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Conjugacy classes of Gauge Groups[1]

Renzo A. Piccinini, Università di Milano,
and Mauro Spreafico, University of Aberdeen

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Preface

Let ξ be a smooth principal G-bundle over a differential manifold B representing a physical theory. An equivariant, fibre preserving bundle automorphism of ξ gives rise to a new bundle with identical physical content and therefore, such an automorphism is a symmetry of the physical theory. It is clear that the set $\mathcal{G}(\xi)$ of all such automorphisms (called *gauge transformations*) has the algebraic structure of a group; it has also a topological structure compatible with the group structure, that is to say, $\mathcal{G}(\xi)$ is a topological group (see Section 2.1) called *gauge group* of ξ.

The gauge group $\mathcal{G}(\xi)$ acts on the connections and curvature forms; the attempt to understand these objects under the action of the gauge group lead to the development of a new theory known as *Gauge Theory*. Born in the framework of Theoretical Physics, gauge theory soon became an important and powerful research instrument also in Differential Geometry, leading to the construction of new invariants. On the other hand, gauge groups are interesting topological and algebraic objects on their own right. Moreover, they can be viewed as particular cases in more general contexts as, for example, in the study of the group of homotopy classes of fibre preserving self-homotopy equivalences of a fibration. Indeed, from this point of view, gauge groups are just a particular case of a far more general abstract situation: they can be regarded as coming from the self-equivalences of an object within a well-defined Category.

The general line of thought in these notes is the classical one followed rather successfuly in Algebraic Topology: we associate algebraic objects to fibre bundles, and through some algebraic properties of the former we try to obtain topological informations about the latter. Now, gauge groups contain interesting informations about the topology of the bundles; thus, it seems rather natural to search for these topological properties through algebraic manipulations of the gauge groups. A direct analysis and classification of gauge groups as algebraic objects is rather complicated; a simpler way seems to be the comparison of the gauge groups of different bundles within a larger group of symmetries. Let us explain this last idea more thoroughly.

Fibre bundles are locally trivial over the open sets U_i of a convenient open covering of B, that is to say, the restriction of ξ over each U_i behaves like a projection $U_i \times G \longrightarrow U_i$. On the other hand, a gauge transformation of ξ is represented locally, by a map $U_i \longrightarrow G$ and thus, it is possible to view $\mathcal{G}(\xi)$ as a topological closed subgroup of the topological group $\mathcal{L} = \prod_i Map(U_i, G)$. Now if B has a *good cover* – an open covering made up of contractible open sets – any principal G-bundle over B can be trivialized over the good cover and therefore, the gauge group of any principal G-bundle over B can be imbedded in $\mathcal{L}$.

The objective of these lecture notes is essentially to make enquiries in the following two directions: (i) what topological informations about bundles one can obtain from the conjugacy of their gauge groups? and (ii) can we use the conjugacy relation introduced previously as a first step towards the classification of gauge groups via the classification of bundles with conjugated gauge groups?

We now give a brief description of the contents of each chapter of these Lecture Notes. In the first chapter we describe the categories we deal with; in particular, we study the category of *weak Hausdorff k-spaces* which is the category of topological spaces we use. Although this category is smaller than the category of *all* topological spaces, it is important to keep in mind that it is still large enough to contain all the interesting spaces of everyday life: manifolds, metric spaces, CW-complexes, etc.; moreover, the category of weak Hausdorff k-spaces satisfies an exponential law, no matter which spaces we consider. For this and other reasons, such a category is extremely useful for homotopy theory. The reader who has not been exposed to this category before might find it very abstruse at a first glance; however, one should not be deterred by this. Indeed, on a first reading, one could very well bypass Section 1.1 and simply assume that all spaces encountered have the right properties. Chapter one also contains a review of the categories of fibre bundles and principal bundles, including their equivariant versions.

Chapter 2 is devoted to the definition and main topological properties of gauge groups; in particular, Section 2.2 is heavily based on results presented in [6].

In Chapter 3 we go deeply into an analysis of the relation between two principal G-bundles which have conjugate gauge groups. We soon obtain that, under certain mild conditions, the set of (equivalence classes) of principal ZG-bundles (here ZG represents the centre of G) is actually an abelian group acting on the set of all (equivalence classes) of principal G-bundles; then we discover that two pricipal G-bundles with conjugate gauge groups are characterised by the fact that they differ by the action of a principal ZG-bundle. This relation becomes clear in the case of vector bundles: two vector bundles (real or complex) ξ and ξ' have conjugate gauge groups if, and only if, $\xi' \cong \xi \otimes \lambda$ where λ is a line bundle. This clearly defined geometrical relation seems to suggest an easy way to enumerate bundles with conjugated gauge groups: just use the classification of line bundles given by the cohomology of the base space. Actually one soon realizes that this is not true in general, because tensor multiplication by a line bundle defines an action which is not free. This problem can be reformulated in an elementary way: given a vector bundle ξ and a line bundle λ over the same base space, are $\xi \otimes \lambda$ and ξ equivalent? Like many basic questions, and despite its simple formulation, this problem is eventually extremely hard to solve; the last section of Chapter 4 is devoted to a particular case (of even dimensional real vector bundles over real projective spaces) which we are able to solve. We also hope to entice the reader to do further work in the direction of the problem we raised.

Up to the end of Chapter 4 we make no use of the Classification Theorem for bundles; that is the technical device we use in Chapter 5. To do this we need several properties of Classifying Spaces; the construction and properties of Classifying Spaces and Universal Bundles are reviewed in the Appendix where the reader can read a reconstruction of the work by J. Milgram and N. Steenrod [42] within the framework of the category of weak Hausdorff k-spaces. Indeed, the Appendix gives a self contained overview of this subject; on the one hand, we state the results which give the necessary tools employed throughout these notes, and on the other hand, we hope to impress upon the interested reader a deeper feeling for the methods and techniques used in the construction of Classifying Spaces. As we said before, Chapter 5 makes use of the Classification Theorem; this theorem is combined with techniques typical of Homotopy Theory to restate, in a far more general context, concepts and results obtained in the previous chapters. Moreover, in this way we can shed new light into the theory we are trying to develop.

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Chapter 1

Preliminaries

The intent of this chapter is to describe the categories which are necessary to develop this work, and to give some of the important properties related to them. We begin by giving a partial list of the categories which we shall use.

$Sets$ - category of sets and functions between sets;

Gr - category of groups and group homomorphisms;

Top - category of all topological spaces (simply called *spaces*) and continuous functions (called *maps*);

CW - category of CW-complexes and maps (not necessarily cellular);

$HTop$ - category of spaces and homotopy classes of maps. This is the *homotopy category associated* to Top;

If in the above categories we consider only *based* objects (i.e., each object is taken together with a distinguished element - a *base point*) and consider only the morphisms which take base points into base points, then we have the based subcategories $Sets_*$, Gr_*, Top_* and CW_*; as for $HTop$, we obtain $HTop_*$ via *based homotopy* i.e., we require that in the definition of homotopy, the entire whisker over the base point of the first space goes to the base point of the second space.

A question of notation: for any category $\mathcal{C}$, we denote by $\mathcal{C}(X,Y)$ the set of all morphisms from an object X to an object Y of $\mathcal{C}$.

We could "combine" some of these categories; for example, we could consider the category $TopGr$ of topological groups; its morphisms are continuous group homomorphisms. Furthermore, we indicate with $\cong$ the equivalences in each category.

1.1 The category $wHk(Top)$

While the previous categories do not need further explanations, there is an important category which we shall have to use for technical reasons and which needs some explaining; we refer to the category $wHk(Top)$ of *weak Hausdorff k-spaces*. Although its definition may seem overwhelming to the person who encounters it for the first time, such a reader should not be deterred by the momentary difficulty; indeed, $wHk(Top)$ contains all the nice spaces one deals with in practice - such as smooth manifolds - and hence, on a first reading, those who are unfamiliar with weak Hausorff k-spaces might simply ignore their definition and admit they are working with "nice spaces".

We define $wHk(Top)$ as follows. Let X be a given space; a subset $A \subset X$ is *compactly closed* if, for every compact Hausdorff space K and every map $f : K \longrightarrow X$, $f^{-1}(A) \subset K$ is closed in K; the space X is said to be a *k-space* if all of its compactly closed subsets are closed. Now define a functor $k : Top \longrightarrow Top$ by associating to each $X \in Top$ the space $k(X)$ with the same underlying set as X but with the topology given by taking as closed sets the compactly closed sets with respect to the topology of X. As for the morphisms, we observe that if Y is a k-space, then a function $f : Y \longrightarrow X$ is a map, if and only if, $f : Y \longrightarrow k(X)$ is continuous; thus, for any morphism $f \in Top(Y, X)$, we simply take $k(f) = f : k(Y) \longrightarrow k(X)$. The functor k is called *k-ification* and its image $k(Top)$ is a full subcategory of Top. The category $k(Top)$ is both complete and cocomplete, with the product of k-spaces given by the rule

$$X \times Y = k(X \times_c Y)$$

where $X \times_c Y$ is the usual cartesian product in Top. A k-space X is said to be *weak Hausdorff* whenever the diagonal $\Delta_X : X \longrightarrow X \times X$ is closed in $X \times X$. The category $wHk(Top)$ is the full subcategory of Top determined by all weak Hausdorff k-spaces and maps. There is a useful characterization of weak Hausdorff k-spaces which reads as follows: if X is a k-space; then X is weak Hausdorff if, and only if, for every map $f : K \longrightarrow X$, with K compact and Hausdorff, $f(K)$ is closed and compact Hausdorff.

We give now the main properties of $wHk(Top)$. To begin with, we note that the previous characterization of weak Hausdorff k-spaces implies that these spaces have separation property T_1; they are not necessarily Hausdorff as the property of Δ being closed in $k(Top)$ is not as strong as being closed in Top (there are examples of k-spaces that are not weak Hausdorff: the *Tychonoff plank* is such an example – see [15, Appendix A1]).

Next, $wHk(Top)$ is closed under the formation of subspaces, finite products and coproducts, and quotients by closed subspaces. Notice that the category $wHk(Top)$ has mapping spaces $Y^X = k(Map(X,Y))$ ($Map(X,Y)$ is endowed with the compact-open topology), and these satisfy the exponential law

$$(Z^X)^Y \cong Z^{X\times Y} \ .$$

Finally, $wHk(Top)$ is closed under the formation of adjunction spaces: given that A is a closed subspace of X, for any map $f : A \longrightarrow B$, the space $B \sqcup_f X$ obtained as a pushout of the diagram $X \longleftarrow A \longrightarrow B$ is also an object of $wHk(Top)$.

As we did before, we could combine the category Gr with $wHk(Top)$ to obtain the category $wHk(Top)Gr$ of weak Hausdorff topological groups.

A space $X \in Top$ has the *initial topology* induced by a map $i : X \longrightarrow Y$ if, for every $Z \in Top$, $f \in Top(Z,X) \iff if \in Top(Z,Y)$. Dually, a space X has the *final topology* induced by a map $p : Y \longrightarrow X$ if, for every space $Z \in Top$, $g \in Top(X,Z) \iff gp \in Top(Y,Z)$. If X has the initial topology induced by $i : X \longrightarrow Y$ in Top and i is injective, we say that i is an *inclusion*; then $U \subset X$ is open if, and only if, $i(U) \subset i(X)$ is open[1]. If X has the final topology induced by $p : Y \longrightarrow X$ and p is surjective, we say that p is a *proclusion*; then $U \subset X$ is open if, and only if, $p^{-1}(U) \subset Y$ is open. It is very easy to show that the composition of two inclusions (resp. proclusions) is an inclusion (resp. proclusion). Furthermore, a finite product of inclusions in Top is an inclusion (see [9, 1.4.1, Corollary to Proposition 3]); the corresponding property for proclusions fails to be true in Top but not in $wHk(Top)$:

Lemma 1.1.1 *Suppose that $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are proclusions in $wHk(Top)$. Then $f \times f' : X \times X' \longrightarrow Y \times Y'$ is a proclusion in $wHk(Top)$.*

[1] The following observation will be useful later on: an injection of Top is closed $\iff$ it is a closed inclusion.

Proof – Since the functor $X \times -$ preserves colimits (see [15, Appendix]) the space $X \times Y'$ has the final topology with respect to $1_X \times f'$; a similar argument applied to $- \times Y'$ shows that $X' \times Y'$ has the final topology with respect to $f \times 1_{Y'}$ and therefore, $X' \times Y'$ has the final topology with respect to $f \times f'$. $\square$

Notice also the following property of the functor k:

Lemma 1.1.2 *The k-ification functor $k : Top \longrightarrow Top$ preserves inclusions.*

Proof – Let $i \in Top(X, Y)$ be an inclusion. Clearly, ki is injective. Now we must show that, given any $Z \in k(Top)$, a function $g : Z \longrightarrow k(X)$ is continuous $\iff$ kig is continuous. In fact, kig is continuous $\iff$ $ig : Z \longrightarrow Y$ is continuous $\iff$ $g : Z \longrightarrow X$ is continuous $\iff$ $g : Z \longrightarrow k(X)$ is continuous. $\square$

We now recall the definition and the main properties of expanding sequences of spaces in $wHk(Top)$; we refer the reader to [15] for further details[2]. An *expanding sequence* of spaces in $wHk(Top)$ is a sequence $\{X_n, n \in \mathbb{N}\}$ of spaces in $wHk(Top)$ such that, for every $n \in \mathbb{N}$, X_n is a closed subspace of X_{n+1}. The *union space* of the expanding sequence is the set $X = \bigcup_{n=0}^{\infty} X_n$ endowed with the final topology with respect to the family of inclusions $X_n \subset X$. The family of spaces $\{X_n, n \in \mathbb{N}\}$ is also called a filtration of X. A map f between two filtered spaces X and Y is said to be *filtered* if there exists a sequence of maps $f_n : X_n \longrightarrow Y_n$ which is *compatible* with the filtrations, that is to say, if $f_{n+1}|_{X_n} = f_n$, for every $n \in \mathbb{N}$. Clearly, a compatible sequence of maps induces a map (the *union map*) between the union spaces.

The main properties of the union space of an expanding sequence, are due to the fact that the topology of the union space is coherent with that determined by the family $\{X_n, n \in \mathbb{N}\}$, and each X_n is a closed subset of X. In particular, this implies that $wHk(Top)$ is closed under the formation of union spaces of expanding sequences; furthermore, if all the inclusions $X_n \subset X_{n+1}$ are closed cofibrations, then the inclusions $X_n \subset X$ are also closed cofibrations.

[2]Unlike [15] we do not require that each space of an expanding sequence is included in the next as a closed cofibration; nevertheless, the results of [15] are still valid.

The following lemmas show how the process of taking the union map of a compatible sequence between filtered spaces, preserves closed inclusions, proclusions and closed cofibrations.

Lemma 1.1.3 *Let* $X = \bigcup_{n=0}^{\infty} X_n$ *and* $Y = \bigcup_{n=0}^{\infty} Y_n$ *be filtered spaces, and let* $f_n : X_n \longrightarrow Y_n$ *be a compatible sequence of closed inclusions (respectively, proclusions). Then, the union map* f *is a closed inclusion (respectively, proclusion).*

Proof – We begin with closed inclusions. Since f is injective, we have only to prove that f is closed. Let C be closed in X; then $f(C) \cap Y_n = f_n(C \cap X_n)$ is closed in Y_n, and hence in Y; it follows that $f(C)$ is closed in Y.

Now we take a look at the proclusions. It is easy to see that f is surjective; it remains to prove that Y has the final topology coinduced by f. If $V \subseteq Y$ is such that $f^{-1}(V)$ is open in X, then $f^{-1}(V) \cap X_n$ is open in X_n for each n. Now set $V_n = V \cap Y_n$ and observe that $f^{-1}(V) = \bigcup_{n=0}^{\infty} f^{-1}(V_n)$ and

$$f^{-1}(V) \cap X_n = f_n^{-1}(V_n)$$

is open in X_n. Since each f_n is a proclusion, V_n is open in Y_n for each n and thus, V is open in Y. $\square$

For the preservation of cofibrations, we need an extra assumption on the maps involved:

Lemma 1.1.4 *Let* $X = \bigcup_{n=0}^{\infty} X_n$ *and* $Y = \bigcup_{n=0}^{\infty} Y_n$ *be filtered spaces, and let* $f_n : X_n \longrightarrow Y_n$ *be a compatible sequence of closed cofibrations; suppose also that all the inclusions* $X_n \subset X_{n+1}$ *and* $Y_n \subset Y_{n+1}$ *are closed cofibrations. Then, the union map* $f : X \longrightarrow Y$ *is a closed cofibration.*

The proof follows the same lines as [15, A.5.5] (the requirement of normality is not needed).

1.2 Relations, actions and quotients

Let R be an equivalence relation on $X \in k(Top)$; because X/R has the final topology induced by the quotient map $p : X \longrightarrow X/R$, then $X/R \in k(Top)$ (see [15, page 242]). This is not quite true in $wHk(Top)$: in fact, we heve the following

Lemma 1.2.1 *Let $f : X \longrightarrow Y$ be a proclusion in $k(Top)$. If X is weak Hausdorff, then Y is weak Hausdorff iff $(f \times f)^{-1}\Delta_Y$ is closed in $X \times X$.*

Let R and R' be two given equivalence relations on X and X' respectively. A map $f \in k(Top)(X, X')$ is said to be *relation preserving* if $x_1 R x_2$ implies $f(x_1)R'f(x_2)$ for each x_1, x_2 in X, and *relation bipreserving* if $x_1 R x_2 \iff f(x_1)R'f(x_2)$ for each x_1, x_2 in X. A relation preserving map $f : X \longrightarrow X'$ defines a unique map $\widehat{f} \in k(Top)(X/R, X'/R')$ such that $p'f = \widehat{f}p$. Notice that if f is relation bipreserving then $\widehat{f}$ is injective; moreover, if f is an injective relation preserving map and $\widehat{f}$ is injective, then f is actually relation bipreserving. Finally, if f is a proclusion, so is $\widehat{f}$. The last statement holds true in $wHk(Top)$ provided X/R and X'/R' are weak-Hausdorff (for example, if R and R' are the relations induced by closed subsets $A \subset X$ and $A' \subset X'$). A similar statement for inclusions canot be made in general; a situation where this can be done will be discussed in Lemma 1.2.4.

We say that $M \in k(Top)$ is a *topological monoid* with multiplication $\tau \in k(Top)(M \times M, M)$ if τ is associative with identity element $u_M \in M$. A topological monoid M *acts* on $X \in k(Top)$ if there exists an *action*[3] $\varphi \in k(Top)(X \times M, X)$ such that $\varphi(\varphi \times 1_M) = \varphi(1_X \times \tau)$ and, for all $x \in X$, $\varphi(x, u_M) = x$. With the previous conditions, X is said to be an *M-space*. An equivalence relation R on X is said to be *consistent* with the action φ if, for every $m \in M$ and $x, x' \in X$, xRx' implies that $(xm)R(x'm)$. If this is the case, the action passes to the quotient, giving an action $\hat{\varphi} : X/R \times M \longrightarrow X/R$.

1.2.1 The category Top_G

Suppose that the topological monoid M is actually a topological group G. The action of G on a G-space $X \in k(Top)$ determines an equivalence relation R on X: $(xRx' \iff (\exists g \in G)\ x' = xg)$ which is *consistent* with the action in the sense that, for every $g \in G$, xRx' implies $(xg)R(x'g)$. The k-space X/R, also denoted by X/G, is said to be an *orbit space*.

In particular, if the G-space X is itself a group, say H, in general the action $r : H \times G \longrightarrow H$ is not a group homomorphism. The next result characterizes this situation.

[3] To simplify the notation, we shall normally write xm for $\varphi(x, m)$.

Lemma 1.2.2 *A right action $r : H \times G \longrightarrow H$ is a group homomorphism $\iff$ r commutes with the right and left translations of H (in other words, if for every g in G and h, h' in H, $hr(h', g) = r(hh', g) = r(h, g)h'$).*

Proof – $\Rightarrow$ For every $g, g' \in G$ and every $h, h' \in H$

$$r((h, g)(h', g')) = r(hh', gg') = r(h, g)r(h', g') \ .$$

In particular, if we take $g' = u_G$ we obtain that $r(hh', g) = r(h, g)h'$, while taking $g = u_G$ we get $r(hh', g') = hr(h', g')$.

$\Leftarrow$ With arbitrary g, g', h, h' as before

$$r((h, g)(h', g')) = r(hh', gg') = r(r(hh', g), g') =$$

$$= r(r(h, g)h', g') = r(h, g)r(h', g') \ .$$

□

As an example, we take G to be a central subgroup of H and r to be the restriction of the group multiplication. Notice that in this example r is free.

Let $\vartheta : G \longrightarrow G'$ be a continuous homomorphism between the topological groups G and G'. Take a G-space X and a G'-space X'; a map $f : X \longrightarrow X'$ is *ϑ-equivariant* (or G, G'-equivariant) if, for every $(x, g) \in X \times G$, $f(xg) = f(x)\vartheta(g)$. The category of G-spaces and G-equivariant maps is denoted by Top_G.

Considering the equivalence relation S in $X \times G$ given by

$$(x_1, g_1)S(x_2, g_2) \iff x_1g_1 = x_2g_2 \iff (\exists g \in G)\ x_2 = x_1g \text{ and } g_2 = g^{-1}g_1$$

(similarly, for $X' \times G'$), we easly get the following relation between equivariance and relation preserving:

Lemma 1.2.3 *Under the conditions set before, the following statements hold true:*
i) if f is ϑ-equivariant, both f and $f \times \vartheta$ are relation preserving;
ii) if f is ϑ-equivariant and injective, then $f \times \vartheta$ is relation bipreserving.
iii) if f is ϑ-equivariant and injective, and ϑ is onto, then f is relation bipreserving.

As pointed out before, ϑ-invariant proclusions induce quotient maps that are proclusions. As for inclusions we have:

Lemma 1.2.4 *Let $\vartheta \in TopGr(G, H)$. Let X be a G-space, Y be an H-space Y, and $f : X \longrightarrow Y$ be a ϑ-equivariant relation bipreserving map. Then, if f is an open or closed inclusion the quotient map $\widehat{f} : X/G \longrightarrow Y/H$ is an inclusion.*

Proof – Since f is relation bipreserving, $\widehat{f}$ is injective. In order to prove that $\widehat{f}$ is an inclusion we show that $\widehat{f} : X/G \longrightarrow \widehat{f}(X/G)$ is open. When f is open this follows immediately, thus take a closed f.

For every $h \in H$, the map $\psi_h : Y \longrightarrow Y$, $\psi_h(y) := y.h$ is a homeomorphism; thus, because $f(X) \subset Y$ is closed, every $f(X).h$ is closed in Y. Let $f(X).H = \bigcup_{h \in H} f(X).h$ with the topology determined by the closed subspaces $f(X).h$, $h \in H$; one now shows that this topology is equivalent to the relative topology induced by Y.

Next, let $p : X \longrightarrow X/G$ be the identification map and let $U \subset X/G$ be an open set; then $V := p^{-1}(U)$ is open in X and, $f(V)$ is open in $f(X)$ since f is an inclusion. Because f is relation bipreserving and ϑ-equivariant $f(X) \cap f(V).H = f(V)$; moreover, for every $h \in H$, $f(V).h = f(X).h \cap f(V).H$ is open in $f(X).h$ and hence, $f(V).H$ is open in $f(X).H$. Using the relative topology of the latter space, this means that there exists an open set B in Y such that $f(V).H = f(X).H \cap B$.

At this point we take the identification map $q : Y \longrightarrow Y/H$ and observe that

$$q(B) \cap \widehat{f}(X/G) = q(B \cap f(X).H) = q(f(V).H) = q(f(V)) = \widehat{f}(U) \ ;$$

since $q(B)$ is open, it follows that $\widehat{f}(U)$ is open and therefore, $\widehat{f}$ is a homeomorphism onto its image $\widehat{f}(X/G)$, that is to say, $\widehat{f}$ is an inclusion. □

The following lemma gives a fundamental propery of spaces with a G-map taking value in G, usefull in dealing with local triviality of homogeneus spaces.

Lemma 1.2.5 *Let X be a G-space and let $f : X \longrightarrow G$ be a G-map; then $X \cong f^{-1}(u_G) \times G$.*

Proof – Define $\phi : f^{-1}(u_G) \times G \longrightarrow X$ as $(x, g) \longmapsto xg$. It is immediate to verify that $f\phi = pr_2$, the natural projection $f^{-1}(u_G) \times G \longrightarrow G$, and that ϕ is bijective. Furthermore, the map $\psi : X \longrightarrow f^{-1}(u_G) \times G$, $x \longmapsto (x(f(x))^{-1}, f(x))$, is a continuous inverse of ϕ. □

Corollary 1.2.6 *If X is a free G-space and $f : X \longrightarrow G$ is a G-map, then $X \cong X/G \times G$.*

We now recall the notion of principal action. Let E be a free G-space and consider the subspace $E^* = \{(x, xg) \mid x \in E,\ g \in G\}$ of $E \times E$. Since the action of G is free, it is possible to define a function $\tau : E^* \longrightarrow G$ (called the *translation function* of the G-space E) by $x\tau(x, x') = x'$, for each $(x, x') \in E^*$, satisfying the following properties:

1. $(\forall x \in E)\ \tau(x, x) = u_G$;

2. $(\forall x, x' \in E)\ \tau(x, x')\tau(x', x) = u_G$;

3. $(\forall x, x', x'' \in E)\ \tau(x, x')\tau(x', x'') = \tau(x, x'')$;

4. $(\forall x \in E,\ g \in G)\ \tau(x, xg) = g$.

The G-action is called *principal* if τ is a continuous function; we shall also say that a free G-space with a principal action is a *principal* G-space. As it will be apparent later on, principal actions play a fundamental role in the definition of principal bundles.

Notice that Lemma 1.2.1 implies the following relation between separability in Top (weak separability in $k(Top)$) of the orbit space and closure of E^*: E/G is Hausdorff iff E^* is closed in $E \times_c E$.

1.2.2 Equivariant cofibrations

Equivariant cofibrations are the natural extensions of cofibrations in the category Top_G of G-spaces. For a question of completeness we state next a proposition which characterizes G-cofibrations (see [47, Chapter I, Section 5] and [15, A.4]).

Proposition 1.2.7 *Let X be a G-space and let A be a G-closed subspace of X. The following statements are equivalent:*

1. *the pair (X, A) has the homotopy extension property in Top_G;*

2. *the space $\widehat{X} = X \times 0 \cup A \times I$ is a strong G-deformation retract of $X \times I$;*

3. *there are G-maps $u : X \longrightarrow I$ and $h : X \times I \longrightarrow X$ (let G act trivially on I) such that: (i) $A = u^{-1}(0)$, (ii) $(\forall x \in X)$ $h(x, 0) = x$, (iii) $(\forall t \in I, x \in A)$ $h(x, t) = x$ and (iv) $(\forall x \in X, u(x) < 1)$, $h(x, 1) \in A$.* [4]

Note that the space $U = u^{-1}([0, 1))$ is an open G-space which retracts to A.

The next result (whose proof is left to the reader) shows the importance of equivariant cofibrations in dealing with quotient spaces.

Proposition 1.2.8 *Let A be a G-closed subspace of a G-space X and let $i : A \longrightarrow X$ be the inclusion map. Then the induced map $\hat{\imath} : A/G \longrightarrow X/G$ is a closed cofibration $\Longleftrightarrow$ i is a G-closed cofibration.*

The next proposition is useful in the theory of principal bundles. First notice that a free G-space X is *locally trivial* if the orbit space $B = X/G$ is covered by open sets U_i for which there exist G-equivariant homeomorphisms $\phi_i : U_i \times G \longrightarrow p^{-1}(U_i)$ over U_i.

Proposition 1.2.9 *Let X be a principal G-space (as defined in Section 1.2) with a G-action $\phi : X \times G \longrightarrow X$. Suppose that there exists a point $x_o \in X$ such that (X, x_oG) is a G-closed cofibration. Then, the quotient map $q : X \longrightarrow X/G$ is locally trivial.*

Proof – Let $\tau : X^* \longrightarrow G$ be the translation map defined by ϕ. For every $x \in X$, the restriction of τ to the subspace $\{(x, xg) | g \in G\} \subset X^*$ is a G-homeomorphism with G; then, xG and x_oG are G-homeomorphic as subspaces of X and so, (X, xG) is a G-closed cofibation. Let $u_x : X \longrightarrow I$ and $h_x : X \times I \longrightarrow X$ be the maps which define (X, xG) as a G-ndr. Take the G-map $f : U_x \longrightarrow G$ defined by $f(x) = h(x, 1)$ for all $x \in U_x$. Corollary 1.2.6 now shows that U_x is G-homeomorphic to $U_x/G \times G$. $\square$

The two previous propositions combined have an important consequence: let H be a closed subgroup of $G \in TopGr$; if (G, H) is a G-closed cofibration, then $q : G \longrightarrow G/H$ is locally trivial. This is the case of a closed subgroup of a Lie group (Lie groups are CW-complexes – see [29]).

[4] In other words, (X, A) is a G-neighborhoud deformation retract (G-ndr).

1.2.3 The category $Top_H \cap Top_G$

Next we investigate the spaces with a double action. This is the general setting where equivariant theories originate. Let G and H be given topological groups. An object $X \in Top_G \cap Top_H$ is a space X togeher with an action of a group H (say, on the left) and an action of a group G (on the right). These actions are said to be *compatible* if they commute with each other, that is to say, if $(hx)g = h(xg)$, for all $x \in X$, $h \in H$ and $g \in G$. Notice that this gives rise to a natural action

$$X \times (H \times G) \longrightarrow X \ , \ (x, (h, g)) \longmapsto hxg \ .$$

The following lemmas characterize the present situation.

Lemma 1.2.10 *Let $X \in Top_G \cap Top_H$ with compatible actions. Then X/G has an H action, X/H has a G action, and the spaces*

$$(X/G)/H \text{ and } (X/H)/G$$

are well defined and homeomorphic.

Proof – By [10, 3.2.5, Proposition 11] G acts on X/H with the action

$$\phi' : X/H \times G \longrightarrow X/H \ , \ (Hx, g) \longmapsto H(xg)$$

and H acts on X/G with the action

$$\psi' : X/G \times H \longrightarrow X/G \ , \ (xG, h) \longmapsto (hx)G \ ;$$

thus, the quotient spaces $(X/H)/G$ and $(X/G)/H$ are well defined.

