Notes on torsion and simple homotopy theory

Preliminary version. Not for divulgation.

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CHAPTER 1

Preliminaries

In this chapter we recall some classical constructions in homotopy theory, that will be used without further comments in the following. Either explicit proofs or detailed references are given. We will briefly review the definition and the main properties of deformation retract and mapping cylinder, and in some more details the definition of CW complexes, including cellular homology, and covering space in the CW category.

1. Homotopy

DEFINITION 1.1. *If $f, g : X \to Y$, then f is* **homotopic** *to g, written $f \sim g$, if there exists a map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$.. The map F is called* **homotopy** *between f and g, sometimes written $f \sim_F g$.*

DEFINITION 1.2. *A map $f : X \to Y$ is an* **homotopy equivalence** *if there exists $g : Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$. We write $X \sim Y$.*

DEFINITION 1.3. *A space X is* **contractible** *if $X \sim *$.*

DEFINITION 1.4. *A space X is* **locally contractible** *if for every $x \in X$, each neighborhood U of x contains an open neighborhood V of x that is contractible to x in U.*

DEFINITION 1.5. *Let A be a subspace of X. If $f, g : X \to Y$, and $f|_A = g|_A$, then f and g are* **homotopic relative** *to A, written $f \sim g$ rel A, if there exists an homotopy $F : X \times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, for all $x \in X$, and $F(a, t) = f(a) = g(a)$ for all $(a, t) \in A \times I$. The map F is called* **relative homotopy** *between f and g, sometimes written $f \sim_F g$ rel A. In the particular case where A is a single point, we say that f and g are* **based homotopic***.*

DEFINITION 1.6. *A map $f : X \to Y$ is an* **homotopy equivalence** *if there exists $g : Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$. We write $X \sim Y$, and we say that X and Y are* **homotopically equivalent***.*

DEFINITION 1.7. *Let (X, A) and (Y, B) two topological pairs. We say that the pair (X, A) is homotopic to the pair (Y, B) and we write $(X, A) \sim (Y, B)$, if they are homotopic in the category of pairs (see ?). This means that there are maps of pair $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$, and homotopies of pairs F and G, i.e. maps such that $fg \sim_F 1_Y$, keeping the image of B in B during the entire homotopy, and $gf \sim_G 1_X$, keeping the image of A in A during the entire homotopy. We aslo say in this case that (X, A) and (Y, B) are* **homotopically equivalent pairs**

It is clear that by taking $A = B = \emptyset$ the pair concepts reduce to the absolute concepts.

DEFINITION 1.8. *Let A be a subspace of X, with inclusion $i : A \to X$. A map $r : X \to A$ that is a left inverse of i (i.e. $ri = 1_A$) is called a* **retraction**. *The subspace A is called a* **retract** *of X.*

Equivalently, $A \subseteq X$ is a retract of X if the identity map $1_A : A \to A$ is extendable to a map $r : X \to A$.

LEMMA 1.1. *If X is Hausdorff, and $A \subseteq X$ a retract of X, then A is closed in X.*

DEFINITION 1.9. *A retraction $r : X \to A$, of a subspace A of X, is a* **deformation retraction** *if ir is homotopic to the identity $1_X : X \to X$. We say that A is a* **deformation retract** *of X in this case.*

DEFINITION 1.10. *A retraction $r : X \to A$, of a subspace A of X, is a* **strong deformation retraction** *if ir is homotopic to the identity $1_X : X \to X$ relatively to A. This means that there exists an homotopy $F : X \times I \to X$ such that*

(1) $F(\_, 0) = 1_X$,
(2) $F(\_, 1) = r$,
(3) $F(a, t) = a$, *for all $a \in A$.*

We say that A is a **strong deformation retract** *of X in this case.*

It is easy to check that if A is a strong deformation retract of X, then the retraction $r : X \to A$ is an homotopy equivalence, the homotopy inverse of which is the inclusion map $i : A \to X$.

DEFINITION 1.11. *Let $i : A \to X$ be the inclusion of a closed subspace. Let $f : A \to Y$ be a map. The push out $Y \sqcup_f X$ is called the* **adjuntion** *or* **attachment** *of X to Y by f. The relevant diagram is*

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ Y & \xrightarrow[\bar{i}]{} & Y \sqcup_f X \end{array}$$

The map $f : X \to Y$ is called the **attaching map** *of the adjunction, and the map $\bar{f} : X \to Y \sqcup_f X$ is called* **characteristic map** *of the adjunction.*

LEMMA 1.2. *Let $q : X \sqcup Y \to X \sqcup_f Y$ be the identification map appearing in the definition of the push out. Then,*

(1) *Y is embedded as a closed subset, homeomorphic to Y, and $q|_Y$ is an homeomorphism,*
(2) *$X - A$ is embedded homeomorphically as an open subset, and $q|_{X-A}$ is an homeomorphism.*

PROOF. [**2**] pg. 128. □

LEMMA 1.3. **Law of vertical composition of adjunctions***: let $A \subset X$ closed, $f : A \to B$, $g : B \to C$, then*

$$\begin{array}{ccc}
A & \xrightarrow{i} & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\
B & \xrightarrow[\bar{i}]{} & B \sqcup_f X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{g}} \\
C & \longrightarrow & C \sqcup_g (B \sqcup_f X) \cong C \sqcup_{gf} X
\end{array}$$

Proof. [**4**] pg. 38. □

Lemma 1.4. **Law of horizontal composition of adjunctions**: *let $A \subset B$ closed, $B \subset X$, closed, $f : A \to Y$, then*

$$\begin{array}{ccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} & & \downarrow \\
Y & \xrightarrow[\bar{i}]{} & Y \sqcup_f B & \xrightarrow[\bar{j}]{} & (Y \sqcup_f B) \sqcup_{\bar{f}} X \cong Y \sqcup_f B
\end{array}$$

Proof. [**4**] pg. 38. □

Remark 1.1. *Let Y be a subspace of Z and $y_0 \in Y$. Then, Z/Y is a push out of the inclusion $Y \to Z$ and the constant map $Y \to y_0$.*

