The Space of Free Loops on a Real Projective Space

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Abstract. Methods from fibrewise topology are used to give a stable splitting of the space of free loops on a real projective space as a wedge of Thom spaces. This splitting is equivariant with respect to the action of the isometry group.

1. Introduction

Suppose that $V \neq 0$ is a finite-dimensional real vector space with an inner product. Let $\mathcal{L}P(V)$ denote the space of continuous free loops on the projective space $P(V)$. The action of the orthogonal group $\mathrm{O}(V)$ on V induces an action of the projective orthogonal group $\mathrm{PO}(V)$ on $P(V)$. Our main result is a $\mathrm{PO}(V)$-equivariant stable splitting of $\mathcal{L}P(V)$.

To describe the stable summands let us write $\mathrm{O}(\mathbb{R}^2, V)$ for the Stiefel manifold of orthogonal 2-frames $b : \mathbb{R}^2 \to V$ in V and $\mathrm{PO}(\mathbb{R}^2, V)$ for the projective Stiefel manifold $\mathrm{O}(\mathbb{R}^2, V)/\{\pm 1\}$. The vector bundle over $\mathrm{PO}(\mathbb{R}^2, V)$ with fibre at $[b]$ the orthogonal complement in V of $b(\mathbb{R}^2)$ will be denoted by ζ.

Theorem 1.1. *There is a* $\mathrm{PO}(V)$*-equivariant stable homotopy equivalence*

$$\big(\mathcal{L}P(V)\big)_+ \simeq P(V)_+ \vee \bigvee_{l=1}^{\infty} \mathrm{PO}(\mathbb{R}^2, V)^{(l-1)\zeta}.$$

Here, and throughout the paper, a subscript $+$ stands for addition of a disjoint basepoint, and the Thom space of a vector bundle ξ over a base B is denoted by B^{ξ}. The stable splitting is understood as a splitting of equivariant spectra. We shall establish a more precise unstable splitting after one suspension in Proposition 3.1.

The methods of this paper are entirely homotopy-theoretic, but some geometric remarks are in order. The space of continuous loops $\mathcal{L}P(V)$ is homotopy equivalent to the manifold M of smooth loops $\omega : S^1 \to P(V)$. Consider the energy functional

$$E : M \to \mathbb{R}, \quad E(\omega) = \frac{1}{2}\int_0^{2\pi} \|\omega'(\theta)\|^2 d\theta.$$

This is a Morse-Bott function with critical submanifolds C_l, $l \in \mathbb{N}$, where C_0 is the space of constant loops and C_l, for $l \geq 1$, is the space of closed geodesics of multiplicity l (so of length πl). One can easily identify C_l with $\mathrm{PO}(\mathbb{R}^2, V)$ for $l \geq 1$.

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The negative bundle of C_l is 0 if $l = 0$ and corresponds to $(l-1)\zeta$ if $l \geq 1$. See, for example, [**9**, Section 2] and [**3**].

The proof of Theorem 1.1 relies on an equivariant refinement of one of the main results in [**1**]. In Section 2 we review some of the constructions in [**1**] and show that they can be carried out equivariantly to obtain a stable splitting for the (based) loop space of the projective space of a real vector space (Proposition 2.2). In the next section we interpret $\mathcal{L}P(V)$ as a fibrewise (based) loop space and deduce the main theorem. The last section describes an $\mathrm{O}(V)$-equivariant stable splitting of the space $\mathcal{L}S(V)$ of continuous free loops on the unit sphere $S(V)$ in V.

2. An equivariant fibrewise stable splitting for projective bundles

Let W be a finite-dimensional real vector space with an inner product. Writing $[x]$ for the element of a projective space determined by a non-zero vector x, we choose the point $[1,0]$ as basepoint in the projective space $P(\mathbb{R}\oplus W)$. It is understood that $\mathrm{O}(W)$ acts trivially on the one-dimensional space $\mathbb{R}$. The space $P(\mathbb{R}\oplus W)$ then has an induced $\mathrm{O}(W)$-action.

Let $l \in \mathbb{N}$ and $l \geq 1$. We define a map

$$\tilde{\gamma}_l : S(W)^l \to \Omega P(\mathbb{R}\oplus W)$$

by setting

$$\tilde{\gamma}_l(x_1, x_2, \ldots, x_l)(t) = \big[\cos(l\pi t), \sin(l\pi t)x_j\big] \quad \text{for } (j-1)/l \leq t \leq j/l.$$

The loop $\tilde{\gamma}_1(x)$ is a closed geodesic lifting to a great semi-circle on the sphere $S(\mathbb{R}\oplus W)$ from $(1,0)$ through $(0,x)$ to $(-1,0)$. The map $\tilde{\gamma}_l$ assigns to an l-tuple of points $(x_1, x_2, \ldots, x_l)$ a piecewise smooth geodesic whose pieces are reparameterized closed paths $\tilde{\gamma}_1(x_j)$. In particular, if $x_1 = x_2 = \ldots = x_l$ this gives a closed geodesic of multiplicity l.