Consider the compositions $p'q$ and $q'p$ of the natural projections

$$X \xrightarrow{q} X/H \xrightarrow{p'} (X/H)/G$$

and

$$X \xrightarrow{p} X/G \xrightarrow{q'} (X/G)/H \ .$$

The maps $p'q$ and $q'p$ define the equivalence relations R and S in X

$$xRx' \iff \exists g \in G, h \in H \mid x' = (hx)g \ ,$$

$$xSx' \iff \exists g \in G, h \in H \mid x' = h(xg) \ .$$

The commutativity of the actions of G and H implies that

$$(\forall x, x' \in X)\ xRx' \ \Leftrightarrow\ xSx'\ .$$

Hence, the sets $X/S = (X/H)/G$ and $X/R = (X/G)/H$ coincide (up to bijection). Furthermore, $p'q = q'p$ and so, the final topology determined by $p'q$ on $(X/H)/G$ coincides with that determined by $q'p$ on the (same) set $(X/G)/H$. Hence, $(X/G)/H \cong (X/H)/G$. □

Let us assume now that the actions of G and H on X coincide when restricted to $N = G \cap H$, with N normal in G and H. The next lemma allows the transformation of this case into the general one given by two independent compatible actions.

Lemma 1.2.11 *Let X be a G-space and N a normal subgroup of G. Then, there is an action of G/N on X/N such that the following diagram commutes:*

$$\begin{array}{ccc} X \times G & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ X/N \times G/N & \xrightarrow[\hat{\phi}]{} & X/N \end{array}$$

Moreover, if X is a free and locally trivial G-space, then X/N is a free and locally trivial G/N-space.

When the actions of G and H coincide on a central subgroup $Z = G \cap H$, the previous lemma allows us to define indipendent actions of G/Z and H/Z on X/Z. Thus, because of Lemma 1.2.10, we obtain an action of H/Z on $(X/Z)/(G/Z) = X/G$ and an action of G/Z on $(X/Z)/(H/Z) = X/H$ such that the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[r, "p"] \arrow[d, "i_G"] & G/Z \arrow[d, dashed, "\hat{\imath}_G = i_{G/Z}"] \\
H \arrow[r, "i_H"'] \arrow[d, "q"'] & X \arrow[r, "p_H"'] \arrow[d, "p_G"] & X/H \arrow[d, dashed, "p_{G/Z} = \widehat{p_G}"] \\
H/Z \arrow[r, dashed, "\hat{\imath}_H = i_{H/Z}"'] & X/G \arrow[r, dashed, "p_{H/Z} = \widehat{p_H}"'] & Y
\end{tikzcd}$$

where: i_G and i_H are the restrictions of the actions of G and H to $\{x_0\} \times G$ and $\{x_0\} \times H$ respectively (for some x_0 fixed in X); p, q, p_G and p_H are the natural projections; $\widehat{p_G}$ and $\widehat{p_G}$ are the induced quotient maps and finally,

$$Y := (X/H)/(G/Z) \cong ((X/Z)/(H/Z))/(G/Z) \cong$$

$$\cong ((X/Z)/(G/Z))/(H/Z) \cong (X/G)/(H/Z) \ .$$

In particular, we have proved the following result:

Lemma 1.2.12 *Let G and H be topological groups whose intersection $G \cap H = Z$ is a central subgroup of both G and H. Let X be a space on which both G and H act compatibly and suppose that the two actions coincide on Z. Then, X/H has a G/Z-action, X/G has a H/Z-action and*

$$(X/H)/(G/Z) \cong (X/G)/(H/Z) \ .$$

Proposition 1.2.13 *Let $X \in Top_H \cap Top_G$ with compatible actions. Suppose that the action of H on X is principal and that the space $X^* = \{(x, hx) | x \in X, h \in H\}$ is closed in $X \times X$. Finally, assume that the natural action of $H \times G$ on X is free. Then, the action of H on X/G is principal.*

Proof – The group G acts on X^* by $((x, x'), g) \longmapsto (xg, x'g)$. Moreover, the translation function $\tau : X^* \longrightarrow H$ determined by the action of H on X is continuous and is relation preserving (with respect to the relation on

X^* determined by the action of G and the relation on H determined by the trivial subgroup $\{u_H\}$); hence, τ defines a unique map $\hat{\tau} : X^*/G \longrightarrow H$ such that the following diagram commutes:

$$\begin{tikzcd} X^* \arrow[r, "\tau"] \arrow[d] & H \arrow[d] \\ X^*/G \arrow[r, "\hat{\tau}"'] & H \end{tikzcd}$$

On the other hand, the inclusion map $i : X^* \longrightarrow X \times X$ is Δ-equivariant relation bipreserving (here Δ is the diagonal homomorphism); hence, by Lemma 1.2.4, it passes to the quotient

$$\begin{tikzcd} X^* \arrow[r, "i"] \arrow[d] & X \times X \arrow[d] \\ X^*/G \arrow[r, "\hat{\imath}"'] & X/G \times X/G \end{tikzcd}$$

and $\hat{\imath}$ is an inclusion. Now we see that $X^*/G \cong (X/G)^*$. $\square$

1.3 Bundles

A *fibre bundle* is a 5-tuple $\xi = (E, p, B, F, G)$ satisfying the following properties:

1. $E, B, F \in Top$, $p \in Top(E, B)$ and G is a topological group acting effectively on the left of F;

2. B is covered by a collection of open sets $\{U_i | i \in J\}$ and for every $i \in J$ there exists a homeomorphism $\phi_i : U_i \times F \longrightarrow p^{-1}(U_i)$ *over* U_i (that is to say, such that $p\phi_i = pr_1$);

3. for every $b \in U_{ij} = U_i \cap U_j \neq 0$ and every $y \in F$, $\phi_i^{-1}\phi_j(b,y) = (b, g_{ij}(b)y)$ and the function $g_{ij} : U_{ij} \longrightarrow G$ is continuous.

The space B is the *base*, F is the *fibre* and G is the *structural group* of the fibre bundle; finally, the maps g_{ij} are its *transition functions.* A fibre bundle with structural group G is also called a G*-bundle.*

The transition functions are the key ingredients to work with fibre bundles. The following formulation of the transition functions is very useful: for every $b \in U_{ij}$, define

$$\phi_{i,b} : \{b\} \times F \longrightarrow p^{-1}(b) \ , \ \phi_{i,b}(y) = \phi_i(b,y) \ ;$$

then, for every $y \in F$, $g_{ij}(b)y = \phi_{i,b}^{-1}\phi_{j,b}(y)$.

It is easy to verify that the transition functions satisfy the following conditions:

TF1 $(\forall b \in U_i) \ g_{ii}(b) = u_G$;

TF2 $(\forall b \in U_{ij}) \ (g_{ij}(b))^{-1} = g_{ji}(b)$;

TF3 $(\forall b \in U_{ijk}) \ g_{ki}(b) = g_{kj}(b)g_{ji}(b)$.

The last condition can be written in a cyclic fashion:

$$(\forall i,j,k \in J)(\forall b \in U_{ijk}) \ g_{ij}(b)g_{jk}(b)g_{ki}(b) = u_G \ .$$

Theorem 1.3.1 *Let G be a topological group which acts effectively on a space F (on the left); let B be a space with an open covering $\{U_i | i \in J\}$ and, for every $i,j \in J$ for which $U_{ij} \neq \emptyset$, we are given maps $g_{ij} : U_{ij} \longrightarrow G$ which satisfy conditions* TF1, TF2 *and* TF3*. Then there exists a fibre bundle $\xi = (E,p,B,F,G)$ with transition functions g_{ij} .*

The preceding result allows us to construct a G-bundle with fibre G out of any given G-bundle ξ: just observe that G acts on itself by multiplication. Such a G-bundle with fibre G is the *principal* G-bundle associated to ξ. We denote a principal G-bundle over B and with total space E simply by the 4-tuple notation (E,p,B,G).

Another important application of Theorem 1.3.1 is the construction of the tangent bundle of a differentiable manifold. Let M be an n-manifold with a C^r-atlas $\{(U_i,\phi_i)|i \in J\}$ $(r \geq 1)$; then

$$h_{ij} = \phi_j\phi_i^{-1} : \phi_i(U_{ij}) \subseteq \mathbb{R}^n \longrightarrow \phi_j(U_{ij}) \subseteq \mathbb{R}^n$$

is a C^r-map. The fibre bundle $\tau(M)$ with base M, fibre $\mathbb{R}^n$ and group $GL(n,\mathbb{R})$ determined by the transition functions

$$g_{ij} : U_{ij} \longrightarrow GL(n,\mathbb{R}) \ , \ b \longmapsto J(h_{ij})(\phi_i(b))$$

where $J(h_{ij})$ is the Jacobian matrix of h_{ij} is the *tangent bundle* of M.

The transition functions are also used to "compare" bundles. Let ξ and ξ' be two G-bundles with the same base B and fibre F; moreover, suppose that both bundles are locally trivial over the same open covering of B (this is always possible by simply intersecting the original open coverings of both bundles). Then we say that ξ and ξ' are *equivalent* if, for every $i \in J$, there exists a map $\rho_i : U_i \longrightarrow G$ such that

$$(\forall b \in U_{ij}) \ g'_{ij}(b) = (\rho_j(b))^{-1} g_{ij}(b) \rho_i(b) \ .$$

We introduce the notation $\xi \cong \xi'$ to indicate that ξ and ξ' are equivalent. Equivalence of bundles is an equivalence relation; we shall not introduce a special notation for the equivalence class of a bundle. It is clear that two G-bundles are equivalent if, and only if, their associated principal G-bundles are equivalent.

Next we study the structure of principal G-bundles more thoroughly.

1.3.1 The category Bun_G

The objects of Bun_G are principal G-bundles; a morphism $f \in Bun_G(\xi,\xi')$ is a G-equivariant map $f : E \longrightarrow E'$ which commutes with the projections p and p'. Actually, any such morphism f turns out to be a G-homeomorphism from E to E'.

Proposition 1.3.2 *Let $\xi = (E,p,B,G)$ be a principal G-bundle. Then, the following hold true:*

1. *E is a principal G-space;*
2. *B is homeomorphic to the orbit space E/G;*
3. *the local homeomorphisms $\phi_i : U_i \times G \longrightarrow p^{-1}(U_i)$ over the open sets U_i are G-equivariant.*

Proof – The action of G on E is given as follows: if $x = \phi_i^{-1}(p(x), g_i)$ and $g \in G$, set $xg = \phi_i(p(x), g_i g)$. The translation function $\tau : E^* \longrightarrow G$ is then given locally as follows: if $x \in p^{-1}(U_i)$, define $\sigma_i(x) = pr_2 \phi_i^{-1}(x) \in G$; now, define $\tau_i(x, x') = (\sigma_i(x))^{-1}\sigma_i(x')$, for every $(x, x') \in p^{-1}(U_i)^*$. Clearly, if x also belongs to $p^{-1}(U_j)$, then $\tau_i(x, x') = \tau_j(x, x')$. Thus, the global function τ is continuous.

It remains to prove that τ satisfies the condition $x\tau(x, x') = x'$, for every $(x, x') \in E^*$. In fact,

$$x\tau_i(x, x') = \phi_i(p(x), \sigma_i(x)\tau_i(x, x')) = \phi_i(p(x), \sigma_i(x')) = x' \ .$$

□

The following proposition is a converse of Proposition 1.3.2

Proposition 1.3.3 *Let E be a locally trivial free G-space. Then,*

1. *E has a continuous translation function;*
2. *the 4-tuple $\xi = (E, p, E/G, G)$ is a principal G-bundle.*

Proof – It is enough to define the transition functions: take the functions $\sigma_i : p^{-1}(U_i) \longrightarrow G$ be defined in the previous proposition and set

$$g_{ij} : U_{ij} \longrightarrow G \ , \ g_{ij}(p(x)) = \sigma_i(x)(\sigma_j(x))^{-1} \ .$$

□

Propositions 1.3.2 and 1.3.3 together show the following:

Theorem 1.3.4 *A 4-tuple $\xi = (E, p, B, G)$ is a principal G-bundle $\iff$ E is a locally trivial free G-space.*

These facts simplify considerably life in the category of principal G-bundles: for example, one can easily change the fibre of a principal G-bundle simply making G act on the product $E \times F$ by $(x, y)g = (xg, g^{-1}y)$, defining $E_F = (E \times F)/G$ and

$$p_F : E_F \longrightarrow B = E/G \ , \ [(x, y)] \longmapsto p(x) \ .$$

Equivalence of principal G-bundles is also easily recognizable: $\xi \cong \xi' \iff$ there exists a G-equivariant map $f : E \longrightarrow E'$ over B.

1.3.2 Homotopy classification of principal G-bundles

At this point a new actor enters the stage: Homotopy Theory. The purpose of this move is to obtain some sort of classification theorem. We begin by saying that a principal G-bundle ξ is *numerable* provided it is locally trivial over an open covering with an open refinement given by a locally finite partition of unity; thus, any principal G-bundle over a paracompact space (e.g., a manifold, CW-complex, etc.) is numerable. We now define the contravariant functor

$$\mathfrak{E}_G : HTop \longrightarrow Set$$

which transforms a space B into the set[5] $\mathfrak{E}_G(B)$ of all equivalence classes of numerable, principal G-bundles over B and the morphism $[f] \in HTop(A, B)$ into the function

$$\mathfrak{E}_G([f]) : \mathfrak{E}_G(B) \longrightarrow \mathfrak{E}_G(B)\ ,\ \xi \longmapsto f^*(\xi)$$

where $f^*(\xi)$ is the numerable, principal G-bundle defined by pullback via any representative f of $[f]$ (note that if $f \cong g$ then $f^*(\xi) \cong g^*(\xi)$).

Next, let $\widetilde{\xi} = (\widetilde{E}, \widetilde{p}, \widetilde{B}, G)$ be a numerable, principal G-bundle and let $[-, \widetilde{B}]$ be the well-known contravariant functor from $HTop$ to Set. It is easy to see that there is a natural transformation

$$\mathcal{T} : [-, \widetilde{B}] \longrightarrow \mathfrak{E}_G$$

such that, for every $B \in HTop$,

$$\mathcal{T}(B) : [B, \widetilde{B}] \longrightarrow \mathfrak{E}_G(B)\ ,\ [f] \longmapsto f^*(\widetilde{\xi})\ .$$

We now can state the following *Classification Theorem*:

Theorem 1.3.5 *If $\widetilde{E}$ is contractible, the functors $\mathfrak{E}_G$ and $[-, \widetilde{B}]$ from $HTop$ to Set are naturally equivalent.*

A principal G-bundle with contractible total space is called *universal G-bundle*; the base space of a universal G-bundle is the *classifying space* of G. There are various ways to construct a classifying space; these constructions are all functorial. In Appendix A we shall describe the Milgram-Steenrod construction of a universal bundle $\xi_G = (E_G, p_G, B_G, G)$ for any topological group G.

[5] The fact that bundles are characterized by the transition functions guarantees that $\mathfrak{E}_G(B)$ is indeed a set.

Remark 1.3.6 Most of the bundles we encounter in this work have a paracompact base; thus, they are numerable and so, unless we have numerability for different reasons (see, for example, Theorem A.3.4), we do not refer explicitly to numerability and assume that **the principal bundles we deal with are numerable**.

1.3.3 Equivariant bundles

The work on spaces with the action of two groups described in Section 1.2.3 naturally leads us to introduce the notion of equivariant bundles. In doing so, we lose the "symmetry" of the actions in favour of a somehow more general situation, where one of the actions is definitely characterized as a principal bundle action, while the other is assumed to be consistent with the whole bundle structure. More precisely, let G be a topological group, let Γ be a compact topological group, and let $\alpha : \Gamma \longrightarrow \mathrm{Aut}(G)$ be a homomorphism into the automorphism group of G such that the left Γ-action

$$\Gamma \times G \longrightarrow G \ , \ (\gamma, g) \longmapsto \alpha_\gamma(g)$$

is continuous.

A (Γ, α, G)-*equivariant bundle* (or just (Γ, α, G)-*bundle* is a principal G-bundle (E, p, B, G) together with a left Γ-action on E and B such that:

1. p is Γ-equivariant;

2. $(\forall \gamma \in \Gamma, \ g \in G, \ x \in E) \ \gamma(xg) = (\gamma x)\alpha_\gamma(g)$.

Observe that the second property just shows that the actions of G and Γ on E are consistent, up to α (compare with Section 1.2.3). If α is trivial, (E, p, B, G) is a Γ-equivariant principal G-bundle.

A (Γ, α, G)-*bundle map* is a principal G-bundle map that is also Γ equivariant.

The semidirect product $\Gamma \times_\alpha G$, defined as the topological product with multiplication

$$(\gamma, g)(\gamma', g') = (\gamma\gamma', \alpha_\gamma(g')g) \ ,$$

acts naturally on the total space E of a (Γ, α, G)-bundle (E, p, B, G):

$$(\forall \gamma \in \Gamma, g \in G, x \in E) \ , \ ((\gamma, g), x) \longmapsto (\gamma x)g \ .$$

The main difficulty in dealing with (Γ, α, G)-bundles is the fact that, in general the action of Γ (and hence of $\Gamma \times_\alpha G$) in E is not free. To handle this problem, we introduce an appropriate notion of local triviality.

We begin with the following result (see [45, Chapter I, Lemma 8.9]).

Lemma 1.3.7 *Let Λ be a closed subgroup of Γ and suppose that the closed subgroup H of $\Gamma \times_\alpha G$ is the graph of a map $s : \Lambda \longrightarrow G$:*

$$H = \{(\lambda, s(\lambda)) \in \Gamma \times_\alpha G \mid \lambda \in \Lambda\}$$

Then $(\Gamma \times_\alpha G), q, \Gamma/\Lambda, G)$ is a principal G-bundle.

The quotient maps

$$q : (\Gamma \times_\alpha G)/H \longrightarrow \Gamma/\Lambda$$

of the type introduced in the previous lemma are called *local objects*. A (Γ, α, G)-bundle (E, p, B, G) is said to be *locally trivial* if B has an open covering by Γ-sets $\{U_i \mid i \in J\}$ such that the restriction $p^{-1}(U_i) \longrightarrow U_i$ admits a (Γ, α, G)-bundle map into a local object.

Unfortunately the complicated and abstract concept we just introduced is necessary to set up the appropriate framework in which one can deal with bundles endowed with the action of an extra group; however, this sort of abstraction has its rewards! Let us just mention the following.

Let (E, p, B, G) be a principal G-bundle and let H be a topological group acting on E consistently with G. We know that the quotient map

$$\widehat{p} : E/H \longrightarrow B/H$$

is well defined; however, we do not know if it is locally trivial. In order to find out a necessary condition for local triviality we study the problem from the equivariant point of view.

Let (E, p, B, G) be a locally trivial (Γ, α, G)-bundle, and let U be one of the open Γ-sets of B over which the bundle is trivial. Then we have an equivariant map $f : p^{-1}(U) \longrightarrow (\Gamma \times_\alpha G)/H$ which induces a quotient map $\widehat{f} : p^{-1}(U)/\Gamma \longrightarrow ((\Gamma \times_\alpha G)/H)/\Gamma$. Notice that the the left (resp. right) action of Γ (resp. G) on $\Gamma \times_\alpha G$ is defined by the restriction of the product to $\Gamma \times \{u_G\}$ (resp. $\{u_\Gamma\} \times G$); since these actions commute with the right action of H, we have corresponding actions on $(\Gamma \times_\alpha G)/H$ (see Lemma 1.2.10); furthermore, as U is Γ-invariant, so is $p^{-1}(U)$. Then, considering the right G-actions θ on $p^{-1}(U)$ and ϕ on $(\Gamma \times_\alpha G))/H$, we can construct the following commutative diagram

$$\begin{tikzcd}
p^{-1}(U) \times G \arrow[rrr, "f \times 1_G"] \arrow[dr] \arrow[ddd, "\theta"'] & & & (\Gamma \times_\alpha G)/H \times G \arrow[dl] \arrow[ddd, "\phi"] \\
& (p^{-1}(U) \times G)/\Gamma \arrow[d, "\widehat{\theta}"'] \arrow[dr, "\widehat{h}"] & ((\Gamma \times_\alpha G)/H)/\Gamma \times G \arrow[d, "\widehat{\phi}"] & \\
& p^{-1}(U)/\Gamma \arrow[r, "\widehat{f}"'] & ((\Gamma \times_\alpha G)/H)/\Gamma & \\
p^{-1}(U) \arrow[rrr, "f"'] \arrow[ur] & & & (\Gamma \times_\alpha G)/H \arrow[ul]
\end{tikzcd}$$

where the composite map $h = \phi(f \times 1_G)$ passes to the quotient, while the map $f \times 1_G$ itself does not. In general, this says nothing about local triviality of the quotient bundle, since we do not have an action of G on $p^{-1}(U)$; however, suppose that the following two conditions hold true:

1. $\alpha : \Gamma \longrightarrow \mathrm{Aut}(G)$ is trivial, and
2. $\Gamma \times_\alpha G = \Gamma \times G$ acts freely on E .

In that case the map $\widehat{h}$ of the diagram factors throught $G \times G$ as $\widehat{h} = \widehat{\phi}(\widehat{f} \times 1_G)$ and $\widehat{\phi}$ is exactly the multiplication in G, implying that

$$\widehat{f} : p^{-1}(U)/\Gamma \longrightarrow (\Gamma \times G)/\Gamma = G$$

is G-equivariant, and thus $(E/H, \widehat{p}, B/H, G)$ is a principal G-bundle (local triviality is established).

A similar situation arises even when the action of the two groups are not completely disjoint as in the previous Lemma 1.2.12; in this case we have the following result.

Proposition 1.3.8 *Let (E, p, B, G) be a principal (Γ, G)-bundle. Suppose that the actions of G and Γ on E coincide on a common subgroup Z of G and Γ which is a central subgroup of both groups; moreover, suppose that the natural action of $\Gamma/Z \times G/Z$ on E/Z is free. Then, the quotient map*

$$\widehat{p} : E/\Gamma \longrightarrow B/\Gamma$$

defines a principal G/Z-bundle $(E/\Gamma, \widehat{p}, B/\Gamma, G/Z)$.

Proof – Because Z is normal in G, we can use Lemmas 1.2.11 and 1.2.12 to reduce p to the $(\Gamma/Z, G/Z)$-bundle $(E/Z, \tilde{p}, B, G/Z)$, and proceed as before. $\square$

We are now going to focus our attention on (Γ, α, G)-bundles (E, p, B, G) for which the homomorphism $\alpha : \Gamma \longrightarrow \mathrm{Aut}(G)$ is trivial and the action of $\Gamma \times G$ on E is free. Such equivariant bundles will be called *principal (Γ, G)-bundles.*

We want to show that in this case local triviality can be defined in a more convenient way, closer to the way it is defined for ordinary bundles. In fact, we say that a principal (Γ, G)-bundle (E, p, B, G) is *locally trivial* if there exists an open covering of B by Γ-sets $\{U_i\}$, and a family of principal (Γ, G)-bundle maps[6] f_i such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "f_i"] \arrow[d, "p"'] & \Gamma \times G \arrow[d, "pr_1"] \\ U_i \arrow[r, "\widehat{f}_i"'] & \Gamma \end{tikzcd}$$

With the aid of the maps f_i we construct a set of homeomorphisms

$$(\phi_i)^{-1} : p^{-1}(U_i) \longrightarrow U_i \times G \ , \ (\phi_i)^{-1} : x \longmapsto (p(x), pr_2 f_i(x))$$

that are Γ-equivariant with respect to the action

$$\Gamma \times U_i \times G \longrightarrow U_i \times G \ , \ (\gamma, (b, g)) \longmapsto (\gamma b, g)$$

[6]Recall the actions of Γ and G on $\Gamma \times G$ are defined by restrictions of the multiplications.

Thus, the set $\{(U_i, \phi_i)\}$ gives a Γ-equivariant local trivialization for p, and hence, we obtain a local trivialization for $\widehat{p} : E/\Gamma \longrightarrow B/\Gamma$:

$$(\widehat{\phi_i})^{-1} : \widehat{p}^{-1}(U_i/\Gamma) = p^{-1}(U_i)/\Gamma \longrightarrow (U_i/\Gamma) \times G$$

$$(\widehat{\phi_i})^{-1} : [x] \longmapsto ([p(x)], pr_2 f_i(x)) \ .$$

Notice that the transition functions satisfy the condition

$$(\forall x \in p^{-1}(U_{ij}))\ \widehat{g_{ij}}\widehat{p}([x]) = g_{ij}p(x) \ .$$

Since the existence of a Γ-equivariant local trivialization is sufficient to guarantee the local triviality of the quotient map we shall be content to consider the previous definition of local triviality for a principal (Γ, G)-bundle as the correct one.

We now revert to fibre bundles. Let F be a space with a left G-action and let $\xi = (E, p, B, G)$ be a principal (Γ, G)-bundle; the fibre bundle with fibre F associated to ξ viewed just as a principal G-bundle is denoted by $\xi[F] = (E \times_G F, p^F, B, F, G)$; the bundles ξ and $\xi[F]$ have the same transition functions (see Theorem 1.3.1). Notice that the action

$$\Gamma \times (E \times_G F) \longrightarrow E \times_G F \ , \ (\gamma, [(x, y)]) \longmapsto [(\gamma x, y)]$$

defines a family of G-bundle maps and the map $p^F : E \times_G F \longrightarrow B$ turns out to be Γ-equivariant. Moreover, if $\{(U_i, \phi_i) | i \in J\}$ is a local trivialization for ξ, we define a local trivialization $\{(U_i, \phi_i^F) | i \in J\}$ for $\xi[F]$ by setting

$$(\phi_i^F)^{-1} : (p^F)^{-1}(U_i) \longrightarrow U_i \times F$$

$$(\phi_i^F)^{-1} : [(x, y)] \longmapsto (p(x), pr_2\phi_i(x)y),$$

It follows that if (E, p, B, G) is a locally trivial principal (Γ, G)-bundle, then the open sets U_i are Γ-sets and the maps ϕ_i^F are Γ-equivariant – with respect to the natural action

$$\Gamma \times (U_i \times F) \longrightarrow U_i \times F \ , \ (\gamma, (b, y)) \longmapsto (\gamma b, y) \ .$$

Thus, we give the following definition: a fibre bundle $\xi = (E, p, B, F, G)$ is said to be a (Γ, G)-*equivariant fibre bundle* (or simply a (Γ, G)-bundle) if the follwing conditions hold true: 1) the compact topological group Γ act freely on E and B (on the left), and the map $p : E \longrightarrow B$ is Γ-equivariant; 2) the opens sets U_i of the atlas $\{(U_i, \phi_i)\}$ of ξ are Γ-sets; 3) the fibre preserving homeomorphisms $\phi_i : U_i \times F \longrightarrow p^{-1}(U_i)$ are Γ-equivariant.

It is now straightforward to verify that the principal bundle associated to a (Γ, G)-fibre bundle is a principal (Γ, G)-bundle.

As we did for principal bundles we can associate a fibre bundle

$$\xi/\Gamma = (E/\Gamma, \widehat{p}, B/\Gamma, F, G)$$

to a (Γ, G)-fibre bundle $\xi = (E, p, B, F, G)$; its atlas is given by $\{(U_i/\Gamma, \widehat{\phi_i^F})\}$, and its transition functions are $\widehat{g_{ij}^F}([p(x)]) = g_{ij}^F p(x)$. The corresponding result to Proposition 1.3.8 is the following:

Proposition 1.3.9 *Let $\xi = (E, q, B, F, G)$ be a (Γ, G)-fibre bundle. Then $\xi/\Gamma = (E/\Gamma, \widehat{p}, B/\Gamma, F, G)$ is a fibre bundle with fibre F and structural group G.*

We now analyse what happens when a principal G-bundle is endowed with the actions of two groups (notice that we could contemplate the case of just one group but with two different actions).

Let (E, p, B, G) be a principal G-bundle which is also a principal (Γ, G)-bundle and a principal (Γ', G)-bundle for two topological groups Γ and Γ' (we are, of course, assuming that $B/\Gamma = B/\Gamma'$). Furthermore, suppose that there exists a group homomorphism $\theta : \Gamma \longrightarrow \Gamma'$. Then we obtain two quotient bundles

$$(E/\Gamma, \widehat{p}, B/\Gamma, G) \text{ and } (E/\Gamma', \widehat{p'}, B/\Gamma', G)$$

over the same base space. It is easy to see that these two bundles are equivalent $\iff$ p has a Γ, Γ'-*equivariant bundle automap* that is to say, a map $f : E \longrightarrow E$ over B such that

$$(\forall \gamma \in \Gamma, x \in E, g \in G) \ f((\gamma x)g) = \theta(\gamma) f(x) g \ .$$

For a (Γ, G) fibre bundle (E, p, B, F, G) all this is translated into the existence of a Γ, Γ'-equivariant G-bundle autoequivalence, namely an automap f of E over B, whose restriction to the fibre takes values in G continuously as a function of the point in the base space; more precisely, if $f_b := f|_{p^{-1}(b)} : p^{-1}(b) \longrightarrow p^{-1}(b)$ is the restriction of f to the fibre over b, then we require that f_b belongs to $G \subseteq \mathrm{Homeo}(F, F)$ and that the map

$$B \longrightarrow G \ , \ b \longmapsto f_b$$

is continuous.

