_0}_{s=2}\zeta_2(s;a,b,x)=-\frac{1}{ab}\left(1+\log b+\psi\left(\frac{x}{b}\right)\right)+\frac{1}{a^2}\sum_{n=0}^\infty\left(\zeta_H\left(2,\frac{bn+x}{a}\right)-\frac{a}{bn+x}\right).$$

Proof. For the first, we start from the first formula given in Proposition 4.1. We must treat the following integral, and this can be done using the definition of the digamma function and [11, 3.415.1].

$$-\frac{i}{b}\int_0^\infty\frac{\psi(\frac{x+iay}{b})-\psi(\frac{x-iay}{b})}{e^{2\pi y}-1}dy$$

$$
\begin{aligned}
&= -\frac{i}{b}\int_0^\infty \sum_{n=0}^\infty \frac{2iy\frac{a}{b}}{\frac{a^2y^2}{b^2}+(\frac{x}{b}+n)^2}\frac{dy}{e^{2\pi y}-1}\\
&= \frac{1}{a}\sum_{n=0}^\infty\left(\log\frac{bn+x}{a}-\frac{a}{2(bn+x)}-\psi\left(\frac{bn+x}{a}\right)\right)\\
&= \frac{1}{a}\sum_{n=0}^\infty\left(\sum_{k=0}^\infty\left(\frac{1}{k+\frac{bn+x}{a}}-\sum_{j=1}^\infty\frac{(-1)^{j+1}}{j}\left(k+\frac{bn+x}{a}\right)^{-j}\right)-\frac{a}{2(bn+x)}\right)\\
&= \frac{1}{a}\sum_{n=0}^\infty\left(\sum_{k=0}^\infty\sum_{j=2}^\infty\frac{(-1)^j}{j}\left(k+\frac{bn+x}{a}\right)^{-j}-\frac{a}{2(bn+x)}\right)\\
&= \frac{1}{a}\sum_{n=0}^\infty\left(\sum_{j=2}^\infty\frac{(-1)^j}{j}\zeta_H\left(j,\frac{bn+x}{a}\right)-\frac{a}{2(bn+x)}\right);
\end{aligned}
$$

alternatively, from the triple sum

$$
\begin{aligned}
&\frac{1}{a}\sum_{n=0}^\infty\left(\sum_{k=0}^\infty\sum_{j=2}^\infty\frac{(-1)^j}{j}\left(k+\frac{bn+x}{a}\right)^{-j}-\frac{a}{2(bn+x)}\right)\\
&= \sum_{n=0}^\infty\left(\sum_{k=0}^\infty\frac{a}{2(bn+ak+x)^2}-\frac{1}{2(bn+x)}\right)+\frac{1}{a}\sum_{n=0}^\infty\sum_{k=0}^\infty\sum_{j=3}^\infty\frac{(-1)^j(ak+bn+x)^{-j}a^j}{j}.
\end{aligned}
$$

For the second, we proceed as follows.

$$
\begin{aligned}
\operatorname*{Res_0}_{s=2}\zeta_2(s;a,b,x) &= \lim_{s\to 2}\left(\zeta_2(s;a,b,x)-\frac{1}{ab}\frac{1}{s-2}\right)\\
&= \lim_{s\to 2}\left(\zeta_2(s;a,b,x)-\frac{1}{a}\frac{b^{1-s}}{s-1}\zeta_H\left(s-1,\frac{x}{b}\right)\right)\\
&\quad+\frac{1}{a}\lim_{s\to 2}\left(\frac{b^{1-s}}{s-1}\zeta_H\left(s-1,\frac{x}{b}\right)-\frac{1}{b}\frac{1}{s-2}\right).
\end{aligned}
$$

Note that the splitting in the two limits is allowed by the equation given in Proposition 3.1. For, that equation shows that the function in the first limit is regular at $s=2$.

The second limit can be computed expanding near $s=2$:

$$
\frac{b^{1-s}}{s-1}\zeta_H\left(s-1,\frac{x}{b}\right)-\frac{b^{-1}}{s-2}=-\frac{1}{b}\left(1+\log b+\psi\left(\frac{x}{b}\right)\right)+O(s-2).
$$

For the first, using the definitions of the zeta functions:

$$
\begin{aligned}
&\lim_{s\to 2}\left(\zeta_2(s;a,b,x)-\frac{b^{1-s}}{a}\frac{1}{s-1}\zeta_H\left(s-1,\frac{x}{b}\right)\right)\\
&\quad=\lim_{s\to 2}\left(\sum_{n=0}^\infty a^{-s}\zeta_H\left(s;\frac{bn+x}{a}\right)-\frac{b^{1-s}}{a}\frac{1}{s-1}\sum_{n=0}^\infty\left(n+\frac{x}{b}\right)^{1-s}\right)
\end{aligned}
$$

$$= \sum_{n=0}^{\infty} \left(a^{-2} \zeta_H \left(2, \frac{bn+x}{a} \right) - \frac{1}{a} \frac{1}{bn+x} \right).$$

Summing up, this gives the thesis. □

Corollary 4.4.

$$\operatorname*{Res_0}_{s=1} \chi(s; a, b) = -\frac{1}{a} \log a + \frac{1}{2} \left(\frac{1}{a} - \frac{1}{b} \right) \log b + \frac{\gamma}{a} + \frac{\gamma}{2b} - \frac{1}{2a} \log 2\pi$$
$$+ \begin{cases} \frac{1}{a} \sum_{n=1}^{\infty} \left(\sum_{j=2}^{\infty} \frac{(-1)^j}{j} \zeta_H \left(j, \frac{bn}{a} \right) - \frac{1}{2} \frac{a}{bn} \right), \\ \frac{1}{2a} \sum_{n=1}^{\infty} \left(\zeta_H \left(2, \frac{bn}{a} + 1 \right) - \frac{a}{bn} \right) + \frac{1}{a} \log \Gamma \left(1 + \frac{a}{b} \right) + \frac{\gamma}{b} \\ \quad + \frac{1}{a} \sum_{j=3}^{\infty} \frac{(-1)^j}{j} \sum_{m,n=1}^{\infty} (am+bn)^{-j} a^j, \quad |a| < 1; \end{cases}$$

$$\operatorname*{Res_0}_{s=2} \chi(s; a, b) = \frac{1}{a^2} \zeta_R(2) + \frac{1}{ab} (\gamma - 1 - \log b) + \frac{1}{a^2} \sum_{n=1}^{\infty} \left(\zeta_H \left(2, \frac{b}{a} n \right) - \frac{a}{bn} \right).$$

Proof. The unique point that is not a direct consequence of the result given in the proposition, is the second part of the first equation. Using the formula given in the proposition with $x = b$ we have

$$\frac{1}{a} \sum_{j=3}^{\infty} \frac{(-1)^j}{j} \zeta_2(j; a, b, b) a^j = \frac{1}{a} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{j=3}^{\infty} \frac{(-1)^j}{j} \left(k + \frac{bn}{a} \right)^{-j}$$
$$= \frac{1}{a} \sum_{n=1}^{\infty} \sum_{j=3}^{\infty} \frac{(-1)^j}{j} \left(\frac{bn}{a} \right)^{-j} + \frac{1}{a} \sum_{k,n=1}^{\infty} \sum_{j=3}^{\infty} \frac{(-1)^j}{j} \left(k + \frac{bn}{a} \right)^{-j}$$
$$= \frac{1}{a} \sum_{j=3}^{\infty} \frac{(-1)^j}{j} \zeta_R(j) \left(\frac{a}{b} \right)^j + \frac{1}{a} \sum_{j=3}^{\infty} \frac{(-1)^j}{j} \sum_{m,n=1}^{\infty} (am+bn)^{-j} a^j.$$