Definition 1.12. *Let $(X, x_0) \in Top_*$. The* **cylinder** *of X is the space $Cyl(X) = X \times I$. The* **cone** *of (X, x_0) is the space*

$$CX := (X \times I)/(X \times \{0\}).$$

The space X is embedded in CX as a closed subspace by the map

$$i : X \to CX,$$
$$i : x \mapsto [(1, x)].$$

Let $j : X \to X \times I$, taking x into $(x, 0)$. Then, the cone CX is the adjunction of $X \times I$ to x_0 via the constant map c_{x_0}, as in the following diagram:

$$\begin{array}{ccc}
X & \xrightarrow{j} & X \times I \\
{\scriptstyle c_{x_0}}\downarrow & & \downarrow{\scriptstyle \bar{c}_{x_0}} \\
x_0 & \xrightarrow[\bar{j}]{} & x_0 \sqcup_{c_{x_0}} X
\end{array}$$

Definition 1.13. *Let $f : X \to Y$. The* **mapping cylinder** *of f is the push out space $M_f := Y \sqcup_f (X \times I)$. The relevant diagram is*

$$\begin{array}{ccc}
X & \xrightarrow{j} & X \times I \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\
Y & \xrightarrow[\bar{j}]{} & M_f := Y \sqcup_f (X \times I)
\end{array}$$

where $j : X \to X \times I$ is the inclusion $j(x) = (x, 0)$.

Definition 1.14. *Let $f : X \to Y$. The* **mapping cone** *of f is the push out space $C_f := X \sqcup_f CX$. The relevant diagram is*

$$\begin{array}{ccc} X & \xrightarrow{\;j\;} & CX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ Y & \xrightarrow[\bar{j}]{} & C_f := Y \sqcup_f CX \end{array}$$

where $j : X \to CX$ is the inclusion $j(x) = [(x, 0)]$.

By definition the mapping cylinder of $f : X \to Y$ is the the quotient space

$$M_f = \frac{(X \times I) \sqcup Y}{(x, 0) = f(x)}.$$

Let denote by q the identification map $(X \times I) \sqcup Y \to M_f$. We denote the class $q(z)$ by $[z]$, where $z \in (X \times I) \sqcup Y$. Note that $\bar{j} = q|_Y$ embeds Y homeomorphically as a closed subset of M_f, and $q|_{X\times(0,1]}$ embeds $X \times (0, 1]$ homeomorphically as an open subset of M_f, we write $q(X \times \{0\}) = X$ and $q(Y) = Y$. In particular, the map $i : X \to M_f$, $i(x) = q(x, 1)$, is a homeomorphism of X onto the upper face $X \times \{1\}$ of M_f. To cut down on symbolism, we identify $i(X)$ with X and $\bar{j}(Y)$ with Y.

A pair of maps (g_1, g_2), $g_1 : X \times I \to Z$, $g_2 : Y \to Z$, satisfying $g_1(x, 0) = g_2(f(x))$, for each $x \in X$, determine a unique map

$$g : M_f \to Z,$$
$$g : \begin{cases} q(x, t) \mapsto g_1(x, t) \\ q(y) \mapsto g_2(y). \end{cases}$$

The **collapsing map** p is defined by

$$p : M_f \to Y,$$
$$p : \begin{cases} q(x, t) \mapsto f(x) \\ q(y) \mapsto y. \end{cases}$$

Lemma 1.5. *For any given $f : X \to Y$, the collapsing map $p : M_f \to Y$ is a strong deformation retract. In particular, $Y \sim M_f$.*

Proof. [**2**] pg. 369. □

Lemma 1.6. *For any given $f : X \to Y$, the inclusion $i : X \to M_f$ satisfies $pi = f$, and is an homotopy equivalence if and only if f is an homotopy equivalence.*

Proof. [**2**] pg. 317. □

The relevant commutative diagram is

$$\begin{array}{ccc} X & \xrightarrow{\;j\;} & X \times I \\ {\scriptstyle f}\downarrow & \searrow^{i} & \downarrow{\scriptstyle \bar{f}} \\ Y & \xleftarrow{\;p\;} & M_f \end{array}$$

where $i = \bar{f}j$.

Proposition 1.1. *If $f \sim g : X \to Y$, then the pair (M_f, X) is homotopic to the pair (M_f, Y).*

Proof. [**2**] pg. 370. □

2. CW complexes

Definition 2.1. *A space is called a* **topological cell***, or simply a* **cell***, of dimension m if it is homeomorphic with B^m. It is called an* **open cell** *of dimension m if it is homeomorphic with $\mathrm{Int}B^m = B^m - \partial B^m$. In each case the integer m is uniquely determined by the space in question. Here $B^m = \{x \in \mathbb{R}^m \mid |x| \leq 1\}$ is the* **unit ball** *or* **unit disc** *of the Euclidean space.*

For a given set J, let $\{S^n_j\}_{j\in J}$ be a set of copies of the $(n-1)$-dimensional sphere, and let $\{B^n_j\}_{j\in J}$ be the family of corresponding n-balls, i.e. $B^n_j = CS^{n-1}_j$, and $S^{n-1}_j = \partial B^n_j$. For a given map

$$f : \bigsqcup_{j\in J} S^{n-1}_j \to A,$$

let

$$X = A \sqcup_f \Big(\bigsqcup_{j\in J} B^n_j\Big),$$

be the adjunction of $\bigsqcup_{j\in J} B^n_j$ to A via f. The relevant commutative diagram is

$$\begin{array}{ccc} \bigsqcup_{j\in J} S^{n-1}_j & \xrightarrow{i} & \bigsqcup_{j\in J} B^n_j \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f} \\ A & \xrightarrow[\bar i]{} & X \end{array}$$

where $i = \bigsqcup_{j\in J} i_j$, and $i_j : S^{n-1}_j \to B^n_j$ is the natural inclusion of the boundary. Note that the map i is a closed cofibration

Lemma 2.1. *There is an homeomorphism*

$$X - A = \bigsqcup_{j\in J} \left(B^n_j - S^{n-1}_j\right),$$

given by the appropriate restriction of the map $\bar f$. The map $\bar i$ is a closed cofibration.