We end this section with an example. Let

$$\tau(S^n) = (TS^n, \mathrm{pr}_1, S^n, \mathbb{R}^n, GL(n, \mathbb{R}))$$

be the tangent bundle to the sphere S^n, $n \geq 1$; its total space TS^n is the space of all pairs $(b, \vec{v}) \in S^n \times \mathbb{R}^{n+1}$ with $< b, \vec{v} >= 0$; the map $\mathrm{pr}_1 : TS^n \longrightarrow S^n$ is just the projection on the first factor; finally, we trivialize it over the open sets

$$U_i := U_{i+} \cup U_{i-} = \{b \in S^n \mid b^i \neq 0\} \ , \ i = 0, \ldots, n \ .$$

The topological group $\mathbb{Z}_2$ acts freely on the base space S^n

$$\phi : \mathbb{Z}_2 \times S^n \longrightarrow S^n \ , \ (\pm 1, b) \longmapsto \pm b \ ;$$

however, there are two free actions of $\mathbb{Z}_2$ on TS^n:

$$\phi_1 : \mathbb{Z}_2 \times TS^n \longrightarrow TS^n \ , \ (\pm 1, (b, \vec{v}) \longmapsto (\pm b, \vec{v})$$

and

$$\phi_2 : \mathbb{Z}_2 \times TS^n \longrightarrow TS^n \ , \ (\pm 1, (b, \vec{v}) \longmapsto (\pm b, \pm \vec{v}) \ ;$$

in either case, the fibre bundle $\tau(S^n)$ can be viewed as a $(\mathbb{Z}_2, GL(n, \mathbb{R}))$-fibre bundle. The pairs of actions (ϕ_1, ϕ) and (ϕ_2, ϕ) and Proposition 1.3.9 now give rise to two quotient bundles

$$\tau(S^n)/\mathbb{Z}_2(\phi_1, \phi) \text{ and } \tau(S^n)/\mathbb{Z}_2(\phi_2, \phi)$$

which, in general, are not equivalent: in fact,

$$\tau(S^n)/\mathbb{Z}_2(\phi_1, \phi) \cong \tau(\mathbb{R}P^n)$$

– the tangent bundle to the real projective space $\mathbb{R}P^n$ – while

$$\tau(S^n)/\mathbb{Z}_2(\phi_2, \phi) \cong \tau(\mathbb{R}P^n) \otimes \gamma_1^n$$

where γ_1^n is the *Hopf bundle* over $\mathbb{R}P^n$, that is to say, γ_1^n is the line bundle associated to the principal $\mathbb{Z}_2$-bundle

$$(S^n, p, \mathbb{R}P^n, \mathbb{Z}_2 = S^0, \mathbb{Z}_2)$$

where p is the map which identifies antipodal points of S^n. The total space E of γ_1^n is given by

$$E = (S^n \times \mathbb{R})/(x, r) \sim (xt, rt) \quad , \quad t \in \mathbb{Z}_2 \ ;$$

moreover, according to [19, Chapter 5, Theorem 7.8 and Remark 7.9], $\gamma_1^n \otimes \gamma_1^n$ is the trivial line bundle over $\mathbb{R}P^n$. The line bundle γ_1^n is also called *canonical line bundle* over $\mathbb{R}P^n$.

Chapter 2

Gauge Groups

2.1 Gauge Groups

Let $\xi = (E, p, B, G)$ be a principal G-bundle and let $\mathcal{G}(\xi)$ be the set of all G-equivariant homeomorphisms over B of E. The set $\mathcal{G}(\xi)$ is endowed with a natural group operation given by composition; moreover, we give to $\mathcal{G}(\xi)$ the subspace topology from the compact-open topology of the space $Map(E, E)$ of all maps from E to itself. At this point we do not know if the algebraic and topological structures of $\mathcal{G}(\xi)$ are or not compatible[1]; we shall see that $\mathcal{G}(\xi)$ is indeed a topological group, but this fact will come out of a different formulation of $\mathcal{G}(\xi)$. The argument is as follows.

Define the G-space $Ad(G)$ to be the topological space G with the left action

$$G \times Ad(G) \longrightarrow Ad(G) \ , \ (g, \bar{g}) \longmapsto g^{-1}\bar{g}g \ .$$

Now take the set $Map_G(E, Ad(G))$ of all G-equivariant maps of E into $Ad(G)$; this set, together with the topology inherited from $Map(E, Ad(G))$ and the group structure given by

$$(f, f') \longmapsto ff' \ , \ (\forall x \in E) \ ff'(x) = f(x)f'(x)$$

(with the inverse map of an arbitrary $f \in Map_G(E, Ad(G))$ defined by $f^{-1}(x) = (f(x))^{-1}$ far all $x \in E$) is a topological group. Finally, we show that the homomorphism of topological groups

$$\mathcal{G}(\xi) \longrightarrow Map_G(E, Ad(G))$$

[1] We call the attention of the reader to [3] which deals with the problem of when the space of self-homeomorphisms of a space is a topological group.

defined by
$$f \longmapsto h_f,\ (\forall x \in E)\ h_f(x) = t(x, f(x)),$$
where t is the associated translation function as defined in section 1.3, is a bicontinuous bijection (its inverse is given by
$$Map_G(E, Ad(G)) \longrightarrow \mathcal{G}(\xi)\ ,\ h \longmapsto f_h\ , \tag{2.1}$$
with f_h defined by: $(\forall x \in E)\ f_h(x) = xh(x)$).

The topological group $\mathcal{G}(\xi)$ is called *gauge group* or *group of gauge transformations* of ξ.

We write explicitly this formulation of $\mathcal{G}(\xi)$:
$$\mathcal{G}(\xi) \cong Map_G(E, Ad(G))\ . \tag{2.2}$$

The next result is an easy consequence of this formulation of $\mathcal{G}(\xi)$.

Theorem 2.1.1 *If ξ is trivial or G is abelian, then*
$$\mathcal{G}(\xi) \cong Map(B, G)\ .$$

We also give a description of the underlying set of a gauge group in terms of the so-called *Hu's criterion* which we describe anon (see [18]). Let ξ and ξ' be principal G-bundles over B and suppose that ξ (resp. ξ') has transition functions g_{ij} (resp. g'_{ij}). Let $Aut(G)$ be the group of all automorphisms of G; define the group homomorphism
$$\rho : G \times G \longrightarrow Aut(G)\ ,\ \rho(g_1, g_2) : h \longmapsto g_1 h {g_2}^{-1}\ .$$
Define the subgroup $G^* \subset Aut(G)$ by means of the exact sequence
$$0 \longrightarrow ZG \xrightarrow{\ \Delta\ } G \times G \xrightarrow{\ \rho\ } G^* \longrightarrow 0$$
where Δ is the diagonal map. Let $\xi^*(\xi, \xi')$ be the bundle with base B, fibre G and structural group G^* determined by the transition functions ${g^*}_{ij}(b) = \rho(g_{ij}(b), g'_{ij}(b))$. This is the *Ehresmann bundle* associated to ξ and ξ'. Let $\Gamma(B, \xi^*(\xi, \xi'))$ be the set of all cross-sections of $\xi^*(\xi, \xi')$. Now, Hu's criterion says that there exists a bijective correspondence between the equivalences of ξ and ξ' and the elements of $\Gamma(B, \xi^*(\xi, \xi'))$. Hence, according to Hu's criterion, there is a bijection
$$\mathcal{G}(\xi) \cong \Gamma(B, \xi^*(\xi, \xi))\ .$$

The bundle $\xi^*(\xi,\xi)$ can be constructed in a more direct way: firstly, take the set of transition functions $\{g_{ij}|i,j\in J\}$ of ξ; next, take the centre $ZG\subset G$ and the quotient map $\pi: G \longrightarrow G/ZG \cong I(G)$, the group of inner automorphisms of G; finally, notice that $I(G)$ acts on the left of G and use the transition functions $\{\pi g_{ij}: U_{ij} \longrightarrow I(G)|i,j\in J\}$ to construct the fibre bundle $F(\xi)=\{\widetilde{E},\widetilde{p},B,G,I(G)\}$. The bundle $F(\xi)$ is the so-called *fundamental bundle* associated to ξ. It is now easy to verify that $\xi^*(\xi,\xi)$ is equivalent to $F(\xi)$; hence, Hu's criterion now tells us that the sets $\mathcal{G}(\xi)$ and $\Gamma(B,F(\xi))$ are isomorphic:

$$\mathcal{G}(\xi)\cong\Gamma(B,F(\xi)) \ . \tag{2.3}$$

There is a third interesting interpretation of the underlying set of a gauge group. Let $\xi[Ad(G)]$ be the fibre bundle with fibre $Ad(G)$ and structural group G obtained from ξ in the usual manner; we write it as

$$\xi[Ad(G)] = (E\times_G Ad(G), p[Ad(G)], B, Ad(G), G) \ .$$

Let $\Gamma(B,\xi[Ad(G)])$ be the set of all cross-sections of $\xi[Ad(G)]$. According to [19, Chapter 4, Theorem 8.1] there is a bijection

$$Map_G(E,Ad(G)) \longrightarrow \Gamma(B,\xi[Ad(G)])$$

which associates to each G-equivariant map $h: B\longrightarrow Ad(G)$ the cross-section

$$s_h : B \longrightarrow E\times_G Ad(G)$$

such that, for every $b\in B$ and every $x\in p^{-1}(b)$, $s_h(b)$ is equal to the equivalence class mod. G of the pair $(x,h(x))$. Thus

$$\mathcal{G}(\xi)\cong\Gamma(B,\xi[Ad(G)]) \ . \tag{2.4}$$

The local triviality of the principal G-bundle ξ gives rise to a very useful formulation of its gauge transformations in terms of the transition functions. As before, we assume ξ to be locally trivial over the open covering $\mathfrak{U} = \{U_i|i\in J\}$ of B and has transition functions g_{ij}.

We begin by defining the *local gauge group* of ξ associated to the open covering $\mathfrak{U}$ as the topological group

$$\mathcal{L}=\prod_{i\in J} Map(U_i,G)$$

endowed with the product topology. Now, for every $i \in J$ define ξ_i to be the restriction of ξ to U_i; since ξ_i is trivial, the gauge group $\mathcal{G}(\xi_i)$ is homeomorphic to the topological group $Map(U_i, G)$, via the function

$$\vartheta_i : \mathcal{G}(\xi_i) \longrightarrow Map(U_i, G) \ , \ \vartheta_i(f_i)(b) = (\phi_{i,b}^{-1} f_{i,b} \phi_{i,b})(u_G)$$

(see Theorem 2.1.1) for every $f_i \in \mathcal{G}(\xi_i)$ and every $b \in U_i$, and where $f_{i,b}$ is the restriction of f_i to the fibre $p^{-1}(b)$.

Next, define the map

$$r_i : \mathcal{G}(\xi) \longrightarrow \mathcal{G}(\xi_i) \ , \ f \longmapsto f|(p^{-1}(U_i)) \ .$$

Lemma 2.1.2 *The function*

$$\vartheta : \mathcal{G}(\xi) \longrightarrow \mathcal{L} \ , \ f \longmapsto \{\vartheta_i r_i(f) | i \in J\} \ .$$

is an embedding of topologial groups.

Having identified $\mathcal{G}(\xi)$ with $\vartheta(\mathcal{G}(\xi)) \subset \mathcal{L}$ we can characterize $\mathcal{G}(\xi)$ using the transition functions of ξ:

Theorem 2.1.3 *The group $\mathcal{G}(\xi)$ coincides with the subgroup*

$$\{\{f_i | i \in J\} \in \mathcal{L} | f_j = g_{ij}^{-1} f_i g_{ij} \text{ on } U_{ij}\} \ .$$

As consequence of the previous theorem, we can see once more that if G is abelian, the gauge group $\mathcal{G}(\xi)$ coincides with $M(B, G)$ (cfr. Theorem 2.1.1).

It is easy to see that if $\xi \cong \xi'$ then $\mathcal{G}(\xi) \cong \mathcal{G}(\xi')$: in fact, let $\ell : E \longrightarrow E'$ be a G-equivariant homeomorphism over B; then

$$Ad_\ell : \mathcal{G}(\xi) \longrightarrow \mathcal{G}(\xi') \ , \ f \longmapsto \ell^{-1} f \ell$$

is an isomorphism. However, note that the converse of this statement is not true: the trivial $\mathbb{Z}$-bundle $S^1 \times \mathbb{Z} \longrightarrow S^1$ and the exponential map bundle $e^{2\pi i} : \mathbb{R} \longrightarrow S^1$ are not equivalent but have homeomorphic gauge groups.

Observe that for any finite set of principal G-bundles over a given space B, one can select a single common local gauge group in such a way that the embedding theorems above hold true for the gauge groups of all the bundles concerned.

Now we give a result on the gauge group of a sum of principal bundles. More precisely, let $\{\xi_k | k = 1, \ldots, n\}$ be a set of principal G_k-bundles over B; suppose that $\mathfrak{U} = \{U_i | i \in J\}$ is an open covering of B over which the ξ_ks are all locally trivial; finally, let $\{g^k_{ij} | i, j \in J\}$ be the set of transition functions of ξ_k. The transition functions

$$\bigoplus_{k=1}^{n} g^k_{ij} : U_{ij} \longrightarrow \bigoplus_{k=1}^{n} G_k$$

define a principal $\bigoplus_{k=1}^{n} G_k$-bundle over B which we denote by $\bigoplus_{k=1}^{n} \xi_k$ and call *sum* of $\xi_1, \ldots, \xi_n$ (if we are dealing with vector bundles, this is the *Whitney sum*).

Theorem 2.1.4 *For any set of principal G_k-bundles $\xi_1, \ldots, \xi_n$ over B,*

$$\mathcal{G}(\bigoplus_{k=1}^{n} \xi_k) \cong \bigoplus_{k=1}^{n} \mathcal{G}(\xi_k) \ .$$

It is interesting to observe that an automorphism φ of the structural group G of a principal G-bundle ξ gives rise to a new principal G-bundle ξ^φ which, in general, is not equivalent to ξ; however, the gauge groups of these bundles are isomorphic, as one can see in the next lemma.

Lemma 2.1.5 *Let $\xi = (E, p, B)$ be a principal G-bundle. An automorphism $\varphi : G \longrightarrow G$ determines a bundle ξ^φ whose gauge group $\mathcal{G}(\xi^\varphi)$ is isomorphic to $\mathcal{G}(\xi)$.*

Proof – Let $\{g_{ij} | i, j \in J\}$ be the set of all tanstion functions of ξ. Then, because the maps φg_{ij} satisfy properties TF1, TF2 and TF3, $\{\varphi g_{ij} | i, j \in J\}$ can be taken as the set of transition functions for the principal G-bundle ξ^φ.

We now prove that ϑ induces an isomorphism

$$\widetilde{\varphi} : \mathcal{G}(\xi) \longrightarrow \mathcal{G}(\xi^\varphi) \ , \ \{f_i\} \longmapsto \{\varphi f_i\}$$

In fact, for every $i, j \in J$ such that $U_{ij} \neq \emptyset$,

$$(\varphi g_{ij})^{-1} (\varphi f_i)(\varphi g_{ij}) = \varphi(g_{ij}^{-1} f_i g_{ij}) = \varphi f_j$$

and so, by Theorem 3.1.1, $\{\varphi f_i\} \in \mathcal{G}(\xi^\varphi)$. To show that $\widetilde{\varphi}$ is surjective, take arbitrarily $\{\widetilde{f}_i\} \in \mathcal{G}(\xi^\varphi)$; from the equality

$$(\varphi g_{ij})^{-1} (\widetilde{f}_i)(\varphi g_{ij}) = \widetilde{f}_j$$

we conclude that

$$\varphi^{-1}\widetilde{f_j} = g_{ij}^{-1}(\varphi^{-1}\widetilde{f}_i)g_ij$$

and so, $\{\varphi^{-1}\widetilde{f}_i\} \in \mathcal{G}(\xi)$. The proof of the injectivity of $\widetilde{\varphi}$ is also easy. □

We conclude this section with a description of the centre of the gauge group of a principal G-bundle $\xi = (E, p, B, G)$. In order to conduct our analysis, we must introduce the following condition:

[C1] $(\forall b_o \in B)\ \eta : \mathcal{G}(\xi) \longrightarrow G\ ,\ f \longmapsto f|p^{-1}(b_o)(u_G)$ is a surjection.

Now take the homeomorphism

$$Map_G(E, Ad(G)) \longrightarrow \mathcal{G}(\xi)$$

introduced in 2.1. On the one hand, we note that this map is centre-preserving and thus,

$$ZMap_G(E, Ad(G)) \cong Z\mathcal{G}(\xi)\ .$$

On the other hand,

$$u \in ZMap_G(E, Ad(G)) \iff$$

$$(\forall f \in Map_G(E, Ad(G)))\ (\forall x \in E)\ f(x)u(x) = u(x)f(x)\ .$$

Thus, if [C1] is satisfied, $u \in ZMap_G(E, ZAd(G)) \cong Map(B, ZG)$ because ZG is abelian. Therefore, we have the following homeomorphism:

$$Z\mathcal{G}(\xi) \cong Map(B, ZG)\ . \tag{2.5}$$

2.2 The topology of gauge groups

In this section we give some results about the homotopy type of $\mathcal{G}(\xi)$. Our first result is a consequence of the interpretation of $\mathcal{G}(\xi)$ given in 2.4 for a particular ξ.

Theorem 2.2.1 *Let ξ to be a principal G-bundle over a sphere S^n with G compact. Let $f : S^n \longrightarrow B_G$ be a classifying map for ξ. Let $Map(S^n, B_G; f)$ be the path-component of $Map(S^n, B_G)$ which contains f. Then $\mathcal{G}(\xi)$ has the weak homotopy type of the loop space $\Omega(Map(S^n, B_G; f))$.*

Proof – Take the fibre bundles

$$\xi[Ad(G)] = (E \times_G Ad(G), p[Ad(G)], S^n, Ad(G), G)$$

and

$$\xi_G[Ad(G)] = (E_G \times_G Ad(G), p_G[Ad(G)], B_G, Ad(G), G)$$

of which the former is equivalent to the fibre bundle induced from the latter by f. Because S^n is paracompact, the map $p[Ad(G)] : E \times_G Ad(G) \longrightarrow S^n$ is a (Hurewicz) fibration with fibre $Ad(G)$ (see [35, Exercise 4.4.5] and [14]); moreover, because G is compact, B_G is paracompact (see the observation at the end of Section A.3) and thus, the map $p_G[Ad(G)] : E_G \times_G Ad(G) \longrightarrow E_G$ is also a fibration.

The preceeding results and the exponential law (see [35, Theorem 1.1.2]; also, cfr. [35, Exercise 2.2.4]) show that the following diagram is a pullback whose vertical arrows are fibrations.

$$\begin{tikzcd}
Map(S^n, E \times_G Ad(G)) \arrow[r] \arrow[d, "q"'] & Map(S^n, E_G \times_G Ad(G)) \arrow[d, "q_G"] \\
Map(S^n, S^n) \arrow[r, "f^*"'] & Map(S^n, B_G)
\end{tikzcd}$$

Note that $f^*(1_{S^n}) = f$; furthermore,

$$q_G{}^{-1}(f) = q^{-1}(1_{S^n}) = \Gamma(B, \xi[Ad(G)]) \cong \mathcal{G}(\xi)$$

(the last bijection by 2.4).

A comparison of the exact sequences of the fibrations p_G and $p_G[Ad(G)]$ with the aid of the map over B_G

$$h : E_G \longrightarrow E_G \times_G Ad(G) \ , \ h(x) = [x, e_G]$$

shows that $E_G \times_G Ad(G)$ is weakly contractible; using [21, Corollary 2.5] we conclude that $Map(S^n, E_G \times_G Ad(G))$ is also weakly contractible and therefore, the exact homotopy sequence of q_G shows that $\mathcal{G}(\xi)$ has the same weak homotopy type as $\Omega(Map(S^n, B_G; f))$. □

The arguments used in the previous result show that if G is compact and B is locally compact Hausdorff, then $\mathcal{G}(\xi)$ is the fibre over $f \in Map(B, B_G)$ of the Hurewicz fibration

$$q_G : Map(B, E_G \times_G Ad(G)) \longrightarrow Map(B, B_G) \ .$$

Actually, without the assumptions on G and B required before, but using different techniques and working within the category of weak Hausdorff k-spaces we obtain a more general result; indeed, it is possible to construct a fibration over $Map(B, B_G; f)$ with fibre $\mathcal{G}(\xi)$ over $\{f\}$ and with contractible total space (see [6, Proposition 3.1]); this leads towards the following result (see [6, Theorem 3.3]):

Theorem 2.2.2 *Let ξ be a principal G-bundle classified by $f : B \longrightarrow B_G$ and let $Map(B, B_G; f)$ be the path-component of $Map(B, B_G)$ containing f. Then the gauge group $\mathcal{G}(\xi)$ has the same homotopy type as the loop space $\Omega(Map(B, B_G; f))$; furthermore, the homotopy equivalence in question preserves the H-space structures of both spaces.*

We use the previous theorem to produce a class of principal G-bundles over a fixed space B with infinitely many non-isomorphic gauge groups. Take $B = S^4$ and $G = SU(2) \cong S^3$; the principal $SU(2)$-bundles over S^4 are classified by self-maps of S^4 because the Hopf bundle $\gamma = (S^7, p, S^4, S^3)$ is 7-universal (see [40, Sections 19.3 and 19.4]). Now take a map $k : S^4 \longrightarrow S^4$ of positive degree k and let ξ_k be the principal $SU(2)$-bundle over S^4 induced from γ via k. According to Theorem 2.2.2 $\mathcal{G}(\xi_k)$ and $\Omega(Map(S^4, B_{SU(2)}; k)$ have the same homotopy type and hence

$$\pi_2(\mathcal{G}(\xi_k)) \cong \pi_2(\Omega(Map(S^4, B_{SU(2)}; k) \ ;$$

on the other hand, $B_{SU(2)} \cong S^4$ and thus, from [43, Lemma 3.10], we conclude that

$$\pi_2(\mathcal{G}(\xi_k) \cong \mathbb{Z}_{24k} \oplus \mathbb{Z}_{12}$$

showing that the second homotopy group of $\mathcal{G}(\xi_k)$ depends on k.

Let b_o be a fixed point of the base space B of ξ and let $\mathcal{G}^1(\xi)$ be the subgroup of $\mathcal{G}(\xi)$ determined by all gauge transformations of ξ whose restriction to $p^{-1}(b_o)$ is the identity map; also, let $Map_*(B, B_G; f)$ be the path component of f of the space $Map_*(B, B_G)$ of base-point preserving maps from B to B_G. The next result is a based version of Theorem 2.2.2 (see [6, Corollary 5.7]).

Theorem 2.2.3 *Let ξ be a principal G-bundle classified by $f : B \longrightarrow B_G$ and let $Map_*(B, B_G; f)$ be the path-component of $Map_*(B, B_G)$ containing f. Then $\mathcal{G}^1(\xi)$ has the homotopy type of the loop space $\Omega(Map_*(B, B_G; f))$; furthermore, the homotopy equivalence in question preserves the H-space structures of both spaces.*

The next result also comes from [6] (see [6, Theorem 6.1]).

Theorem 2.2.4 *Suppose that the path-components of $Map(B, B_G)$ (resp. $Map_*(B, B_G)$) have the same homotopy type; then $\mathcal{G}(\xi)$ (resp. $\mathcal{G}^1(\xi)$) has the homotopy type of $Map(B, G)$ (resp. $Map_*(B, G)$). Furthermore, the homotopy equivalences connecting the respective spaces preserve the multiplicative structures.*

The interest of the theorem rests mostly on the based case: indeed, a sufficient condition for the path components of $Map_*(B, B_G)$ to be of the same homotopy type is that B is an associative COH-space (for example, B is a suspension). A sufficient condition for the path components of $Map(B, B_G)$ to have the same type is to require that B_G is an associative H-space (see [44, page 31]). For example, B_G could be an Eilenberg-MacLane space or a group; however, the latter hypothesis is uninteresting because B_G has a continuous multiplication induced from that of $E_G \iff G$ is abelian (see Proposition A.5.2) and then, by Theorem 2.1.1, $\mathcal{G}(\xi) \cong Map(B, G)$, thus bypassing 2.2.4.

The spaces $\mathcal{G}^1(\xi)$ and $\mathcal{G}(\xi)$ are more intimately related than just by the relation subspace/space; in fact, let

$$\eta : \mathcal{G}(\xi) \longrightarrow G \ , \ f \longmapsto f|p^{-1}(b_o)(u_G) \ ;$$

the following theorem holds true (see [6, Proposition 5.8]).

Theorem 2.2.5 *The map $\eta : \mathcal{G}(\xi) \longrightarrow G$ is a Hurewicz fibration with fibre $\mathcal{G}^1(\xi)$ over u_G.*

This result will play an interesting role in the development of Chapter 3 (see Condition [C1] there); it is also useful in computing certain gauge groups.

Theorem 2.2.4 compares the homotopy type of $\mathcal{G}(\xi)$ (resp. $\mathcal{G}^1(\xi)$) to that of the mapping space $Map(B, G)$ (resp. $Map_*(B, G)$), provided we are prepared to assume that the path-components of the space $Map(B, B_G)$

(resp. $Map_*(B, B_G)$) have the same homotopy type; if this is not the case, we still can compare the homotopy groups of the gauge groups and the mapping spaces (within a certain range). What we have in mind is the following theorem.[2]

Theorem 2.2.6 *Suppose that G is (n-1)-connected (with $n > 1$) and that B is a CW-complex of dimension $m < 2n$. Then, for every $0 \leq j \leq 2n - m - 1$,*

$$\pi_j(\mathcal{G}(\xi)) \cong \pi_j(Map(B, G)) \ ,$$
$$\pi_j(\mathcal{G}^1(\xi)) \cong \pi_j(Map_*(B, G)) \ .$$

Proof – Let $\rho : B_G \longrightarrow \Omega\Sigma B_G$ be the adjoint to the identity map of the (reduced) suspension ΣB_G into itself. The hypothesis on the connectivity of G implies that B_G is n-connected and hence, ρ is a $(2n+1)$-equivalence (for the definition of n-equivalence see [35, Exercise 6.2.2] or [39, page 404]; for the proof of the result quoted above see [39, Corollary 10, Ch.8,Sec.5]). The map ρ induces a map

$$\rho' : Map(S^1, B_G) \longrightarrow Map(S^1, \Omega\Sigma B_G) \ , \ g \longmapsto \rho g$$

such that the next diagram commutes:

$$\begin{tikzcd} Map(S^1, B_G) \arrow[r, "\rho'"] \arrow[d, "ev"'] & Map(S^1, \Omega\Sigma B_G) \arrow[d, "ev"] \\ B_G \arrow[r, "\rho"'] & \Omega\Sigma B_G \end{tikzcd}$$

(Here ev is the evaluation map at the base point of S^1.) But the columns of the diagram are fibrations with fibres $Map_*(S^1, B_G)$ and $Map_*(S^1, \Omega\Sigma B_G)$ (over the canonical base points of B_G and $\Omega\Sigma B_G$, respectively); then, the long exact sequences of homotopy groups corresponding to these fibrations and the five lemma imply that ρ' is a $2n$-equivalence.

Let $\mathcal{L}(B, Map(S^1, B_G); f)$ be the space of all lifts of f to $Map(S^1, B_G)$; it is clear that ρ' induces a map

$$\mathcal{L}(\rho') : \mathcal{L}(B, Map(S^1, B_G); f) \longrightarrow \mathcal{L}(S^1, Map(S^1, \Omega\Sigma B_G); \rho f) \ .$$

[2]For a more general result see [6, Theorem 6.3].

We are going to show that $\mathcal{L}(\rho')$ is a $(2n-m)$-equivalence. In fact, take S^j with $0 \leq j \leq 2n-m$, and form the maps

$$\widetilde{f} := f\mathrm{pr}_2 : S^j \times B \longrightarrow B_G$$

and

$$\widetilde{f}' : S^j \vee B \longrightarrow Map(S^1, B_G) \ , \ \widetilde{f}' := (f\mathrm{pr}_2)|(S^j \times \{b_o\}) \cup \theta$$

where θ is a selected base-point of $\mathcal{L}(B, Map(S^1, B_G); f)$; next, apply [39, Theorem 12, Ch. 7, Sec. 8] to the commutative diagram

$$\begin{tikzcd}
S^j \vee B \arrow[r, "\widetilde{f}'"] \arrow[d] & Map(S^1, B_G) \arrow[r, "\rho'"] \arrow[d] & Map(S^1, \Omega\Sigma B_G) \arrow[d] \\
S^j \times B \arrow[r, "\widetilde{f}"] & B_G \arrow[r, "\rho"'] & \Omega\Sigma B_G
\end{tikzcd}$$

to obtain the stated result. This means that, for every $j \leq 2n-m-1$,

$$\pi_j(\mathcal{L}(B, Map(S^1, B_G; f)) \cong \pi_j(\mathcal{L}(B, Map(S^1, \Omega\Sigma B_G; \rho f)) \ .$$

A double application of the exponential law proves that

$$\Omega Map(B, B_G; f) \cong \mathcal{L}(B, Map(S^1, B_G); f)$$

and thus, for every $j \leq 2n-m-1$,

$$\pi_j(\Omega Map(B, B_G; f)) \cong \pi_j(\Omega Map(B, \Omega\Sigma B_G; \rho f)) \ .$$

On the other hand, because $\Omega\Sigma B_G$ is an associative H-space with inverse, the path components of $Map(B, \Omega\Sigma B_G)$ have the same homotopy type; hence, if c denotes the constant map of B to the base point of $\Omega\Sigma B_G$,

$$(\forall j \leq 2n-m-1) \ \ \pi_j(\Omega Map(B, B_G; f)) \cong \pi_j(\Omega Map(B, \Omega\Sigma B_G; c)) \ .$$

Now we observe that

$$\pi_j(\Omega Map(B, \Omega\Sigma B_G; c)) \cong \pi_j(Map(B, \Omega B_G), c) \cong \pi_j(Map(B, G), c) \ .$$

Theorem 2.2.2 completes the proof. The proof of the based case follows the same lines. $\square$

We give an application of Theorem 2.2.6. Let $\xi = (E, p, S^m, G)$ be a principal G-bundle with G $(n-1)$-connected and $m < 2n$ (we assume that n is strictly larger than 1). Then,

$$(\forall j \leq 2n - m - 1)\ \pi_j(\mathcal{G}(\xi)) \cong \pi_{j+m}(G) \oplus \pi_j(G)\ .$$

To see this, take the fibration

$$ev : Map(S^m, G) \longrightarrow G\ ,\ \ h \longmapsto h(\mathbf{e}_o)$$

(where $\mathbf{e}_o$ is the base-point of S^m); its fibre (over the identity $u_G \in G$) is $Map_*(S^m, G)$ and moreover, ev has a section. This implies that

$$(\forall j > 0)\ \pi_j(Map(S^m, G), c) \cong \pi_{j+m}(G) \oplus \pi_j(G)$$

and thus, we obtain the statement using 2.2.6.

We now make two remarks about the gauge group of a principal G-bundle ξ over a sphere S^n, with $n \geq 1$. The first of these is that as a consequence of Theorem 2.2.4 we conclude that $\mathcal{G}^1(\xi)$ and $\Omega^n G$ have the same type. Next, we notice that the long exact sequence of homotopy groups of the fibration $\eta : \mathcal{G}(\xi) \longrightarrow G$ and the previous observation show that the homotopy groups of $\mathcal{G}(\xi)$ are related to those of G by an exact sequence of the type

$$\begin{array}{c} \ldots \longrightarrow \pi_k(G) \longrightarrow \pi_{k+n-1}(G) \longrightarrow \\ \longrightarrow \pi_{k-1}(\mathcal{G}(\xi)) \longrightarrow \pi_{k-1}(G) \longrightarrow \ldots \end{array} \tag{2.6}$$

We conclude this section with a discussion about the gauge groups of Lorentz bundles. The topological group

$$O(n-1, 1) = \{L \in GL(n, \mathbb{R}) | L^t \mu_n L = \mu_n\}$$

where μ_n is the *Minkowski matrix*

$$\mu_n = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$$

is a *Lorentz group*. The group $O(n-1,1)$ has an important subgroup, namely the subgroup $O^{\uparrow}(n-1,1)$ of all *orthochronous transformations* that is to say, of the matrices $L \in O(n-1,1)$ with $\det L = 1$ and $L_{n,n} > 0$; geometrically the *special* Lorentz group $O^{\uparrow}(n,1)$ is the connected component of $O(n-1,1)$ containing the identity element.