The first term simplifies as follows

$$\sum_{j=3}^{\infty} \frac{(-1)^j}{j} \zeta_R(j) \left(\frac{a}{b} \right)^j = \log \Gamma \left(1 + \frac{a}{b} \right) + \gamma \frac{a}{b} - \frac{1}{2} \frac{a^2}{b^2} \zeta_H(2),$$

and using the following series of equalities, this gives the result,

$$\sum_{n=1}^{\infty} \left(\zeta_H \left(2, \frac{bn}{a} + 1 \right) - \frac{a}{bn} \right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \left(k + \frac{bn}{a} \right)^{-2} - \frac{a}{bn} \right)$$
$$= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \left(k + \frac{bn}{a} \right)^{-2} - \frac{a}{bn} \right) + \sum_{n=1}^{\infty} \left(\frac{bn}{a} \right)^{-2} - \frac{a^2}{b^2} \zeta_R(2)$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\left(k+\frac{bn}{a}\right)^{-2}-\frac{a}{bn}\right)-\frac{a^2}{b^2}\zeta_R(2)$$
$$= \sum_{n=1}^{\infty}\left(\zeta_H\left(2,\frac{bn}{a}\right)-\frac{a}{bn}\right)-\frac{a^2}{b^2}\zeta_R(2). \qquad \square$$

Remark 4.5. It can be interesting to compare the results when $a=b=1$. In fact,

$$\chi(s;1,1)=\zeta_R(s)+\zeta_R(s-1),$$

thus

$$\operatorname*{Res_0}_{s=1}\chi(s;1,1)=\gamma+\zeta_R(0)=\gamma-\frac{1}{2},$$
$$\operatorname*{Res_0}_{s=2}\chi(s;1,1)=\zeta_R(2)+\gamma=\frac{\pi^2}{6}+\gamma.$$

For the first we can use Corollary 4.2; we have

$$\operatorname*{Res_0}_{s=1}\chi(s;1,1)=\frac{3}{2}\gamma-\frac{1}{2}\log 2\pi-i\int_0^{\infty}\frac{\psi(iy+1)-\psi(-iy+1)}{e^{2\pi y}-1}\,dy,$$

and numerical computation agrees with $\gamma-\frac{1}{2}$. For the second, by Corollary 4.4, we have

$$\operatorname*{Res_0}_{s=2}\chi(s;1,1)=2\zeta_R(2)+\sum_{n=1}^{\infty}\left(\zeta_H(2,n+1)-\frac{1}{n}\right)-1+\gamma,$$

and we can compute (see [26]) that

$$\sum_{n=1}^{\infty}\left(\zeta_H(2,n+1)-\frac{1}{n}\right)=1-\zeta_R(2).$$

5. Integral representation of $\zeta_2'(0;a,b,x)$, $\chi_2'(0;a,b)$ and $\Gamma_2(x;a,b)$

We can use the equations obtained in Section 3 in order to give some formulas for the derivative at $s=0$ of the functions $\zeta_2(s;a,b,x)$ and $\chi(s;a,b)$. As a consequence, this will also give a formula for the function $\Gamma_2(x;a,b)$.

Proposition 5.1.

$$\zeta_2'(0;a,b,x)=\left(-\frac{1}{2}\zeta_H\left(0,\frac{x}{a}\right)+\frac{a}{b}\zeta_H\left(-1,\frac{x}{a}\right)-\frac{1}{12}\frac{b}{a}\right)\log a+\frac{1}{2}\log\Gamma\left(\frac{x}{a}\right)$$
$$-\frac{1}{4}\log 2\pi-\frac{a}{b}\zeta_H\left(-1,\frac{x}{a}\right)-\frac{a}{b}\zeta_H'\left(-1,\frac{x}{a}\right)+i\int_0^{\infty}\log\frac{\Gamma(\frac{x+iby}{a})}{\Gamma(\frac{x-iby}{a})}\frac{dy}{e^{2\pi y}-1}.$$

Proof. Using Proposition 3.1

$$\begin{aligned}
\zeta_2'(0;a,b,x) &= -\frac{1}{2}\zeta_H\left(0,\frac{x}{a}\right)\log a + \frac{1}{2}\zeta_H'\left(0,\frac{x}{a}\right) + \frac{a}{b}\zeta_H\left(-1,\frac{x}{a}\right)\log a \\
&\quad - \frac{a}{b}\zeta_H\left(-1,\frac{x}{a}\right) - \frac{a}{b}\zeta_H'\left(-1,\frac{x}{a}\right) \\
&\quad - i\log a\int_0^\infty \frac{\zeta_H(0,\frac{x+iby}{a}) - \zeta_H(0,\frac{x-iby}{a})}{e^{2\pi y}-1}\,dy \\
&\quad + i\int_0^\infty \frac{\zeta_H'(0,\frac{x+iby}{a}) - \zeta_H'(0,\frac{x-iby}{a})}{e^{2\pi y}-1}\,dy.
\end{aligned}$$

The two integrals can be simplified as follows. For the first one:

$$\begin{aligned}
-i\log a\int_0^\infty \frac{\zeta_H(0,\frac{x+iby}{a}) - \zeta_H(0,\frac{x-iby}{a})}{e^{2\pi y}-1}\,dy &= -i\log a\int_0^\infty \frac{-2i\frac{b}{a}y}{e^{2\pi y}-1}\,dy \\
&= -2\frac{b}{a}\log a\int_0^\infty \frac{y}{e^{2\pi y}-1}\,dy \\
&= -\frac{1}{12}\frac{b}{a}\log a,
\end{aligned}$$

where we have used [11, 9.611.1 and 9.71]. For the second one:

$$i\int_0^\infty \frac{\zeta_H'(0,\frac{x+iby}{a}) - \zeta_H'(0,\frac{x-iby}{a})}{e^{2\pi y}-1}\,dy = i\int_0^\infty \log\frac{\Gamma(\frac{x+iby}{a})}{\Gamma(\frac{x-iby}{a})}\frac{dy}{e^{2\pi y}-1}. \qquad \square$$

Corollary 5.2.

$$\begin{aligned}
\chi'(0;a,b) &= \frac{1}{2}\log b + \left(\frac{1}{4} - \frac{1}{12}\frac{a}{b} - \frac{1}{12}\frac{b}{a}\right)\log a - \frac{3}{4}\log 2\pi + \frac{1}{12}\frac{a}{b} - \frac{a}{b}\zeta_R'(-1) \\
&\quad + i\int_0^\infty \log\frac{\Gamma(1+\frac{iby}{a})}{\Gamma(1-\frac{iby}{a})}\frac{dy}{e^{2\pi y}-1}.
\end{aligned}$$

Observe that similar formulas hold exchanging a and b.

Using the previous results and Proposition 1.1, we can also obtain the following integral representation formula for the double Gamma function.

Corollary 5.3.

$$\log\Gamma_2(x;a,b) = \frac{1}{2}\log 2\pi - \frac{1}{2}\log b - \frac{1}{2}\left(\frac{x}{a}+\frac{x}{b}-1-\frac{x^2}{ab}\right)\log a$$
$$+\frac{1}{2}\log\Gamma\left(\frac{x}{a}\right) - \frac{a}{b}\zeta_H'\left(-1,\frac{x}{a}\right) + \frac{a}{b}\zeta_R'(-1) + \frac{1}{2}\frac{x}{b}\left(\frac{x}{a}-1\right)$$
$$+ i\int_0^\infty \log\frac{\Gamma(\frac{x+iby}{a})\Gamma(1-\frac{iby}{a})}{\Gamma(\frac{x-iby}{a})\Gamma(1+\frac{iby}{a})}\frac{dy}{e^{2\pi y}-1}.$$

6. Series representation of $\zeta_2(s;a,b,x)$ and $\chi_2(s;a,b)$

In the case under study, the method of [29] is particularly useful because it gives an exact result, namely a decomposition of the zeta function as an infinite sum of the same zeta function shifted by positive integer values. This is an interesting kind of functional equation. See also the works of Carletti and Monti Bragadin [6,7]. One interesting application of this result is due to the simple behavior of the series at zero, that allows to obtain a formula for the derivative at zero and consequently also a formula for $\Gamma_2(x;a,b)$, as it will be shown in the next section.

