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A Generalization of the Euler Gamma Function

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Abstract. We define a generalized Euler gamma function $\Gamma_\beta(z)$, where the product is taken over powers of integers rather than integers themselves. Studying the associated spectral functions and in particular the zeta function, we obtain the main properties of $\Gamma_\beta(z)$ and its asymptotic expansion for large values of the argument.

Key words: Euler gamma function, spectral function, asymptotic expansion.

1. Introduction. Consider the regularized Weierstrass product

$$F_\beta(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^\beta}\right) e^{\sum_{j=1}^{p} \frac{(-1)^j}{j} \frac{z^j}{n^{j\beta}}},$$

where β is real and positive, $p = [1/\beta]$, and $\gamma_\beta = -\log F_\beta(1)$. We define a function $\Gamma_\beta(z)$ by the formula $1/\Gamma_\beta(z) = ze^{\gamma_\beta z} F_\beta(z)$. This is an analytic function of z in the entire complex plane except for simple poles at the points $z = 0, -1^\beta, -2^\beta, \dots$, and one has $\Gamma_\beta(1) = 1$. Here we announce the main properties of Γ_β; details and complete proofs will be published elsewhere. Since $\Gamma_1(z) = \Gamma(z)$, we see that Γ_β is a natural generalization of the Euler function. We show that all properties that characterize the gamma function except for those based on the "linearity" of terms in the defining sequence (see Sec. 2) hold for Γ_β in a very natural, simple way. In particular, we write out a functional equation, the Taylor series, and the asymptotic expansion for large z. Remarkably, we obtain this asymptotic expansion by proving the existence of a full asymptotic expansion for the associated heat function (Lemma 3). This function has the form $\sum_{n=1}^{\infty} e^{-a_n t}$ for some sequence a_n, and in general one has little chance of obtaining a full expansion for such functions. Already the case of the 2-sphere is hard [4]. For $\beta > 1$, the Euler formula for the gamma function can be generalized as

$$F_\beta(z) = \lim_{m\to\infty} \frac{(1^\beta + z)(2^\beta + z)\cdots(m^\beta + z)}{1^\beta 2^\beta \cdots m^\beta},$$

and this expression can be extended to $\beta < 1$ with appropriate regularizing factors. This formula suggests that neither the functional equation nor the integral representation of the gamma function is likely to admit a generalization to Γ_β, since they are based on the "linearity" of the general term.

Apart from purely academic interest [6, 8], the study of such functions is motivated by the fact that they arise when dealing with the regularized determinant [5, 9] of real powers of the Laplace operator on the circle* [3]: one has $\det_\zeta(-\Delta^\beta_{S^1} + a) = \Gamma_\beta(a)$, which generalizes the Lerch formula (Lemma 3). In particular, the asymptotic expansions of $\log \Gamma_\beta$ specify the behavior if the mass term is small or large.

2. Spectral functions. Consider the sequence $S_{\beta,a} = \{a_n = n^\beta + a\}$, where $\beta \neq 1$ is given.** The sequence $S_{\beta,a}$ has the finite convergence exponent $1/\beta$ and the only accumulation point at infinity. We assign the following spectral functions to $S_{\beta,a}$: the zeta function $\zeta_\beta(s,a) = \sum_{n=1}^{\infty} (n^\beta + a)^{-s}$, the regularized Weierstrass product $F_\beta(z)$, which is an entire function of order $1/\beta$ and genus $p = [1/\beta]$, the heat function $f_\beta(t,a) = \sum_{n=0}^{\infty} e^{-(n^\beta + a)t}$, and the resolvent function defined by the

*In the standard case, $F_2(z) = \frac{\operatorname{sh} \pi\sqrt{z}}{\pi\sqrt{z}}$.

**Note that for $\beta = 1$ the general term of the homogeneous sequence $S_{1,0}$ is linear in the variable n; this property is in fact responsible for some of the main properties of the gamma function, and that is why the latter do not hold for the general nonlinear case $\beta \neq 1$.