We define a *Lorentz bundle* to be a principal bundle over an n-dimensional manifold B with structure group $O^{\uparrow}(n-1,1)$. We wish to investigate the homotopy groups of the gauge group of a Lorentz bundle; the key step in pursuing this investigation is the next theorem which shows that the special Lorentz group $O^{\uparrow}(n-1,1)$ has the same homotopy type as the special orthogonal group $SO(n-1)$. The proof we are going to present is due to M. Marcolli (see [23]). Before we go into the theorem we make a few observations. The Minkowski matrix μ_n defines a bilinear form in $\mathbb{R}^n$ of signature -1:

$$< x, y >_{\mu_n} = \sum_{i=1}^{n} x_i y_i - x_n y_n = x^t \mu_n y \ ;$$

moreover, $O(n-1,1)$ is the group of all transformations of $\mathbb{R}^n$ which maintain this form (see [46]). The *pseudosphere* P in $\mathbb{R}^n$ is the set

$$P = \{x \in \mathbb{R}^n | x^t \mu_n x = \pm\}$$

with the topology induced from $\mathbb{R}^n$; the following result holds true:

Lemma 2.2.7 *The special Lorentz group $O^{\uparrow}(n-1,1)$ is the subgroup of all Lorentz transformations which map the upper layer of P, namely*

$$P^- = \{x \in \mathbb{R}^n | x^t \mu_n x = -1, x_n > 0\} \ ,$$

into itself.

Theorem 2.2.8 *$O^{\uparrow}(n-1,1)$ and $SO(n-1)$ have the same homotopy type.*

Proof – Let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ with $\vec{e}_n \in P^-$. A transformation defined by a matrix

$$L_\zeta = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & cosh\zeta & sinh\zeta \\ 0 & sinh\zeta & cosh\zeta \end{pmatrix}$$

is said to be a *hyperbolic rotation*. Observe that $L_\zeta \in O^{\uparrow}(n-1,1)$, for every $\zeta \in \mathbb{R}$; moreover, the hyperbolic rotations just defined form a subgroup $\mathfrak{H}$ of

$O^{\uparrow}(n-1,1)$. Because the elements of $O^{\uparrow}(n-1,1)$ map P^{-} into itself (see Lemma 2.2.7), we conclude that every $L \in O^{\uparrow}(n-1,1)$ can be written as a product

$$L = R_1 L_{\zeta} R_2$$

with

$$R_i = \begin{pmatrix} R^i_{n-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$R^i_{n-1} \in SO(n-1)$, $i = 1, 2$.

The proof of the theorem is concluded with the observation that the group $\mathfrak{H}$ is contractible: just define

$$F : \mathfrak{H} \times I \longrightarrow \mathfrak{H} \ , \ (L_{\zeta}, t) \longmapsto L_{t\zeta} \ .$$

□

Corollary 2.2.9 *Let ξ be a Lorentz bundle over S^3. Then*

$$\pi_k(\mathcal{G}(\xi)) \cong \begin{cases} \mathbb{Z} & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \ . \end{cases}$$

Proof – From the previous theorem we conclude that $O^{\uparrow}(2,1)$ and $SO(2)$ have the same homotopy type; next, because of 2.2.4 $\mathcal{G}^1(\xi)$ and $\Omega^3(O^{\uparrow}(2,1))$ are of the same type; finally, use the long exact sequence 2.6, and the fact that $\pi_k(SO(2)) = 0$ for every $k \neq 1$ and $\pi_1(SO(2)) \cong \mathbb{Z}$. □

We indicate two other ways to show the previous result.

1) The long sequence of homotopy groups associated to the bundle

$$(E_{SO(2)}, p, B_{SO(2)}, SO(2))$$

implies that

$$\pi_{q+1}(B_{SO(2)}) \cong \pi_q(SO(2)) \cong \begin{cases} \mathbb{Z} & \text{if } q = 1 \\ 0 & \text{otherwise} \ . \end{cases}$$

and therefore, $B_{SO(2)}$ has the type of an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. This implies that

$$\mathfrak{E}_{O^{\uparrow}(2,1)}(S^3) \cong [S^3, B_{SO(2)}] \cong H^2(S^3, \mathbb{Z}) \cong 0$$

and thus, ξ is trivial, implying that $\mathcal{G}(\xi) \cong Map(S^3, SO(2))$ and so, the homotopy groups of $\mathcal{G}(\xi)$ are as stated in the Corollary.
2) Use the long exact sequence of homotopy groups associated to the fibration $(\mathcal{G}(\xi), \eta, O^{\uparrow}(2,1))$ with fibre $\mathcal{G}^1(\xi) \cong Map_*(S^3, O^{\uparrow}(2,1))$.

2.3 The classifying space of $\mathcal{G}(\xi)$

The various constructions of classifying spaces for a topological group G (e.g., the Milgram-Steenrod construction of Appendix A) give rise to a classifying space $B_{\mathcal{G}(\xi)}$. We now give a useful description of the homotopy type of $B_{\mathcal{G}(\xi)}$.

Theorem 2.3.1 *Let $\xi = (E, p, B, G)$ be a principal G-bundle classified by a map $f : B \longrightarrow B_G$; suppose also that B has the type of a finite CW-complex and that G is a CW-complex. Then $B_{\mathcal{G}(\xi)}$ and $Map(B, B_G; f)$ have the same homotopy type.*

Proof – We first observe that $B_{\mathcal{G}(\xi)}$ has the same weak homotopy type as $Map(B, B_G; f)$: this follows from [26, Corollary 7.7]. To prove that they have the same type it suffices to show that they have the type of CW-complexes and then use Whitehead's Realizability Theorem (see [15, Theorem 2.5.1]). Next, we notice that both $\mathcal{G}(\xi)$ and $Map(B, B_G; f)$ have the type of CW-complexes. Because G is a CW-complex, the Milgram-Steenrod construction yields a classifying space B_G with the type of a CW-complex; then, because B has the type of a compact CW-complex, we conclude from [15, Theorem 5.3.4] that $Map(B, B_G)$ has the type of a CW-complex; finally, we use [15, Proposition 1.4.11] to obtain that $Map(B, B_G; f)$ has the type of a CW-complex. Next, we use Theorem 2.2.2 and [15, Theorem 5.3.4] to prove that $B_{\mathcal{G}(\xi)}$ has the type of a CW-complex. $\square$

Chapter 3

Fundamental equivalence and conjugation of gauge groups

3.1 Conjugation in the local gauge group

As we have noticed in Chapter 2, equivalent principal G-bundles over a space B give rise to isomorphic gauge groups; more precisely, if $\ell : \xi \longrightarrow \xi'$ is a G-equivariant homeomorphism, the isomorphism between the gauge groups $\mathcal{G}(\xi)$ and $\mathcal{G}(\xi')$ is given by the "adjoint map" $Ad_\ell(f) = \ell^{-1} f \ell$. If we assume ξ and ξ' to be locally trivial over the same open covering (recall that we can do this simply by intersecting the open covers of ξ and ξ'), then $\mathcal{G}(\xi)$ and $\mathcal{G}(\xi')$ can both be viewed as subgroups of the common local gauge group $\mathcal{L}$; in this case, $\mathcal{G}(\xi)$ and $\mathcal{G}(\xi')$ are *conjugate* subgroups of $\mathcal{L}$. We indicate this fact by writing $\mathcal{G}(\xi) \sim_C \mathcal{G}(\xi')$.

At this point, it is natural to ask whether stable equivalence also induces conjugation of the gauge groups. The answer to this question is given in the negative by the following example. The principal $SO(4)$-bundles $\xi = (S^4 \times \mathbb{R}, p, S^4, SO(4))$ and $\xi' = (TS^4, \pi, S^4, SO(4))$ are stably equivalent (there is a trivial line bundle ε such that $\xi \oplus \varepsilon \cong \xi' \oplus \varepsilon$). If $\mathcal{G}(\xi) \sim_C \mathcal{G}(\xi')$ they would have the same topological structure as subgroups of $\mathcal{L}$; then, because of Theorem 2.1.1, both gauge groups $\mathcal{G}(\xi)$ and $\mathcal{G}(\xi')$ would be homeomorphic to the function space $Map(S^4, SO(4))$. However, because of [21, Corollary 2.5] we obtain that

$$\pi_1(\mathcal{G}(\xi)) \cong \pi_1(Map(S^4, SO(4)) \cong \pi_1(SO(4)) \oplus \pi_5(SO(4)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

and using [38] we obtain that

$$\pi_1(\mathcal{G}(\xi')) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \ .$$

We would like to develop further this conjugacy relation. Our analysis will be conducted from a local point of view; hence, we shall assume that whenever we talk about "conjugate" gauge groups, the principal G-bundles which produce them are localy trivial over a common open covering $\mathfrak{U} = \{U_i | i \in J\}$. Moreover, these gauge groups are viewed as subgroups of the common local gauge group $\mathcal{L} = \prod_{i\in J} Map(U_i, G)$.

For the benefit of the reader, we put both Lemma 2.1.2 and Theorem 2.1.3 under a single roof:

Theorem 3.1.1 *Let ξ be a principal G-bundle over a space B with an open covering $\mathfrak{U} = \{U_i | i \in J\}$. Then, the map*

$$\vartheta : \mathcal{G}(\xi) \longrightarrow \mathcal{L} = \prod_{i\in J} Map(U_i, G) \ , \ f \longmapsto \{\vartheta_i r_i(f) | i \in J\}$$

is an embedding of topologial groups. Furthermore, the group $\mathcal{G}(\xi)$ coincides with the subgroup

$$\{\{f_i | i \in J\} \in \mathcal{L} | f_j = g_{ij}^{-1} f_i g_{ij} \text{ on } U_{ij}\} \ .$$

We wish to observe that the conjugation of two gauge groups does not depend on the transition functions selected. In fact, assume that ξ (resp. ξ') have transition functions g_{ij} and $\widetilde{g_{ij}}$ (resp. g'_{ij} and $\widetilde{g'_{ij}}$). Then, for every $i \in J$, we can find maps $h_i : U_i \longrightarrow G$ and $k_i : U_i \longrightarrow G$ such that

$$\widetilde{g_{ij}} = h_j^{-1} g_{ij} h_i \ , \ \widetilde{g_{ij}} = k_j^{-1} g_{ij} k_i$$

and the functions

$$\mathcal{G}(\xi) \longrightarrow \mathcal{G}(\xi) \ , \ \{f_i\} \longmapsto \{h_i^{-1} f_i h_i\}$$

and

$$\mathcal{G}(\xi') \longrightarrow \mathcal{G}(\xi') \ , \ \{f'_i\} \longmapsto \{k_i^{-1} f'_i k_i\}$$

are inner automorphisms. Now assume that $\mathcal{G}(\xi)$ and $\mathcal{G}(\xi')$ are conjugate; then, we can find an element $\{\ell_i\} \in \mathcal{L}$ such that

$$\mathcal{G}(\xi') = \{\ell_i\}^{-1} \mathcal{G}(\xi) \{\ell_i\}$$

and in particular, we may suppose that, for every $i \in J$, $f'_i = {\ell_i}^{-1} f_i \ell_i$. Hence, writing $q_i = {h_i}^{-1} f_i k_i$ for every $i \in J$, we obtain the equality

$$ {k_i}^{-1} f'_i k_i = (h_i q_i)^{-1} f_i (h_i q_i) $$

proving that the conjugation relation is maintained by the change of transition functions.

Remark 3.1.2 For technical reasons we shall require from now, and to the end of the section, that any point $b \in B$ is **non-degenerate**, that is to say, the inclusion $\{b\} \longrightarrow B$ is a closed cofibration (this is the case whenever B is a manifold or a CW-complex).

As before, let ZG be the centre of G and let $\pi : G \longrightarrow G/ZG = I(G)$ be the quotient map. We indicate by $\mathfrak{E}_{ZG}(B)$ the set of equivalence classes of principal ZG-bundles over B and by $\mathfrak{E}_G(B)$ the set of equivalence classes of principal G-bundles over B.

Lemma 3.1.3 *The set $\mathfrak{E}_{ZG}(B)$ is an abelian group which acts on the set $\mathfrak{E}_G(B)$.*

Proof – We first describe the action. Take a principal ZG-bundle λ over B with transition functions $\{c_{ij} | i, j \in J\}$, and an arbitrary principal G-bundle ξ over B with transition functions g_{ij}. Now observe that the composite functions $c_{ij} g_{ij} : U_{ij} \longrightarrow G$ satisfy properties TF1, TF2 and TF3 (because the elements of ZG commute with all elements of G) and thus, they define a new principal G-bundle which we denote by $\lambda \odot \xi$. This operation is independent of the representative on each class. In particular, if $\xi = \lambda'$ is a principal ZG-bundle over B, then $\lambda \odot \lambda' \in \mathfrak{E}_{ZG}(B)$ and indeed, the operation $\odot$ gives $\mathfrak{E}_{ZG}(B)$ an abelian group structure. The unit element is given by the trivial line bundle. $\square$

Observe that if $\xi' = \lambda \odot \xi$ then, the fundamental bundles $F(\xi)$ and $F(\xi')$ associated to ξ and ξ' (see Chapter 1 for the definition) coincide; moreover, the gauge groups $\mathcal{G}(\xi)$ and $\mathcal{G}(\xi')$ coincide as sets. This last statement's converse is not true in general; however, it is correct if ξ satisfies condition [C1] introduced in section 2.1 and which we reproduce here for completeness:
[C1] $(\forall b_o \in B)\ \eta : \mathcal{G}(\xi) \longrightarrow G\ ,\ f \longmapsto f|p^{-1}(b_o)(u_G)$ is a surjection.
In fact, if $\{f_i\} \in \mathcal{G}(\xi) = \mathcal{G}(\xi')$, then, for every $i, j \in J$ such that $U_{ij} \neq \emptyset$,

$$ f_j = {g_{ij}}^{-1} f_i g_{ij} \text{ and } f_j = {g'_{ij}}^{-1} f_i g'_{ij}\ . $$

These equalities prove that, for every $i \in J$,

$$(g'_{ij}g_{ij}^{-1})f_i = f_i(g'_{ij}g_{ij}^{-1}) \ .$$

Condition [C1] then shows that for every $g \in G$ and every $b_o \in B$, $(g'_{ij}g_{ij}^{-1})(b_o)$ commutes with g, that is to say,

$$g'_{ij}g_{ij}^{-1} : B \longrightarrow ZG$$

and thus, $\xi' = \lambda \odot \xi$.

We are going to prove that conjugacy of gauge groups is related to the fundamental groups associated and to the action introduced in Lemma 3.1.3 provided that condition [C1] is satisfied, together with the requirement that all maps $U_i \longrightarrow I(G)$ can be lifted to G (see condition [C2] below).

Let us spell out our new condition:

[C2] $(\forall i \in J)(\forall \bar{h}_{ii} : U_i \longrightarrow I(G))(\exists h_i : U_i \longrightarrow G)\ \bar{h}_i = \pi h_i$.

Theorem 3.1.4 *Suppose that ξ satisfies conditions* [C1] *and* [C2]. *For any principal G-bundle ξ' over B with transition functions g'_{ij} the following statements are equivalent:*

1. $F(\xi') \cong F(\xi)$;
2. $\mathcal{G}(\xi') \sim_C \mathcal{G}(\xi)$;
3. *there exists a principal ZG-bundle over B such that*
$$\xi' \cong \lambda \odot \xi \ .$$

Proof – 1. $\Rightarrow$ 2. The hypothesis implies that for every $i \in J$ there exists a map $\bar{h}_i : U_i \in I(G)$ such that

$$\pi g'_{ij} = \bar{h}_j^{-1}(\pi g_{ij})\bar{h}_j \ .$$

Because of condition [C2] every $\bar{h}_i$ admits a lift $h_i : U_i \longrightarrow G$ and so we can define

$$h_* : \mathcal{L} \longrightarrow \mathcal{L} \ , \ \{f_i\} \longmapsto \{h_i\}^{-1}\{f_i\}\{h_i\} = \{h_i^{-1}f_ih_i\}$$

which, together with Lemma 2.1.2, shows that $\mathcal{G}(\xi')$ is conjugate to $\mathcal{G}(\xi)$.
2. $\Rightarrow$ 3. If $\mathcal{G}(\xi')$ is conjugate to $\mathcal{G}(\xi)$, for every $i \in J$, there exixts h_i : $U_i \longrightarrow G$ such that

$$\mathcal{G}(\xi') = \{h_i\}^{-1}\mathcal{G}(\xi)\{h_i\} \ .$$

Hence,

$$(\forall\{f'_i\}\in\mathcal{G}(\xi'))(\exists\{f_i\}\in\mathcal{G}(\xi))(\forall i\in J)\ f'_i=h_i^{-1}f_ih_i\ ;$$

in other words, for every $\{f_i\}\in\mathcal{G}(\xi)$ and every pair of indices $i,j\in J$ such that $U_{ij}\neq\emptyset$,

$$h_j{}^{-1}f_jh_j=g'_{ij}{}^{-1}h_i{}^{-1}f_ih_jg'_{ij}\ .$$

From this equality we deduce that

$$h_j{}^{-1}g_{ij}{}^{-1}f_ig_{ij}h_j=g'_{ij}{}^{-1}h_i{}^{-1}f_ih_ig'_{ij}$$

and therefore,

$$h_ig'_{ij}h_j{}^{-1}g_{ij}{}^{-1}f_i=f_ih_ig'_{ij}h_j{}^{-1}g_{ij}{}^{-1}\ .$$

This implies that $(\forall\{f_i\}\in\mathcal{G}(\xi))(\forall b\in U_{ij})$,

$$(h_ig'_{ij}h_j{}^{-1}g_{ij}{}^{-1})(b)f_i(b)=f_i(b)(h_ig'_{ij}h_j{}^{-1}g_{ij}{}^{-1})(b)\ .$$

Now we use condition [C1] to conclude that, for every $g\in G$ and every $b\in U_{ij}$,

$$(h_ig'_{ij}h_j{}^{-1}g_{ij}{}^{-1})(b)g=g(h_ig'_{ij}h_j{}^{-1}g_{ij}{}^{-1})(b)\ ;$$

thus, for every $i,j\in J$ such that $U_{ij}\neq\emptyset$, we have the maps

$$c_{ij}=h_ig'_{ij}h_j{}^{-1}g_{ij}{}^{-1}:U_{ij}\longrightarrow ZG$$

and these satisfy conditions TF1, TF2 and TF3 of transition functions, thus defining a principal ZG-bundle λ over B; furthermore, from the definition of these maps we conclude that $\xi'\cong\lambda\odot\xi$.

3. $\Rightarrow$ 1. The hypothesis now indicates that, for every $i\in J$, there exists a map $h_i:U_i\longrightarrow G$ such that

$$g'_{ij}=h_i{}^{-1}c_{ij}g_{ij}h_j\ . \qquad (3.1)$$

Then, for every $g\in G$ and every $b\in U_{ij}$, we have that

$$(h_ig'_{ij}h_j{}^{-1}g_{ij}{}^{-1})(b)g=g(h_ig'_{ij}h_j{}^{-1}g_{ij}{}^{-1})(b)\ .$$

A straightforward computation now shows that

$$\pi g'_{ij}=\bar{h}_j{}^{-1}(\pi g_{ij})\bar{h}_j$$

and therefore, the bundles $F(\xi')$ and $F(\xi)$ are equivalent. $\square$

Two principal G-bundles over B which satisfy the equivalent conditions of Theorem 3.1.4 are said to be *fundamentally equivalent*. Notice that fundamental equivalence is an equivalence relation; furthermore, as seen in the beginning of this section, two equivalent principal G-bundles are fundamentally equivalent; however, the converse is not true: as observed in Section 2.1, the trivial $\mathbb{Z}$-bundle $S^1 \times \mathbb{Z} \longrightarrow S^1$ and the exponential map bundle $e^{2\pi i} : \mathbb{R} \longrightarrow S^1$ are not equivalent but have isomorphic gauge groups.

At times it is possible to "reduce" the structural group of a bundle; next we give a first result on the behaviour of fundamental equivalence with respect to this "reduction" (definition below). Let ξ be a principal G-bundle over B with transition functions $\{g_{ij} | i, j \in J\}$ and let H be a closed subgroup of G with inclusion map $\iota : H \longrightarrow G$; we say that ξ *admits a reduction to the structural group* H if there is a principal H-bundle $\widetilde{\xi}$ over B with transition functions $\{\widetilde{g_{ij}} | i, j \in J\}$ such that ξ and the principal G-bundle over B with transition functions $\{\iota\widetilde{g_{ij}} | i, j \in J\}$ are equivalent.

Lemma 3.1.5 *Let ξ and ξ' be two principal G-bundles which admit a reduction to a closed subgroup $H \subset G$. Suppose that $ZH \subset ZG$. Then, if the reductions of ξ and ξ' to H are fundamentally equivalent, so are ξ and ξ'.*

Proof – It follows by comparing the transition functions via relations like 3.1. □

In Chapter 4 we shall take up again this theme.

We now observe that conditions [C1] and [C2] are not so unusual. For example, they are clearly satisfied whenever G is abelian. Furthermore, condition [C1] holds true if G is path-connected because in that case the map $\eta : \mathcal{G}(\xi) \longrightarrow G$ is a fibration (see [6]). As for condition [C2], we prove the following:

Lemma 3.1.6 *Suppose that the open sets U_i of the distinguished open covering are all contractible. Then condition* [C2] *holds true.*

Proof – Take the exact sequence of groups

$$0 \longrightarrow ZG \longrightarrow G \longrightarrow I(G) \longrightarrow 0$$

and the exact sequence of sheafs

$$0 \longrightarrow \mathcal{S}_{ZG} \longrightarrow \mathcal{S}_G \longrightarrow \mathcal{S}_{I(G)} \longrightarrow 0$$

obtained from the pre-sheafs of continuous functions from the open subsets of B into these groups. According to the non-abelian version of [17, Theorem 2.4.2] we obtain, for every open subset of B, an exact sequence

$$0 \longrightarrow \Gamma(U, \mathcal{S}_{ZG}) \longrightarrow \Gamma(U, \mathcal{S}_G) \longrightarrow \Gamma(U, \mathcal{S}_{I(G)})$$

where $\Gamma(U, -)$ indicates that we are dealing with the sections of the appropriate sheaf over U. The cokernel of the last homomorphism of the above exact sequence is the cohomology group $H^1(U, \mathcal{S}_{ZG})$ (see [17, Theorem 2.10.1]); the result now follows from the fact that the open sets U_i are contractible. $\square$

The hypothesis of Lemma 3.1.6 is satisfied when B is a CW-complex or a manifold with a riemannian metric because in both cases B can be covered by contractible open sets: the reference for the former case is [13, Proposition 6.7] while for the latter is [8, Theorem 5.1]. These open covers are called *good covers* by some authors - see [8].

In the present context, where conjugation of gauge groups is beeing studied, the existence of a good cover of B has interesting consequences. We start with the following:

Lemma 3.1.7 *Let $\mathfrak{U} = \{U_i | i \in J\}$ be a good cover of B. Then any principal G-bundle over B is locally trivial over $\mathfrak{U}$.*

Proof – Let $\xi = (E, p, B, G)$ be a principal G-bundle. Since the inclusion map $\iota_i : U_i \longrightarrow B$ is homotopic to a constant map, the induced principal G-bunde $(p^{-1}(U_i), p, U_i, G)$ is trivial (see [19, Chapter 4, Theorem 9.9]). Now E is a localy trivial principal G-space over $\mathfrak{U}$ and so, by Theorem 1.3.4, ξ is a principal G-bundle, locally trivial over $\mathfrak{U}$. $\square$

Under these circumstances, if $\mathfrak{U}$ is a fixed good cover of B, the topological group

$$\mathcal{L} = \prod_{i \in J} Map(U_i, G)$$

contains as subgroups the gauge groups of all principal G-bundles with base space B. Hence, we can divide the set of the gauge groups of all principal G-bundles over B into conjugacy classes. We denote by $\mathfrak{C}\mathcal{G}(\xi)$ the conjugacy class of the gauge group $\mathcal{G}(\xi)$. We shall take up again this theme later on.

3.2 Isomorphism and conjugacy of gauge groups

It is clear that if $\mathcal{G}(\xi) \sim_C \mathcal{G}(\xi')$, then these gauge groups are isomorphic; however, if two principal G-bundles over B have isomorphic gauge groups, it not necessarily true that the gauge groups are conjugate. In this section we describe an example of that assertion, given by M. Marcolli in [24].

Take G to be the dihedral group of order 8 given by a clockwise ninety degree rotation of the square around its centre (call it r), its compositions r^2, r^3, $r^4 = 1$ and, the four flips (two around the diagonals, called m and n, and two around the horizontal and vertical axis, called h and v). We endow this group with the discrete topology. Let $\varphi : G \longrightarrow G$ be the isomorphism defined by

$$\begin{array}{ccc} r & \longmapsto & r \\ m & \longmapsto & h \\ n & \longmapsto & v \\ h & \longmapsto & m \\ v & \longmapsto & n\ . \end{array}$$

We now take $B = S^1$ with the open covering $\{U_1, U_2\}$ where U_1 is the open arc containing the point $(-1, 0)$ and limited by the points $\frac{\pi}{2} - \epsilon$ and $3\frac{\pi}{2} + \epsilon$, and U_2 is the open arc containing the point $(1, 0)$ and limited by $\frac{\pi}{2} + \epsilon$ and $3\frac{\pi}{2} - \epsilon$.

The intersection U_{12} is the union of the open arcs

$$U = \{\theta | \frac{\pi}{2} - \epsilon < \theta < \frac{\pi}{2} + \epsilon\}$$

and

$$V = \{\theta | 3\frac{\pi}{2} - \epsilon < \theta < 3\frac{\pi}{2} + \epsilon\}\ ;$$

Define the transition funtion $g_{12} : U_{12} \longrightarrow G$ by the following condition: for every $b \in U_{12}$,

$$g_{12}(b) = \{ \begin{array}{ll} g_U(b) = h\ , & \text{if } b \in U \\ g_V(b) = 1\ , & \text{if } b \in V\ . \end{array}$$

The set of transition functions $\{g_{ij} | i, j = 1, 2\}$ gives rise to a principal G-bundle ξ and the set $\{\varphi g_{ij} | i, j = 1, 2\}$, to the principal G-bundle ξ^φ. Observe that the bundles ξ and ξ^φ are not equivalent; however, according to

Lemma 2.1.5, these two bundles have isomorphic gauge groups (indeed, the patient reader can verify, by direct computation, that $\mathcal{G}(\xi) \cong \{1, r^2, v, h\}$ and $\mathcal{G}(\xi^\varphi) \cong \{1, r^2, n, m\}$, both isomorphic to the so-called "Viergruppe").

Let us prove that $\mathcal{G}(\xi)$ and $\mathcal{G}(\xi^\varphi)$ cannot be conjugate. Suppose that they were so. Then, we could find an element $\{h_i | i \in J\} \in \mathcal{L}$ such that

$$\mathcal{G}(\xi^\varphi) = \{h_i\}\mathcal{G}(\xi)\{h_i\}^{-1} \ ;$$

thus, for every $\{f_i\} \in \mathcal{G}(\xi)$, the element $\{f_i^\varphi\}$ defined by

$$f_i^\varphi = h_i f_i h_i^{-1}$$

belongs to $\mathcal{G}(\xi^\varphi)$. But, according to Theorem 2.1.3, for every $i, j \in J$, we have the following equalities in U_{ij}:

$$f_j = g_{ji} f_i g_{ij} \text{ and } f_j^\varphi = g_{ji}^\varphi f_i^\varphi g_{ij}^\varphi \ ;$$

therefore,

$$h_i g_{ij} f_j g_{ji} h_i^{-1} = g_{ij}^\varphi h_j f_j h_j^{-1} g_{ji}^\varphi$$

or, in other words,

$$(h_j^{-1} g_{ji}^\varphi h_i g_{ij}) f_j = f_j (h_j^{-1} g_{ji}^\varphi h_i g_{ij}) \ .$$

This means that for every $i, j \in J$, the element $\lambda_{ji} = h_j^{-1} g_{ji}^\varphi h_i g_{ij}$ commutes with the j^{th} component of any element of $\mathcal{G}(\xi)$ and thus, in the present context in which G is the dihedral group of order eight, $\lambda_{ji} : U_{ij} \longrightarrow G$ takes values in the subgroup $S = \{1, r^2, v, h\} \cong \mathcal{G}(\xi)$. Moreover, observe that one can write

$$g_{ji}^\varphi = h_j \lambda_{ji} g_{ji} h_i^{-1} \ .$$

The previous considerations give rise to the following two equalities:

$$g_U^\varphi = h_1 \lambda_U g_U h_2^{-1} \text{ and } g_V^\varphi = h_2 \lambda_V g_V h_1^{-1} \ ;$$

however, a straighforward computation shows that it is not possible to find elements $h_1, h_2 \in G$ and $\lambda_U, \lambda_V \in S$ such that

$$m = h_2 \lambda_V \lambda_U h_2^{-1} \text{ and } h_1 = h_2 \lambda_V \ .$$

Notice that in this example [C1] is not valid and hence we cannot use Theorem 3.1.4; however, the reader can prove with no difficulty that the fundamental bundles $F(\xi)$ and $F(\xi')$ associated, respectively, to ξ and ξ', are equivalent!

Chapter 4

Conjugacy classes of vector bundles

4.1 Characteristic Classes

In this chapter we study the conjugacy classes of the gauge groups of real and complex vector bundles over a fixed base space B which is assumed to be a connected, smooth, paracompact manifold endowed with a Riemannian metric.

We recall that real (resp. complex) n-vector bundles have structural group $GL(n,\mathbb{R})$ (resp. $GL(n,\mathbb{C})$). However, because we are taking our vector bundles to be smooth over a riemannian manifold, we can endow their total spaces with a Riemannian structure, and therefore, we can reduce their structural groups to $O(n)$ and $U(n)$, respectively, in the real and complex cases (see [8, Sections 6 and 20]). Therefore, we shall deal with principal bundles having for structural group either the orthogonal group $O(n)$ or the unitary group $U(n)$.