We will use the method introduced in [29] to deal with double series. We refer to that work for definitions and complete proofs. Consider the double sequence $S = \{am+bn+x\}_{m,n=0}^\infty$. The relative genus are $(p_0,p_1,p_2) = (2,1,1)$. We decompose on the sequence $U = \{bn+x\}_{n=0}^\infty$, with order $r_0 = 1$ and genus $q = 1$. Decomposability can be checked using Theorem 4.2 of [29]. The main spectral function appearing in order to apply the spectral decomposition technique of [29] is the logarithmic Gamma function associated to the sequence $\tilde S_n = \{\frac{am+bn+x}{bn+x}\}_{m=0}^\infty$, namely the function

$$\log\Gamma(\lambda,\tilde S_n) = -\log\prod_{m=0}^\infty\left(1+\frac{(bn+x)(-\lambda)}{am+bn+x}\right)e^{-\frac{(bn+x)(-\lambda)}{am+bn+x}}$$
$$= -\log(1-\lambda)e^{\lambda} - \log\prod_{m=1}^\infty\left(1+\frac{\frac{bn+x}{a}(-\lambda)}{\frac{bn+x}{a}+m}\right)e^{-\frac{\frac{bn+x}{a}(-\lambda)}{\frac{bn+x}{a}+m}}.$$

Observing that

$$-\frac{\frac{bn+x}{a}(-\lambda)}{\frac{bn+x}{a}+m} = -\frac{\frac{bn+x}{a}(-\lambda)}{m} + \frac{(\frac{bn+x}{a})^2(-\lambda)}{m(\frac{bn+x}{a}+m)},$$

we have

$$\log\Gamma(\lambda,\tilde S_n) = -\log(1-\lambda) - \lambda - \log\prod_{m=1}^\infty\left(1+\frac{\frac{bn+x}{a}(-\lambda)}{\frac{bn+x}{a}+m}\right)e^{-\frac{\frac{bn+x}{a}(-\lambda)}{m}}$$
$$-\sum_{m=1}^\infty\frac{(\frac{bn+x}{a})^2}{m(\frac{bn+x}{a}+m)}(-\lambda),$$

that can be rewritten as

$$\log\Gamma(\lambda,\tilde S_n) = -\lambda + \gamma\frac{bn+x}{a}(-\lambda) + \log\Gamma\left(\frac{bn+x}{a}(1-\lambda)\right)$$
$$- \log\Gamma\left(\frac{bn+x}{a}\right) - \sum_{m=1}^{\infty}\frac{(\frac{bn+x}{a})^2}{m(\frac{bn+x}{a}+m)}(-\lambda). \tag{4}$$

Next, by Lemma 3.4 of [29], we have

$$\zeta_2(s;a,b,x) = \frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}}\frac{e^{-\lambda t}}{-\lambda}\mathcal{T}(s,\lambda)\,d\lambda\,dt, \tag{5}$$

where $\Lambda_{\theta,c} = \{\lambda\in\mathbb{C} \mid |\arg(\lambda-c)| = \frac{\theta}{2}\}$, $0<c<x$, $0<\theta<\pi$, and

$$\mathcal{T}(s,\lambda) = \sum_{n=0}^{\infty}(bn+x)^{-s}\log\Gamma(\lambda,\tilde S_n).$$

We can do the λ integration appearing in Eq. (5) for all the terms appearing in formula for the function $\mathcal{T}(\lambda,s)$ as follows. First, use Eq. (4) in the definition of $\mathcal{T}(\lambda,s)$. This gives

$$\mathcal{T}(s,\lambda) = \sum_{n=0}^{\infty}(bn+x)^{-s}\left(-\lambda+\gamma\frac{bn+x}{a}(-\lambda)\right) \tag{6}$$
$$+ \sum_{n=0}^{\infty}(bn+x)^{-s}\log\Gamma\left(\frac{bn+x}{a}(1-\lambda)\right) \tag{7}$$
$$- \sum_{n=0}^{\infty}(bn+x)^{-s}\left(\log\Gamma\left(\frac{bn+x}{a}\right) + \sum_{m=1}^{\infty}\frac{(\frac{bn+x}{a})^2}{m(\frac{bn+x}{a}+m)}(-\lambda)\right). \tag{8}$$

Using Eq. (18) of Appendix A, we see that the unique term that gives a non-zero λ-integral is the one in the second line (7). Second, we can decompose the term (7) using the classical series representation for the Euler Gamma function, namely [11, 8.343], that converges uniformly in the domain allowed for λ,

$$\log\Gamma\left(\frac{bn+x}{a}(1-\lambda)\right)$$
$$= \left(\frac{bn+x}{a}(1-\lambda)-\frac{1}{2}\right)\log\left(\frac{bn+x}{a}(1-\lambda)\right) \tag{9}$$
$$- \frac{bn+x}{a}(1-\lambda) + \frac{1}{2}\log 2\pi \tag{10}$$
$$+ \frac{1}{2}\sum_{k=1}^{\infty}\frac{k}{(k+1)(k+2)}\sum_{j=1}^{\infty}\left(\frac{bn+x}{a}(1-\lambda)+j\right)^{-k-1}. \tag{11}$$

Let consider the terms appearing in the three lines independently. The term in the first line (9) can be rewritten as

$$\left(\frac{bn+x}{a}(1-\lambda)-\frac{1}{2}\right)\log\left(\frac{bn+x}{a}(1-\lambda)\right)=\left(\frac{bn+x}{a}(1-\lambda)-\frac{1}{2}\right)\log\frac{bn+x}{a}$$
$$+\frac{bn+x}{a}(1-\lambda)\log(1-\lambda)-\frac{1}{2}\log(1-\lambda).$$

The λ-integral of the term in the first line is zero, again by Eq. (18). The complete integrals of the terms in the second line can be done using Eq. (20). We obtain,

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}}\frac{e^{-\lambda t}}{-\lambda}\frac{bn+x}{a}(1-\lambda)\log(1-\lambda)\,d\lambda\,dt=\frac{bn+x}{a}\frac{\Gamma(s-1)}{s},$$
$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}}\frac{e^{-\lambda t}}{-\lambda}\left(-\frac{1}{2}\log(1-\lambda)\right)d\lambda\,dt=\frac{1}{2}\frac{\Gamma(s)}{s}.$$

The term in the second line (10) has zero λ-integral by Eq. (18). The general term appearing in the sum in the last line (11) can be computed using Eq. (22). We obtain

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}}\frac{e^{-\lambda t}}{-\lambda}\left(\frac{bn+x}{a}(1-\lambda)+j\right)^{-k-1}d\lambda\,dt$$
$$=(bn+x)^s a^{k+1}(bn+aj+x)^{-s-k-1}\frac{\Gamma(s+k+1)}{\Gamma(k+1)s}.$$

Summing up we obtain

$$\begin{aligned}
\zeta_2(s;a,b,x)&=\frac{s}{\Gamma(s)}\sum_{n=0}^\infty(bn+x)^{-s}\left(\frac{bn+x}{a}\frac{\Gamma(s-1)}{s}+\frac{1}{2}\frac{\Gamma(s)}{s}\right.\\
&\quad\left.+\frac{1}{2}\sum_{k=1}^\infty\frac{k}{(k+1)(k+2)}\sum_{j=1}^\infty(bn+x)^s a^{k+1}(bn+aj+x)^{-s-k-1}\frac{\Gamma(s+k+1)}{\Gamma(k+1)s}\right)\\
&=\frac{1}{a}\frac{1}{s-1}\sum_{n=0}^\infty(bn+x)^{1-s}+\frac{1}{2}\sum_{n=0}^\infty(bn+x)^{-s}\\
&\quad+\frac{1}{2}\sum_{k=1}^\infty\frac{k}{(k+2)!}\frac{\Gamma(s+k+1)}{\Gamma(s)}a^{k+1}\sum_{n=0}^\infty\sum_{j=1}^\infty(bn+aj+x)^{-s-k-1}\\
&=\frac{b^{1-s}}{a}\frac{1}{s-1}\zeta_H\left(s-1,\frac{x}{b}\right)+\frac{1}{2}b^{-s}\zeta_H\left(s,\frac{x}{b}\right)\\
&\quad+\frac{1}{2}\sum_{k=1}^\infty\frac{k}{(k+2)!}\frac{\Gamma(s+k+1)}{\Gamma(s)}a^{k+1}\sum_{n=0}^\infty\sum_{j=1}^\infty(bn+aj+x)^{-s-k-1}.
\end{aligned}$$

Recalling the representation of the binomial coefficients in terms of Euler Gamma functions, we get the following result.