Proof. [**4**] pg. 154. □

For each $j \in J$, $\bar f(B^n_j) = \bar e^n_j$ is a compact subspace of X (closed if A is Hausdorff). The subspaces $\bar e^n_j$ are the **n-cells** of X. The restriction of $\bar f$ to an open ball $B^n_j - S^{n-1}_j$ is homeomorphic onto e^n_j, an **open n-cell** of X, whose closure coincides with $\bar e^n_j$. The map

$$\bar f_j = \bar f|_{B^n_j} : B^n_j \to X,$$

is a **characteristic map** for the cell $\bar e^n_j$; the map

$$f_j = f|_{S^{n-1}_j} : S^{n-1}_j \to A,$$

which glues the cell $\bar e^n_j$ to A is an **attaching map** for the cell $\bar e^n_j$. The pair (X, A) is called an **adjunction of n-cells** (see Definition 1.11).

A CW complex is the direct limit of the sequences of inclusions $i_n : X_n \to X_{n+1}$ of a sequence of adjunctions of discs $X_{n+1} := X_n \sqcup_f \left(\bigsqcup_{j\in J} B^n_j\right)$, where $f : \bigsqcup_{j\in J} \partial B^n_j \to X_n$. We present a more concrete definition, covering the more general case of a relative CW complex.

Definition 2.2. *A pair (X, A) is called a* **relative CW complex** *if there exists a sequences of spaces*

$$X^{-1} = A \subseteq X^0 \subseteq X^1 \subseteq \dots,$$

such that:

(1) *X^0 is obtained from A by adjunction of 0-cells (i.e., X is the topological sum of A and a discrete space);*
(2) *for every $n \geq 1$, the pair (X^n, X^{n-1}) is an adjunction of n-cells;*
(3) *X is the union space of the sequences (see Example **??**)*

$$X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \dots,$$

this is the final topology (weak topology) coinduced by the family of the inclusions of X^n in the union $\bigcup_{n=-1}^{\infty} X^n$.

If the sequence $X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \dots$, is **stationary** at n, namely if $X^{n-1} \neq X^n$, and $X^k = X^n$, for all $k \geq n$, we say that the relative CW complex (X, A) has **dimension** is n, otherwise that the complex has infinite dimension. The space X^n is called the n**-skeleton** of the complex.

If $A = \emptyset$, then $X^{-1} = \emptyset$ and X^0 is a discrete space. In this case X is called a **CW complex**. The collection of the cells and the characteristic maps is called a **CW decomposition** of the space X. Note that from a set theoretical point of view, a CW complex is just the disjoint union of its open cells; furthermore, while the closed cells are closed (and compact) subsets of X, the open cells are not necessarily open subsets of X (indeed, and open cell of X is not open if it intersects the boundary of a cell of higher dimension). A CW complex with a finite number of cells is said to be a **finite CW complex**; such a CW complex is clearly a compact space.

A CW complex is a filtered space, according to the following definition.

Definition 2.3. *If X is a space, a* **filtration** *of X is a sequence*

$$\dots \subseteq X_0 \subseteq X_2 \subseteq X_2 \subseteq \dots,$$

with $X_n = \emptyset$ for $n < 0$, of subsepaces of X whose union is X. A space X together with a filtration of X, is called a filtered space. If X and Y are filtered spaces, a map $f : X \to Y$ such that $f(X_n) \subseteq Y_n$ for all n is said to be filtration-preserving

The next two results follow from the very definition.

Corollary 2.1. *CW complexes are Hausdorff and normal.*

Corollary 2.2. *Let X be a CW complex with open cells e_j. A map $f : X \to Y$ is continuous if and only iff $f|_{\bar{e}_j}$ is continuous for each e_j. A map $F : X \times I \to Y$ is continuous if and only if $F|_{e_j \times I}$ is continuous for each e_j.*

The following is an equivalent definition of a CW complex (see [**5**] 8.24, [**3**] Section 38).

Definition 2.4. *Let X be an Hausdorff space and $\{e_j^n\}_{n \in \mathbb{N}, j \in J}$ a family of (disjoint) topological open cells (i.e. $e_j^n = B^n - \partial B^n$ is homeomorphic to the open n-ball (n-disc), see Definition 2.1). Let $X^n = \bigcup_{j \in J} e_j^n$, and assume:*

(1) *$X = \bigsqcup_{j,n} e_j^n$, i.e. $X = \bigcup_{j,n} e_j^n$, and $e_j \cap e_k = \emptyset$, whenever $j \neq k$;*
(2) *for each cell e_j^n there is a relative homeomorphism $f_j^n : (B^n, \partial B^n) \to (e_j^n \cup X^{n-1}, X^{n-1})$, i.e. there is a map $f_j^n : B^n \to X$ such that:*

(a) $f_j^n|_{B^n-\partial B^n}$ *is an homeomorphism onto* e_j^n,
(b) $f_j^n(\partial B^n) \subseteq X^{n-1}$;

(3) *each* $\bar{e}_j^n$ *is contained in the union of finitely many* e_k^m;
(4) X *has the weak topology determined by the family* $\{\bar{e}_j^n\}$, *i.e. a set* $U \subseteq X$ *is closed in* X *if and only if* $U \cap \bar{e}_j^n$ *is closed in* $\bar{e}_j^n$ *for all* e_j^n.

Theorem 2.1. *Definitions 2.2 and 2.4 are equivalent.*

Proof. [**5**] 8.24 or [**3**] 38.2. □

We now give some properties of CW complex.

Lemma 2.2. *CW complexes are locally path connected and locally contractible.*

Proof. [**5**] 8.25. □

Lemma 2.3. *The topology of a CW complex is the weak topology induced by the family of its closed cells.*

Lemma 2.4. *Let* K *be a compact subset of a CW complex* X. *Then* K *is contained in a finite union of open cells of* X.

Proof. [**4**] pg. 163. □

Lemma 2.5. *Let* X *be a CW complex, then every open cell is open in* X^n, *and* $X^n - X^{n-1}$ *is open in* X^n.

Proof. [**5**] pg. 203. □

Next, we give the definition of subcomplex, and of cellular maps.