Furthermore, as we have seen in Section 3.1 we can take a *good cover* for B, that is to say, an open covering $\mathfrak{U} = \{U_i | i \in J\}$ with all the U_i's contractible. Hence, because of Lemma 3.1.7 all principal $O(n)$- or $U(n)$-bundles over B are locally trivial over $\mathfrak{U}$; this means that the gauge group of any principal $O(n)$- or $U(n)$-bundle over B is a subgroup of the local gauge group $\mathcal{L}$ and thus, we can arrange all these gauge groups into conjugacy classes.

The contents of this chapter are based on [25].

We begin by observing that Theorem 3.1.4 takes on the following format:

Theorem 4.1.1 *$G(\xi') \sim_C G(\xi)$ if, and only if, there exists a line bundle λ over B such that $\xi' \cong \xi \otimes \lambda$.*

Proof – It is enough to show that under the present circumstances, conditions [C1] and [C2] of Section 3.1 hold true. Condition [C2] is satisfied because of Lemma 3.1.6. As for [C1], we distinguish two cases. In the unitary case, [C1] follows from the connectivity of $U(n)$ and the fact that $\eta : G(\xi) \longrightarrow U(n)$ is a fibration. In the real case, the map η is onto $SO(n)$ – the path-component of $O(n)$ which contains the unit element – and this is a sufficient condition for the equivalence of the three statements of Theorem 3.1.4. □

Using the terminology introduced in Chapter 3, we can say that two (real or complex) vector bundles ξ and ξ' as above are *fundamentally equivalent* if, and only if, there exists a line bundle λ such that $\xi' \cong \xi \otimes \lambda$.

At this point we wish to study the relationship between the characteristic classes of two fundamentally equivalent vector bundles. A description of the general theory of characteristic classes can be found in the books by D. Husemoller [19] and J. Milnor and J. Stasheff [30].

4.1.1 Complex vector bundles

Our first result follows from [7, page 493]:.

Lemma 4.1.2 *Let ξ and ξ' be two fundamentally equivalent complex vector bundles of rank n over B. Then the Chern classes of ξ and ξ' are related by the formula*

$$c_k(\xi') = \sum_{j=0}^{k} \binom{n-j}{k-j} c_j(\xi) c_1(\lambda)^{k-j} \ . \tag{4.1}$$

Corollary 4.1.3 *Suppose that the Riemannian manifold B is also endowed with a CW-complex structure. Let ξ and ξ' be two fundamentally equivalent complex vector bundles over B. Suppose that $nc_1(\lambda) \neq 0$ in $H^2(B, \mathbb{Z})$. If the restrictions of ξ and ξ' to the 2-skeleton $B^{(2)}$ are equivalent, then $\xi \cong \xi'$.*

Proof – From the previous lemma we conclude that

$$c_1(\xi') = c_1(\xi) + nc_1(\lambda) \ .$$

Take the inclusion $\iota : B^{(2)} \longrightarrow B$ and notice that

$$0 = c_1(\lambda|B^{(2)}) = \iota^*(c_1(\lambda)) \in H^2(B^{(2)}, \mathbb{Z}) \ ;$$

but ι^* is injective because $H^2(B, B^2; \mathbb{Z}) = 0$ since $B/B^{(2)}$ has no cells in dimension less than 3. $\square$

We are going to show with an example that the condition on the first Chern class of λ is necessary. Let η be the complexification of the Hopf bundle γ_1^5 over $\mathbb{R}P^4$ and form the complex 2-bundle $\xi = \eta \oplus \eta$ over $\mathbb{R}P^4$. The bundle η is not trivial; however, using [19, Chapter 5, Theorem 7.8] we conclude that ξ is fundamentally equivalent to the trivial 2-bundle ε_2 over $\mathbb{R}P^4$ (take $\lambda = \eta$); this same theorem shows that $2c_1(\eta) = 0$. Now, according to [1, Theorem 7.3], the restrictions of ξ and ε_2 to $\mathbb{R}P^2$ are equivalent.

Since ξ is a smooth vector bundle, the Chern classes $c_i(\xi)$ can be expressed - modulo torsion - in terms of symmetric invariant polynomials in the curvature 2-form Ω. As we shall see, this method yields interesting and new results which are not visible just from Lemma 4.1.2. In order to conduct our analysis in an ordered fashion, we give a quick review of the principal definitions and ideas about connections and curvature in complex vector bundles; our main sources of informations are the books of M. Nakahara [32] and Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick [12]. First, some notation: $\mathfrak{L}(U(n))$ denotes the Lie algebra of $U(n)$ and, for a given open set $U_i \in \mathfrak{U}$, $T(U_i)$ and $T^*(U_i)$ represent, respectively, the tangent and cotangent spaces to U_i. Next, recall that the Lie algebra $\mathfrak{L}(U(n))$ is isomorphic to the tangent space to $U(n)$ at the unit element I_n. There are several equivalent ways to define a connection on a principal $U(n)$-bundle $\xi = (E, p, B, U(n))$, the first of which is very geometrical: a *connection* on ξ is a unique splitting of the tangent space T_xE, for every $x \in E$, as a direct sum of a vertical subspace V_xE and a horizontal subspace H_xE, such that horizontal spaces on the same fibre are related by a linear map induced by a right action $R_g x = xg$; one also requires that smooth vector fields on E separate into smooth horizontal and vertical vector fields. In practice, we need to obtain the splitting $T_xE = H_xE \oplus V_xE$ in a systematic fashion; this can be done with the Lie algebra valued one-form $\omega \in \mathfrak{L}(U(n)) \otimes \Omega^1(E)$ [1]: this is just a projection of T_xE onto $V_xE \cong \mathfrak{L}(U(n))$. The *curvature two-form* Ω is the covariant derivative of the connection one-form ω; hence, $\Omega \in \mathfrak{L}(U(n)) \otimes \Omega^2(E)$. Connection

[1] We indicate the space of smooth r-forms over a manifold M by $\Omega^r(M)$.

and curvature are related by the following *Cartan structure equation*:

$$(\forall X, Y \in T_x E)\ \Omega(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)] \ ,$$

which is also written as

$$\Omega = d\omega + \omega \wedge \omega \ .$$

As in the case of gauge transformations, it is convenient to view connections and curvatures in terms of the open sets of $\mathfrak{U}$; accordingly, for each $i \in J$, take a cross-section $\sigma_i : U_i \longrightarrow p^{-1}(U_i)$ and define the *local (gauge) potential*

$$\mathcal{A}_i := {\sigma_i}^*(\omega) \in \mathfrak{L}(U(n)) \otimes \Omega^1(U_i) \ .$$

The set $\mathcal{A} = \{\mathcal{A}_i | i \in J\}$ is called *(Yang-Mills) potential*. Conversely, given an $\mathfrak{L}(U(n))$-valued one-form $\mathcal{A}_i$ on U_i and a cross-section $\sigma_i : U_i \longrightarrow p^{-1}(U_i)$, there exists a one-form $\omega_i \in \mathfrak{L}(U(n)) \otimes \Omega^1(U_i)$ such that $\mathcal{A}_i = {\sigma_i}^*(\omega_i)$ (see [32, Theorem 10.5]). In order to define ω uniquely out of the local potentials (in other words, so that $\omega_i = \omega_j$ on U_{ij}), the local potentials must satisfy the following compatibility conditions:

$$\mathcal{A}_j = g_{ij}^{-1} \mathcal{A}_i g_{ij} + g_{ji} d(g_{ij}) \ . \tag{4.2}$$

The local form $\mathcal{F}_i$ of the curvature Ω at U_i is given by

$$\mathcal{F}_i = {\sigma_i}^*(\Omega) \ ,$$

where σ_i is a cross-section at U_i. The two-forms $\mathcal{F}_i$ are called *local fields*; $\mathcal{F} = \{\mathcal{F}_i | i \in J\}$ is called *(Yang-Mills) field strength*. In this context, we also write $\Omega_i = p^*(\mathcal{F}_i)$.

The equations corresponding to the Cartan structure equations are:

$$\mathcal{F}_i = d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i \tag{4.3}$$

with the local transformation conditions

$$\mathcal{F}_j = g_{ij}^{-1} \mathcal{F}_i g_{ij} \tag{4.4}$$

for every $i, j \in J$.

Now we are ready to talk about Chern classes. Let G be a group of complex square matrices (say $GL(n,\mathbb{C})$ or $U(n)$) and let $\mathfrak{L}G$ be its Lie algebra. We say that a symmetric, k-linear mapping

$$f : \mathfrak{L}G \times \ldots \times \mathfrak{L}G \longrightarrow \mathbb{R}$$

is Ad-G *invariant* if, for every $g \in G$ and every k-tuple $(V_1, ..., V_k) \in \mathfrak{L}G \times \ldots \times \mathfrak{L}G$,

$$f(\mathrm{Ad}_g V_1, \ldots, \mathrm{Ad}_g V_k) = f(V_1, \ldots, V_k) \ ,$$

where $\mathrm{Ad}_g V_i = g^{-1} V_i g$. The following is the main theorem of the theory of characteristic classes (see [12, Theorem 1, V BIS] or [32, Theorem 11.1]):

Theorem 4.1.4 *(Chern-Weil) Let f be an* Ad-G *invariant symmetric k-linear mapping of $\mathfrak{L}G$ to $\mathbb{R}$, let $\xi = (E, p, B, G)$ be a principal G-bundle and let Ω be the curvature two-form of a connection ω of ξ. The exterior differential form $f(\Omega)$ of degree $2k$ defined by*

$$f(\Omega)(v_1, \ldots, v_k) = \frac{1}{2k!} \sum_{\sigma} \mathrm{sign}(\sigma) f(\Omega(v_{\sigma(1)}, v_{\sigma(2)}) \ldots (\Omega(v_{\sigma(2k-1)}, v_{\sigma(2k)}))$$

where σ is a permutation of the set $\{1, 2, \ldots, k\}$, has the following properties:
(i) $f(\Omega)$ projects to a unique $2k$-form γ_k on B (i,e., $p^*(\gamma_k) = f(\Omega)$) such that $d(\gamma_k) = 0$;
(ii) the element $[\gamma_k]$ of the De Rham cohomology group $H^{2k}_{DR}(B, \mathbb{R})$ is independent of the choice of the curvature two-form Ω.

The reader should note that this theorem could be stated in terms of potentials and relative field strengths.

If we choose a basis for $\mathfrak{L}G$ and, for every $V \in \mathfrak{L}G$, we set

$$f(V) := f(V, \ldots, V)$$

then, $f(V)$ is a polynomial of degree k in the components of V. The algebra of Ad-G-invariant symmetric multilinear mappings is identified with the algebra of Ad-G-invariant polynomials. At this point notice that the coefficients $f_k(V)$ of the characteristic polynomial of a matrix $V \in \mathfrak{L}G$, namely

$$\det(\lambda I_n - \frac{i}{2\pi} V) = \sum_{k=0}^{n} f_k(V) \lambda^{n-k} \ , \tag{4.5}$$

are Ad-G-invariant polynomials. In particular, if $V = \Omega$, the closed $2k$-form γ_k such that $p^*(\gamma_k) = f_k(\Omega)$ gives rise to the k^{th}-*Chern class* of ξ:

$$c_k(\xi) = [\gamma_k] \in H^{2k}_{DR}(B, \mathbb{R}) \ .$$

The *total Chern class* of ξ is given by

$$c(\xi) = 1 + c_1(\xi) + \ldots + c_n(\xi) \in \bigoplus_{j=0}^{n} H^{2j}_{DR}(B, \mathbb{R}) \ .$$

From the general properties of the characteristic polynomials, we conclude that $f_1(\Omega) = \mathrm{tr}(\frac{i}{2\pi}\Omega)$ and that $f_n(\Omega) = \det(\frac{i}{2\pi}\Omega)$. We also note that the defining equation 4.5 yields the total Chern class of ξ simply by replacing the variable λ by 1. Finally, we observe that from 4.5 we retrieve the axioms – see [19, Chapter 17, Section 3] – which characterize Chern classes (valued in the real cohomology ring of the base space).

The set $\mathfrak{A}_\xi$ of potentials (or connections) on ξ is not empty: indeed, there are infinitely many connections on a smooth vector bundle (with paracompact base - see [12, Theorem, pg. 363]). Actually $\mathfrak{A}_\xi$ is an *affine space*, that is to say, a vector space with a selected origin: hence, if $\mathcal{A}, \mathcal{A}' \in \mathfrak{A}_\xi$, so is $(1-t)\mathcal{A} - t\mathcal{A}'$, for every $t \in \mathbb{R}$ (see [33, Chapter VIII]). Our next result shows that there is a bijection between the affine spaces of all connections of two fundamentally equivalent n-bundles ξ and ξ':

Lemma 4.1.5 *If $\xi' \cong \lambda \otimes \xi$, then there is a bijection*

$$\theta : \mathfrak{A}_{\xi'} \longrightarrow \mathfrak{A}_\xi \ .$$

Proof – Let g_{ij}, g'_{ij} and c_{ij} be, respectively, the transition functions of the vector bundles ξ, ξ' and λ. As in Theorem 3.1.4 (3.$\Rightarrow$ 1), the hypothesis implies that there exists a set of maps $\{h_i : U_i \longrightarrow S^1 | i \in J\}$ such that

$$g'_{ij} = {h_i}^{-1} c_{ij} g_{ij} h_j \ . \tag{4.6}$$

Now take a partition of unity $\{\rho_i | i \in J\}$ subordinated to the open covering $\mathfrak{U}$ of B and, for every $i \in J$, define the 1-form

$$\theta_i = \sum_{k \in J} \rho_k c_{ik} dc_{ki} \ . \tag{4.7}$$

Take arbitrarily $\mathcal{A} \in \mathfrak{A}_\xi$ and, for every $i \in J$, define

$$\mathcal{A}'_i = {h_i}^{-1} \mathcal{A}_i h_i + {h_i}^{-1} dh_i + \theta_i \ ; \tag{4.8}$$

we wish to prove that the local one-forms $\mathcal{A}'_i$ define a connection $\mathcal{A}' \in \mathfrak{A}_{\xi'}$ that is to say, according to 4.2, the one-forms $\mathcal{A}'_i$ must satisfy the equation

$$\mathcal{A}'_i = g'_{ij} \mathcal{A}'_j g'_{ji} + g'_{ij} d(g'_{ji}) \ .$$

Using equalities 4.6 and 4.8, the right hand side of the previous formula gives rise to the following sequence of equalities:

$$g'_{ij}\mathcal{A}'_j g'_{ji} + g'_{ij} d(g'_{ji}) =$$

$$= {h_i}^{-1} c_{ij} g_{ij} h_j ({h_j}^{-1}\mathcal{A}_j h_j + {h_j}^{-1} dh_j + \sum_{k\in J} \rho_k c_{jk} dc_{kj}) {h_j}^{-1} c_{ji} g_{ji} h_i +$$

$$+ ({h_i}^{-1} c_{ij} g_{ij} h_j ({h_j}^{-1}) d({h_j}^{-1} c_{ji} g_{ji} h_i) =$$

$$= {h_i}^{-1} g_{ij} \mathcal{A}_j g_{ji} h_i + {h_i}^{-1} g_{ij}((dh_j){h_j}^{-1}) g_{ji} h_i +$$

$$+ {h_i}^{-1} g_{ij} h_j (\sum_{k\in J} \rho_k c_{jk} dc_{kj}) {h_j}^{-1} g_{ji} h_i + {h_i}^{-1} g_{ij}(h_j(d{h_j}^{-1})) g_{ji} h_i +$$

$$+ {h_i}^{-1} g_{ij}(c_{ij} dc_{ij}) g_{ji} h_i + {h_i}^{-1} g_{ij} dg_{ji} h_i + {h_i}^{-1} dh_i \ .$$

Since $h_j(dh_j)^{-1} + (dh_j){h_j}^{-1} = 0$, the second and fourth terms of the last sum cancel out; moreover, the adjoint action of any element of $U(n)$ is trivial on $\mathfrak{L}(U(n))$ and thus, we conclude that

$$\mathcal{A}'_i = {h_i}^{-1}({g_{ji}}^{-1}\mathcal{A}_j g_{ji} + g_{ij}(dg_{ji})) h_i + {h_i}^{-1}(dh_i) + \sum_{k\in J} \rho_k c_{jk} dc_{kj} + c_{ij}(dc_{ji}) \ .$$

Because $\sum_{k\in J} \rho_k = 1$, we can write the last two summands as follows:

$$\sum_{k\in J} \rho_k c_{jk} dc_{kj} + c_{ij}(dc_{ji}) = \sum_{k\in J} \rho_k c_{jk} dc_{kj} + (\sum_{k\in J} \rho_k) c_{ij}(dc_{ji})$$

by the cyclicity property of the transition functions (see [TF3]) and the product property of the exterior derivative, we conclude that

$$\sum_{k\in J} \rho_k c_{jk} dc_{kj} + (\sum_{k\in J} \rho_k) c_{ij}(dc_{ji}) = \sum_{k\in J} \rho_k c_{ik} dc_{ki} \ .$$

Therefore,

$$g'_{ij}\mathcal{A}'_j g'_{ji} + g'_{ij} d(g'_{ji}) = {h_i}^{-1}\mathcal{A}_i h_i + {h_i}^{-1} dh_i + \theta_i = \mathcal{A}'_i$$

concluding our proof. □

The next theorem compares the Chern characters of two complex n-bundles which are fundamentally equivalent when their structural groups can be reduced to a subgroup H with discrete centre. We already know from Lemma 3.1.5 that if the reductions to H of ξ and ξ' are fundamentally equivalent, then ξ and ξ' themselves are fundamentally equivalent and therefore, their Chern classes are related by equation 4.1; however, the next result shows that in this case the Chern classes of ξ actually coincide with those of ξ'.

Theorem 4.1.6 *Let ξ and ξ' be two complex n-bundles with structural group $U(n)$ reduced to a subgroup H such that ZH is discrete. If these bundles are fundamentally equivalent with respect to the structural group H then, for every $i = 1, \ldots, n$, $c_i(\xi) = c_i(\xi')$.*

Proof – Because the Lie algebra of ZH is trivial, the one-forms θ_i defined in 4.7 vanish. This implies that

$$\mathcal{A}'_i = {h_i}^{-1}\mathcal{A}_i h_i + {h_i}^{-1} dh_i$$

and therefore, the local fields (and corresponding local curvatures) are related by $\mathcal{F}'_i = \mathrm{Ad}_{h_i}\mathcal{F}_i$. The statement of the theorem now follows from the definition of the total Chern class. $\square$

We wish to observe that the conclusion of Theorem 4.1.6 above does not follow from equalities 4.1 because the central step in the proof of theorem is to show that there is a flat connection on the line bundle λ and this implies that its Chern class is a torsion element. We also note that the converse of the above theorem is false: in fact, on the one hand, λ has a flat connection $\Longleftrightarrow$ it has locally constant transition functions (see [20, page 6]) and on the other hand, one can easily construct a line bundle with locally constant transition functions that are not contained in any finite order subgroup of $U(1)$.

4.1.2 Real vector bundles

Our first result has to do with the Pontrjagin classes of two fundamentally equivalent real vector bundles and is a direct consequence of Theorem 4.1.6.

Proposition 4.1.7 *Two fundamentally equivalent real n-bundles have the same Pontrjagin classes in the De Rham cohomology ring of their base space.*

Proof – The Pontrjagin classes $p_i(\xi)$ of a real n-bundle ξ are obtained from its Chern classes by complexification:

$$p_i(\xi) := (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \ .$$

Now, by complexification, we can embed $O(n)$ into $U(n)$. Since $\xi \otimes \mathbb{C}$ and $\xi' \otimes \mathbb{C}$ are fundamentally equivalent over $O(n)$ and $ZO(n) = \mathbb{Z}_2$, Theorem 4.1.6 shows that the Chern characters of the complexified bundles are equal; thus, we have the result stated. □

At this point we want to observe that the previous proposition might be false if we work in the integral cohomology ring of the base space, as we can see from the following example. Let γ_1^5 be the Hopf bundle over $\mathbb{R}P^4$ and let $\xi = \gamma_1^5 \oplus \gamma_1^5$; because of [19, Chapter 5, Theorem 7.8], ξ is fundamentally equivalent to the trivial real 2-bundle ε_2 over $\mathbb{R}P^4$ (take $\lambda = \gamma_1^5$). However, ξ has a non-vanishing top Pontrjagin class: this can be seen by taking the complexification $\eta = \gamma_1^5 \otimes \mathbb{C}$ and observing that because $\eta \cong \overline{\eta}$, $c_1(\eta)$ is a non-zero element of order 2 in $H^2(\mathbb{R}P^4, \mathbb{Z})$.

Next, we recall that to each real n-vector bundle ξ over a space B we can associate a sequence of cohomology classes

$$w_i(\xi) \in \check{H}^i(B, \mathbb{Z}_2)$$

$i = 0, \ldots, n$ which satisfies certain axioms listed in [30], for example. These are the so-called *Stifel-Whitney classes* of ξ. Our first result is similar to Lemma 4.1.2:

Lemma 4.1.8 *Let ξ and ξ' be two fundamentally equivalent real vector bundles of rank n over B. Then the Stiefel-Whitney classes of ξ and ξ' are related by the formula*

$$w_k(\xi') = \sum_{j=0}^{k} \begin{pmatrix} n-j \\ k-j \end{pmatrix} w_j(\xi) w_1(\lambda)^{k-j} \tag{4.9}$$

with coefficients taken modulo 2.

Proof – See [7, page 497]. □

The first and second Stiefel-Whitney classes have a particular geometric interest; in support to this statement we give here the following two results:

1) if M is a riemannian manifold and $\tau = (TM, \pi, M)$ is its tangent bundle, then M is orientable if, and only if, $w_1(\tau) = 0$; 2) if M is orientable, then there exists a *Spin* bundle over M if, and only if, $w_2(\tau) = 0$ (see [32, Theorems 11.21 and 11.23]). We wish to compare the first two Stiefel-Whitney classes of two fundamentally equivalent real vector bundles, However, before we do this, we recall briefly an alternative method of defining Stiefel-Whitney classes for smooth vector bundles over riemannian manifolds; this method is based on properties of the Čech cohomology of B (unlike the Chern classes, the Stiefel-Whitney classes cannot be constructed via the curvature two-form Ω associated to a curvature form ω of ξ).

We begin by taking a good cover $\mathfrak{U} = \{U_i | i \in J\}$ of B, and for every $n \geq 1$, let $U_{i_0,\ldots,i_n}$ be the intersection $U_{i_0} \cap \ldots \cap U_{i_n} \neq \emptyset$. A *Čech n-cochain* is a map

$$f(i_0, \ldots, i_n) : U_{i_0,\ldots,i_n} \longrightarrow \mathbb{Z}_2$$

which is totally symmetric under any permutation σ of $i_0, \ldots, i_n$. Next, consider the multiplicative group $C^n(\mathfrak{U}, \mathbb{Z}_2)$ of all Čech n-cochains, for all possible non-empty intersections of $n+1$ open sets in $\mathfrak{U}$. Finaly, define the coboundary operator

$$\delta_n : C^n(\mathfrak{U}, \mathbb{Z}_2) \longrightarrow C^{n+1}(\mathfrak{U}, \mathbb{Z}_2)$$

$$(\delta_n f)(i_0, \ldots, n+1)) = \prod_{k=0}^{n+1} f(i_0, \ldots, \widehat{i_k}, \ldots, i_{n+1})$$

(as usual, the symbol $\widehat{}$ means that the variable under it has been cancelled). It is easy to prove that $\delta^2 = 1$. Hence,

$$(\forall n \geq 0) \ \mathrm{im}\ \delta_n = B^n(\mathfrak{U}, \mathbb{Z}_2) \subset \ker \delta_{n+1} = Z^n(\mathfrak{U}, \mathbb{Z}_2) \ ;$$

by definition, the n^{th}-*Čech cohomology group* of $\mathfrak{U}$ is the quotient group

$$\check{H}^n(\mathfrak{U}, \mathbb{Z}_2) = Z^n(\mathfrak{U}, \mathbb{Z}_2)/B^n(\mathfrak{U}, \mathbb{Z}_2) \ .$$

Now let $\mathfrak{V} = \{V_j | j \in J\}$ be another good cover of B and suppose that $\mathfrak{V}$ *refines* $\mathfrak{U}$, that is to say, there exists a "refinement" function $\alpha : J \longrightarrow I$ such that

$$(\forall j \in J) \ V_j \subset U_{\alpha(j)} \ .$$

Such a refinement function induces a chain complex homomorphism C^*: for each $n \geq 0$, define

$$C^n(\alpha) : C^n(\mathfrak{U}, \mathbb{Z}_2) \longrightarrow C^n(\mathfrak{V}, \mathbb{Z}_2)$$

$$(C^n(\alpha)f)(j_0,\ldots,j_n) = f(\alpha(j_0),\ldots,\alpha(j_n)) \ ;$$

thus, α induces homomorphisms

$$\alpha_n : \check{H}^n(\mathfrak{U},\mathbb{Z}_2) \longrightarrow \check{H}^n(\mathfrak{V},\mathbb{Z}_2)$$

for each $n \geq 0$. This homomorphism between the cohomology groups is actually independent of the refinement function, because two refinement functions from $\mathfrak{V}$ to $\mathfrak{U}$ produce homotopic chain complex functions. All this means that the set $\{\check{H}^*(\mathfrak{U},\mathbb{Z}_2|\mathfrak{U}\}$ is a direct system of graded abelian groups; the Čech cohomology of B is then defined as the direct limit

$$\check{H}^*(B,\mathbb{Z}_2) = \lim_{\mathfrak{U}} \check{H}^*(\mathfrak{U},\mathbb{Z}_2) \ .$$

Let ξ be a principal $O(n)$-bundle over a riemannian manifold M covered by a good cover $\mathfrak{U}$. In order to construct the first Stiefel-Whitney class $w_1(\xi)$ we consider the transition functions $g_{ij} : U_{ij} \longrightarrow O(n)$ and define

$$f_1(i,j) : U_{ij} \longrightarrow \mathbb{Z}_2$$

by the condition: for every $b \in U_{ij}$, $f_1(i,j)(b) = \det(g_{ij}(b))$. Clearly, $f_1(i,j) = f_1(j,i)$ and hence, $f_1(i,j) \in C^1(\mathfrak{U},\mathbb{Z}_2)$. On the other hand, from the cyclicity condition $g_{ij}g_{jk}g_{ki} = u_G$, we conclude that

$$(\delta f_1)(i,j,k) = \det(g_{ij})\det(g_{jk})\det(g_{ki}) = 1$$

and therefore, $f_1 \in Z^1(\mathfrak{U},\mathbb{Z}_2)$. We define $w_1(\xi)$ to be the image in $\check{H}^1(B,\mathbb{Z}_2)$ of the cohomology class $[f_1] \in \check{H}^1(\mathfrak{U},\mathbb{Z}_2)$.

Now we define the second Stiefel-Whitney class of ξ. Suppose that the principal $O(n)$-bundle ξ is *orientable* that is to say $\Longleftrightarrow$ the structure group $O(n)$ can be reduced to $SO(n)$ (see [8, Chapter I,Proposition 6.4]). Recall also that, for every positive integer n there is a universal covering map

$$\phi : Spin(n) \longrightarrow SO(n)$$

with fibre $\mathbb{Z}_2$; the simply connected space $Spin(n)$ is actually a topological group (see [19, Chapter 12]). Now, for every $b \in U_{ij}$, take a lift $\widetilde{g}_{ij}(b)$ of $g_{ij}(b)$ and define

$$f_2(i,j,k) : U_{ijk} \longrightarrow \mathbb{Z}_2 \ , \ b \longmapsto \widetilde{g}_{ij}(b)\widetilde{g}_{jk}(b)\widetilde{g}_{ki}(b) \ .$$

In this case too, f_2 is a cocycle and we define $w_2(\xi)$ to be the image in $\check{H}^2(B,\mathbb{Z}_2)$ of the cohomology class $[f_2] \in \check{H}^2(\mathfrak{U},\mathbb{Z}_2)$.

Now we are ready to prove the following two results which give a more accurate comparison of the first two Stiefel-Whitney classes of two fundamentally equivalent real vector bundles.

Theorem 4.1.9 *Let ξ and ξ' be two real, even dimensional, fundamentally equivalent vector bundles over B. Then, $w_1(\xi') = w_1(\xi)$.*

Proof – We know that, because ξ and ξ' are fundamentally equivalent, the transition functions of these two vector bundles are related by an equation of the type

$$g'_{ij} = {h_i}^{-1} c_{ij} g_{ij} h_j \ .$$

(see 4.6). But we are dealing with vector bundles whose structural group is $O(2m)$ and so, $\det(c_{ij}) = 1$. Then, from the equation above we conclude that

$$\det(g'_{ij}) = \det(g_{ij}) \det(h_i)^{-1} \det(h_j)$$

and therefore, the cocycles $f_1(i,j)$ and $f'_1(i,j)$ differ by a coboundary:

$$f'_1(i,j) = f_1(i,j)(\delta \bar{f})(i,j) \ .$$

□

We can obtain this from Equation 4.9, provided we take $k = 1$, n even, and we write $\mathbb{Z}_2$ additively.

Theorem 4.1.10 *Let ξ and ξ' be two real, orientable $2m$-vector bundles for which there exists a line bundle λ such that $\xi' \cong \xi \otimes \lambda$. If $w_2(\oplus_{2m}\lambda) = 1$, then $w_2(\xi) = w_2(\xi')$.*

Proof – The transition functions of both bundles take values in $SO(n)$; moreover, because ξ and ξ' are fundamentally equivalent, these transition functions are related by a relation like 3.1, namely:

$$g'_{ij} = {h_i}^{-1} c_{ij} g_{ij} h_j \ .$$

Lift the maps h_i and c_{ij} to $Spin(2m)$:

$$\widetilde{h_i} : U_i \longrightarrow Spin(2m) \ ,$$

$$\widetilde{g_{ij}} : U_{ij} \longrightarrow Spin(2m) \; ;$$

these liftings give rise to a lifting of g'_{ij}, namely

$$\widetilde{g'_{ij}} = \widetilde{h_i}^{-1} \widetilde{c_{ij}} \widetilde{g_{ij}} \widetilde{h_j} \; .$$

But from $w_2(\xi) = 1$ we conclude that

$$\widetilde{g_{ij}} \widetilde{g_{jk}} \widetilde{g_{ki}} = 1$$

and so,

$$\widetilde{g'_{ij}} \widetilde{g'_{jk}} \widetilde{g'_{ki}} = \widetilde{c_{ij}} \widetilde{c_{jk}} \widetilde{c_{ki}} \; .$$

But $\widetilde{c_{ij}} \widetilde{c_{jk}} \widetilde{c_{ki}} = 1$ because $w_2(\oplus_{2m}\lambda) = 1$ by assumption; thus, $w_2(\xi') = 1$. $\square$

4.2 The isotropy group of a vector bundle

Let ξ be a (principal) real or complex vector bundle over B and let $\mathfrak{I}(\xi)$ be the set of all line bundles over B such that $\lambda \otimes \xi \cong \xi$; with the tensor product operation this set becomes a group, indeed a subgroup of the group $\mathfrak{E}_{O(1)}(B)$ (resp. $\mathfrak{E}_{U(1)}(B)$) of all classes of equivalent principal real (resp. complex) line bundles over B. The group $\mathfrak{I}(\xi)$ is the *isotropy group* of ξ.