Proposition 6.1.

$$
\begin{aligned}
\zeta_2(s;a,b,x) &= \frac{b^{1-s}}{a}\frac{1}{s-1}\zeta_H\left(s-1,\frac{x}{b}\right)+\frac{1}{2}b^{-s}\zeta_H\left(s,\frac{x}{b}\right)\\
&\quad+\frac{1}{2}s\sum_{k=1}^{\infty}\frac{k}{(k+1)(k+2)}\binom{s+k}{k}\zeta_2(s+k+1;a,b,x+a)a^{k+1}\\
&=\frac{b^{1-s}}{a}\frac{1}{s-1}\zeta_H\left(s-1,\frac{x}{b}\right)+\frac{1}{2}b^{-s}\zeta_H\left(s,\frac{x}{b}\right)\\
&\quad+\frac{1}{2}sa^{-s}\sum_{k=1}^{\infty}\frac{k}{(k+1)(k+2)}\binom{s+k}{k}\sum_{n=0}^{\infty}\zeta_H\left(s+k+1,\frac{bn+x}{a}+1\right).
\end{aligned}
$$

Corollary 6.2.

$$
\begin{aligned}
\chi(s;a,b) &= a^{-s}\zeta_R(s)+\frac{b^{1-s}}{a}\frac{1}{s-1}\zeta_R(s-1)+\frac{1}{2}b^{-s}\zeta_R(s)\\
&\quad+\frac{1}{2}sa^{-s}\sum_{k=1}^{\infty}\frac{k}{(k+1)(k+2)}\binom{s+k}{k}\sum_{n=1}^{\infty}\zeta_H\left(s+k+1,\frac{bn}{a}+1\right)\\
&=a^{-s}\zeta_R(s)+\frac{b^{1-s}}{a}\frac{1}{s-1}\zeta_R(s-1)+\frac{1}{2}b^{-s}\zeta_R(s)\\
&\quad+\frac{1}{2}s\sum_{k=1}^{\infty}\frac{k}{(k+1)(k+2)}\binom{s+k}{k}a^{k+1}\sum_{m,n=1}^{\infty}(am+bn)^{-s-k-1}.
\end{aligned}
$$

7. Series representation of $\zeta_2'(0;a,b,x)$, $\chi_2'(0;a,b)$ and $\Gamma_2(x;a,b)$

We can use the equations obtained in Section 6 in order to give more formulas for the derivative at $s=0$ of the functions $\zeta_2(s;a,b,x)$ and $\chi(s;a,b)$. As a consequence, this will also give a formula for the function $\Gamma_2(x;a,b)$.

Proposition 7.1.

$$
\begin{aligned}
\zeta_2'(0;a,b,x) &= \left(\frac{b}{a}\zeta_H\left(-1,\frac{x}{b}\right)-\frac{1}{2}\zeta_H\left(0,\frac{x}{b}\right)-\frac{a}{12b}\right)\log b-\frac{b}{a}\zeta_H\left(-1,\frac{x}{b}\right)-\frac{b}{a}\zeta_H'\left(-1,\frac{x}{b}\right)\\
&\quad+\frac{1}{2}\zeta_H'\left(0,\frac{x}{b}\right)+\frac{1}{12}\sum_{n=0}^{\infty}\left(\zeta_H\left(2,\frac{bn+x+a}{a}\right)-\frac{a}{bn+x}\right)-\frac{a}{12b}\psi\left(\frac{x}{b}\right)\\
&\quad+\frac{1}{2}\sum_{k=2}^{\infty}\frac{k}{(k+1)(k+2)}\zeta_2(k+1;a,b,x+a)a^{k+1}\\
&=\left(\frac{b}{a}\zeta_H\left(-1,\frac{x}{b}\right)-\frac{1}{2}\zeta_H\left(0,\frac{x}{b}\right)-\frac{a}{12b}\right)\log b-\frac{b}{a}\zeta_H\left(-1,\frac{x}{b}\right)-\frac{a}{12b}\psi\left(\frac{x}{b}\right)\\
&\quad+\frac{1}{2}\zeta_H'\left(0,\frac{x}{b}\right)-\frac{b}{a}\zeta_H'\left(-1,\frac{x}{b}\right)+\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\frac{k\zeta_H(k+1,\frac{bn+x}{a}+1)}{(k+1)(k+2)}-\frac{1}{6}\frac{a}{bn+x}\right).
\end{aligned}
$$

Proof. First we prove the first equation. We decompose the sum appearing in the equation given in Proposition 6.1 as follows.

$$\begin{aligned} S(s;a,b,x) &= \frac{1}{12}s(s+1)\zeta_2(s+2;a,b,x+a)a^2 \\ &\quad + \frac{1}{2}s\sum_{k=2}^{\infty}\frac{k}{(k+1)(k+2)}\binom{s+k}{k}\zeta_2(s+k+1;a,b,x+a)a^{k+1} \\ &= S_1(s;a,b,x)+S_2(s;a,b,x). \end{aligned}$$

We deal with the two terms independently. Some care is necessary to treat the first one, due to the pole of the double zeta function. Namely, near $s=0$

$$S_1(s;a,b,x) = \frac{a^2}{12}R_1 + \frac{a^2}{12}(R_1+R_0)s + O\left(s^2\right),$$

where

$$R_i = \operatorname*{Res}_{s=2}{}_i\, \zeta_2(s;a,b,x+a),$$

that have been computed in Proposition 2.3 and in Proposition 4.1 or 4.3, respectively. Thus

$$\begin{aligned} S_1(s;a,b,x) &= \frac{a}{12b} + \frac{s}{12}\left(\sum_{n=0}^{\infty}\left(\zeta_H\left(2,\frac{bn+x+a}{a}\right) - \frac{a}{bn+x+a}\right) - \frac{a}{b}\left(\log b + \psi\left(\frac{x+a}{b}\right)\right)\right) \\ &\quad + O\left(s^2\right). \end{aligned}$$

For the second term, we have already observed that it is vanishing at $s=0$, and we compute

$$\begin{aligned} S_2(s;a,b,x) &= \frac{1}{2}s\sum_{k=2}^{\infty}\frac{k}{(k+1)(k+2)}\binom{s+k}{k}\zeta_2(s+k+1;a,b,x+a)a^{k+1} \\ &= \frac{1}{2}\sum_{k=2}^{\infty}\frac{k}{(k+1)(k+2)}\zeta_2(k+1;a,b,x+a)a^{k+1}s + O\left(s^2\right), \end{aligned}$$

near $s=0$.

This gives

$$\begin{aligned} \zeta_2'(0;a,b,x) &= \left(\frac{b}{a}\zeta_H\left(-1,\frac{x}{b}\right) - \frac{1}{2}\zeta_H\left(0,\frac{x}{b}\right) - \frac{a}{12b}\right)\log b - \frac{b}{a}\zeta_H\left(-1,\frac{x}{b}\right) - \frac{b}{a}\zeta_H'\left(-1,\frac{x}{b}\right) \\ &\quad + \frac{1}{2}\zeta_H'\left(0,\frac{x}{b}\right) + \frac{1}{12}\sum_{n=0}^{\infty}\left(\zeta_H\left(2,\frac{bn+x+a}{a}\right) - \frac{a}{bn+x+a}\right) - \frac{a}{12b}\psi\left(\frac{x+a}{b}\right) \\ &\quad + \frac{1}{2}\sum_{k=2}^{\infty}\frac{k}{(k+1)(k+2)}\zeta_2(k+1;a,b,x+a)a^{k+1}. \end{aligned}$$

The term in the third line can be simplified as follows. First, recalling the definition of the digamma function, we have that

$$\sum_{n=0}^{\infty}\left(\frac{1}{n+x/b}-\frac{1}{n+(x+a)/b}\right)-\psi\left(\frac{x+a}{b}\right)=\gamma+\frac{b}{x}-\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+x/b}\right)$$
$$=-\psi\left(\frac{x}{b}\right).$$