Let U be a codimension 1 subspace of W. Consider the embedding $S(\mathbb{R}\oplus U) \subseteq \mathbb{R}\oplus U$ with trivial normal bundle $\mathbb{R}\times S(\mathbb{R}\oplus U)$. The Pontrjagin-Thom construction gives a map

$$f : \Sigma U^+ = (\mathbb{R}\oplus U)^+ \to \big(\mathbb{R}\times S(\mathbb{R}\oplus U)\big)^+ = \Sigma S(\mathbb{R}\oplus U)_+, \tag{2.1}$$

where a superscript $+$ is used for one-point compactification. This map f splits the suspension of the stereographic projection $S(\mathbb{R}\oplus U)_+ \to U^+$. Making the appropriate identifications, we obtain maps

$$f_k : \Sigma(kU)^+ \to \Sigma(S(\mathbb{R}\oplus U)^k)_+$$

by defining $f_1 = f$ and $f_k = (f_{k-1}\wedge \mathrm{id})\circ(\mathrm{id}\wedge f)$ for $k \geq 2$.

Viewing U as the tangent space at a point of the sphere $S(W)$, we now apply this construction fibrewise to the tangent bundle $TS(W)$. We write $T \to S$ for the tangent bundle $TS(W) \to S(W)$ for ease of notation and obtain fibrewise maps

$$\Sigma_S(kT)^+_S \to \Sigma_S(S(\mathbb{R}\oplus T)^k_S)_{+S}$$

over S, where $\mathbb{R}$ stands for the one-dimensional real trivial bundle and the subscript S indicates the appropriate fibrewise construction.

Collapsing the basepoint sections S to a point and identifying $\mathbb{R}\oplus T$ with the trivial bundle with fibre W gives maps

$$\alpha_k : \Sigma\big(S(W)^{kTS(W)}\big) \to \Sigma(S(W)^{k+1})_+$$

for $k \geq 1$. Additionally we define α_0 to be the identity on $\Sigma S(W)_+$.

Now define for $l \geq 1$ the map

$$\gamma_l : \Sigma\big(S(W)^l\big)_+ \to \Sigma\big(\Omega P(\mathbb{R} \oplus W)\big)_+$$

to be $(\Sigma\tilde{\gamma}_l)\alpha_{l-1}$. Let γ_0 be the suspension of the pointed map $S^0 \to \Omega P(\mathbb{R} \oplus W)_+$ that takes -1 to the constant loop $t \mapsto [1,0]$. Observe that the construction of the γ_l is $\mathrm{O}(W)$-equivariant.

PROPOSITION 2.2. *The map*

$$\gamma = \gamma_0 \vee \bigvee_{l=1}^{\infty} \gamma_l : \Sigma S^0 \vee \bigvee_{l=1}^{\infty} \Sigma\big(S(W)^{(l-1)TS(W)}\big) \to \Sigma\big(\Omega P(\mathbb{R} \oplus W)\big)_+$$

is an $\mathrm{O}(W)$*-equivariant homotopy equivalence.*

PROOF. We check first of all that the map γ is a non-equivariant homotopy equivalence. There is nothing to do if $W = 0$.

If $\dim W = 1$ the equivalence is clear. For the loop space is homotopically a disjoint union of points indexed by the degree. The lth component of the wedge corresponds to the points $\{l, -l\}$.

If $\dim W > 1$ the loop space has two components Ω_0 and Ω_1 indexed by the fundamental group $\pi_1(P(\mathbb{R} \oplus W)) = \mathbb{Z}/2$. The γ_l map into $\Sigma(\Omega_0)_+$ or $\Sigma(\Omega_1)_+$ according to whether l is even or odd. An integral homology calculation for each component shows that γ is an equivalence. We refer to [**1**] for details.

By [**6**] (or [**8**]), for any compact Lie group G an equivariant map between two G-CW-complexes (or G-ANRs respectively) is an equivariant homotopy equivalence if it induces (ordinary) homotopy equivalences on the fixed point sets of all closed subgroups of G. The spaces with which we are concerned all have the equivariant homotopy type of $\mathrm{O}(W)$-CW-complexes. Indeed, the sphere $S(W)$ and the projective space $P(\mathbb{R} \oplus W)$ are $\mathrm{O}(W)$-CW-complexes, and it follows from an equivariant generalization in [**11**] of Milnor's theorems on spaces having the homotopy type of a CW-complex that the loop space $\Omega P(\mathbb{R} \oplus W)$ has the equivariant homotopy type of an $\mathrm{O}(W)$-CW-complex. Alternatively, one can check that the spaces involved in this proof are of the equivariant homotopy type of $\mathrm{O}(W)$-ANRs.