The following observation shows that the isotropy group of a vector bundle ξ can assume a key role in the game of characterizing the conjugacy class $\mathfrak{C}\mathcal{G}(\xi)$. Suppose that $\mathfrak{I}(\xi) = 0$; then, according to Theorem 3.1.4, there exists a bijective correspondence between the set $\mathfrak{C}\mathcal{G}(\xi)$ and the set $\mathfrak{E}_{O(1)}(B)$ (resp. $\mathfrak{E}_{U(1)}(B)$). Thus, if $\mathfrak{I}(\xi) = 0$ the Classification Theorem 1.3.5 implies that:

1. Real Case: $\mathfrak{C}\mathcal{G}(\xi) \cong H^1(B, \mathbb{Z}_2)$;

2. Complex Case: $\mathfrak{C}\mathcal{G}(\xi) \cong H^2(B, \mathbb{Z})$.

As we shall see after the statement of Theorem 4.2.5, the isotropy group of a trivial bundle is not necessarily trivial.

Now we give a simple example of a non-trivial vector bundle whose isotropy group is trivial.

Theorem 4.2.1 *Let ξ be a real vector bundle of odd dimension over B. Then $\mathfrak{I}_{O(1)}(\xi) = 0$.*

Proof – We begin by observing that $\lambda \otimes \xi \cong \xi \iff \lambda$ is trivial. In fact, from equations 4.9 we conclude that

$$w_1(\xi) = (\dim\ \xi) w_1(\lambda) + w_1(\xi)\ ;$$

because the Stiefel-Whitney classes are elements of the $\mathbb{Z}_2$-cohomology ring of B and $\dim \xi$ is odd, it follows that $w_1(\xi) = 0$ and hence, $\lambda = 1$ is the trivial line bundle over B. □

The analysis of the first Stiefel-Whitney class conducted in the previous theorem clearly yields no results if ξ is even-dimensional; this method is also fruitless in case ξ is an even-dimensional trivial real bundle. For complex vector bundles we use the Chern characters and conclude that $\mathfrak{I}(\xi) = 0$ whenever ξ is trivial and $H^2(B, \mathbb{Z})$ is torsion-free.

Now we give an example of a real vector bundle ξ with non-trivial *isotropy* (that is to say, $\mathfrak{I}(\xi) \neq 0$). Suppose that $n = 2m$ with $m \geq 1$; construct the vector bundle over $\mathbb{R}P^n$

$$\xi = m\gamma_1^n \oplus m$$

(here $m\gamma_1^n$ stands for the Whitney sum of γ_1^n with itself m-times and the second factor m stands for the trivial m-dimensional bundle over $\mathbb{R}P^n$). The dimension of ξ is n and it is immediate to see that

$$\xi \otimes \gamma_1^n \cong \xi\ .$$

4.2.1 Isotropy and stable equivalence

Our first objective is to prove that if an n-real vector bundle over a CW-complex with only one top cell (for example, a projective space) has non-trivial isotropy, then any n-bundle in its stable class also has non-trivial isotropy. Let us recall that two real vector bundles ξ and ξ' are *stably equivalent* (and we write $\xi \sim_S \xi'$) if there exist trivial vector bundles, say m and n such that $\xi \oplus m$ and $\xi' \oplus n$ are equivalent.[2]

In order to prove what we just anounced we must make use of the concept of *cooperation* of a COH-space on a based space due to B. Eckmann and P. J. Hilton (see [16]), which we briefly review here for the reader's benefit. Let

[2] As we did with equivalence classes of bundles, we do not differentiate notationaly a vector bundle and its stable class.

A be a COH-space with comultiplication $\nu : A \longrightarrow A \vee A$; we say that A *cooperates* on a based space X if there exists a map $p : X \longrightarrow A \vee X$ such that

1. if $q : A \vee X \longrightarrow X$ is the projection, then qp and 1_X are based-homotopic;

2. $(1_A \vee p)p$ and $(\nu \vee 1_X)p$ are based-homotopic.

CW-complexes with just one top cell are a natural source of examples of cooperation of a sphere on a space. In fact, suppose that $X = Y \sqcup_\phi D^n$ is defined by attaching the n-disk D^n to the CW-complex Y (note that dimY < n) via the map $\phi : S^{n-1} \longrightarrow Y$. The canonical map $p : X \longrightarrow S^n \vee X$ obtained by pinching the cone $CS^{n-1} \cong D^n$ half-way through its height is a cooperation.

Notice that $X = Y \sqcup_\phi D^n$ is homeomorphic to the mapping cone C_ϕ; now take the long sequence of spaces determined by ϕ

$$S^{n-1} \xrightarrow{\phi} Y \xrightarrow{\overline{\imath}} C_\phi \xrightarrow{\overline{c_{y_o}}}$$

$$\xrightarrow{\overline{c_{y_o}}} \sum S^{n-1} \xrightarrow{\sum \phi} \sum Y \longrightarrow \dots \tag{4.10}$$

where $\overline{c_{y_o}}$ is induced by the constant map onto the base point $y_o \in Y$. Let Z be an arbitrary based space and construct the long exact sequence of based spaces and groups associated to the sequence 4.10 (see [35, Theorem 3.2.1]):

$$\dots \longrightarrow [\Sigma Y, Z]_* \xrightarrow{\Sigma\phi_*} [S^n, Z]_* \xrightarrow{\overline{c_{y_0}}_*}$$

$$[X, Z]_* \xrightarrow{\overline{\imath}_*} [Y, Z]_* \xrightarrow{\phi_*} [S^{n-1}, Z]_* \,. \tag{4.11}$$

Before proceeding further, we introduce some notation. Firstly, we indicate the free homotopy class of a map, say $f : X \longrightarrow Z$, simply by its representative f; secondly, if in the previous example the spaces X, Z and the map f are based, we indicate the based homotopy class of f by f_*.

Take arbitrarily $f_* \in [X, Z]_*$ and $\alpha \in \pi_n(Z, z_0)$ and define $f_*^\alpha \in [X, Z]_*$ to be the based homotopy class of the composite map

$$X \xrightarrow{p} S^n \vee X \xrightarrow{\alpha \vee f} Z \vee Z \xrightarrow{\nu} Z,$$

where ν is the folding map. We now state the key result that makes possible the result on isotropy and stability we have anounced at the end of the previous section (for its proof, see [16, Corollary 15.5]):

Theorem 4.2.2 *Let $f_*, g_* \in [X, Z]_*$ be given. Then,*

$$\bar{\imath}_*(f_*) = \bar{\imath}_*(g_*) \iff (\exists \alpha \in \pi_n(Z, z_o)) \;\; g_* = f_*^\alpha \;.$$

We wish to extend this theorem to sets of free homotopy classes of maps with $Z = B_{O(n)}$. To this end, we recall another well-known result of Homotopy Theory: since the base point $y_o \in Y \subset X$ is non degenerate (we assume implicitly that we work with well pointed based spaces) and $B_{O(n)}$ is path-connected, there exists an action of $\pi_1(B_{O(n)}, w_o)$ on $[X, B_{O(n)}]_*$ and

$$[X, B_{O_n}]_* / \pi_1(B_{O(n)}, w_o) \cong [X, B_{O_n}]$$

(see [35, Corollary 7.1.13]). Notice that because $\pi_1(B_{O(n)}, w_o) \cong \mathbb{Z}_2 = \{-1, 1\}$ this action becomes particularly simple; we indicate it by writing, for every $f_* \in [X, B_{O_n}]_*$, $-1.f_* = -f_*$. An analysis of the proof of [35, Corollary 7.1.13] shows that the function

$$[X, B_{O_n}]_* \xrightarrow{\bar{\imath}_*} [Y, B_{O_n}]_*$$

is $\pi_1(B_{O(n)})$-equivariant, that is to say, $\bar{\imath}_*(\pm f_*) = * \pm \bar{\imath}_*(f_*)$, for every $f_* \in [X, B_{O_n}]_*$.

With this we can extend Theorem 4.2.2 to the free case: Suppose that $f, g \in [X, B_{O_n}]$ are such that

$$\bar{\imath}_*(f) = \bar{\imath}_*(g) \;. \tag{4.12}$$

Because $B_{O(n)}$ is path connected and the base point $y_o \in X$ is non degenerate, within the free homotopy classes f and g there are based representatives of f and g, respectively; thus, without loss of generality, we may assume that the maps $f, g : X \longrightarrow B_{O(n)}$ are based, that is to say, $f(y_o) = g(y_o) = w_o$. In view of the format of the orbits of $[X, B_{O_n}]_*/\mathbb{Z}_2$, equality 4.12 means that either $\bar{\imath}_*(f_*) = \bar{\imath}_*(g_*)$ (in such case $g_* = f_*^\alpha$) or $\bar{\imath}_*(f_*) = \bar{\imath}_*(-g_*)$ (in such case $-g_* = f_*^\alpha$), for some $\alpha \in \pi_n(Z, z_o)$); hence, $g = f^\alpha$. The converse also holds true, showing that Theorem 4.2.2 can indeed be extended to the free case if $Z = B_{O(n)}$.

Now we can prove the following result.

Theorem 4.2.3 *Let ξ and η be two stably equivalent real n-dimensional bundles on a connected CW-complex with only one top cell X, and suppose that $\xi\otimes\lambda \cong \xi$ for some line bundle λ. Then, $\eta\otimes\lambda \cong \eta$. In particular, the isotropy groups of ξ and η coincide.*

Proof – For the proof the reader is referred to the next diagram

$$
\begin{tikzcd}
 & X \arrow[d, "p"] & \\
 & S^n \vee X \arrow[dl, "1\times\Delta"'] \arrow[dr, "\Delta"] & \\
(S^n \vee X) \times (S^n \vee X) \arrow[d, "(\alpha\vee f)\times(c_0\vee l)"'] & & S^n \vee (X \times X) \arrow[d, "\alpha\vee(f\times l)"] \\
(B_{O_n} \vee B_{O_n}) \times (B_{O_1} \vee B_{O_1}) \arrow[d, "\nu\times\nu"'] & & B_n \vee (B_{O_n} \times B_{O_1}) \arrow[d, "1\vee\psi"] \\
B_{O_n} \times B_{O_1} \arrow[dr, "\psi"'] & & B_{O_n} \vee B_{O_n} \arrow[dl, "\nu"] \\
 & B_{O_n} &
\end{tikzcd}
$$

Assume that $X = Y \sqcup_\phi D^n$; moreover, suppose that ξ, η and λ are classified by $f : X \longrightarrow B_{O(n)}$, $g : X \longrightarrow B_{O(n)}$ and $\ell : X \longrightarrow B_{O(1)}$, respectively. Since ξ and η are stably equivalent and $\dim Y < n$, the restrictions to $\xi|Y$ and $\eta|Y$ are equivalent (see [19, Chapter 9, Theorem 1.5]); hence, according to the free version of Theorem 4.2.2, there exists $\alpha \in \pi_n(B_{O(n)}, w_o)$ such that $g = f^\alpha$; if we indicate by ξ^α the bundle classified by f^α, we can write that

$\eta = \xi^\alpha$. With this notation in mind, we observe that the bundle $(\xi \otimes \lambda)^\alpha$ is classified by the map

$$X \xrightarrow{p} S^n \vee X \xrightarrow{\alpha \vee h} B_{O_n} \vee B_{O_n} \xrightarrow{\nu} B_{O_n}$$

where $h := \psi(f \times \ell)\Delta$ and ψ is the action of $B_{O(1)}$ on $B_{O(n)}$ (see Theorem A.6.6); moreover, the bundle $\xi^\alpha \otimes \lambda$ is classified by the map $\psi(f^\alpha \times \ell)\Delta$.

It is now a straightforward matter to prove that the diagram at the beginning of the proof is commutative. This means that $(\xi \otimes \lambda)^\alpha \cong \xi^\alpha \otimes \lambda$ and therefore,

$$\eta \otimes \lambda \cong \xi^\alpha \otimes \lambda \cong (\xi \otimes \lambda)^\alpha \cong \xi^\alpha \cong \eta \ .$$

Hence,

$$\lambda \in \mathfrak{I}(\xi) \iff \lambda \in \mathfrak{I}(\eta) \ .$$

$\square$

Remark 4.2.4 The condition that the two vector bundles ξ and η of the statement must have the same dimension is essential: indeed, for $n = 2m$, $\xi = m\gamma_1^n \oplus m$ and $\eta = m\gamma_1^n$ are stably equivalent with $\xi \otimes \gamma_1^n \cong \xi$ and $\eta \otimes \gamma_1^n \cong m$.

4.2.2 Vector bundles on real projective spaces

In this section we will study the isotropy group of a real even-dimensional vector bundle over a real projective space $\mathbb{R}P^n$. Since line bundles over $\mathbb{R}P^n$ are classified by $H^1(\mathbb{R}P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2$, these isotropy groups are subgroups of $\mathbb{Z}_2$; one is then tempted to believe that such an analysis is relatively simple but, as we shall see, this is not so!

We begin by reviewing some basic acts about topological K-theory. The reader can find all the necessary definitions in [19] or [22]; here we only recall the following.

Let $\mathfrak{V}(B)$ be the set of all equivalence classes of real vector bundles over a finite CW-complex B; the Whitney sum and tensor product of vector bundles give to this set the structure of a semi-ring; the (Grothendieck) ring $KO(B)$ obtained by symmetrization of $\mathfrak{V}(B)$ is the so-called *real K-ring* of B. If B is based, say with base point b_o, define $\widetilde{KO}(B)$ as the kernel of the homomorphism

$$\iota^! : KO(B) \longrightarrow KO(\{b_o\}) \cong \mathbb{Z}$$

induced by the inclusion map $\iota : \{b_o\} \longrightarrow B$.

The collapsing map $B \longrightarrow \{b_o\}$ shows that

$$KO(B) \cong \widetilde{KO}(B) \oplus \mathbb{Z} \ .$$

It is well-known that there exists a surjection

$$\alpha : \mathfrak{V}(B) \longrightarrow \widetilde{KO}(B) \ , \ \xi \longmapsto \xi - \mathrm{dim}\xi$$

and

$$\xi \sim_S \eta \iff \alpha(\xi) = \alpha(\eta)$$

(that is to say, the stable equivalence classes of real vector bundles over B can be identified with the elements of $\widetilde{KO}(B)$ – see [19, Chapter 9, Theorem 3.8]).

At this point we can describe the reduced KO-ring of a real projective space (see [22, Chapter 4, Proposition 3.12] or [19, Chapter 16, Proposition 12.5]).

Theorem 4.2.5 *The ring $\widetilde{KO}(\mathbb{R}P^n)$ is generated by the element $x = \gamma_1^n - 1$ subject to the relations*

$$x^2 + 2x = 0 \textit{ and } x^{f(n)+1} = 0$$

where $f(n)$ is the number of integers q with $q = 0, 1, 2, 4$ (mod. 8) and $0 < q \leq n$. In particular, the group $\widetilde{KO}(\mathbb{R}P^n)$ is cyclic of order $2^{f(n)}$.

This theorem shows in particular, that if ξ is an r-dimensional real vector bundle over $\mathbb{R}P^n$, there exists an integer s, $0 \leq s < 2^{f(n)}$, such that

$$\xi = s\gamma_1^n + r - s \ .$$

Furthermore, it also shows that the isotropy groups of the trivial vector bundle of dimension $2^{f(n)}$ over $\mathbb{R}P^n$ is not trivial: it contains γ_1^n !

Remark 4.2.6 The integer valued function $f(n)$ is given by $f(n + 8) = f(n) + 4$ and in particular, $f(8q) = 4q$. The following table will be useful later on:

$$\begin{array}{ccccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ f(n) & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\ 2^{f(n)} & 2 & 4 & 4 & 8 & 8 & 8 & 8 & 16 \end{array} \tag{4.13}$$

(see [22, Chapter 3, Remark 3.13]).

After this hiatus we resume the normal course of our discourse with the following result.

Theorem 4.2.7 *Let ξ be a real r-dimensional bundle over $\mathbb{R}P^n$ (with r even[3]) which is stably equivalent to $s\gamma_1^n$ with $0 \leq s < 2^{f(n)}$. Then*

$$\xi \sim_S \xi \otimes \gamma_1^n \iff \textit{either } s = \frac{1}{2}r \textit{ or } s = \frac{1}{2}(r + 2^{f(n)}) \ .$$

Proof – Because of Theorem 4.2.5

$$\xi \oplus 2^{f(n)} = s\gamma_1^n \oplus 2^{f(n)} + r - s$$

and so,

$$\xi \otimes \gamma_1^n \oplus 2^{f(n)} = \gamma_1^n \otimes (\xi \oplus 2^{f(n)}) =$$
$$= \gamma_1^n \otimes (s\gamma_1^n \oplus (2^{f(n)} + r - s)) = s \oplus (2^{f(n)} + r - s)\gamma_1^n \ .$$

Therefore,

$$\xi \oplus 2^{f(n)} = (\xi \otimes \gamma_1^n) \oplus 2^{f(n)} \iff s \equiv 2^{f(n)} + r - s \ (\mathrm{mod}\ 2^{f(n)}) \ .$$

□

We now recall that if B is a CW-complex with dim $B = n < \infty$, and ξ and η are two real r-bundles over B with $r \geq n + 1$, then

$$\xi \sim_S \eta \iff \xi \sim \eta$$

(see [19, Ch.9, Theorem 1.5]). This fact and the previous Theorem have an immediate consequence:

Corollary 4.2.8 *Let ξ be a real r-dimensional bundle over $\mathbb{R}P^n$ (with r even) such that $\xi \sim_S s\gamma_1^n$, for $0 \leq s < 2^{f(n)}$. Then*

$$\xi \sim \xi \otimes \gamma_1^n \iff \textit{either } s = \frac{1}{2}r \textit{ or } s = \frac{1}{2}(r + 2^{f(n)}) \ .$$

If the dimension of ξ is less or equal to n we have a weaker result:

Corollary 4.2.9 *Let ξ be a real r-dimensional bundle over $\mathbb{R}P^n$, with $r \leq n$ even; suppose that $\xi \otimes \gamma_1^n \cong \xi$. Then, either $\xi \sim_S \frac{1}{2}n\gamma_1^n$ or $\xi \sim_S \frac{1}{2}(n+2^{f(n)})\gamma_1^n$.*

[3] See Theorem 4.2.1.

The previous Theorem and its corollaries are significant steps towards the proof of the following interesting and highly non-trivial result:

Theorem 4.2.10 *Let ξ be a real vector bundle of dimension n over $\mathbb{R}P^n$, with n even. Then*

$$\mathfrak{I}(\xi) \cong \begin{cases} \mathbb{Z}_2 \cong \{1, \gamma_1^n\} & \Longleftrightarrow \quad \xi \cong \frac{1}{2}n\gamma_1^n \text{ or } \xi \cong \frac{1}{2}(n + 2^{f(n)})\gamma_1^n \ , \\ 0 & \qquad\quad \text{otherwise} \ . \end{cases}$$

Proof – We are going to give a proof of the "easy cases" of this theorem and we will sketch the argument for the remaining "hard case"; indeed, this part of the proof transcends the scope of these notes, and so the reader is referred to [11, Theorem 1.1].

Suppose that for $s = \frac{1}{2}n$ or $s = \frac{1}{2}(n + 2^{f(n)})$, we can show that there exists a real n-bundle ξ over $\mathbb{R}P^n$ which is stably equivalent to $s\gamma_1^n$ and is such that $\xi \otimes \gamma_1^n \cong \xi$. Then by Theorem 4.2.3 the same will hold true for any n-bundle stably equivalent to ξ, thus proving the Theorem. As we have already noted, the existence of such an n-bundle ξ is clear for $s = \frac{1}{2}n$: just take $\xi = \frac{1}{2}n\gamma_1^n \oplus \frac{1}{2}n$.

The problem is to establish the existence of such a vector bundle whenever $s = \frac{1}{2}(n+2^{f(n)})$. The cases $n = 2$ and $n = 6$ are easy to deal with. In fact, on the one hand we note that in both cases $s = \frac{1}{2}(n + 2^{f(n)}) = n + 1$ (refer back to Table 4.13); on the other hand, we recall that for every positive integer n

$$\tau(\mathbb{R}P^n) \sim_S (n+1)\gamma_1^n$$

(see [19, page 17]). We must show that $\tau(\mathbb{R}P^n) \otimes \gamma_1^n \cong \tau(\mathbb{R}P^n)$. This fact can be established using the results of Section 1.3.3; see in particular the example at the end of that Section. We recall that $\tau(\mathbb{R}P^n)$ is equivalent to the vector bundle obtained as the quotient of the $(\mathbb{Z}_2, GL(n, \mathbb{R}))$-equivariant bundle $\tau(S^n)$ by the action (ϕ_1, ϕ) and that $\tau(\mathbb{R}P^n) \otimes \gamma_1^n$ is the quotient bundle of $\tau(S^n)$ by the action (ϕ_2, ϕ). Now, for $n = 2, 6$ there exists a fibre preserving equivariant map $f : TS^n \longrightarrow TS^n$ given by $f(b, \vec{v}) = (b, b \times \vec{v})$ where $b \times \vec{v}$ is either the usual cross-product in $\mathbb{R}^3$ or the Cayley-product in $\mathbb{R}^7$. This map induces the equivalence $\tau(\mathbb{R}P^n) \otimes \gamma_1^n \cong \tau(\mathbb{R}P^n)$ we are seeking.[4]

[4]This is similar to the fact that the only almost complex spheres are precisely S^2 and S^7 (see [22, Chapter 3]).

To handle the general case, we must introduce a usefull tool, namely *twisted complex structures*. A twisted complex structure on a real vector bundle ξ over a space B is like a complex structure, except that the pure imaginary scalars live in a real euclidean line bundle λ over B, instead of in a constant "imaginary axis $i\mathbb{R}$". Thus in place of $\mathbb{C}$ we define $\mathbb{C}_\lambda$ to be the bundle of fields with underlying real bundle $1 \oplus \lambda$ and whose fibrewise multiplication is determined by setting $v^2 = -1$ for any v in λ with $||v|| = 1$. Like ordinary complex structures, and like the analogous twisted symplectic structures in [5], twisted complex structures have a useful desuspension property (see Proposition 4.2.11 below).

Although a complex structure on ξ involves a scalar action of $\mathbb{C}$ on the fibres of ξ, the usual definition of complex structure concentrates on the action of the pure imaginary scalars: we define a complex structure on ξ to be a fibrewise linear map $J : \xi \longrightarrow \xi$ such that $J^2 = -1$. The analogous definition of twisted complex structures shows immediately why we are interested in them here. Let ξ be a real vector bundle, λ a real euclidean line bundle over the same base, and let ξ_b , λ_b denote the fibres over a point $b \in B$. Then we introduce the following definition. A λ-twisted complex structure on ξ is a fibrewise linear map $J : \lambda \otimes \xi \longrightarrow \xi$ such that $J^2 = -1$; more precisely, for every $b \in B$, $u \in \xi_b$, and $v \in \lambda_b$ such that $||v|| = 1$, we require that $J(v \otimes J(v \otimes u)) = -u$.

Thus if ξ has a λ-twisted complex structure, then $\lambda \otimes \xi$ and ξ are equivalent in a special way.

When λ is trivial a λ-twisted complex structure on ξ is just a complex structure on ξ. Since any line bundle λ is locally trivial, locally a twisted complex structure is the same as a complex structure.

Given a λ-twisted complex structure J on ξ, we get a corresponding fibrewise scalar action of $\mathbb{C}_\lambda$ on ξ, and in particular this gives each fibre of ξ a complex structure. For if we choose a unit vector v in λ_b, we may denote a point in $(1 \oplus \lambda)_b$ by $x \oplus yv$ where x , $y \in \mathbb{R}$, and define the scalar action by

$$(x \oplus yv) \cdot u = xu + yJ(v \otimes u) \ .$$

This is well-defined since if we use $-v$ in place of v the recipe gives

$$(x \oplus (-y)(-v)) \cdot u = xu + (-y)J((-v) \otimes u) = xu + J(v \otimes u) \ .$$

We call this scalar action a $\mathbb{C}_\lambda$-structure and ξ equipped with a $\mathbb{C}_\lambda$-structure is called a $\mathbb{C}_\lambda$-bundle.

The simplest case is $1 \oplus \lambda$, which admits a natural $\mathbb{C}_\lambda$-structure.

As a second example, reconsider the situation of the tangent bundle described in Section 1.3.3. If $n = 2$ (resp. 6), a well defined $\mathbb{C}_{\gamma_1^n}$-structure

$$J : \tau(\mathbb{R}P^n) \otimes \gamma_1^n \longrightarrow \tau(\mathbb{R}P^n)$$

is given on $\tau(\mathbb{R}P^n)$ by the vector product (resp. Cayley number product)

$$J(b, \vec{v}) = (b, b \times \vec{v}) \ .$$

Notice that $J^2 = -1$, as expected.

The Whitney sum of $\mathbb{C}_\lambda$-bundles for a fixed λ is again a $\mathbb{C}_\lambda$-bundle; over a finite-dimensional base space we can "stabilise" $\mathbb{C}_\lambda$-bundles by adding multiples of $\mathbb{C}_\lambda$, in the same sense that complex bundles are stabilised by adding multiples of the trivial bundle $\mathbb{C}$. In particular the following "twisted desuspension" theorem holds true.

Proposition 4.2.11 *Let λ be a real line bundle over a $2m$-dimensional CW-complex B, and let ζ be a $\mathbb{C}_\lambda$-bundle of complex dimension $m + N$ over B, with $N \geq 0$. Then there is a (unique) $\mathbb{C}_\lambda$-bundle η of complex dimension m over B such that ζ and $\eta \oplus N\mathbb{C}_\lambda$ are isomorphic as $\mathbb{C}_\lambda$-bundles.*

Notice that, as in the ordinary case, λ-twisted complex structures are related to non-degenerate skew-symmetric fibrewise maps $\xi \otimes \xi \longrightarrow \lambda$ (we may pass from one to the other by making a choice of euclidean metric on ξ). This shows up the similarity with the twisted symplectic structures in [5].

The next result establishes a connection between the existence of $\mathbb{C}_{\gamma_1^n}$ structures for bundles over the real projective spaces and equivariant bundle theory.

Proposition 4.2.12 *Let ξ be a $(\mathbb{Z}_2, Gl(n, \mathbb{R}))$-bundle over S^n with the actions (ϕ_1, ϕ) and (ϕ_2, ϕ) as at the end of Section 1.3.3. Let us assume that there exists an equivariant bundle autoequivalence f of ξ. Then, if f is an involution, that is to say, $f^2 = -1$, the corresponding quotient bundles have the $\mathbb{C}_{\gamma_1^n}$-structure $J = \hat{f}$.*

Now let us go back to the theorem. We first deal with the case $n = 2$, 4 or 6 mod 8.

Suppose we can show that the trivial bundle $2^{f(n)}$ over $\mathbb{R}P^n$ admits a $\mathbb{C}_{\gamma_1^n}$-structure. By Proposition 4.2.11 then $2^{f(n)}$ is isomorphic as a $\mathbb{C}_{\gamma_1^n}$-bundle to $\eta \oplus \frac{1}{2}(2^{f(n)} - n)$ for some $\mathbb{C}_{\gamma_1^n}$-bundle η of complex dimension $\frac{1}{2}n$. Let ξ be the real n-plane bundle underlying η. Then

$$\xi \oplus \frac{1}{2}(2^{f(n)} - n)\gamma_1^n \oplus \frac{1}{2}(2^{f(n)} - n) \cong 2^{f(n)},$$

and since $\xi \oplus 2^{f(n)} \cong s\gamma_1^n \oplus (n - s + 2^{f(n)})$ for some s, an easy calculation shows that ξ is stably equivalent to $\frac{1}{2}(n + 2^{f(n)})\gamma_1^n$. Furthermore, $\xi \otimes \gamma_1^n \cong \xi$ since ξ admits a $\mathbb{C}_{\gamma_1^n}$-structure.

We aim to show that for $n = 2$, 4 or 6 mod 8, the trivial bundle $2^{f(n)}$ admits a $\mathbb{C}_{\gamma_1^n}$-structure. At this point we could simply quote from Proposition (7.1) of [2]. However, we shall give a self-contained argument using Clifford algebra bundles and modules, and equivariant theory.

We refer to [4] as a good reference for Clifford algebras and modules, and will use their notation here.

First, recall that $2^{f(n)} = d_n$, the dimension of the irreducible real module for the Clifford algebra $Cl_n = Cl(\mathbb{R}^n)$; therefore, $\mathbb{R}^{2^{f(n)}}$ is a real module for Cl_n. Now, if $n = 2 \bmod 4$ (i.e. $n = 2, 6 \bmod 8$), the volume element ω in Cl_{n+1} has positive square, $\omega^2 = 1$, and is central in Cl_{n+1}; moreover, as $n+1$ is odd, ωb belongs to the even subalgebra $Cl^0_{n+1} \cong Cl_n$, and hence acts on the module $\mathbb{R}^{2^{f(n)}}$. This means that on the trivial bundle $(S^n \times \mathbb{R}^{2^{f(n)}}, \mathrm{pr}_1, S^n, \mathbb{R}^{2^{f(n)}}, GL(n, \mathbb{R}))$, we can define the map $f(b, \vec{v}) = (b, \omega b \cdot \vec{v})$, where b is immersed as an odd element of Cl_{n+1}, and the $\cdot$ means the Clifford multiplication. This is an automap of $S^n \times \mathbb{R}^{2^{f(n)}}$, that is linear on the fibre and equivariant as $f(-b, -\vec{v}) = (-b, \omega b \cdot \vec{v})$; furthermore, it is an involution, since

$$f(f(b, \vec{v})) = (b, \omega b \omega b \cdot \vec{v}) = (b, \omega^2 b^2 \cdot \vec{v}) = (b, -\vec{v}) \ .$$

Now, this does not work for $n = 0 \bmod 4$, when $\omega^2 = -1$; anyway, we can solve the case $n = 4 \bmod 8$ by a similar argument, observing that, for these values of n, we have $2^{f(n)} = 2^{f(n+1)}$. Actually, this is always true for $n = 2$, 4, 6 mod 8, and therefore, the following is a unified proof for all these cases.