Second, note the following series of equivalences:

$$\begin{aligned}
&\frac{1}{12}\sum_{n=0}^{\infty}\left(\zeta_H\left(2,\frac{bn+x+a}{a}\right)-\frac{a}{bn+x+a}\right)-\frac{a}{12b}\psi\left(\frac{x+a}{b}\right)\\
&=\frac{1}{12}\sum_{n=0}^{\infty}\left(\zeta_H\left(2,\frac{bn+x+a}{a}\right)-\frac{a}{bn+x}\right)\\
&\quad+\frac{1}{12}\sum_{n=0}^{\infty}\left(\frac{a}{bn+x}-\frac{a}{bn+x+a}\right)-\frac{a}{12b}\psi\left(\frac{x+a}{b}\right)\\
&=\frac{1}{12}\sum_{n=0}^{\infty}\left(\zeta_H\left(2,\frac{bn+x+a}{a}\right)-\frac{a}{bn+x}\right)-\frac{a}{12b}\psi\left(\frac{x}{b}\right).
\end{aligned}$$

Next, we obtain the second equation from the first. For, consider the following

$$\begin{aligned}
&\frac{1}{2}\sum_{k=2}^{\infty}\frac{k}{(k+1)(k+2)}\zeta_2(k+1;a,b,x+a)a^{k+1}\\
&=\frac{1}{2}\sum_{k=2}^{\infty}\frac{k}{(k+1)(k+2)}\sum_{m,n=0}^{\infty}(am+bn+x+a)^{-k-1}a^{k+1}\\
&=\frac{1}{2}\sum_{k=2}^{\infty}\frac{k}{(k+1)(k+2)}\sum_{n=0}^{\infty}\zeta_H\left(k+1,\frac{bn+x}{a}+1\right)\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\frac{k}{(k+1)(k+2)}\zeta_H\left(k+1,\frac{bn+x}{a}+1\right)-\frac{1}{6}\zeta_H\left(2,\frac{bn+x}{a}+1\right)\right)\\
&=\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}\frac{k}{(k+1)(k+2)}\zeta_H\left(k+1,\frac{bn+x}{a}+1\right)-\frac{1}{6}\frac{a}{bn+x}\right)\\
&\quad-\frac{1}{12}\sum_{n=0}^{\infty}\left(\zeta_H\left(2,\frac{bn+x}{a}+1\right)-\frac{a}{bn+x}\right). \qquad \square
\end{aligned}$$

Corollary 7.2.

$$\chi'(0;a,b)=\frac{1}{2}\log a+\left(\frac{1}{4}-\frac{b}{12a}-\frac{a}{12b}\right)\log b-\frac{3}{4}\log 2\pi+\frac{b}{12a}+\frac{a\gamma}{12b}-\frac{b}{a}\zeta_R'(-1)$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{k\zeta_H(k+1,\frac{bn}{a}+1)}{(k+1)(k+2)}-\frac{1}{6}\frac{a}{bn}\right)$$

$$= \frac{1}{2}\log a + \left(\frac{1}{4} - \frac{b}{12a} - \frac{a}{12b}\right)\log b - \frac{3}{4}\log 2\pi + \frac{b}{12a} + \frac{a\gamma}{12b} - \frac{b}{a}\zeta_R'(-1)$$
$$+ \frac{1}{12}\sum_{n=1}^{\infty}\left(\zeta_H\left(2, \frac{bn}{a}+1\right) - \frac{a}{bn}\right)$$
$$+ \frac{1}{2}\sum_{k=2}^{\infty}\frac{k}{(k+1)(k+2)}a^{k+1}\sum_{m,n=1}^{\infty}(am+bn)^{-k-1}.$$

Corollary 7.3.

$$\log\Gamma_2(x; a.b) = -\frac{1}{2}\log a + \frac{1}{2}\log 2\pi + \frac{1}{2}\left(1-\frac{x}{b}\right)\left(\frac{x}{a}-1\right)\log b - \frac{1}{2a}\left(1-\frac{x}{b}\right)x$$
$$+ \frac{1}{2}\log\Gamma\left(\frac{x}{b}\right) - \frac{b}{a}\left(\zeta_H'\left(-1, \frac{x}{b}\right) - \zeta_R'(-1)\right)$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{k(\zeta_H(k+1, \frac{bn+x}{a}+1) - \zeta_H(k+1, \frac{bn+b}{a}+1))}{(k+1)(k+2)}.$$

Note that the values of $\Gamma_2(b; a, b)$ follows immediately from this equation. However it will be reobtained in the following section together with other particular values of the Gamma function.

8. Properties of the Barnes Gamma function as a function of *x*: Product representation, functional equation, Taylor expansion, asymptotic expansion, and particular values

In this section we study the function $\Gamma_2(x; a, b)$ as a function of x. We start by giving a Lerch type formula for the zeta function $\zeta(s, S_x)$, and a product representation for the Barnes double Gamma function. This representation will make clear that we can extend the domain of definition to complex values of the variable x. Next, we give the principal analytic properties of the double Gamma function, following the natural line of investigation dictated by the analogy with the classical Euler Gamma function, and already followed when investigating other generalizations [23,25,27].

Our first result follows by direct application of the generalized Lerch formula given in Proposition 2.9 of [27].

Proposition 8.1.

$$\zeta'(0, S_x) = \chi'(0; a, b) - \operatorname*{Res}_{s=1}{}_0\,\chi(s; a, b)x + \frac{1}{2}\left(\operatorname*{Res}_{s=2}{}_0\,\chi(s; a, b) + \operatorname*{Res}_{s=2}{}_1\,\chi(s; a, b)\right)x^2 + \log\Gamma(x, S_0).$$

Note that we have explicit formulas for the residues in the previous sections. As promised, we give now a product definition for the Barnes Gamma function. This follows from the definition and from the two Lerch formulas, Propositions 1.1 and 8.1, that also give the following relation on the Gamma functions:

Lemma 8.2.

$$\log\Gamma_2(x; a, b) = -\log x - \operatorname*{Res}_{s=1}{}_0\,\chi(s; a, b)x + \frac{1}{2}\left(\operatorname*{Res}_{s=2}{}_0\,\chi(s; a, b) + \operatorname*{Res}_{s=2}{}_1\,\chi(s; a, b)\right)x^2 + \log\Gamma(x, S_0).$$

Proposition 8.3 *(Product representation).*

$$\Gamma_2(x;a,b)=\frac{e^{-\operatorname*{Res}_{s=1}{}_0\,\chi(s;a,b)x+\frac{1}{2}(\operatorname*{Res}_{s=2}{}_0\,\chi(s;a,b)+\operatorname*{Res}_{s=2}{}_1\,\chi(s;a,b))x^2}}{x}\prod_{(m,n)\in\mathbb{N}_0^2}^{\infty}\frac{e^{\frac{x}{am+bn}-\frac{x^2}{2(am+bn)^2}}}{1+\frac{x}{am+bn}}.$$

From this equation it is apparent that the function $\Gamma_2(x;a,b)$ (originally defined for real x) has an analytic extension to the whole complex x-plane except at the points $x=-(am+bn)$, $(m,n)\in\mathbb{N}_0^2$, where it is has a pole.

Remark 8.4. Note that the poles of the function $\Gamma_2(x;a,b)$ are not necessarily simple poles, in fact, for example when $a=b=1$, they are all double, as can be seen using Proposition 8.5 below.

The case $a=b=1$ deserves some comments. Recalling that $\chi(s;1,1)=\zeta_R(s)+\zeta_R(s-1)$, after some computation we obtain

$$\Gamma_2(x;1,1)=\frac{e^{-(\gamma-\frac{1}{2})x+\frac{1}{2}(\frac{\pi^2}{6}+\gamma+1)x^2}}{x}\prod_{(m,n)\in\mathbb{N}_0^2}^{\infty}\frac{e^{\frac{x}{m+n}-\frac{x^2}{2(m+n)^2}}}{1+\frac{x}{m+n}},$$

and we see that $\Gamma_2(x;1,1)$ corresponds to the particular case $m=1$ of the multiple Gamma function $\Gamma_m(x+1)$ studied in [27] (see also [30]). In fact, comparing with the definition of the corresponding function $\Gamma_1(x+1)$, given at the beginning of Section 3.2 of [27], we have

$$\log\Gamma_2(x;1,1)=\log\Gamma_1(x+1)-\log x.$$

Moreover, we have the following relation with the Barnes G-function [5], [32, p. 264].