It remains to show that γ induces equivalences on fixed point spaces of subgroups. Let K be a closed subgroup of $\mathrm{O}(W)$. Then the fixed point subspace $P(\mathbb{R} \oplus W)^K$ is a disjoint union of projective spaces

$$P(\mathbb{R} \oplus W)^K = P(\mathbb{R} \oplus W^K) \amalg \coprod_{\chi \neq 1} P(W_\chi)$$

indexed by the characters $\chi : K \to \{1, -1\}$. The spaces W_χ are the χ-isotypical summands of the K-representation W. Because a loop is fixed if and only if it is pointwise fixed we have $\big(\Omega P(\mathbb{R} \oplus W)\big)^K = \Omega\big(P(\mathbb{R} \oplus W^K)\big)$. Since $S(W)^K = S(W^K)$ and because the fixed point space of $TS(W)$ is $TS(W^K)$, the fixed subspace of the domain of γ is obtained by replacing W by W^K. As all our constructions are natural in W, the restriction of γ to K-fixed points is the corresponding map for W^K instead of W. So we have a homotopy equivalence on the K-fixed points, as required to complete the proof. □

We obtain a fibrewise version of this proposition by using the Borel construction. Let G be a compact Lie group, and let ξ be a G-vector bundle over a compact G-ENR B. Assume ξ has fibre W and let E be the associated G-principal

$\mathrm{O}(W)$-bundle. Then the fibrewise loop space $\Omega_B P(\mathbb{R}\oplus\xi)$ of the projective bundle $E\times_{\mathrm{O}(W)} P(\mathbb{R}\oplus W)$ is equal to

$$\Omega_B P\big(E\times_{\mathrm{O}(W)}(\mathbb{R}\oplus W)\big) = E\times_{\mathrm{O}(W)} \Omega P(\mathbb{R}\oplus W).$$

Abbreviating the $\mathrm{O}(W)$-equivariant homotopy equivalence γ from Proposition 2.2 to $\gamma : X\to Y$, we obtain a G-equivariant fibrewise homotopy equivalence

$$E\times_{\mathrm{O}(W)} X \to E\times_{\mathrm{O}(W)} Y.$$

Collapsing the basepoint sections yields a G-homotopy equivalence

$$(E\times_{\mathrm{O}(W)} X)/B \to (E\times_{\mathrm{O}(W)} Y)/B$$

and so establishes the following corollary.

Corollary 2.3. *Let G be a compact Lie group, and let ξ be a G-vector bundle over a compact G-ENR B. Then there is a G-equivalence*

$$\Sigma\big(\Omega_B P(\mathbb{R}\oplus\xi)\big)_+ \simeq \Sigma B_+ \vee \bigvee_{l=1}^{\infty} \Sigma\big(S(\xi)^{(l-1)T_B S(\xi)}\big),$$

where $T_B S(\xi)$ is the fibrewise tangent bundle of the unit sphere bundle $S(\xi)$ of ξ. □

3. Splitting the space of free loops on a projective space

Consider the trivial bundle $P(V)\times V\to P(V)$. There is a homeomorphism from $\mathcal{L}P(V)$ to the fibrewise loop space of the associated projective bundle:

$$\mathcal{L}P(V) \xrightarrow{\approx} \Omega_{P(V)} P\big(P(V)\times V\big) : \omega\mapsto(\omega(0),\omega).$$

Here, the basepoint in each fibre is given by the diagonal map. Thus the fibre over $L\in P(V)$ is just $P(V)$ with basepoint L. When we make the canonical identification

$$P(V) = P(L^*\otimes V) = P(L^*\otimes(L\oplus L^\perp)) = P(\mathbb{R}\oplus(L^*\otimes L^\perp))$$

(given explicitly by the mapping $[y,z]\mapsto[x^*(y), x^*\otimes z]$, where $x^*\in L^*\setminus\{0\}$, $y\in L$, and $z\in L^\perp$), the basepoint $L\in P(V)$ corresponds to the usual basepoint $[1,0]$ in $P(\mathbb{R}\oplus(L^*\otimes L^\perp))$. Globally, this identifies $P(P(V)\times V)$ with $P(\mathbb{R}\oplus(H^*\otimes H^\perp))$, where H is the canonical line bundle over $P(V)$. Recognizing $H^*\otimes H^\perp$ as the tangent bundle $TP(V)$ we have a $\mathrm{PO}(V)$-equivariant homeomorphism $\mathcal{L}P(V)\approx\Omega_{P(V)}P(\mathbb{R}\oplus TP(V))$.