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formula $R_\beta(\lambda,a)=\sum_{n=1}^{\infty}\frac{1}{\lambda-n^\beta-a}=-\zeta_\beta(1,a-\lambda)$ for $\beta>1$ and for λ lying in an arbitrary sector that does not contain the numbers a_n. For arbitrary β, we can only introduce the derivatives $R_\beta^{(p)}(\lambda,a)=(-1)^p p!\sum_{n=1}^{\infty}(\lambda-n^\beta-a)^{-p-1}=-p!\,\zeta_\beta(p+1,a-\lambda)$.

By using the Plana theorem in the same way as in [7], we obtain a full asymptotic expansion of the heat function.

Lemma 1. *The asymptotic expansion*

$$f_\beta(t,0)\sim\Gamma\big(1+\frac{1}{\beta}\big)t^{-1/\beta}+\frac{1}{2}+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}c_m t^m$$

is valid as $t\to 0^+$, where

$$c_{2m}=2\sum_{j=0}^{m-1}(-1)^{j+1}\binom{2m}{2j+1}\cos^{2j+1}\frac{\pi}{2}\beta\sin^{2m-2j-1}\frac{\pi}{2}\beta\int_0^\infty\frac{y^{2m\beta}}{e^{2\pi y}-1}\,dy,$$

$$c_{2m+1}=2\sum_{j=0}^{m}(-1)^{j+1}\binom{2m+1}{2j}\cos^{2j}\frac{\pi}{2}\beta\sin^{2m+1-2j}\frac{\pi}{2}\beta\int_0^\infty\frac{y^{(2m+1)\beta}}{e^{2\pi y}-1}\,dy.$$

3. The zeta function. The main properties of the generalized Hurwitz zeta function $\zeta_\beta(s,a)$ can be obtained by standard methods [2], but we present a simple proof of how the main zeta invariants are related to Weierstrass products [9].

Lemma 2. *The function $\zeta_\beta(s,a)$ has a regular analytic continuation into the entire complex plane except for simple poles at the points $s=\frac{1}{\beta}-j$ for all $j=0,1,2,\dots$ if $1/\beta\notin\mathbb{N}$ and for $0\leqslant j\leqslant\frac{1}{\beta}-1$ if $1/\beta\in\mathbb{N}_{0,1}$. Moreover,*

$$\operatorname{Res}_1\big(\zeta_\beta(s,a),\,s=\frac{1}{\beta}-j\big)=\frac{(-1)^j}{j!}\,\frac{\Gamma(1+\frac{1}{\beta})}{\Gamma(\frac{1}{\beta}-j)}a^j.$$

Corollary 1. *The coefficients in the expansion of the heat function satisfy the alternative formula $c_m=\zeta_\beta(-m,0)=\zeta(-\beta m)$.*

Lemma 3. *One has*

$$\zeta_\beta(0,a)=\begin{cases}-\frac{1}{2}, & \frac{1}{\beta}\notin\mathbb{N},\\ -\frac{1}{2}+(-1)^{1/\beta}a^{1/\beta}, & \frac{1}{\beta}\in\mathbb{N}_{0,1},\end{cases}$$

$$\zeta_\beta'(0,a)=-\frac{\beta}{2}\log 2\pi+\sum_{j=1}^{p-1}\frac{(-1)^j}{j}\,\zeta(\beta j)a^j-\log F_\beta(a)$$
$$+\begin{cases}\frac{(-1)^p}{p}\zeta(\beta p)a^p, & \frac{1}{\beta}\notin\mathbb{N},\ p\neq 0\\ (-1)^{1/\beta}[(\beta+1)\gamma+\psi(1/\beta)]a^{1/\beta}, & \frac{1}{\beta}\in\mathbb{N}_{0,1}.\end{cases}$$