Proposition 8.5.

$$G(x+1)=\frac{\Gamma(x)}{\Gamma_2(x;1,1)},$$

where $G(x+1)=G_1(x+1)$ is the Barnes G-function, as defined classically (see for example [27, 3.2] *or* [19, (1.13)]*).*

Proposition 8.6 *(Functional equation).*

$$\begin{aligned}\Gamma_2(x+b;a,b)&=a^{\zeta_H(0,\frac{x}{a})}e^{-\zeta_H'(0,\frac{x}{a})}\Gamma_2(x;a,b)\\&=\frac{\sqrt{2\pi}\,a^{\frac{1}{2}-\frac{x}{a}}}{\Gamma(\frac{x}{a})}\Gamma_2(x;a,b).\end{aligned}$$

Proof. We have that

$$\begin{aligned}\zeta_2(s;a,b,x+b)&=\sum_{m,n=0}^{\infty}\big(am+b(n+1)+x\big)^{-s}\\&=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(am+bn+x)^{-s}\end{aligned}$$

$$= \sum_{m,n=0}^{\infty}(am+bn+x)^{-s} - \sum_{m=0}^{\infty}(am+x)^{-s}$$
$$= \zeta_2(s;a,b,x) - a^{-s}\zeta_H\left(s,\frac{x}{a}\right).$$

Therefore, taking the derivative and applying the Lerch formula 1.1, we obtain

$$\log \Gamma_2(x+b;a,b) + \chi'(0;a,b) = \log \Gamma_2(s;a,b) + \chi'(0;a,b) + \zeta_H\left(0,\frac{x}{a}\right)\log a - \zeta_H'\left(0,\frac{x}{a}\right),$$

and the thesis. □

Corollary 8.7.

$$\Gamma_2(x+1;a,1) = a^{\zeta_H(0,\frac{x}{a})}\mathrm{e}^{-\zeta_H'(0,\frac{x}{a})}\Gamma_2(x;a,1)$$
$$= \frac{\sqrt{2\pi}a^{\frac{1}{2}-\frac{x}{a}}}{\Gamma(\frac{x}{a})}\Gamma_2(x;a,1).$$

Corollary 8.8.

$$\frac{1}{\Gamma_2(x;a,b)\Gamma_2(-x;a,b)} = \sin_2(x;a,b),$$

where

$$\sin_2(x;a,b) = -x^2\mathrm{e}^{(\operatorname{Res}_{s=2}\chi(s;a,b)+\operatorname{Res}_{s=2}\chi(s;a,b))x^2}\prod_{(m,n)\in\mathbb{N}_0^2}\left(1+\frac{x^2}{am+bn}\right)\mathrm{e}^{\frac{x^2}{(am+bn)^2}}.$$

The function $s_2(x;1,1)$ is similarly strictly related with the double sine function, see the works of Kurokawa (for example [13] and references therein), see also Section 3.2 of [27].

Proposition 8.9 *(Series expansion). For $x<1$,*

$$\log \Gamma_2(x;a,b) = -\log x - \operatorname{Res}_{s=1}\chi(s;a,b)x + \frac{1}{2}\Big(\operatorname{Res}_{s=2}\chi(s;a,b) + \operatorname{Res}_{s=2}\chi(s;a,b)\Big)x^2$$
$$+ \sum_{j=3}^{\infty}\frac{(-1)^j}{j}\chi(j;a,b)x^j.$$

Proof. This follows directly from Lemma 8.2. □

Using the results of Sections 2 and 4, Remark 2.6 and Proposition 2.14 of [29], we have the following result.

Lemma 8.10. *For large λ with $|\arg(-\lambda)|<\pi$,*

$$\log \Gamma(-\lambda, S_0) \sim a_{2,1}(-\lambda)^2\log(-\lambda) + a_{1,1}(-\lambda)\log(-\lambda) + a_{0,1}\log(-\lambda) + \sum_{h=0}^{\infty}a_{2-h,0}(-\lambda)^{2-h},$$

where

$$a_{2,1} = -\frac{1}{2}e_{-2} = -\frac{1}{2ab},$$
$$a_{1,1} = e_{-1} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right),$$
$$a_{0,1} = -e_0 = \frac{3}{4} - \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a}\right),$$
$$a_{2,0} = \frac{1}{2}\left(\frac{1}{2}e_{-2} - \operatorname*{Res_0}_{s=2} \chi(s; a, b)\right) = \frac{1}{2}\left(\frac{1}{2ab} - \operatorname*{Res_0}_{s=2} \chi(s; a, b)\right),$$
$$a_{1,0} = -\left(e_{-1} - \operatorname*{Res_0}_{s=1} \chi(s; a, b)\right) = -\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) + \operatorname*{Res_0}_{s=1} \chi(s; a, b),$$
$$a_{0,0} = -\chi'(0; a, b),$$
$$a_{2-h,0} = \Gamma(h-2)e_{h-2}, \quad h > 2.$$

Using Lemmas 8.2 and 8.10 we get the following expansion.

Proposition 8.11 *(Asymptotic expansion). For large x with $|\arg(x)| < \pi$,*

$$\log \Gamma_2(x; a, b) \sim -\frac{1}{2ab}x^2 \log x + \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)x \log x - \left(\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a}\right)\right) \log x$$
$$+ \frac{3}{4ab}x^2 - \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)x - \chi'(0; a, b) + \sum_{h=3}^{\infty} \Gamma(h-2)e_{h-2}x^{2-h}.$$

We conclude with some particular values.

Proposition 8.12 *(Particular values).*

$$\Gamma_2(a; a, b) = \Gamma_2(a; b, a) = \sqrt{\frac{2\pi}{b}}, \tag{12}$$
$$\Gamma_2(a + b; a, b) = \frac{2\pi}{\sqrt{ab}}, \tag{13}$$
$$\Gamma_2(a + 1; a, 1) = \Gamma_2(a + 1; 1, a) = \frac{2\pi}{\sqrt{a}}, \tag{14}$$
$$\Gamma_2(x + 1; 1, 1) = \frac{\sqrt{2\pi}}{\Gamma(x)}\Gamma_2(x; 1, 1), \tag{15}$$
$$\Gamma_2(a; a, 1) = \Gamma_2(1; 1, 1) = \sqrt{2\pi}, \tag{16}$$
$$\Gamma_2(2; 1, 1) = 2\pi. \tag{17}$$

Proof. All formulas follow from Proposition 8.6, Corollary 8.7, and Eq. (1). □

9. Asymptotic expansions for small and large *a*

We present in this section asymptotic expansions of the functions $\zeta_2(s;a,b,x)$ and $\log\Gamma_2(x;a,b)$, for small and large a. Let us start with small a. Expansions for these functions for small a have been given by Matsumoto in [15], where an estimate for the remainder is also proved. We present here a different proof, based on a simple application of the technique introduced in [29], already used in Section 6. We also note that an even simpler proof can be obtained using the method that we apply in the proof of Proposition 9.7 below.