Setting $\xi = TP(V)$ and $G=\mathrm{PO}(V)$ in Corollary 2.3 gives:

Proposition 3.1. *There is a $\mathrm{PO}(V)$-equivariant homotopy equivalence*

$$\Sigma\big(\mathcal{L}P(V)\big)_+ \simeq \Sigma P(V)_+ \vee \bigvee_{l=1}^{\infty} \Sigma\big(S\big(TP(V)\big)^{(l-1)\tau}\big),$$

where τ is the fibrewise tangent bundle of $S\big(TP(V)\big)\to P(V)$. □

To complete the proof of Theorem 1.1 we need to see that τ over $S\big(TP(V)\big)$ is the same as ζ over $\mathrm{PO}(\mathbb{R}^2,V)$. The spaces $\mathrm{O}(\mathbb{R}^2,V)$ and $S\big(TS(V)\big)$ are easily identified via the map $b\mapsto\big(b(1,0),b(0,1)\big)$. Taking the quotient modulo the action of the group $\{\pm1\}$ gives the required identification. □

REMARK 3.2. Suppose V has a complex structure, and let $\mathrm{U}(V)$ be its unitary group. Then the lth wedge summand, for $l \geq 1$, in Proposition 3.1 has a stable $\mathrm{U}(V)$-equivariant decomposition

$$S\big(TP(V)\big)^{(l-1)\tau} \simeq P(V)^{(l-1)\eta} \vee P(V)^{l\eta},$$

where η is the pull-back of the tangent bundle of the complex projective space of V. This can be seen by splitting $TP(V)$ as the sum of η and the trivial one-dimensional bundle and using f, (2.1), to split $S(\mathbb{R} \oplus \eta)$ over $P(V)$.

4. Free loops on spheres

Similar methods give a decomposition of the free loop space of the sphere. The map γ in Proposition 2.2 restricts to an $\mathrm{O}(W)$-equivalence

$$\Sigma S^0 \vee \bigvee_{m=1}^{\infty} \Sigma\big(S(W)^{(2m-1)TS(W)}\big) \to \Sigma\big(\Omega S(\mathbb{R} \oplus W)\big)_+,$$

and the argument of Section 3 gives:

THEOREM 4.1. *There is an* $\mathrm{O}(V)$*-equivariant stable homotopy equivalence*

$$\big(\mathcal{L}S(V)\big)_+ \simeq S(V)_+ \vee \bigvee_{m=1}^{\infty} \mathrm{O}(\mathbb{R}^2, V)^{(2m-1)\zeta}.$$

□

Non-equivariantly there is a finer decomposition. Suppose $V = \mathbb{R}^2 \oplus U$. As an easy special case of Miller's stable splitting of Stiefel manifolds (see [**10**]) we have

$$\mathrm{O}(\mathbb{R}^2, V)_+ \simeq S^0 \vee P(\mathbb{R}^2)^{H \otimes U} \vee \Sigma(2U)^+,$$

where H is the Hopf bundle over $P(\mathbb{R}^2)$. Since ζ is stably trivial, the mth summand for $m \geq 1$ in Theorem 4.1 is stably

$$\mathrm{O}(\mathbb{R}^2, V)^{(2m-1)\zeta} \simeq \big((2m-1)U\big)^+ \vee \big((2m-1)U\big)^+ \wedge P(\mathbb{R}^2)^{H \otimes U} \vee \Sigma\big((2m+1)U\big)^+.$$

The 0th summand is $S(V)_+ \simeq S^0 \vee \Sigma U^+$.

It is interesting to compare this decomposition with the Carlsson-Cohen splitting, [**4**],

$$\big(\mathcal{L}S(V)\big)_+ \simeq S^0 \vee \bigvee_{n=1}^{\infty} (S^1 \times_{\mathbb{Z}/n} nU)^+,$$

where $\mathbb{Z}/n$ permutes the summands in nU cyclically. (See also [**2**] and [**7**].) For $n = 2m$ even the nth summand is

$$\big((2m-1)U\big)^+ \wedge P(\mathbb{R}^2)^{H \otimes U}.$$

However, for $n = 2m - 1$ odd it is a wedge

$$\Sigma\big((2(m-1)+1)U\big)^+ \vee \big((2m-1)U\big)^+$$

of two pieces coming from the $(m-1)$st and mth summands in Theorem 4.1.

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