Simply observe that for these values of n, $\mathbb{R}^{2^{f(n)}} = \mathbb{R}^{f(n+1)}$, and since the second is a module for Cl_{n+1}, every $b \in \mathbb{R}^{n+1}$ acts by Clifford multiplication on $\mathbb{R}^{2^{f(n)}}$ as desired. Then, define the automap $f(b, \vec{v}) = (b, b \cdot \vec{v})$ of $\mathbb{R}P^n \times \mathbb{R}^{2^{f(n)}}$, where again b is view as an element of Cl_{n+1}.

The argument just described can not be generalized to the values of n that are divisible by 8. This follows from the computation of the twisted K-groups performed in Section 7 and 8 of [11] for the projective spaces. On the other hand, one can give a direct proof that holds true in general for n divisible by 4. As stated before, this requires sophisticated tools, and therefore here we give just a sketch of the argument.

Let $n = 4k$ for some integer k, and for any non-negative integers r, s with $r+s \geq 2k$, consider the vector bundle $\zeta = 2r \oplus 2s\gamma_1^n$ over $\mathbb{R}P^{4k}$. Then ζ is the real bundle underlying the complex bundle $r \oplus s\gamma_1^n$. The latter desuspends uniquely to a complex bundle η of complex dimension $2k$, that is to say,

$$\eta \oplus (r + s - 2k) \cong r \oplus s\gamma_1^n.$$

Now, since $2^{f(4k)} = 2^{2k+e}$, where $e = 0, 1$ for even, odd k, if we specialize to the case $s = k + 2^{2k-2+e}$, $r = 0$, we get, for the real bundle ξ underlying η

$$\xi \oplus \frac{1}{2}(2^{f(n)} - n) \cong \frac{1}{2}(2^{f(n)} + n)\gamma_1^n,$$

that is, ξ is stably equivalent to $\frac{1}{2}(2^{f(n)} + n)\gamma_1^n$ as desired.

Observe that η and $\eta \otimes \gamma_1^n$ are stably equivalent and for an odd k they are actually equivalent. This gives at once an easy proof for the case $n = 4 \bmod 8$.

Now, the real problem is to prove that ξ and $\xi \otimes \gamma_1^n$ are equivalent for any k. The proof is based on a desuspension argument for $Spin^c$ structures, similar to the one described before for $\mathbb{C}_\lambda$-bundles.

The idea is that if two bundles are stably equivalent and admit a $Spin^c$ structure, and if the stable equivalence preserves in some sense the $Spin^c$ structure, then the obstruction to desuspend the equivalence lies in the $\mathbb{Z}_2$-equivariant K theory of the base space. The precise statement (whose proof can be found in [11]) is the following.

Proposition 4.2.13 *Let X be a connected closed manifold of even dimension $2m$, with non trivial first Stiefel-Withney class $w_1(X)$ and such that $w_2(X)$ is the reduction of an integral class in $H^2(X, \mathbb{Z})$. Let ξ and ξ' be two $2m$-dimensional $Spin^c$ bundles over X, and suppose that there exists a stable equivalence $\xi \sim_f \xi'$ under which the $Spin^c$ structures correspond. Then, f desuspends to an equivalence $\xi \cong \xi'$ $\iff$ the Euler classes of the $\mathbb{Z}_2$-equivariant bundles associated to ξ and ξ' are equal in $K^0_{\mathbb{Z}_2}(X)$.*

Now, the complex structures on η and $\eta \otimes \gamma_1^n$, define the desired *Spin*^{c} structure on ξ and $\xi \otimes \gamma_1^n$. Since ξ and $\xi \otimes \gamma_1^n$ are stably equivalent, it follows that (with the orientation given by their *Spin*^{c} structures) they are stably oriented equivalent. Moreover, by Corollary 9.3 of [11], there exists a stably oriented equivalence between them that preserves their *Spin*^{c} structures.

There remains to prove the equality of Euler classes; this is more technical, and must be done by explicit computation. The desired results are in Lemma 10.6 of [11]. $\square$

Chapter 5

Homotopy and fundamental equivalence

5.1 Introductory material

In Chapter 3 we introduced the concept of *fundamental equivalence* between two principal G-bundles ξ and ξ' over a space B; we arrived at this notion through three different routes, namely by comparing: (i) the fundamental bundles associated to ξ and ξ', (ii) the gauge groups of ξ and ξ' as subgroups of a common local group, and (iii) the transition functions of ξ and ξ. In Chapter 4 we tried to gain some insight about fundamental equivalence between smooth vector bundles; we discovered that if we select and fix a good cover for B, say $\mathfrak{U} = \{U_i | i \in J\}$, then there is a bijection between the set of all classes of fundamentally equivalent principal $O(n)$-bundles (resp. $U(n)$-bundles) over B and the set of all conjugacy classes of all the gauge groups of such bundles (viewed as subgroups of the local gauge group $\mathcal{L} = \prod_{i \in J} Map(U_i, G)$). In neither of these two chapters we made a systematic use of the homotopy classification of bundles; we do this in the present chapter.

In order to develop this chapter we need to recall some properties of classifying spaces; the proof of these properties can be found in Appendix A. We first recall that the construction of classifying spaces is functorial; furthermore, we need the following properties: (i) the classifying space B_G of a topological group G is the base space of a principal G-bundle $\xi_G = (E_G, p_G, B_G, G)$ with contractible total space E_G; (ii) because the centre ZG

of the group G is abelian, the classifying space B_{ZG} is a topological abelian group with multiplication ψ (Proposition A.5.2); (iii) the free action r of ZG on G obtained by restricting the multiplication of G defines a free action

$$B(r) : B_G \times B_{ZG} \longrightarrow B_G\ .$$

(see Theorems A.6.6 and A.6.7). Finally, we make the following observation. Let $I(G) = G/ZG$ be the group of inner automorphisms of G. If we apply the functor

$$\mathcal{B} : TopGr \longrightarrow Top\ ,\ G \longmapsto B_G$$

to the central sequence

$$0 \longrightarrow ZG \xrightarrow{\ i\ } G \xrightarrow{\ \pi\ } I(G) \longrightarrow 0$$

we obtain the exact sequence of based spaces and maps (Theorem A.6.1)

$$0 \longrightarrow B_{ZG} \xrightarrow{\ B_i\ } B_G \xrightarrow{\ B_\pi\ } B_{I(G)} \longrightarrow 0. \tag{5.1}$$

Remark 5.1.1 For technical reasons we now assume that the generic structural group G of all bundles considered in this chapter are **compact**; we also observe that since our version of the Milgram-Steenrod construction of classifying spaces is done within the framework of weak-Hausdorff k-spaces, the group G is automatically Hausdorff, because all k-spaces are T_1. We also assume that the identity element u_G of G is non-degenerate. Last, but not least, we suppose that the pair (G, ZG) is a ZG-equivariant closed cofibration.

The exact sequence 5.1 yields an exact sequence of sets or groups of homotopy classes of maps as follows. Because of the assumptions made in Remark 5.1.1, the 4-tuple

$$(B_G, B_\pi, B_{I(G)}, B_{ZG})$$

is a principal B_{ZG}-bundle by Theorem A.7.2; moreover, because of Remark A.7.3, the map $B_\pi : B_G \longrightarrow B_{I(G)}$ is a Hurewicz fibration with fibre B_{ZG}. This fibration yields, for every based space Y, an exact sequence of based sets and groups

$$\ldots \longrightarrow [B, \Omega B_{ZG}]_* \xrightarrow{\ \Omega B_{i*}\ } [B, \Omega B_G]_* \xrightarrow{\ \Omega B_{\pi*}\ } [B, \Omega B_{I(G)}]_* \xrightarrow{\ j_*\ }$$

$$\xrightarrow{j_*} [B, B_{ZG}]_* \xrightarrow{B_{i*}} [B, B_G]_* \xrightarrow{B_{\pi*}} [B, B_{I(G)}]_* \tag{5.2}$$

(see [35, Theorem 3.1.4]). Note that if B is a non-based space and $Y = B \sqcup \{*\}$ is the disjoint union of B and a singleton space $\{*\}$, then $[Y, \; - \;]_* = [B, \; - \;]$. With this stratagem we obtain the free version of the exact sequence 5.2:

$$\ldots \longrightarrow [B, \Omega B_{ZG}] \xrightarrow{\Omega B_{i*}} [B, \Omega B_G] \xrightarrow{\Omega B_{\pi*}} [B, \Omega B_{I(G)}] \xrightarrow{j_*}$$

$$\xrightarrow{j_*} [B, B_{ZG}] \xrightarrow{B_{i*}} [B, B_G] \xrightarrow{B_{\pi*}} [B, B_{I(G)}] \tag{5.3}$$

5.2 Conjugacy classes of gauge groups

Let B be a space with a fixed good cover $\mathfrak{U} = \{U_i | i \in J\}$; according to Lemma 3.1.7, every principal G-bundle over B is locally trivial over $\mathfrak{U}$ and hence, we can subdivide the set of gauge groups of all principal G-bundles over B into conjugacy classes $\mathfrak{C}\mathcal{G}(\xi)$, where $\mathcal{G}(\xi)$ is the gauge group of the principal G-bundle $\xi = (E, p, B, G)$; we denote by $\mathfrak{C}(B, G)$ the set of all these conjugacy classes.

Let us assume that ξ is classified by a map $f : B \longrightarrow B_G$ and satisfies conditions [C1] and [C2] introduced in Sections 2.1 and 3.1 respectively, but which we reproduce here for the reader's benefit:
[C1] $(\forall b_o \in B)\ \eta : \mathcal{G}(\xi) \longrightarrow G\ ,\ f \mapsto f|p^{-1}(b_o)(u_G)$ is a surjection ;
[C2] $(\forall i \in J)(\forall \bar{h}_{ii} : U_i \longrightarrow I(G))(\exists h_i : U_i \longrightarrow G)\ \bar{h}_i = \pi h_i$;
then, the following theorem is an immediate consequence of Theorem 3.1.4:

Theorem 5.2.1 *The function*

$$\Xi : \mathfrak{C}(B, G) \longrightarrow [B, B_{I(G)}]$$

$$\mathfrak{C}\mathcal{G}(\xi) \longmapsto [B_\pi f]$$

is injective.

The next result (which also follows from Theorem 3.1.4) is more interesting:

Theorem 5.2.2 *Suppose that ξ is classified by a map $f : B \longrightarrow B_G$ and satisfies conditions* [C1] *and* [C2]*; moreover, assume that one of the following hipotheses holds true:*

1. *the set $[B, B_{I(G)}]$ has only one element;*
2. *ZG is trivial ;*
3. *the sequence of topological groups*

$$0 \longrightarrow ZG \xrightarrow{i} G \xrightarrow{\pi} I(G) \longrightarrow 0$$

   splits ;

4. *B is the (reduced) suspension of a simply connected space X and ZG is discrete.*

Then , $\Xi : \mathfrak{C}(B, G) \longrightarrow [B, B_{I(G)}]$ is bijective.

Proof – The stated conclusion is an immediate consequence of either hypotheses 1. or 2.

Assume 3. to be valid: a section $\sigma : I(G) \longrightarrow G$ produces a map B_σ such that $B_\pi B_\sigma = 1_{B_{I(G)}}$.

Now suppose that 4. is true. Since ZG is a discrete subgroup of G, the projection map $\pi : G \longrightarrow I(G)$ is a covering projection; hence, every map from a simply connected space X into $I(G)$ can be lifted to G over π. This means that $\pi_* : [X, G] \longrightarrow [X, I(G)]$ is surjective. Now consider the following commutative diagram:

$$\begin{tikzcd}
{[\sum X, B_G]} \arrow[r, "B_{\pi *}"] \arrow[d] & {[\sum X, B_{I(G)}]} \arrow[d] \\
{[X, \Omega B_G]} \arrow[r, "\Omega B_{\pi *}"] \arrow[d] & {[X, \Omega B_{I(G)}]} \arrow[d] \\
{[X, G]} \arrow[r, "\pi_*"'] & {[X, I(G)]}
\end{tikzcd}$$

where the two upper vertical maps are given by adjointness (so they are bijective) and the other two vertical maps are induced by the homotopy equivalences $\Omega B_G \cong G$ and $\Omega B_{I(G)} \cong I(G)$. This diagram shows that $B_{\pi *}$ is surjective, that is to say, every map $\sum X \longrightarrow B_{I(G)}$ has a lift over B_π. $\Box$

The last consequence of Theorem 3.1.4 which we wish to state as a theorem is the following:

Theorem 5.2.3 *Let $B = \sum X$ satisfying condition* [C2] *and suppose that G is path-connected. Also, let τ be the trivial principal G-bundle over B. Then the gauge groups $\mathcal{G}(\xi)$ and $\mathcal{G}(\tau)$ are conjugate $\iff$ ξ admits a reduction of its strucure group G to ZG.*

Proof – Consider the tail end of the exact sequence 5.3

$$[B, B_{ZG}] \xrightarrow{B_{i*}} [B, B_G] \xrightarrow{B_{\pi *}} [B, B_{I(G)}]$$

and notice that because B is an associative COH-space, this is an exact sequence of groups (see [35, Theorem 1.2.8]). The result now follows from Theorem 3.1.4 and [19, Chapter 6, Theorem 5.1]. $\Box$

We give now some examples. Suppose that $B = S^4$ and $G = SU(2) \cong S^3$. Since $ZSU(2) \cong \mathbb{Z}_2$ is discrete, we conclude from Theorem 5.2.2 that

$$\mathfrak{C}(S^4, SU(2)) \cong [S^4, B_{I(SU(2))}] \cong \pi_3(\mathbb{R}P^3) \cong \mathbb{Z} \ .$$

Hence, there are infinitely many conjugacy classes of gauge groups of principal $SU(2)$ bundles over S^4. It is worth noticing that because

$$\mathfrak{E}_{SU(2)}(S^4) \cong [S^4, B_{SU(2)}] \cong \mathbb{Z}$$

we can say that if ξ and ξ' are two principal $SU(2)$-bundles over S^4 then

$$\xi \cong \xi' \iff \mathcal{G}(\xi) \sim_C \mathcal{G}(\xi') \ .$$

The reader should recall that in general, equivalence of principal bundles is not equivalent to neither isomorphism or conjugacy of the corresponding gauge groups (see the example given in section 3.2).

In the next example we take $B = S^2$ and $G = SU(2) \times S^1$. Because the homomorphism

$$B_{\pi *} : [S^2, B_G] \cong \mathbb{Z} \longrightarrow [S^2, B_{I(G)}] \cong \mathbb{Z}_2$$

is trivial, the set $\mathfrak{C}(S^2, SU(2) \times S^2)$ has only one element and hence, by Theorem 5.2.3, any principal $SU(2) \times S^1$-bundle ξ over S^2 has a reduction to $Z(SU(2) \times S^1) \cong \mathbb{Z}_2 \times S^1$. In particular, $\mathcal{G}(\xi) \cong Map(S^2, SU(2) \times S^1)$.

5.3 Fundamental action and classifying spaces

The observations made before imply that, for every space B, the based set $[B, B_{ZG}]$ is an abelian group with multiplication

$$\psi_* : [B, B_{ZG}] \times [B, B_{ZG}] \longrightarrow [B, B_{ZG}] \ , \ ([\ell_1], [\ell_2]) \longmapsto [\psi(\ell_1 \times \ell_2)\Delta]$$

(where Δ is the diagonal map). In what follows we shall indicate the homotopy classes by any of their representatives and hence, the multiplication ψ_* will be written simply as

$$\psi_*(\ell_1, \ell_2) = \psi(\ell_1 \times \ell_2)\Delta \ .$$

It is easy to see that the there is an action $B(r)_*$ of the group $[B, B_{ZG}]$ on the set $[B, B_G]$ given by

$$B(r)_*(f, \ell) = B(r)(f \times \ell)\Delta \ .$$

In the sequel this action will be known as the *fundamental action* (of the group $[B, B_{ZG}]$ on $[B, B_G]$).

The next proposition shows that $B(r)_*$ is actually the action introduced in Lemma 3.1.3; a preparatory lemma is necessary.

Lemma 5.3.1 *Let $\xi = (E, p, B, G)$ and $\xi' = (E', p', B', G')$ be principal bundles; let $\mu : G \longrightarrow G'$ be a group homomorphism and let $E_\mu : E \longrightarrow E'$ be a μ-equivariant map. Then, there exists a common local trivialization of the two bundles such that their transition functions are related by the formula $\mu g_{ij} = g'_{ij} B_\mu$, where B_μ is the map induced by E_μ at the base space level.*

Proof – Let $\{U'_i, \phi'_i\}$ be a local trivialization for p'. Then $\{B_\mu^{-1}(U'_i)\}$ is an open covering of B over which we can obtain (by refinement) an open trivialization $\{U_i, \phi_i\}$ of p. The following commutative diagram

$$\begin{tikzcd}
p^{-1}(U_i) \arrow[r, "E_\mu"] & p'^{-1}(U'_i) \\
U_i \times G \arrow[u, "\phi_i"] \arrow[r, "B_\mu \times \mu"'] & U'_i \times G' \arrow[u, "\phi'_i"']
\end{tikzcd}$$

can now be used to define a new family of local homeomorphisms $\{\psi'_i\}$ for p': in fact, $E_\mu\phi_i(x, u_G)$ and $\phi'_i(B_\mu(x), u'_G)$ belong to the same fibre and hence define an element $\hat{g}'$ in G' giving rise to the desired trivialization $\psi'_i(x, g') := \phi'_i(x, g'\hat{g}')$. The definition of the transition functions via the local homeomorphisms $\{\phi_i\}$ and $\{\psi'_i\}$ completes the proof. $\square$

Proposition 5.3.2 *Two principal G-bundles ξ and ξ' over B classified by maps $f, f' : B \longrightarrow B_G$ are fundamentally equivalent $\iff$ $f' = B(r)_*(f, \ell)$ for some ℓ in $[B, B_{ZG}]$.*

Proof – We indicate the set of transition functions of a generic principal G-bundle ξ by $\{g_{ij}^\xi\}$. With this notation in mind, if $f' = B(r)_*(f, \ell)$, then $g_{ij}^{\xi'} = g_{ij}^{\xi_G} B(r)(f \times \ell)\Delta$.

Now apply Lemma 5.3.1 to the following diagram

$$\begin{tikzcd}
G \times ZG \arrow[r, "r"] \arrow[d, "i_G \times i_{ZG}"'] & G \arrow[d, "i_G"] \\
E_G \times E_{ZG} \arrow[r, "E(r)"] \arrow[d, "p_G \times p_{ZG}"'] & E_G \arrow[d, "p_G"] \\
B_G \times B_{ZG} \arrow[r, "B(r)"'] & B_G
\end{tikzcd}$$

to obtain the equality

$$g_{ij}^{\xi'} = s(g_{ij}^{\xi_G} f \times g_{ij}^{\xi_{ZG}} \ell)\Delta = g_{ij}^{\xi} c_{ij}$$

where the maps $c_{ij} = g_{ij}^{\xi_{ZG}} \ell$ take values in ZG; we conclude the proof using Theorem 3.1.4. □

Corollary 5.3.3 *Let ξ be a numerable principal G-bundles over a space B, and λ be a numerable principal ZG-bundle over the same space. Suppose that these bundles are classifyed by the maps f and l, respectively; then the bundle $\xi \odot \lambda$ is classifyed by the map $B(r)_*(f, \ell)$.*

5.4 An exact sequence

It is well-known that if $p : E \longrightarrow B$ is a Hurewicz fibration with fibre F and E contractible, then F and ΩB have the same homotopy type; for the reader's benefit and in order to give detailed proofs of the statements we are going to make later on, we now explain how we arrive at such a homotopy eqivalence.

Consider the diagram

$$\begin{tikzcd}
\Omega B \arrow[r, "j"] & L_p \arrow[r, "\bar{p}"] \arrow[d, "\bar{\epsilon}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}B \arrow[d, "\epsilon_1"] \\
F \arrow[r, "i"'] \arrow[ur, "h"] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

where the square is a pullback, $\mathcal{P}B$ is the space of all paths originating at the base point $x_o \in B$, j is the fibre inclusion, and $h(x) = (x, c_{x_0})$ is a homotopy equivalence whose homotopy inverse is constructed as follows. Take the homotopy

$$H : L_p \times I \longrightarrow B \ , \ ((x, \alpha), t) \mapsto \alpha(1 - t)$$

and lift it to a homotopy $\widetilde{H} : L_p \times I \longrightarrow E$. The map $s = \widetilde{H}(\ - \ , 1)$ takes values in the fibre F and one can prove that sh and hs are homotopic to the appropriate identity maps (see [35, Lemma 3.1.3] for details). Now suppose that E is contractible, and take a based contraction $K : E \times I \longrightarrow E$ to the base point $x_o \in E$ (we assume that x_o is non-degenerate). Then ΩB is a weak deformation retract of L_p: in fact, the homotopy $K'((x,\alpha),t) = (K(x,\frac{1-t}{1+t}),\beta_t)$ of L_p, where

$$\beta_t(s) = \begin{cases} \alpha\left(\frac{2s}{1+t}\right) & 0 \leq s \leq \frac{1+t}{2} \\ pK\left(x, \frac{2s-1-t}{1+t}\right) & \frac{1+t}{2} \leq s \leq 1, \end{cases}$$

yields a retraction $\rho((x,\alpha)) = K'((x,\alpha),0)$; finally, we observe that $\delta = sj : \Omega B \longrightarrow F$ is a homotopy equivalence with homotopy inverse ρh.

In the present context, there is a further point which deserves our attention. Suppose that F is an H-space with a multiplication $\nu : F \times F \longrightarrow F$ which has a strict identity. Moreover, assume that there is an action $\psi : F \times E \longrightarrow E$ such that the following diagram commutes:

$$\begin{tikzcd} F \times F \arrow[r, "\nu"] \arrow[d] & F \arrow[d] \\ F \times E \arrow[r, "\psi"] \arrow[d, "\mathrm{pr}_2"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

Then, under these conditions, the map $\delta : \Omega B \longrightarrow F$ is an H-map (see [6, Lemma 3.2]).[1]

Because $B_\pi : B_G \longrightarrow B_{I(G)}$ and $p_{I(G)} : E_{I(G)} \longrightarrow B_{I(G)}$ are Hurewicz fibrations with fibres B_{ZG} and $I(G)$, respectively, we have the following H-space preserving homotopy equivalences:

$$\delta : \Omega B_{I(G)} \longrightarrow B_{ZG} \ , \ \delta_{I(G)} : \Omega B_{I(G)} \longrightarrow I(G) \ .$$

[1] Recall that ΩB is an H-space with multiplication given by composition of loops (with strict unit).

We can summarize the previous information in the following double pullback diagram:

$$\begin{tikzcd}
\mathcal{P}B_{I(G)} \arrow[d] & L_{B_p} \arrow[l] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] & \Omega B_{I(G)} \arrow[l, "j_\pi"'] \arrow[d, "\delta"] & = & \Omega B_{I(G)} \arrow[r, "j_{I(G)}"] \arrow[d, "\delta_{I(G)}"] & L_{p_{I(G)}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{P}B_{I(G)} \arrow[d] \\
B_{I(G)} & B_G \arrow[l, "B_\pi"] & B_{ZG} \arrow[l] \arrow[ul, "h_\pi"'] & & I(G) \arrow[r] \arrow[ur, "h_{I(G)}"'] & E_{I(G)} \arrow[r, "p_{I(G)}"'] & B_{I(G)}
\end{tikzcd}$$

The retraction $\rho_{I(G)} : L_{p_{I(G)}} \longrightarrow \Omega B_{I(G)}$ and $s_\pi : L_{B_\pi} \longrightarrow B_{ZG}$ (homotopy inverse of h_π) combine to give a homomorphism

$$v = s_\pi j_\pi \rho_{I(G)} h_{I(G)} : I(G) \longrightarrow B_{ZG} \ .$$

Going back to the Puppe sequence 5.3 and using the appropriate H-homotopy equivalences defined before, we obtain the following exact sequence of based spaces and groups:

$$\begin{gathered}
\ldots \longrightarrow [B, ZG] \xrightarrow{i_*} [B, G] \xrightarrow{\pi_*} [B, I(G)] \xrightarrow{v_*} \\
\xrightarrow{v_*} [B, B_{ZG}] \xrightarrow{B_{i*}} [B, B_G] \xrightarrow{B_{\pi*}} [B, B_{I(G)}] \ .
\end{gathered} \tag{5.4}$$

Remark 5.4.1 From this point on, the space B is taken to be paracompact.

Because of the Classification Theorem 1.3.5, the function

$$B_{\pi*} : [B, B_G] \longrightarrow [B, B_{I(G)}]$$

establishes the transition from principal G-bundles over B to their associated fundamental bundles (see the definition in Section 2.1). This, together with Theorem 3.1.4, implies the following result:

Proposition 5.4.2 *Take arbitrarily $f, f' \in [B, B_G]$. Then*

$$B_{\pi*}(f) = B_{\pi*}(f') \iff (\exists \ell \in [B, B_{I(G)}]) f = B(r)_*(f', \ell) \ .$$

Using the language introduced in Chapter 3, we can paraphrase the previous proposition by saying that two principal G-bundles ξ and ξ' over B, classified respectively by $f, f' : B \longrightarrow B_G$ are *fundamentally equivalent* $\iff B_{\pi*}(f) = B_{\pi*}(f')$.

5.5 The isotropy of the fundamental action

From various examples given in Chapter 4 we conclude that the fundamental action $B(r)_*$ is not free; the obstruction to freenes is given by eventual non-trivial isotropy groups. Let us discuss this point in a more general context than that of Chapter 4. We recall from Lemma 3.1.3 that there is an action

$$\odot : \mathfrak{E}_G(B) \times \mathfrak{E}_{ZG}(B) \longrightarrow \mathfrak{E}_G(B) \ , \ (\xi, \lambda) \longmapsto \xi \odot \lambda \ ;$$

then, the isotropy group of a principal G-bundle ξ over B is given by

$$\mathfrak{I}(\xi) = \{\lambda \in \mathfrak{E}_{ZG}(B) \mid \xi \odot \lambda = \xi\} \ ;$$

on the other hand, if ξ is classified by $f : B \longrightarrow B_G$, from what we did before, we conclude that

$$\mathfrak{I}(\xi) \cong \{\ell \in [B, B_{ZG}] \mid B(r)_*(f, \ell) = f\} \ .$$

Proposition 5.5.1 *Let τ be a trivial G-bundle over B; then*

$$\mathfrak{I}(\tau) \cong \ker \ (B_{i*} : [B, B_{ZG}] \longrightarrow [B, B_G]) \ .$$

Proof – The bundle τ is classified by a constant map $c_o : B \longrightarrow B_G$; notice that $B(r)(c_o, \ell)\Delta = B_i \ell$: in fact, these two functions produce equivalent pullback bundles from the universal G-bundle ξ_G (this can be seen using Lemma 5.3.1 and comparing the transition functions of the pullback bundles). □

The next result shows that the isotropy group of a principal G-bundle depends on the equivalence class of fundamental equivalence of ξ; more precisely,

Proposition 5.5.2 *If ξ and ξ' are fundamentally equivalent then $\mathfrak{I}(\xi) = \mathfrak{I}(\xi')$.*

Now suppose that the base space B is a suspension, $B = \Sigma A$. In this case, the classifying sequence 5.4 is an exact sequence of groups and group homomorphisms; hence, we have the following:

Proposition 5.5.3 *The isotropy group of a principal G-bundle over a suspension $B = \Sigma A$ is isomorphic* ker B_{i*} *(that is to say, it coincides with the isotropy group of a trivial G-bundle over B).*

Proof – Let ξ be a real vector bundle represented by a map $f : B \longrightarrow B_G$; let $\lambda \in \mathfrak{I}(\xi)$ be represented by $\ell : B \longrightarrow B_{ZG}$. According to the definitions, $\xi \odot \lambda$ is represented by $\psi(f \times \ell)\Delta$. On the other hand, $B = \Sigma A$ is a COH-space with comultiplication $\nu : \Sigma A \longrightarrow \Sigma A \vee \Sigma A$ and thus, $[\Sigma A, B_G]$ is a group. The product $f \bullet B_i\ell$ is given by $\sigma(f \vee B_i\ell)\nu$, where σ is the folding map. But

$$\psi(f \times \ell)\Delta \;\sim\; \sigma(f \vee B_i\ell)\nu$$

(cfr. Theorem 4.2.3). Hence, $f \bullet B_i\ell \sim f$ and so, $B_i\ell$ is homotopic to the constant map, that is to say, $\lambda \in \mathfrak{I}(\tau)$ (see Proposition 5.5.1). $\square$

Appendix A

The Milgram-Steenrod classifying space

The contents of this appendix are taken freely from our paper *The Milgram-Steenrod construction of classifying spaces for topological groups* [36].

A classifying space for a topological group G is the base space of a principal G bundle (E_G, p_G, B_G) with contractible total space E_G. Classifying spaces have been constructed using several different methods; amongst the most popular constructions we recall the *Milnor construction*, the *Dold-Lashof-Fuchs construction* (see [34] for an up-dated version) and the *Milgram-Steenrod contruction*. The latter was introduced in [28] and then, taken up and reformulated by N. Steenrod (see [42]). The objective of this appendix is to review the contents of Steenrod's paper in the light of more streamlined techniques and, as a consequence, obtain new results not contained in [42] about the *classifying space functor* $\mathcal{B}$ which transforms a topological group G into a space B_G. In fact, we shall prove that $\mathcal{B}$ is exact and preserves products, normality, closed inclusions, proclusions and closed cofibrations; moreover, we shall see that if Z is a central subgroup of a topological group G such that the pair (G, Z) is a Z-equivariant closed cofibration, the map $B_q : B_G \longrightarrow B_{G/Z}$ induced by the quotient map $q : G \longrightarrow G/Z$ gives rise to a locally trivial principal B_Z-bundle.

Finally observe that, while in his version of Milgram's construction, Steenrod selected as his basic category the category $\mathcal{CG}$ of compactly generated topological spaces defined in [41], here we are working in the more convenient (and larger) category $wHk(Top)$ of weak Hausdorff k-spaces. As we shall see in the sequel, this gives many advantages and more elegant proofs.