Proposition 9.1. *For small a, we have the asymptotic expansion*

$$\zeta_2(s;a,b,x)\sim\frac{b^{1-s}}{a}\frac{1}{s-1}\zeta_H\left(s-1,\frac{x}{b}\right)+\frac{1}{2}b^{-s}\zeta_H\left(s,\frac{x}{b}\right)$$
$$+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k}\frac{\Gamma(s+2k-1)}{\Gamma(s)\Gamma(2k)}b^{1-s-2k}\zeta_H\left(s+2k-1,\frac{x}{b}\right)a^{2k-1}.$$

Proof. Suppose we apply the same approach as in Section 7, as far as we obtain the following representation of the zeta function

$$\zeta_2(s;a,b,x)=\frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}\mathcal{T}(s,\lambda)\,d\lambda\,dt,$$

where

$$\mathcal{T}(s,\lambda)=\sum_{n=0}^{\infty}(bn+x)^{-s}\log\Gamma(s,\tilde{S}_n),$$

and (see Eq. (4))

$$\log\Gamma(\lambda,\tilde{S}_n)=-\lambda+\gamma\frac{bn+x}{a}(-\lambda)+\log\Gamma\left(\frac{bn+x}{a}(1-\lambda)\right)$$
$$-\log\Gamma\left(\frac{bn+x}{a}\right)-\sum_{m=1}^{\infty}\frac{(\frac{bn+x}{a})^2}{m(\frac{bn+x}{a}+m)}(-\lambda).$$

Next, instead of using the series representation of the Euler Gamma function, we use the asymptotic expansion for $a<1$:

$$\log\Gamma\left(\frac{bn+x}{a}(1-\lambda)\right)\sim\left(\frac{bn+x}{a}(1-\lambda)-\frac{1}{2}\right)\log\left(\frac{bn+x}{a}(1-\lambda)\right)$$
$$-\frac{bn+x}{a}(1-\lambda)+\frac{1}{2}\log 2\pi$$
$$+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}(bn+x)^{1-2k}(1-\lambda)^{1-2k}a^{2k-1}.$$

We can perform all λ-integrations as in Section 6 using the equations provided in Appendix A. After some computations this gives the thesis. $\square$

Remark 9.2. Note that the coefficients can be written in a different form using the equation

$$-\frac{\Gamma(s+2k-1)}{\Gamma(s)\Gamma(2k)} = \binom{-s}{2k-1},$$

that follows easily from classical properties of the binomial coefficient and of the Euler Gamma function.

Corollary 9.3. *For small a,*

$$\chi(s;a,b) \sim \zeta_R(s)a^{-s} + \frac{1}{2}b^{-s}\zeta_R(s) + \frac{1}{a}\frac{b^{1-s}}{s-1}\zeta_R(s-1)$$
$$-\sum_{k=1}^{\infty}\binom{-s}{2k-1}\frac{B_{2k}}{2k}\zeta_R(s+2k-1)b^{1-s-2k}a^{2k-1},$$

where the series is a finite sum whenever s is a non-positive integer.

Corollary 9.4. *For small a,*

$$\zeta_2'(0;a,b,x) \sim b\left(\zeta_H\left(-1,\frac{x}{b}\right)\log b - \zeta_H\left(-1,\frac{x}{b}\right) - \zeta_H'\left(-1,\frac{x}{b}\right)\right)a^{-1}$$
$$-\frac{1}{2}\zeta_H\left(0,\frac{x}{b}\right)\log b + \frac{1}{2}\zeta_H'\left(0,\frac{x}{b}\right) - \frac{1}{12b}\left(\log b + \psi\left(\frac{x}{b}\right)\right)a$$
$$-\sum_{k=2}^{\infty}\frac{(-1)^{2k-1}}{2k-1}\frac{B_{2k}}{2k}b^{1-2k}\zeta_H\left(2k-1,\frac{x}{b}\right)a^{2k-1}.$$

Corollary 9.5. *For small a,*

$$\chi'(0;a,b) \sim b\left(\frac{1}{12} - \frac{1}{12}\log b - \zeta_R'(-1)\right)a^{-1} + \frac{1}{2}\log a + \frac{1}{4}\log b - \frac{3}{4}\log 2\pi$$
$$-\frac{1}{12b}(\log b - \gamma)a - \sum_{k=2}^{\infty}\frac{(-1)^{2k-1}}{2k-1}\frac{B_{2k}}{2k}b^{1-2k}\zeta_R(2k-1)a^{2k-1}.$$

Corollary 9.6. *For small a,*

$$\log\Gamma_2(x;a,b) \sim b\left(\frac{x}{2b}\left(1-\frac{x}{b}\right)\log b + \frac{1}{2}\left(\frac{x}{b}-1\right)\frac{x}{b} - \zeta_H'\left(-1,\frac{x}{b}\right) + \zeta_R'(-1)\right)a^{-1}$$
$$-\frac{1}{2}\log a - \frac{1}{2}\left(1-\frac{x}{b}\right)\log b + \frac{1}{2}\log\Gamma\left(\frac{x}{b}\right) + \frac{1}{2}\log 2\pi$$
$$-\frac{1}{12b}\left(\psi\left(\frac{x}{b}\right)+\gamma\right)a$$
$$-\sum_{k=2}^{\infty}\frac{(-1)^{2k-1}}{2k-1}\frac{B_{2k}}{2k}b^{1-2k}\left(\zeta_H\left(2k-1,\frac{x}{b}\right) - \zeta_R(2k-1)\right)a^{2k-1}.$$

Next, we consider the expansions for large a. Such expansions were also given by Matsumoto in the cited papers. However, we present here an elementary proof. As observed at the beginning of this section, the method used in the following proof can be used to prove the previous expansions for small a as well.

Proposition 9.7. *For large a,*

$$\zeta_2(s;a,b,x) \sim b^{-s}\zeta_H\left(s,\frac{x}{b}\right) + \frac{1}{b}\frac{a^{1-s}}{s-1}\zeta_R(s-1) + a^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}\zeta_R(s+j)\zeta_H\left(-j,\frac{x}{b}\right)b^j a^{-j},$$

where the series is a finite sum whenever s is a non-positive integer.

Proof. For $\mathrm{Re}(s) > 2$, applying the Plana theorem as in Proposition 3.1, we have

$$\begin{aligned}\zeta_2(s;a,b,x) &= \frac{1}{2}\sum_{m=0}^{\infty}(am+x)^{-s} + \sum_{m=0}^{\infty}\int_0^{\infty}(am+bt+x)^{-s}\,dt + I(s;a,b,x)\\ &= \frac{1}{2}a^{-s}\zeta_H\left(s,\frac{x}{a}\right) + \frac{1}{b}\frac{a^{1-s}}{s-1}\zeta_H\left(s-1,\frac{x}{a}\right) + I(s;a,b,x),\end{aligned}$$

where

$$\begin{aligned}I(s;a,b,x) &= ia^{-s}\int_0^{\infty}\frac{\zeta_H(s,\frac{x+iby}{a}) - \zeta_H(s,\frac{x-iby}{a})}{e^{2\pi y}-1}\,dy\\ &= i\int_0^{\infty}\frac{(x+iby)^{-s}-(x-iby)^{-s}}{e^{2\pi y}-1}\,dy\\ &\quad + i\sum_{m=1}^{\infty}\int_0^{\infty}\frac{(am+x+iby)^{-s}-(am+x-iby)^{-s}}{e^{2\pi y}-1}\,dy.\end{aligned}$$

For the first term,

$$\begin{aligned}i\int_0^{\infty}\frac{(x+iby)^{-s}-(x-iby)^{-s}}{e^{2\pi y}-1}\,dy &= 2\int_0^{\infty}\left(\left(\frac{x}{b}\right)^2+y^2\right)^{-\frac{s}{2}}\sin\left(s\arctan\frac{by}{x}\right)\frac{dy}{e^{2\pi y}-1}\\ &= b^{-s}\left(\zeta_H\left(s,\frac{x}{b}\right) - \frac{1}{2}x^{-s}b^s - \frac{x^{1-s}}{s-1}b^{s-1}\right).\end{aligned}$$