Definition 2.5. *Let* $\mathcal{F}$ *be a family of open cells of a CW complex* X, *and let* Z *be the union of the cells of* $\mathcal{F}$. *We say that* Z *is a* **subcomplex** *of* X *if for every open cell* $e \in \mathcal{F}$, $\bar{e} \in Z$ *and* Z *has the topology induced by the closure of all cells in* $\mathcal{F}$. *We write* $Z \leq X$ *and we call the pair* (X, Z) *a* **CW pair**.

Proposition 2.1. *Arbitrary union and intersections of subcomplexs of a CW complex* X *are subcomplexes of* X.

Proof. [**4**] pg. 164. □

Lemma 2.6. *Let* X *be a CW complex, let* $\mathcal{F}$ *be a family of open cells of* X, *and let* Z *be the union of the cells in* $\mathcal{F}$. *Then,* Z *is a subcomplex of* X *if and only if* Z *is a CW complex determined by the skeleta* $Z^n = Z \cap X^n$, $n \geq 0$.

Proof. [**4**] pg. 164. □

Note that CW complex satisfies an homotopy extension property ([**5**] 8.27) by construction, in particular this implies the following useful result.

Proposition 2.2. *Let* L *be a subcomplex of a CW complex* K. *Then, the following assertions are equivalent:*

(1) L *is a strong deformation retract of* K,
(2) *the inclusion map* $i : L \to K$ *is a homotopy equivalence,*
(3) $\pi_n(K, L) = 0$ *for all* $n \leq \dim(K - L)$, *where* $\dim(K - L)$ *means the dimension of the top cell in* $K - L$.

PROOF. The fact that (1) implies (2) and that (2) implies (3) are elementary. From (3) to (1) one proceeds inductively using the hypothesis and the homotopy extension property. In particular, see for example [**4**] 6.2.5 and 6.2.6, where it is proved that adjunction of n-cells produces $(n-1)$-connected spaces, for example (X, X^n) is $(n-1)$-connected. □

PROPOSITION 2.3. *Let Z be a subcomplex of a CW complex X. Then the quotient space X/Z is a CW complex.*

PROOF. [**4**] pg. 171. □

DEFINITION 2.6. *A map $f : X \to Y$ between two CW complexes is called* **cellular** *if it takes the n-skeleton X^n of X into the n-skeleton Y^n of Y. In particular, if each cell of X is sent into a cell of Y. A map of CW pairs $f : (X, A) \to (Y, B)$ is cellular if $f(X^n \cup A) \subseteq (Y^n \cup B)$ (note this does not imply that $f|_A$ is cellular).*

REMARK 2.1. *Note that a cellular map not necessarily sends cells into cells. for example consider the decomposition of the circle with one or two 1-cells.*

DEFINITION 2.7. *A* **CW** *or* **cellular isomorphism** *is an homeomorphism such that the image of each cell is a cell.*

LEMMA 2.7. *A cellular homeomorphism with cellular inverse is a cellular isomorphism.*

Adjunction of CW complexes are CW complexes as long as the attaching map is cellular. More precisely, we have:

PROPOSITION 2.4. *Let A be a subcomplex of a CW complex X and let $f : A \to Y$ be a cellular map. Then $Y \sqcup_f X$ is a CW complex containing Y as a subcomplex, and whose cells are those of $X - A$ and those of Y*

PROOF. [**4**] pg. 168. □

COROLLARY 2.3. *Let $f : X \to Y$ be a cellular map between CW complexes. Then the mapping cylinder M_f is a CW complex, with cells which are either cells of Y or which are of the form $e \times \{1\}$ or $e \times (0, 1)$, where e is an arbitrary cell of X.*

THEOREM 2.2. **Cellular approximation theorem**. *Any map $f : (X, A) \to (Y, B)$ between CW pairs (or even between relative CW) is homotopic rel A to a cellular map.*

PROOF. [**6**] □

LEMMA 2.8. *A cellular map $f : X \to Y$ between CW complexes is a homotopy equivalence if and only if X is a strong deformation retract of the mapping cylinder M_f.*

PROOF. Exercise 1.

□

3. Cellular homology theory

We now show how to compute the singular homology of a CW complex. The symbol H_n will denote singular homology in general, but if it happens that the space in question is a triangulable CW complex, then H_n can also be taken to denote simplicial homology, since there is a natural isomorphism between singular and simplicial theory. The ring of coefficients is $\mathbb{Z}$.

Recall that if given a triple $B \subseteq A \subseteq X$ of spaces, one has a short exact sequence of chain complexes (where S_k denotes the singular chain complexes)

$$0 \longrightarrow \frac{S_n(A)}{S_n(B)} \longrightarrow \frac{S_n(X)}{S_n(B)} \longrightarrow \frac{S_n(X)}{S_n(A)} \longrightarrow 0$$

In particular if X is a (simplicial) complex, A a subcomplex of X and B a subcomplex of A, this holds for simplicial chain complexes. This gives rise to the following exact sequence, called the **homology exact sequence of a triple** (X, A, B):

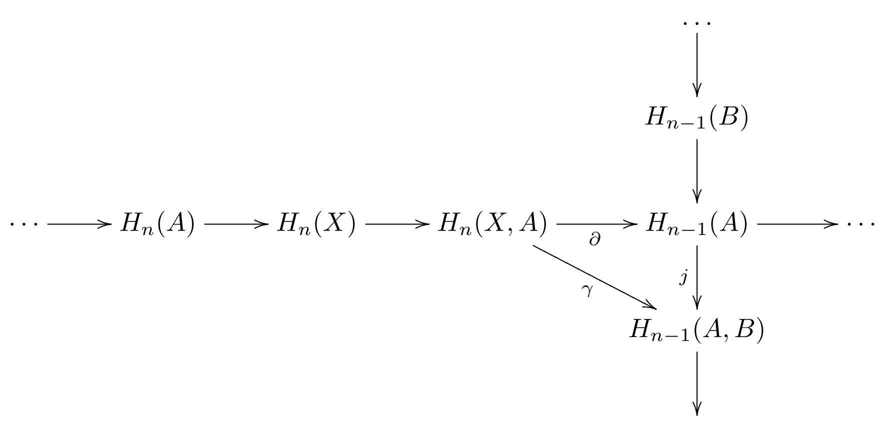
$$\cdots \longrightarrow H_n(A,B) \xrightarrow[\alpha]{} H_n(X,B) \xrightarrow[\beta]{} H_n(X,A) \xrightarrow[\gamma]{} H_{n-1}(A,B) \longrightarrow \cdots$$

where α and β are induced by the inclusions, and γ is the composite

$$H_n(X,A) \xrightarrow[\partial]{} H_{n-1}(A) \xrightarrow[j_*]{} H_{n-1}(A,B),$$

where $j : (A, *) \to (A, B)$, and ∂ is the boundary homomorphism in the homology exact sequence of the pair (X, A).