Proof. For $|a|<1$, we expand the powers of the binomial in the definition of the zeta function and obtain $\zeta_\beta(s,a)=\sum_{j=0}^{\infty}\binom{-s}{j}\zeta_\beta(s+j,0)a^j$, where $\zeta_\beta(s,0)=\zeta(\beta s)$. For $1/\beta\notin\mathbb{N}$, we can evaluate both the series and its s-derivative at $s=0$. For $1/\beta\in\mathbb{N}_{0,1}$, there is some trouble with the pth term, which can be computed with the use of the following expansions at $s=0$:

$$\binom{-s}{j}\zeta_\beta(s+j,0)=\frac{(-1)^j}{j}\{R_1(j)+[R_0(j)+(\gamma+\psi(j))R_1(j)]s\}+O(s^2),$$

where

$$R_k(j)=\operatorname{Res}_k(\zeta_\beta(s,0),\,s=j)=\begin{cases}\gamma, & k=0,\\ 1/\beta, & k=1.\end{cases}$$

Using Lemma 3, we generalize the Choi–Quine factorization lemma [1] as follows.

Corollary 2. *One has*

$$\zeta_{2\beta}(0,a^2)=\frac{1}{2}[\zeta_\beta(0,ia)+\zeta_\beta(0,-ia)],$$

$$\zeta'_{2\beta}(0,a^2)=\zeta'_\beta(0,ia)+\zeta'_\beta(0,-ia)-\begin{cases}2(-1)^{p/2}\sum_{j=1}^{p/2}\frac{1}{2j-1}a^p & \text{for even } p=1/\beta\in\mathbb{N}_{0,1},\\ 0 & \text{otherwise.}\end{cases}$$

4. Properties of the gamma function. Our first results follow from the definition.*

Proposition 1. $\Gamma_\beta(iz)\Gamma_\beta(-iz)=e^{\gamma_{2\beta}z^2}\Gamma_{2\beta}(z^2)$.

Proposition 2. *If* $\beta>1$, *then*

$$\left(1-\frac{x}{z}\right)e^{-\gamma_\beta x}\prod_{n=1}^{\infty}\left(1-\frac{x}{n^\beta+z}\right)=\frac{\Gamma_\beta(z)}{\Gamma_\beta(z-x)}\,.$$

As a consequence, one can evaluate the general class of infinite products of the form $\Pi=\prod_{n=1}^{\infty}P(n^\beta)/Q(n^\beta)$, where $P(x)$ and $Q(x)$ are two polynomials of the same degree J and Q has no roots of the form n^β. Straightforward calculations give

$$\Pi=\prod_{j=1}^{J}\frac{b_j\Gamma_\beta(-b_j)}{a_j\Gamma_\beta(-a_j)}\,e^{\gamma_\beta(b_j-a_j)},$$

where a_j and b_j are the roots of P and Q, respectively.

The Taylor series of $\log\Gamma_\beta$ can also be obtained from Lemma 3. Alternatively, we can use the regularized resolvent function

$$\widetilde{R}_\beta(-z,0)=\sum_{n=1}^{\infty}\left[\frac{1}{-z-n^\beta}+\sum_{j=0}^{p-1}\frac{(-z)^j}{n^{(j+1)\beta}}\right],$$

which will be important in the proof of Proposition 4. The series is uniformly convergent in each bounded domain together with the first p derivatives, and so we can differentiate and, using an appropriate normalization, obtain $(d^k/dz^k)\log F_\beta(z)=(-1)^k\widetilde{R}_\beta^{(k-1)}(-z,0)$. Passing to the limit, we have $\widetilde{R}_\beta^{(k)}(-z,0)=0$ for $0\leqslant k\leqslant p-1$. This proves the following assertion.

Proposition 3. *For small* z, *one has* $\log\Gamma_\beta(z)=-\log z-\gamma_\beta z+\sum_{j=p+1}^{\infty}\frac{(-1)^j}{j}\zeta(\beta j)z^j$.