For the second term,

$$
\begin{aligned}
& i\sum_{m=1}^{\infty}\int_0^{\infty}\frac{(am+x+iby)^{-s}-(am+x-iby)^{-s}}{e^{2\pi y}-1}\,dy \\
&\quad = i\sum_{m=1}^{\infty}(am)^{-s}\int_0^{\infty}\sum_{j=0}^{\infty}\binom{-s}{j}\big((x+iby)^j-(x-iby)^j\big)(am)^{-j}\frac{dy}{e^{2\pi y}-1} \\
&\quad \sim -2\sum_{j=1}^{\infty}\binom{-s}{j}b^j\sum_{m=0}^{\infty}(am)^{-s-j}\int_0^{\infty}\left(\left(\frac{x}{b}\right)^2+y^2\right)^{\frac{j}{2}}\sin\left(j\arctan\frac{by}{x}\right)\frac{dy}{e^{2\pi y}-1} \\
&\quad = \sum_{j=1}^{\infty}\binom{-s}{j}a^{-s-j}b^j\zeta_R(s+j)\left(\zeta_H\left(-j,\frac{x}{b}\right)-\frac{1}{2}x^jb^{-j}+\frac{x^{j+1}}{j+1}b^{-j-1}\right),
\end{aligned}
$$

where note that the series is only asymptotic and can be extended to include the term with $j=0$. Summing up

$$
\begin{aligned}
I(s;a,b,x) &= b^{-s}\zeta_H\left(s,\frac{x}{b}\right)-\frac{1}{2}x^{-s}-\frac{1}{b}\frac{x^{1-s}}{s-1}+a^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}\zeta_R(s+j)\zeta_H\left(-j,\frac{x}{b}\right)a^{-j}b^j \\
&\quad -\frac{1}{2}a^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}\zeta_R(s+j)a^{-j}x^j+\frac{a^{-s}}{b}\sum_{j=0}^{\infty}\binom{-s}{j}\frac{\zeta_R(s+j)}{j+1}a^{-j}x^{j+1}.
\end{aligned}
$$

Now consider the following equivalences:

$$
\frac{1}{2}a^{-s}\zeta_H\left(s,\frac{x}{a}\right)=\frac{1}{2}x^{-s}+\frac{1}{2}a^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}\zeta_R(s+j)x^ja^{-j},
$$

and

$$
\frac{a^{1-s}}{s-1}\zeta_H\left(s-1,\frac{x}{a}\right)=\frac{x^{1-s}}{s-1}+\frac{a^{1-s}}{s-1}\zeta_R(s-1)+a^{-s}\sum_{j=0}^{\infty}\binom{1-s}{j+1}\frac{\zeta_R(s+j)}{s-1}a^{-j}x^{j+1}.
$$

Using this equalities and the expression for $I(s;a,b,x)$, we obtain

$$
\begin{aligned}
\zeta_2(s;a,b,x) &= b^{-s}\zeta_H\left(s,\frac{x}{b}\right)+\frac{1}{b}\frac{a^{1-s}}{s-1}\zeta_R(s-1)+a^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}\zeta_R(s+j)\zeta_H\left(-j,\frac{x}{b}\right)b^ja^{-j} \\
&\quad +\frac{a^{-s}}{b}\sum_{j=0}^{\infty}\left(\binom{-s}{j}\frac{1}{j+1}+\binom{1-s}{j+1}\frac{1}{s-1}\right)\zeta_R(s+j)a^{-j}x^{j+1},
\end{aligned}
$$

and the thesis. □

Corollary 9.8. *For large a,*

$$\chi(s;a,b)\sim\zeta_R(s)a^{-s}+b^{-s}\zeta_R(s)+\frac{1}{b}\frac{a^{1-s}}{s-1}\zeta_R(s-1)+a^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}\zeta_R(s+j)\zeta_R(-j)b^ja^{-j},$$

where the series is a finite sum whenever s is a non-positive integer.

Using these equations, it is immediate to obtain the value of the zeta functions at negative integers. This result was given in [14, Theorem 5], and for k even in [2, (2.6)].

Corollary 9.9. *For* $-k=-1,-2,-3,\dots,$

$$\zeta_2(-k;a,b,x)=-\frac{b^kB_k(\frac{x}{b})}{k+1}+k!\sum_{j=1}^{k+2}\frac{B_j}{j!}\frac{B_{k-j+2}(\frac{x}{b})}{(k-j+2)!}b^{k-j+1}a^{j-1}.$$

From these equations follow the expansions of the derivative at $s=0$ and the expansion of the Gamma function.

Proposition 9.10. *For large a,*

$$\begin{aligned}\zeta_2'(0;a,b,x)\sim{}&-\frac{1}{12b}a\log a+\frac{1}{b}\left(\frac{1}{12}-\zeta_R'(-1)\right)a+\frac{1}{2}\zeta_H\left(0,\frac{x}{b}\right)\log a-\zeta_H\left(0,\frac{x}{b}\right)\log b\\&+\zeta_H'\left(0,\frac{x}{b}\right)-\frac{1}{2}\zeta_H\left(0,\frac{x}{b}\right)\log 2\pi+b\zeta_H\left(-1,\frac{x}{b}\right)a^{-1}\log a\\&-\gamma b\zeta_H\left(-1,\frac{x}{b}\right)a^{-1}+\sum_{j=2}^{\infty}\frac{(-1)^j}{j}\zeta_R(j)\zeta_H\left(-j,\frac{x}{b}\right)b^ja^{-j}.\end{aligned}$$

Corollary 9.11. *For large a,*

$$\begin{aligned}\chi'(0;a,b)={}&-\frac{1}{12b}a\log a+\frac{1}{b}\left(\frac{1}{12}-\zeta_R'(-1)\right)a+\frac{1}{4}\log a+\frac{1}{2}\log b-\frac{3}{4}\log 2\pi\\&-\frac{b}{12}a^{-1}\log a+\frac{\gamma b}{12}a^{-1}+\sum_{j=2}^{\infty}\frac{(-1)^j}{j}\zeta_R(j)\zeta_R(-j)b^ja^{-j}.\end{aligned}$$

Corollary 9.12. *For large a,*

$$\begin{aligned}\log\Gamma_2(x;a,b)\sim{}&-\frac{1}{2}\frac{x}{b}\log a+\left(\frac{x}{b}-1\right)\log b+\log\Gamma\left(\frac{x}{b}\right)+\frac{1}{2}\frac{x}{b}\log 2\pi-\frac{x}{2}\left(\frac{x}{b}-1\right)a^{-1}\log a\\&+\frac{\gamma x}{2}\left(\frac{x}{b}-1\right)a^{-1}+\sum_{j=2}^{\infty}\frac{(-1)^j}{j}\zeta_R(j)\left(\zeta_H\left(-j,\frac{x}{b}\right)-\zeta_R(-j)\right)b^ja^{-j}.\end{aligned}$$

Remark 9.13. As observed by Matsumoto in [15], the above expansions extend analytically to complex values of the parameter a in the opportune domains.

Appendix A

We give in this appendix some complex contour integrals used in the text. Let $\Lambda_{\theta,c} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - c)| = \frac{\theta}{2}\}$, $0 < \theta < \pi$, $0 < c < 1$, a real, $k = 0, 1, 2, \ldots$. Then we have the following formulas.

$$\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \mathrm{e}^{-\lambda}(-\lambda)^a \, d\lambda = 0, \tag{18}$$

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda}\frac{1}{(1-\lambda)^a}\, d\lambda\, dt = \frac{\Gamma(s+a)}{\Gamma(a)s}. \tag{19}$$

See [26].

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda}\frac{\log(1-\lambda)}{(1-\lambda)^a}\, d\lambda\, dt = \frac{\Gamma(s+a)}{\Gamma(a)s}\big(\psi(a) - \psi(s+a)\big). \tag{20}$$

Take logarithmic derivative of Eq. (19).

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda}(1-\lambda)^k\log(1-\lambda)\, d\lambda\, dt = (-1)^{k+1}k!\frac{\Gamma(s-k)}{s}. \tag{21}$$

This is a particular case of (20), with integer a.

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda}\frac{1}{(\alpha-\beta\lambda)^a}\, d\lambda\, dt = \alpha^{-s-a}\beta^s\frac{\Gamma(s+a)}{\Gamma(a)s}. \tag{22}$$

For, with $x = \frac{\beta}{\alpha}$, and then $x\lambda = \mu$,

$$\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda}\frac{1}{(\alpha-\beta\lambda)^a}\, d\lambda\, dt = \alpha^{-a}\frac{\beta^s}{\alpha^s}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda}\frac{1}{(1-\lambda)^a}\, d\lambda\, dt,$$

and we can use Eq. (19).

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