Note that this can be also understood by composing the two exact sequences of the two pairs (X, B) and (B, A) as described in the following diagram:



The construction is natural, in the sense that a map of triple $f : (X, A, B) \to (Y, C, D)$ induces a homomorphism of the corresponding exact homology sequences.

Definition 3.1. *Let (K, L) be a CW pair. Let*

$$C_n(K,L) := H_n(K^n \cup L, K^{n-1} \cup L),$$

and let d_n be defined by the composite

$$H_n(K^n \cup L, K^{n-1} \cup L) \xrightarrow[\partial]{} H_{n-1}(K^{n-1} \cup L) \xrightarrow[j_*]{} H_{n-1}(K^{n-1} \cup L, K^{n-2} \cup L),$$

The chain complex : $\mathcal{C}(K, L) = \{C_n(K, L), d_n\}$ is called the **cellular chain complex** *of the pair (K, L).*

Exercise 2. Verify that $d^2 = 0$.

Lemma 3.1. *Let e^n be an n-cell of $K - L$, and let ϕ be a characteristic map for e^n. Then, the map of pairs $\phi : (B^n, \partial B^n) \to (\bar{e}^n, \partial e^n)$ induces an isomorphism in relative homology.*

Proof. This is clear since it is a relative homeomorphism, see also [**3**] 39.1. □

Recalling that $(K^n \cup L, K^{n-1} \cup L)$ is the adjoint space (see beginning of Section 2)

$$\begin{array}{ccc} \bigsqcup_{j \in J_n} S_j^{n-1} & \xrightarrow{\ i\ } & \bigsqcup_{j \in J_n} B_j^n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ K^{n-1} \cup L & \xrightarrow[\bar{i}]{} & K^n \cup L \end{array}$$

and denoting by $q : (K^{n-1} \cup L) \sqcup \bigsqcup_{j \in J_n} B_j^n$ the quotient map in the definition of adjunction, we have the following result:

Lemma 3.2. *The map q induces an isomorphism in relative homology:*

$$H_k(K^n \cup L, K^{n-1} \cup L) \cong H_k(\bigsqcup_{j \in J_n} B_j^n, \bigsqcup_{j \in J_n} \partial B_j^n).$$

Proposition 3.1. *Let (K, L) be a CW pair. Let e_j be an open cell of $K - L$ and ϕ_j a characteristic map for e_j. Then,*

1. *$H_k(K^n \cup L, K^{n-1} \cup L) \cong 0$ if $k \neq n$,*
2. *$C_n(K, L) = H_n(K^n \cup L, K^{n-1} \cup L)$ is free with basis the elements $(\phi_j)_*(\omega_n)$, where ω_n is a fixed generator for $H_n(B^n, \partial B^n)$,*
3. *if c is a singular n-cycle of K mod L representing $[c] \in H_n(K^n \cup L, K^{n-1} \cup L)$ and if $|c|$ does not include the n-cell e_{j_0}, then $n_{j_0} = 0$ in the expression $[c] = \sum_{j \in J_n} n_j (\phi_j)_*(\omega_n)$, n_j in the coefficient ring.*

The generator $(\phi_j)_*(\omega_n)$ of $H_n(K^n \cup L, K^{n-1} \cup L)$ is called a **fundamental cycle** *of the n-cell e_j.*

Proof. [**3**] 39.4 for the first two points, [**6**] pg. 58 for all the points. □

Proposition 3.2. *Let X be filtered by the subspaces $X_0 \subseteq X_1 \subseteq \dots$. Assume that $H_k(X_n, X_{n-1}) = 0$ for $n \neq k$. Suppose also that given any compact set C in X, there is an n such that $C \subseteq X_n$ (in particular this holds for a CW complex). Let $\mathcal{C}(X)$ be the chain complex associated to X as in definition 3.1. Then there is an isomorphism*

$$T : H_n(\mathcal{C}(X)) \to H_n(X).$$

T is natural with respect to homomorphisms induced by filtration-preserving maps and takes the homology class of a cycle $\sum_{j \in J} n_j (\phi_j)_*(\omega_n) \in C_n(X)$ onto the homology class of the cycle $\sum_{j \in J} n_j \bar{\phi}_j \in S_n(X)$, where $\bar{\phi}_j$ is a singular chain representing $(\phi_j)_*(\omega_n)$..

Proof. [**3**] 39.4. □

Proposition 3.3. *Let X be filtered by the subspaces $X_0 \subseteq X_1 \subseteq \dots$. Suppose that X is the space of a simplicial complex K, and each subspace X_n is the space of a subcomplex of K of dimension at most n. Let $H_{\mathrm{simp},n}$ denote simplicial homology, and assume that $H_{\mathrm{simp},k}(X_n, X_{n-1}) = 0$ for $n \neq k$. Then $H_n(X_n, X_n - 1)$ equals* *a subgroup of $C_{\mathrm{simp},n}(K)$, and the isomorphism of Proposition 3.2 is induced by inclusion. Indeed, $H_n(X_n, X_n - 1)$ is the subgroup of $C_{\mathrm{simp},n}(K)$ consisting of all n-chains of K carried by X_n whose boundaries are carried by X_{n-1}.*

PROOF. [**3**] 39.5. ☐

Definition 3.1 is natural in the following sense (easy to verify): a map of pairs $f : (K, L) \to (K', L')$ induces a chain map $f_\# : \mathcal{C}(K, L) \to \mathcal{C}(K', L')$ and thus a homomorphism in homology.