Our last result is the asymptotic expansion for large values of the argument. The proof comprises two steps. First, we use the Mellin transform as in the proof of Lemma 2 to express the p-resolvent function $R_\beta^{(p)}(\lambda,0)$ as the Laplace transform of the heat function; thus the asymptotic expansion of $R_\beta^{(p)}(\lambda,0)$ is given by a substitution using the expansion of the heat function provided by Lemma 1. Since asymptotic expansions can be integrated, from this expansion we obtain the expansion for the logarithm of the Weierstrass product, up to finitely many constants b_j. Here some care is necessary to deal with the possible $1/z$ term. We need to distinguish two cases depending on whether $1/\beta\in\mathbb{N}_{0,1}$ or not.** The second step is to determine the still unknown constants b_j. To this end, we need to introduce the following analytic representation of the zeta function:

$$\zeta_\beta(s,0)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{c,\theta}}e^{-\lambda t}\widetilde{R}_\beta(\lambda,0)\,d\lambda\,dt,$$

where $\Lambda_{c,\theta}=\{\lambda\in\mathbb{C}:|\arg(\lambda-c)|=\theta/2\}$ becomes a contour of Hankel type for some positive real $c<1$ and $0<\theta<\pi$. Integrating by parts first in λ and then in t, we obtain $\log F_\beta$ instead of the regularized resolvent and also an extra s factor. The presence of the latter provides a zero

*Proposition 1 can also be obtained from Corollary 2.

**Note that in the first case one necessarily has $\beta<1$, and hence the second case contains the case $\beta>1$, $p=0$. Note also that $1/\beta$ can never vanish, since β is assumed to be finite.

of second order at $s = 0$ and allows one to use the standard technique* (e.g., see [2]) and the known expansion of $\log F_\beta$ to compute the residues at $s = 0$. By comparing them with the values of the Riemann zeta function and its derivative, we find the constants b_j. We obtain the following assertion.**

Proposition 4. *For large z with $\arg(z) \neq 0$, one has*

$$\log \Gamma_\beta(z) \sim -\frac{1}{2}\log z + \frac{\beta}{2}\log 2\pi + \sum_{m=1}^{\infty} \frac{(-1)^m}{m}\zeta(-\beta m) z^{-m} - \gamma_\beta z$$

$$+ \begin{cases} \displaystyle\sum_{j=1}^{p} \frac{(-1)^{j+1}}{j}\zeta(\beta j) z^j + \frac{(-1)^{p+1}\pi}{\sin \pi(1/\beta - p)} z^{1/\beta} & \text{if } \dfrac{1}{\beta} \notin \mathbb{N},\ p = \left[\dfrac{1}{\beta}\right] \neq 0, \\ \displaystyle -\frac{\pi}{\sin(\pi/\beta)} z^{1/\beta} & \text{if } \dfrac{1}{\beta} \notin \mathbb{N},\ p = \left[\dfrac{1}{\beta}\right] = 0, \\ \displaystyle\sum_{j=1}^{p-1} \frac{(-1)^{j+1}}{j}\zeta(\beta j) z^j + (-1)^{p+1} z^p[\log z + \gamma\beta] & \text{if } \dfrac{1}{\beta} = p \in \mathbb{N}_{0,1}. \end{cases}$$

To conclude, note that the constant γ_β is a natural generalization of the Euler constant $\gamma = \gamma_1$. One can see this by writing out the defining series in closed form or by observing that $\gamma_\beta = \sum_{k=p+1}^{\infty} \frac{(-1)^k}{k}\zeta(\beta k)$. From the preceding results, we also obtain

$$-\gamma_\beta = \lim_{z\to 0} \frac{d}{dz} \log z\Gamma_\beta(z) = \lim_{z \to 0}\left(\frac{d}{dz}\log \Gamma_\beta(z) + \frac{1}{z}\right),$$

in accordance with the classical case and independently from the functional equation. On the other hand, for arbitrary β it is no longer true that the constant γ_β coincides with the nonsingular part of the corresponding zeta function $\zeta_\beta(s, 0)$ at $s = 1$.

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*Some care is necessary to modify the contour.

**Note that the last formula covers also the case $p = 1$.