PROPOSITION 3.4. *Cellular homology is a functor from CW pair.*

THEOREM 3.1. *There is a natural equivalence T between the cellular homology functor and the singular homology functor (simplicial homology functor for triangulable CW complexes). Namely, for every CW complex pair (K, L) there is an isomorphism $T_{(K,L)} : H_*(\mathcal{C}(K, L)) \to H_*(K, L)$, and for every cellular map $f : (K, L) \to (K', L')$ the following diagram commutes for each n*

$$\begin{array}{ccc} H_n(\mathcal{C}(K,L)) & \xrightarrow{T_{(K,L)}} & H_n(K,L) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H_n(\mathcal{C}(K',L')) & \xrightarrow{T_{(K',L')}} & H_n(K',L') \end{array}$$

The isomorphism $T_{(K,L)}$ takes the homology class of a cycle $\sum_{j \in J} n_j (\phi_j)_*(\omega_n) \in C_n(K, L)$ onto the homology class of the cycle $\sum_{j \in J} n_j \bar{\phi}_j \in S_n(K, L)$, where $\bar{\phi}_j$ is a singular chain representing $(\phi_j)_*(\omega_n)$.

PROOF. [**6**] pg. 65. ☐

We conclude with few words on orientation of cells and cellular chains.

DEFINITION 3.2. *Let e be a topological n-cell (see Definition 2.1). The group $H_n(\bar{e}, \partial e)$ is infinite cyclic. The two generators of this group are called the two* **orientations** *of the cell e. An* **oriented** *n-cell is a topological cell e together with an orientation of e.*

We have seen that the cellular chain group $\mathcal{C}_n(X) = H_n(X_n, X_{n-1})$ is a free abelian group. One obtains a basis for it by orienting each open n-cell e_j of X and passing to the corresponding element of $H_n(X_n, X_{n-1})$, that is, by taking the image of the orientation under the homomorphism induced by inclusion $H_n(\bar{e}_j, \partial e_j) \to H_n(X_n, X_{n-1})$.

The homology of the chain complex $\mathcal{C}(X)$ is isomorphic with the singular homology of X. In the special case where X is a triangulable CW complex triangulated by a complex K, we interpret these comments as follows: the fact that X_n and X_{n-1} are subcomplexes of K implies that each open n-cell e_j is a union of open simplices of K, so that e_j is the polytope of a subcomplex of K. The group $H_{\mathrm{simp},n}(\bar{e}_j, \partial e_j)$ equals the group of n-chains carried by $\bar{e}_j$ whose boundaries are carried by ∂e_j. The cellular chain group $\mathcal{C}(X)$ equals the group of all simplicial n-chains of X carried by X_n whose boundaries are carried by X_{n-1}.

We conclude resuming the situation: given a CW pair (K, L), the CW decomposition canonically defines a chain complex

$$\mathcal{C}(K, L) = \{\mathcal{C}_n(K, L), d_n\},$$

where

$$\mathcal{C}_n(K,L) = \sum_{j\in J_n} H_n(\bar{e}_j, \partial e_j),$$

of free abelian groups, i.e. free abelian $\mathbb{Z}$-modules. By ordering and orienting the n-cells we obtain a basis of this chain complex, given by the set of the n-fundamental cycles. We call this base the **geometric basis** of $\mathcal{C}(K,L)$. Note that the ambiguities in fixing this basis are encoded by the action of the group

$$\prod_{n\geq 0} S_{J_n} \times \mathbb{Z}/2^{J_n},$$

where S_J denotes the group of permutations of a set J.

EXERCISE 3.1. *Give a cellular decomposition of the real projective space/plane and compute its homology.*

Exercise 3.

3.1. Covering spaces in the CW category. The next three lemmas follow by classical theory of covering spaces, see for example [**5**].

LEMMA 3.3. *If K is a CW complex for any subgroup G of $\pi_1(K)$ there is a covering space $(\hat{K}, p)$ of K with $p_*(\pi_1(\hat{K})) = G$. In particular, K has a universal covering space.*

A covering in the CW category $p : \hat{K} \to K$ is a covering with $\hat{K}$ and K CW complexes and p CW map. When dealing with CW complexes, nothing is lost assuming it is a CW covering by the following result.

LEMMA 3.4. *Let $p : \hat{K} \to K$ be a covering, with K a CW complex. Then, the family of cells $\hat{e}_j$, that are lifts of the cells e_j of K gives a cell structure for $\hat{K}$ with respect to which $\hat{K}$ is a CW complex. If $\phi_j : B^n \to K$ is a characteristic map for e_j, $\hat{e}_j$ is a lift of e_j, and $\hat{\phi}_j : B^n \to \hat{K}$ is lift of ϕ such that $\hat{\phi}_j(x) \in \hat{e}_j$ for some $x \in B^n - \partial B^n$, then $\hat{\phi}_j$ is a characteristic map for $\hat{e}_j$.*

LEMMA 3.5. *If $p : \hat{K} \to K$ is a covering and $f : L \to K$ is a cellular map which lifts to $\hat{f} : L \to \hat{K}$, then $\hat{f}$ is cellular. If f is also a covering in the CW category, so is $\hat{f}$.*

Since a CW covering that is also a homeomorphism is a cellular isomorphism, Lemma 3.5 implies that the universal covering space of a CW complex K is unique up to cellular isomorphism.

LEMMA 3.6. *Let (K, L) be a pair of connected CW complexes, and $p : \tilde{K} \to K$ the universal covering of K. Let $\tilde{L} = p^{-1}(L)$. If $i_* : \pi_1(L) \to \pi_1(K)$ is a isomorphism, then $p|_{\tilde{L}} : \tilde{L} \to L$ is the universal covering of L. If L is a strong deformation retract of K, then $\tilde{L}$ is a strong deformation retract of $\tilde{K}$.*

PROOF. $\tilde{L}$ is a closed set which is the union of cells of $\tilde{K}$ (the lifts of the cells of L). Thus $\tilde{L}$ is a subcomplex of $\tilde{K}$. Clearly, $p|_{\tilde{L}}$ is a covering of L. We show that if i_* is isomorphism, then $\tilde{L}$ is simply connected. First, by the five Lemma applied to the exact homotopy sequences of $(\tilde{K}, \tilde{L})$ and (K, L), it follows that $\pi_q(\tilde{K}, \tilde{L}) \cong \pi_q(K, L)$ for all $q \geq 1$. By the exact homotopy sequence of the pair (K, L), $\pi_1(K, L) = 0$. Combining with the previous isomorphism, $\pi_1(\tilde{K}, \tilde{L}) = 0$, and by connectedness of

$\tilde{K}$ and the exact homotopy sequence of $(\tilde{K}, \tilde{L})$, $\tilde{L}$ is connected. Commutativity of the diagram

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
\pi_1(\tilde{L}) & \longrightarrow & \pi_1(\tilde{K}) = 0 \\
\downarrow & & \downarrow \\
\pi_1(L) & \xrightarrow[\cong]{} & \pi_1(K).
\end{array}$$

shows that $\tilde{L}$ is 1-connected. Eventually, if L is a strong deformation result of K, by Proposition 3.1, $\pi_q(K, L) = 0$, implying by the previous isomorphism that $\pi_q(\tilde{K}, \tilde{L}) = 0$, and hence by the same proposition the last statement. □

LEMMA 3.7. *Let $f : K \to L$ be a cellular map between connected CW complexes such that $f_* : \pi_1(K) \to \pi_1(L)$ is an isomorphism. If $\tilde{K}, \tilde{L}$ are universal covering spaces of K, L, and $\tilde{f} : \tilde{K} \to \tilde{L}$ is a lift of f, then $M_{\tilde{f}}$ is a universal covering space of M_f.*

PROOF. Exercise 4. □

We conclude reviewing in some details the standard identification of the group of covering transformation with the fundamental group of a CW complex K.

PROPOSITION 3.5. *Let $p : \tilde{K} \to K$ the universal covering of a CW complex K. Fix a base point $x_0 \in K$ and $\tilde{x}_0 \in p^{-1}(x_0) \in \tilde{K}$. Then, the function*

$$\begin{aligned}
\theta_{x_0, \tilde{x}_0} :& \pi_1(K, x_0) \to Cov(\tilde{K}, K), \\
\theta_{x_0, \tilde{x}_0} :& \alpha \mapsto g_\alpha,
\end{aligned}$$

*where g_α is the unique covering homeomorphism with $g_\alpha(\tilde{x}_0) = \tilde{\alpha}(1)$ (by Lemma **??**), is an isomorphism of groups.*

PROOF. We provide an explicit description of the action of g_α points of $\tilde{K}$, not in the fibre of x_0. Let $a \in \alpha$, $a : (I, \partial I) \to (K, x_0)$ be a loop. By Proposition **??** or Proposition **??**, there exists a unique lift $\tilde{a}$ of a with $\tilde{a}(0) = \tilde{x}_0$. If $\tilde{y} \in \tilde{K}$ and $b : (I, 0, 1) \to (\tilde{K}, \tilde{x}_0, \tilde{y})$ is any path, then

$$g_\alpha(\tilde{y}) = \widetilde{a * pb}(1),$$

where $*$ denotes sum of loops. For we have the following picture. In K the loop a starts and ends at x_0, and the path pb starts at x_0 and ends at some $y_0 = p(y)$. In $\tilde{K}$, there is the lift of a, that is the path $\tilde{a}$ starting at $\tilde{x}_0 = a(0)$, and ending $\tilde{x}_1 = a(1)$, and there are the lifts of pb: $\widetilde{pb}$ starting at $\tilde{x}_0$, and ending at say $\tilde{y}_0$, and $\widehat{pb}$ starting at $\tilde{x}_1$ and ending at $\tilde{y}_1$. By unicity of lift, $\widetilde{pb} = b$ and hence $\tilde{y}_0 = \tilde{y}$.

Since g_α is an homeomorphism, and $g_\alpha(\tilde{x}_0) = \tilde{x}_1$ by hypothesis, it follows that $g_\alpha(b) = g_\alpha(\widetilde{pb}) = \widehat{pb}$. Thus

$$g_\alpha(\tilde{y}) = g_\alpha(b(1)) = \widehat{pb}(1).$$

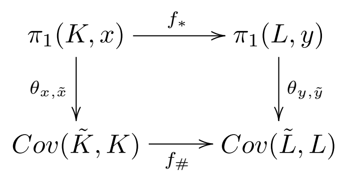
It is now clear by direct investigation that $\widehat{pb}(1) = \widetilde{a * pb}(1)$, and this completes the proof of the formula.

It is now easy to see that θ is isomorphism. For example, given $\alpha, \beta \in \pi_1(K, x_0)$, and identifying paths with classes,

$$g_\alpha g_\beta(\tilde{x}) = g_\alpha(\tilde{\beta}(1)) = \widetilde{\alpha * (p\tilde{\beta})}(1) = \widetilde{\alpha * \beta}(1) = g_{\alpha\beta}(\tilde{x}).$$

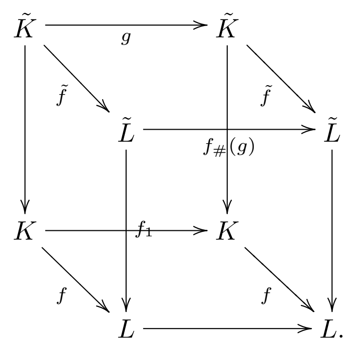
□

If $p : \tilde{K} \to K$ and $q : \tilde{L} \to L$ are universal coverings, any map $f : (K, x) \to (L, y)$ induces an homomorphism f_* on the fundamental groups and an homomorphism $f_\#$ on the covering transformation groups. The following diagram obviously commutes



Lemma 3.8. *If $\tilde{f} : \tilde{K} \to \tilde{L}$ covers f, then $\tilde{f}g = f_\#(g)\tilde{f}$, for all $g \in Cov(\tilde{K}, K)$.*

Proof. The following diagram is useful:



Both the maps cover f, so it suffices to show that they agree at a point. Let $\alpha = \theta^{-1}_{x,\tilde{x}}(g)$, since $\tilde{f}\tilde{\alpha}(0) = \tilde{y} = \widetilde{f\alpha}(0)$, then

$$\tilde{f}g(\tilde{x}) = \tilde{f}\tilde{\alpha}(1) = \widetilde{f\alpha}(1) = \theta_{y,\tilde{y}}(f_*(\alpha))(\tilde{y}) = \theta_{y,\tilde{y}}(f_\#(\theta^{-1}_{x,\tilde{x}}(g)))(\tilde{y}) = f_\#(g)(\tilde{y}).$$

□

3.2. Fundamental properties of the universal cover of a CW complex. Let (K, L) be a CW pair, and $p : \tilde{K} \to K$ the universal covering space of K. Then, by results of Section 3, the cellular chain complex $\mathcal{C}(\tilde{K}, \tilde{L})$ (where $\tilde{L} = p^{-1}(L)$) is a free $\mathbb{Z}$-module with properties described in Proposition 3.1 , with natural basis given at the end of the section. We now show that $\mathcal{C}(\tilde{K}, \tilde{L})$ is canonically a free $\mathbb{Z}\pi_1(K)$-module, and we describe a natural bases for it.

Recall that $\pi_1(K) \cong Cov(\tilde{K}, K)$, the group of covering transformations of $\tilde{K}$, namely homeomorphisms $g : \tilde{K} \to \tilde{K}$ such that $pg = p$. If $g \in Cov(\tilde{K}, K)$ then it is a cellular isomorphism of $\tilde{K}$ by Lemma 3.5, and it induces the homomorphism

$$g_* : \mathcal{C}_n(\tilde{K}, \tilde{L}) \to \mathcal{C}_n(\tilde{K}, \tilde{L}),$$

with $dg_* = g_* d$ (where d is the boundary operator), for each n. Let define an action of $Cov(\tilde{K}, K)$ on $\mathcal{C}(\tilde{K}, \tilde{L})$:

$$\cdot : Cov(\tilde{K}, K) \times \mathcal{C}(\tilde{K}, \tilde{L}) \to \mathcal{C}(\tilde{K}, \tilde{L}),$$
$$\cdot : (g, c) \mapsto g \cdot c := g_*(c).$$

Clearly, $d(g \cdot c) = g \cdot d(c)$. This makes $\mathcal{C}(\tilde{K}, \tilde{L})$ a $\mathbb{Z}\pi_1(K)$-complex (i.e. a complex of $\mathbb{Z}\pi_1(K)$-modules) if we extend the action linearly, namely if we define

$$\left(\sum_{\alpha \in A} n_\alpha g_\alpha \right) \cdot c := \sum_{\alpha \in A} n_\alpha g_\alpha \cdot c.$$

The following proposition shows that $\mathcal{C}(\tilde{K}, \tilde{L})$ is indeed a free $\mathbb{Z}\pi_1(K)$-complex with a natural class of bases. These bases are obtained by lifting and orienting the geometric basis of the complex $\mathcal{C}(K, L)$ (see the end of Section 3).

PROPOSITION 3.6. *Let (K, L) be a CW pair, $p : \tilde{K} \to K$ the universal covering space of K, and $\tilde{L} = p^{-1}(L)$. For each n-cell $e_j \in K - L$ fix a characteristic map $\phi_j : B^n \to L$, and a lift $\tilde{\phi}_j : B^n \to \tilde{K}$. Then, the set*

$$\{(\tilde{\phi}_j)_*(\omega_n)\}_{j \in J, e_j \in K - L},$$

is a basis for $\mathcal{C}_n(\tilde{K}, \tilde{L})$ as a $\mathbb{Z}\pi_1(K)$-module for each n, and thus the union of these sets for all $n \geq 0$, is a basis for $\mathcal{C}_n(\tilde{K}, \tilde{L})$ as $\mathbb{Z}\pi_1(K)$-complex.

PROOF. Fix a base point $\star \in I^n - \partial I^n$ for each $n > 1$. For each $y \in p^{-1}\phi_j(\star)$, let $\tilde{\phi}_{j,y}$ be the unique lift of ϕ_j with $\tilde{\phi}_{j,y}(\star) = y$ (use the Lifting Theorem). Since the covering is the universal covering, $Cov(\tilde{K}, K)$ acts freely and transitively on each fibre $p^{-1}(x)$. Thus each $\tilde{\phi}_{j,y}$ is uniquely expressible as $\tilde{\phi}_{j,y} = g\tilde{\phi}_j$ for some $g \in Cov(\tilde{K}, K)$. Then,

$$\{\tilde{\phi}_{j,y} \mid y \in p^{-1}\phi_j(\star)\} = \{g\tilde{\phi}_j \mid g \in Cov(\tilde{K}, K)\}.$$

By Proposition 3.1 and Lemma 3.4, $\mathcal{C}(\tilde{K}, \tilde{L})$ is a free $\mathbb{Z}$-module with basis

$$\begin{aligned} \{(\tilde{\phi}_{j,y})_*(\omega_n) \mid y \in p^{-1}\phi_j(\star)\} =& \{(g\tilde{\phi}_j)_*(\omega_n \mid g \in Cov(\tilde{K}, K)\} \\ =& \{g_*(\tilde{\phi}_j)_*(\omega_n) \mid g \in Cov(\tilde{K}, K)\} \\ =& \{g \cdot (\tilde{\phi}_j)_n(\omega_n) \mid g \in Cov(\tilde{K}, K)\}, \end{aligned}$$

where $j \in J$ varies over the given characteristic maps for $K - L$. Thus each chain $c \in \mathcal{C}(\tilde{K}, \tilde{L})$ is uniquely representable as a finite sum

$$c = \sum_{j \in J, \alpha \in A} n_{j,\alpha} g_\alpha \cdot (\tilde{\phi}_j)_*(\omega_n) = \sum_{j \in J} \left(\sum_{\alpha \in A} n_{j,\alpha} g_\alpha \right) \cdot (\tilde{\phi}_j)_*(\omega_n),$$

where $\sum_{\alpha \in A} n_{j,\alpha} g_\alpha \in \mathbb{Z}\pi_1(K)$. This proves that the set

$$\{(\tilde{\phi}_j)_*(\omega_n)\}_{j \in J, e_j \in K - L},$$

is a basis for $\mathcal{C}_n(\tilde{K}, \tilde{L})$ as a $\mathbb{Z}\pi_1(K)$-module for each n.

□

EXERCISE 3.2. *Give the cellular chain complex of the universal covering space of a circle S^1 with coefficients in $\mathbb{Z}\pi_1(S^1)$.*

Exercise 5.

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