A.1 Enlargement of actions and contractions

Suppose that we are given a closed subspace $A \subset X \in wHk(Top)$ and an action $\phi : A \times G \longrightarrow A$ of a *weak Hausdorff k-group* G (that is, an object of $TopGr_* \cap wHk(Top_*)$). Then $A \times G$, $X \times G \in wHk(Top)$ and the pushout space $\overline{X}$ of the diagram

$$X \times G \xleftarrow{i \times 1_G} A \times G \xrightarrow{\phi} A$$

is also a weak Hausdorff k-space (see [15, Corollary A.1.4]). Going quickly over the construction of the pushout, we recall that one produces a commutative diagram

$$\begin{tikzcd} A \times G \arrow[r, "\phi"] \arrow[d, "i \times 1_G"'] & A \arrow[d, "\overline{i \times 1_G}"] \\ X \times G \arrow[r, "\bar{\phi}"'] & \overline{X} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in which the maps $\bar{\phi}$ and $\overline{i \times 1_G}$ are the compositions of the appropriate inclusion maps with a proclusion $p : A \sqcup (X \times G) \longrightarrow \overline{X}$. Notice that because $\phi : A \times G \longrightarrow A$ is an epimorphism, the elements of $\overline{X}$ can be represented by classes $[x, g]$ given by the equivalence relation on $X \times G$ obtained by extending ϕ:

$$(x,g)R(x',g') \;\Leftrightarrow\; \begin{cases} (x,g) = (x',g') & \text{if } (x,g),(x',g') \in (X \setminus A) \times G \\ xg = x'g' & \text{if } (x,g),(x',g') \in A \times G\ ; \end{cases}$$

then $\overline{X}$ is the quotient space $(X \times G)/R$ and $\bar{\phi}$ is the quotient map.

Proposition A.1.1 *The space $\overline{X}$ is a G-space; the action $\Phi : \overline{X} \times G \longrightarrow \overline{X}$ extends ϕ.*

Proof – The functor $- \times G$ preserves colimits (see [15, Appendix A.1]) and therefore, $\overline{X} \times G$ is the pushout of the diagram

$$(X \times G) \times G \xleftarrow{(i \times 1_G) \times 1_G} (A \times G) \times G \xrightarrow{\phi \times 1_G} A \times G$$

Let $\tau_G : G \times G \longrightarrow G$ be the multiplication of G. The maps

$$\bar{\phi}(1_X \times \tau_G) : (X \times G) \times G \longrightarrow \overline{X}$$

$$(\overline{i \times 1_G})\phi : A \times G \longrightarrow \overline{X}$$

are such that

$$(\overline{i \times 1_G})\phi(\phi \times 1_G) = \bar{\phi}(1_X \times \tau_G)(i \times 1_G \times 1_G)$$

and therefore, by the universal property of pushouts, there exists a unique map

$$\Phi : \overline{X} \times G \longrightarrow \overline{X}$$

rendering commutative the resulting diagram. It is a straightforward matter to prove that Φ is an action defined by $\Phi([x, g], g') = [x, gg']$, for every $[x, g] \in \overline{X}$ and every $g' \in G$. The action Φ gives rise to an equivalence relation $\overline{R}$ on $\overline{X}$. □

The adjunction space $\overline{X}$ is the *enlargement of the G-action* of A (to X).

Remark A.1.2 - To carry on this construction within the category $\mathcal{CG}$ of compactly generated topological spaces we must require that both pairs (X, A) and $(G, \{u_G\})$ are closed cofibrations; this is another instance where there is a distinct advantage to work in $wHk(Top)$.

Proposition A.1.3 *The space X is homeomorphic to a closed subset of $\overline{X}$.*

Proof – The restriction to $X \times \{u_G\}$ of the map

$$\bar{\phi} : X \times G \xrightarrow{i_2} A \sqcup (X \times G) \xrightarrow{p} \overline{X}$$

is induced by the homeomorphism $\phi(-, u_G) : A \longrightarrow A$; hence it is a homeomorphism onto the closed set $\bar{\phi}(X \times \{u_G\})$ of $\overline{X}$. □

Our objective now is to clarify the question whever the inclusion $(\overline{X}, X)$ is a closed cofibration; to do this we shall give an auxiliary result which relates relative homeomorphisms and adjunction spaces. We first recall that a map of pairs $(\phi', \phi) : (X, A) \longrightarrow (Y, B)$ in $wHk(Top)$ is a *relative homeomorphism* if A is closed in X, the map ϕ' is proclusive and induces a homeomorphism $(X \setminus A) \longrightarrow (Y \setminus B)$.

Lemma A.1.4 *A map of pairs* $(\phi', \phi) : (X, A) \longrightarrow (Y, B)$ *is a relative homeomorphism* $\iff$ $Y \cong B \sqcup_\phi X$*, with* ϕ *proclusive.*

Proof – Let $(\phi', \phi) : (X, A) \longrightarrow (Y, B)$ be a relative homeomorphism and let Z be the adjunction space of X to B via ϕ. Consider the following commutative diagram where the square is a pushout:

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \arrow[ddr] & \\
X \arrow[r, "\bar{\phi}"] \arrow[drr, "\phi'"] & Z \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "f"] & \\
 & & Y
\end{tikzcd}$$

By the universal property of pushouts and the definition of relative homeomorphism, the sets $Z \cong Y$; thus, the map f is bijective. Since $\bar{\phi} = f^{-1}\phi'$ and ϕ' is a proclusion, f^{-1} is continuous and hence, $Z \cong Y$ as topological spaces.

The converse is given by [15, Proposition A.4.8 (iii)] . □

Proposition A.1.5 *Suppose that* (X, A) *and* $(G, \{u_G\})$ *are closed cofibrations in* $wHk(Top)$*; then* $(\overline{X}, X)$ *is a closed cofibration in* $wHk(Top)$*.*

Proof – The following pairs are closed cofibrations: $(X \times G, A \times G)$, $(X \times G, X \times \{u_G\} \cup A \times G)$ and $(\overline{X}, A)$ (see [15, Proposition A.4.2 (iii), (iv) and Proposition A.4.8 (ii)] , respectively). Now apply Lemma A.1.4 to the map of pairs

$$(X \times G, X \times \{u_G\} \cup A \times G) \longrightarrow (\overline{X}, X)$$

to end the proof. □

We conclude our observations about the enlargement of actions noticing that the universal property of pushouts proves easily that we can interpret the enlargement of an action as a functor:

Proposition A.1.6 *For every $\vartheta \in (TopGr \cap wHk(Top))(G,H)$ and every map of pairs $(f,u) : (X,A) \longrightarrow (Y,B)$ in $wHk(Top)$ such that A is a G-space, B is an H-space and u is ϑ-equivariant, then $f\times\vartheta : X\times G \longrightarrow Y\times H$ is ϑ-equivariant with respect to the trivial actions $1_X\times\tau_G$, $1_Y\times\tau_H$ and induces a ϑ-equivariant quotient mapping $\overline{f} : \overline{X} \longrightarrow \overline{Y}$.*

We note explicitly that

$$(\forall [x,g] \in \overline{X})\ ,\ \overline{f}([x,g]) = [f(x), \vartheta(g)]\ .$$

Now we start to talk about contractions. Regard the unit interval $I = [0,1]$ as a topological monoid under ordinary multiplication and define an *I-action* as a map $h' : X \times I \longrightarrow X$ such that $h'(-,1) = 1_X$ and $h'(x,tt') = h'(h'(x,t),t')$, for every $x \in X$ and $t,t' \in I$. Let $X \in wHk(Top)$ be based, with base point x_0; a *contraction of X to x_0* is an I-action $h' : X\times I \longrightarrow X$ that factors through the smash product $h' : X \times I \xrightarrow{p} X \wedge I \xrightarrow{h} X$ (we take $0 \in I$ as base point). In particular, the ordinary multiplication on I is a contraction of I to 0. Denote the base point of $X \wedge I$ by $\overline{x_0}$; then the trivial action

$$\tau : (X \times I) \times I \longrightarrow X \times I\ ,\ ((x,t),s) \longmapsto (x,ts)$$

induces a contraction of $X \wedge I$ to $\overline{x_0}$, called *canonical contraction*. In fact, τ factors through $X \vee I$ and gives rise to a map

$$c' : (X \wedge I) \times I \longrightarrow X \wedge I\ ,\ (x \wedge t, s) \longmapsto x \wedge ts$$

which is the desired contraction.

Suppose that we are given $x_0 \in A \subset X$ (with A closed, $X \in wHk(Top)$) and a contraction $h' = hp : A \times I \longrightarrow A$ of A to x_0. The adjunction space $\widetilde{X} := A \sqcup_h (X \wedge I) \in wHk(Top)$ is the *enlargement to X of the contraction*[1] $h' : A \times I \longrightarrow A$. Let $\widetilde{x_0}$ be the base point of $\widetilde{X}$.

Arguments similar to those used for the proofs of propositions A.1.1, A.1.3, A.1.5 and A.1.6 show the following results:

Proposition A.1.7 *The space $\widetilde{X}$ has a contraction $\widetilde{h}' : \widetilde{X} \times I \longrightarrow \widetilde{X}$ that extends the contraction $h' : A \times I \longrightarrow A$.*

Proposition A.1.8 *The space X is homeomorphic to a closed subset of $\widetilde{X}$.*

[1] If we stay in the category $\mathcal{CG}$, we must require that (X,A), $(X,\{x_0\})$ and $(A,\{x_0\})$ are n.d.r.'s in order to guarantee that $\widetilde{X} \in \mathcal{CG}$ (see [42, Lemma 3.3]).

Proposition A.1.9 *If the pairs*

$$(X, A)\ ,\ (X, \{x_0\}) \text{ and } (A, \{x_0\})$$

of $wHk(Top)$ are closed cofibrations, then

$$(\widetilde{X}, A)\ ,\ (\widetilde{X}, x_0) \text{ and } (\widetilde{X}, X)$$

are closed cofibrations in $wHk(Top)$.

Proposition A.1.10 *Let $(f, u) : (X, A) \longrightarrow (Y, B)$ be a based map of pairs of weak Hausdorff based k-spaces and let $h' : A \times I \longrightarrow A$, $k' : B \times I \longrightarrow B$ be contractions such that $k(u \wedge 1_I) = uh$. Then there exists a unique based map $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ which commutes with the extensions of the contractions h' and k' .*

The last proposition shows that we can view the process of enlargement of contractions functorially.

A.2 Enlargement of inclusions, proclusions, cofibrations

In this section we prove that the functorial processes of enlargement of actions and contractions preserve closed inclusions, proclusions and closed cofibrations.

Proposition A.2.1 *Let $X, Y \in wHk(Top)$ with closed subsets $A \subset X$ and $B \subset Y$; the corresponding inclusions are indicated by*

$$i : A \longrightarrow X\ ,\ j : B \longrightarrow Y\ .$$

Let G, H be weak Hausdorff k-groups acting on A and B respectively, with actions

$$\phi : X \times G \longrightarrow X$$

and

$$\psi : Y \times H \longrightarrow Y\ .$$

Take $\vartheta \in TopGr \cap wHk(Top_*)(G, H)$ and a map of pairs $(f, u) : (X, A) \longrightarrow (Y, B)$ with $u : A \longrightarrow B$ ϑ-equivariant; finally, suppose that ϑ and f are *closed inclusions and $f(X) \cap B = u(A)$. Then the induced map $\overline{f} : \overline{X} \longrightarrow \overline{Y}$ is an inclusion, and $\overline{f}(\overline{X}) \cap \overline{r}(Y) = \overline{f}\,\overline{s}(X)$, where $\overline{s} : X \longrightarrow \overline{X}$ and $\overline{r} : Y \longrightarrow \overline{Y}$ are the inclusions obtained in the construction of $\overline{X}$ and $\overline{Y}$ (see Proposition A.1.3).*

Proof – The diagram below portrays the relationship between the various spaces of the proposition:

$$\begin{tikzcd}
A \times G \arrow[rrr, "\phi"] \arrow[dr, "u \times \vartheta"] \arrow[ddd, "i \times 1_G"'] & & & A \arrow[dl, "u"'] \arrow[ddd] \\
& B \times H \arrow[r, "\psi"] \arrow[d, "j \times 1_H"'] & B \arrow[d] & \\
& Y \times H \arrow[r, "\bar{\psi}"'] & \overline{Y} & \\
X \times G \arrow[rrr, "\bar{\phi}"'] \arrow[ur, "f \times \vartheta"] & & & \overline{X} \arrow[ul, "\overline{f}"']
\end{tikzcd}$$

Since u is ϑ-equivariant and injective, $u \times \vartheta$ is relation bipreserving (see Lemma 1.2.3); then $f \times \vartheta$ is relation bipreserving (with respect to the extended equivalence relations) and consequently, $\overline{f}$ is injective.

It remains to prove that $\overline{f}$ is closed. We begin by observing that in view of [9, 1.10.1, Propositions 2 and 4], both $f \times \vartheta$ and $u \times \vartheta$ are closed maps. Now take a closed subset $C \subset \overline{X}$; set $K = \bar{\phi}^{-1}(C)$ and take $(f \times \vartheta)(K) \subset Y \times H$. Because of [14, VI.6.2]

$$\overline{f}(C) = \bar{\psi}(f \times \vartheta)(K) \text{ closed} \iff \psi((f \times \vartheta)(K) \cap (B \times H)) \text{ closed} .$$

Hence, we must prove that $\psi((f \times \vartheta)(K) \cap (B \times H))$ is closed in B. Notice that because $\bar{\phi}(K) = C$ is closed, [14, VI.6.2] can be used again to prove that $\phi(K \cap (A \times G))$ is closed. The injectivity of u, f, ϑ and the assumption $f(X) \cap B = u(A)$ imply that

$$(f \times \vartheta)(K) \cap (B \times H) = (u \times \vartheta)(K \cap (A \times G))$$

and thus,

$$\psi((f \times \vartheta)(K) \cap (B \times H)) = u\phi(K \cap (A \times G))$$

is closed, because u is a closed map. The last statement follows from simple set theoretical considerations. □

Corollary A.2.2 *In the situation of the proposition, if u is relation bipreserving with respect to the equivalence relations defined in A and B by the actions of G and H respectively, then $\overline{f}$ and $\overline{f} \times \vartheta$ are relation bipreserving.*

Proof – The first statement is proved by direct computation; as for the second, notice that $\overline{f}$ is ϑ-equivariant and injective and then use Lemma 1.2.3. □

Before we prove a result similar to Proposition A.2.1 for the enlargement of a contraction, we give an auxiliary lemma.

Lemma A.2.3 *If $f \in wHk(Top_*)((X, x_0), (Y, y_0))$ is a closed inclusion, then $f \wedge 1 : X \wedge I \longrightarrow Y \wedge I$ is a closed inclusion.*

Proof – Construct the commutative diagram

$$\begin{tikzcd}
X \vee I \arrow[rrr, "c_*"] \arrow[dr, "f \vee 1_I"] \arrow[ddd, "i"'] & & & (x_0,0) \arrow[dl] \arrow[ddd, "\overline{\imath}"] \\
& Y \vee I \arrow[r, "c'_*"] \arrow[d, "j"'] & (y_0,0) \arrow[d, "\overline{\jmath}"] & \\
& Y \times I \arrow[r, "\overline{c'_*}"'] & Y \wedge I & \\
X \times I \arrow[rrr, "\overline{c_*}"'] \arrow[ur, "f \times 1_I"] & & & X \wedge I \arrow[ul, "f \wedge 1_I"']
\end{tikzcd}$$

and observe that $X \wedge I$ and $Y \wedge I$ are quotient spaces of $X \times I$ and $Y \times I$ by the equivalence relations R and S respectively, defined by

$$(x,t)R(x',t') \;\Leftrightarrow\; x = x' = x_0 \text{ or } t = t' = 0,$$

$$(y,t)S(y',t') \;\Leftrightarrow\; y = y' = y_0 \text{ or } t = t' = 0\ .$$

Since f is injective, $f \times 1_I$ is relation bipreserving and thus, $f \wedge 1_I$ is injective.

Let $C \subset X \wedge I$ be closed; because $f \times 1_I$ is a closed map, $(f \times 1_I)(\overline{c_*}^{-1}(C))$ is closed in $Y \times I$, implying that $(f \wedge 1_I)(C)$ is closed in $Y \wedge I$ (use [14] VI.6.2). □

Proposition A.2.4 *Let $(f,u) : (X,A,x_0) \longrightarrow (Y,B,y_0)$ be a based map of pairs in $wHk(Top_*)$ such that A is a closed subset of X with a contraction h' to x_0, B is a closed subset of Y with a contraction k' to y_0, and u is an I-mapping*[2]*. Suppose that f is a closed inclusion and that*

[2] i.e., for every $(x,t) \in A \times I$, $u(xt) = u(x)t$

$f(X) \cap (B) = u(A)$. Then the quotient map $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ is a closed inclusion, and $\widetilde{f}(\widetilde{X}) \cap \widetilde{r}(Y) = \widetilde{f}\widetilde{s}(X)$, where $\widetilde{s}$ and $\widetilde{r}$ are the inclusions $(\widetilde{X}, X)$ and $(\widetilde{Y}, Y)$ respectively.

Proof – Take the commutative diagram

$$\begin{tikzcd}
A \wedge I \arrow[rrr, "h"] \arrow[dddd, "i \wedge 1_I"'] \arrow[dr, "u \wedge 1_I"] & & & A \arrow[dl, "u"'] \arrow[dddd, "\overline{i \wedge 1_I}"] \\
& B \wedge I \arrow[r, "k"] \arrow[d, "j \wedge 1_I"'] & B \arrow[d, "\overline{j \wedge 1_I}"] & \\
& Y \wedge I \arrow[r, "\bar{k}"'] & \widetilde{Y} & \\
& & & \\
X \wedge I \arrow[rrr, "\bar{h}"'] \arrow[uur, "f \wedge 1_I"] & & & \widetilde{X} \arrow[uul, "\widetilde{f}"']
\end{tikzcd}$$

recall that $f \wedge 1_I$ is a closed inclusion and proceed as in Proposition A.2.1.
$\square$

Now consider the preservation of proclusions.

Proposition A.2.5 *Let $\vartheta : G \longrightarrow H$ be in $TopGr \cap wHk(Top)$; let $(f, u) : (X, A) \longrightarrow (Y, B)$ be a mapping of pairs in $wHk(Top)$ such that A is a closed subset of X and a G-space with action ϕ, B is a closed subset of Y and an H-space with action ψ, and u is ϑ-equivariant. Suppose that ϑ and f are proclusions. Then the quotient map $\overline{f} : \overline{X} \longrightarrow \overline{Y}$ is a proclusion.*

Proof – Just recall that a relation preserving proclusion $f \in k(Top)(X, X')$ defines a unique proclusion $\widehat{f} \in k(Top)(X/R, X'/R')$. $\square$

Proposition A.2.6 *Let $(f, u) : (X, A, x_0) \longrightarrow (Y, B, y_0)$ be a based map of pairs in $wHk(Top_*)$ such that A is a closed subset of X with a contraction h' to x_0, B is a closed subset of Y with a contraction k' to y_0, and u is an I-mapping. Suppose that f is a proclusion. Then the quotient map $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ is a proclusion.*

The main tool to deal with closed cofibration is the following lemma whose proof is straightforward.

Lemma A.2.7 *Let B, B', X and X' be spaces; let $A \subset X$ and $A' \subset X'$ be closed subspaces; finally, let $\phi : A \longrightarrow B$ and $\phi' : A' \longrightarrow B'$ be given maps. Now construct the adjunction spaces $\overline{X} := A \sqcup_\phi X$ and $\overline{X'} := A' \sqcup_{\phi'} B'$. If $(f, u) : (X, A) \longrightarrow (X', A')$ and $v : B \longrightarrow B'$ are closed cofibrations, so is the induced map $\bar{f} : \overline{X} \longrightarrow \overline{X'}$.*

This lemma is helpful in proving a modified version of proposition A.2.1 for closed cofibrations. The corresponding result for the enlargement of a contration follows in a similar way after recalling that the wedge product preserves closed cofibrations.

A.3 The construction of the classifying space

Let G be a weak Hausdorff k-group. We are going to construct two expanding sequences in $wHk(Top_*)$ with homeomorphic union spaces.

We begin by taking

$$E_{-1} := \emptyset \ , \ D_0 := \{u_G\}$$

and the maps

$$\phi_{-1} : \emptyset \times G \longrightarrow \emptyset \ , \ h'_0 : \{u_G\} \times I \longrightarrow \{u_G\} \ .$$

Notice that we can view ϕ_{-1} as a group action, h'_0 as a contraction, and E_{-1} as a closed subset of D_0; then construct by enlargement

$$E_0 := E_{-1} \sqcup_{\phi_{-1}} (D_0 \times G) \text{ and } D_1 := D_0 \sqcup_{h_0} (E_0 \wedge I) \ .$$

The weak Hausdorff k-spaces E_0, D_0 and D_1 are based, with base point u_G. According to Proposition A.1.1 there exists a G-action $\phi_0 : E_0 \times G \longrightarrow E_0$.

Moreover, in view of Proposition A.1.8, E_0 is homeomorphic to a closed subset of D_1; from this we construct E_1 as the enlargement of the action ϕ_0 namely, $E_1 := E_0 \sqcup_{\phi_0} (D_1 \times G)$. On the other hand, there is a contraction $h'_1 : D_1 \times I \longrightarrow D_1$ (see Proposition A.1.7) and we know that D_1 is homeomorphic to a closed subset of E_1 (see Proposition A.1.3); then we construct the enlargement of the contraction $D_2 := D_1 \sqcup_{h_1} (E_1 \wedge I)$. Up to now we have a finite string of inclusions

$$E_{-1} \subset D_0 \subset E_0 \subset D_1 \subset E_1 \subset D_2 \ ;$$

we construct an infinite string

$$(\mathbf{ED}) \quad E_{-1} \subset D_0 \subset E_0 \subset D_1 \subset E_1 \subset \ldots \subset D_n \subset E_n \subset D_{n+1} \subset \ldots$$

by induction :

$$E_{n-1} := E_{n-2} \sqcup_{\phi_{n-2}} (D_{n-1} \times G) \ ,$$

$$D_n := D_{n-1} \sqcup_{h_{n-1}} (E_{n-1} \wedge I) \ .$$

Because of Propositions A.1.1 and A.1.7 we can also construct the following actions and contractions:

$$\phi_{n-2} : E_{n-2} \times G \longrightarrow E_{n-2} \ ,$$

$$h_{n-1} := D_{n-1} \wedge I \longrightarrow D_{n-1}$$

The next diagram is useful to understand the construction of E_{n-1} and D_n:

$$\begin{tikzcd}
 & D_{n-1} \times I \arrow[r, "p_{n-1}"] \arrow[d] & D_{n-1} \wedge I \arrow[r, "h_{n-1}"] \arrow[d] & D_{n-1} \arrow[d] \\
(D_{n-1} \times G) \times I \arrow[r] & E_{n-1} \times I \arrow[r] \arrow[dl, phantom, very near start, "\llcorner"] & E_{n-1} \wedge I \arrow[r] & D_n \arrow[ul, phantom, very near start, "\ulcorner"] \\
(E_{n-2} \times G) \times I \arrow[r, "\phi_{n-2} \times 1_I"'] \arrow[u] & E_{n-2} \times I \arrow[u] & &
\end{tikzcd}$$

The reader should observe that if the pair $(G, \{u_G\})$ is a closed cofibration in $wHk(Top_*)$, then all the inclusions in the infinite sequence of spaces **(ED)** above are closed cofibrations: in fact, (E_0, D_0) is a closed cofibration under the assumption on $(G, \{u_G\})$; for the other inclusions we proceed by induction and use Propositions A.1.5 and A.1.9.

We obtained two expanding sequences of spaces in $wHk(Top_*)$,

$$E_{-1} \subset E_0 \subset E_1 \subset \ldots \subset E_n \subset \ldots \quad \text{and}$$

$$D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_n \subset \ldots \ ,$$

whose union spaces are homeomorphic:

$$E_G := \bigcup_{n=-1}^{\infty} E_n = \bigcup_{n=0}^{\infty} D_n$$

(by definition, the topology of E_G is determined by either of the families $\{E_n | n \geq -1\}$ or $\{D_n | n \geq 0\}$). Note that $E_G \in wHk(Top)$ by [15, Proposition A.5.2]. Furthermore, for every $n \geq 0$, the inclusion $i_n : D_{n-1} \longrightarrow D_n$ is nulhomotopic (take the homotopy $i_n h_{n-1} p_{n-1} : D_{n-1} \times I \longrightarrow D_n$); hence $E_G \cong \bigcup_{n=0}^{\infty} D_n$ is weakly contractible.

The union space of the expanding sequence $\{E_n \times G | n \geq -1\}$ is homeomorphic to $E_G \times G$ (see [15, Proposition A.5.1]); this fact determines the existence of an action

$$\phi := \bigcup_{n=0}^{\infty} \phi_n : E_G \times G \longrightarrow E_G \ .$$

By the same token $\bigcup_{n=-1}^{\infty} (D_n \times I) = E_G \times I$. Moreover, because I is compact Hausdorff, there is a bijection between the sets of based maps

$$wHk(Top_*)(X \wedge I, Y) \rightleftharpoons wHk(Top_*)(X, Map_*(I, Y))$$

that is to say, $- \wedge I$ is a left adjoint functor and thus, preserves colimits; therefore $\bigcup_{n=-1}^{\infty} (D_n \wedge I) \cong E_G \wedge I$. Under these circumstances, we can define a contraction

$$h := \bigcup_{n=0}^{\infty} h_n : E_G \wedge I \longrightarrow E_G$$

and thus E_G is actually a contractible space.

Now we construct the classifying space B_G. For every $n \geq -1$, let B_n be the orbit space determined by the action $\phi_n : E_n \times G \longrightarrow E_n$ and let

$p_n : E_n \longrightarrow B_n$ be the identification map. It is immediate to verify that B_n is the push out of the triad

$$B_{n-1} \xleftarrow{p_n} E_{n-1} \longrightarrow D_n,$$

and that $\{B_n | n \geq -1\}$ is an expanding sequence of spaces. Then, we define the *(Milgram-Steenrod) classifying space* B_G of G to be the union space of the sequence $\{B_n | n \geq -1\}$. Notice that B_G is homeomorphic to the orbit space determined by the action ϕ of G on E_G and the identification map $p_G : E_G \longrightarrow B_G$ is the union map $p_G = \bigcup_{n=-1}^{\infty} p_n$.

The process just described, which gives rise to the based weak Hausdorff k-spaces E_G and B_G associated to a weak Hausdorff k-group G, is known as the *Milgram-Steenrod construction*. The next results show that the Milgram-Steenrod construction is functorial.

Lemma A.3.1 *Let $\vartheta : G \longrightarrow H$ be a homomorphism between two weak Hausdorff k-groups. Then ϑ determines an equivariant map*

$$E_\vartheta : E_G \longrightarrow E_H \ .$$

Proof – The proof is done by induction and using Propositions A.1.6 and A.1.10. By the Milgram-Steenrod construction E_G is the union space of the expanding sequence $\{E_n | n \geq -1\}$ with the action $\phi := \bigcup_{n=0}^{\infty} \phi_n$. Likewise, E_H is the union space of the expanding sequence $\{\widehat{E_n} | n \geq -1\}$ with action $\widehat{\phi} := \bigcup_{n=0}^{\infty} \widehat{\phi_n}$. The first steps of the induction are trivial. Assume that we have defined the equivariant map $E_{\vartheta,n-1} : E_{n-1} \longrightarrow \widehat{E_{n-1}}$; the next step in the construction is depicted by the following commutative diagram with pushout squares:

$$
\begin{tikzcd}
E_{n-1} \times G \arrow[rrr, "\phi_{n-1}"] \arrow[dr, "E_{\vartheta,n-1} \times \vartheta"] \arrow[ddd] & & & E_{n-1} \arrow[dl, "E_{\vartheta,n-1}"'] \arrow[ddd] \\
& \widehat{E_{n-1}} \times H \arrow[r, "\widehat{\phi_{n-1}}"] \arrow[d] & \widehat{E_{n-1}} \arrow[d] & \\
& \widehat{D_n} \times H \arrow[r] & \widehat{E_n} & \\
D_n \times G \arrow[rrr] \arrow[ur, "D_{\vartheta,n-1} \times \vartheta"] & & & E_n \arrow[ul, "E_{\vartheta,n}"']
\end{tikzcd}
$$

$\square$

Corollary A.3.2 *Let $\vartheta : G \longrightarrow H$ be a homomorphism between two weak Hausdorff k-groups. Then ϑ determines a map $B_\vartheta : B_G \longrightarrow B_H$.*

Proof – The existence of B_ϑ is clear. We only wish to notice, for the sake of completeness, that $B_\vartheta = \bigcup_{n=0}^{\infty} B_{\vartheta,n}$, where the maps $B_{\vartheta,n} : B_n \longrightarrow \widehat{B_n}$ are induced by the corresponding maps $E_{\vartheta,n}$. $\square$

Let us put together the lemma and its corollary. Denote by $\mathcal{EG}$ the category of weak Hausdorff k-spaces together with the action of a weak Hausdorff k-group and equivariant k-maps.

Theorem A.3.3 *The Milgram-Steenrod construction defines two functors*

$$\mathcal{E} : TopGr_* \cap wHk(Top_*) \longrightarrow \mathcal{EG} \ , \ G \longmapsto E_G$$

$$\mathcal{B} : TopGr_* \cap wHk(Top_*) \longrightarrow wHk(Top_*) \ , \ G \longmapsto B_G \ .$$

With the aid of the mapping track we can replace p_G by a fibration whose fibre is weakly homotopic to ΩB_G; however, under a mild assumption on the pair $(G, \{u_G\})$ we can obtain much more (see [42, Theorem 8.3] and [27, Theorem 4.2]):

Theorem A.3.4 *Let G be a weak Hausdorff k-group with identity element u_G such that $(G, \{u_G\})$ is a closed cofibration in $wHk(Top_*)$. Then (E_G, p_G, B_G) is a numerable principal G-bundle with contractible total space E_G.*[3]

Notice that if $(G, \{u_G\})$ is a closed cofibration, then the pairs

$$(D_n, E_{n-1})\ ,\ (E_n, D_n)\ ,\ (E_n, E_{n-1})\ ,\ (D_n, D_{n-1})\ ,$$

$$(B_n, B_{n-1})\ ,\ (E_G, E_n)\ ,\ (E_G, D_n) \text{ and } (B_G, B_n)$$

are closed cofibrations; moreover,

$$(E_n, E_{n-1}) \text{ and } (E_G, E_n)$$

are G-closed cofibrations for each n.

Remark A.3.5 If G is a compact weak Hausdorff k-group, then E_G and B_G are paracompact and normal.

In fact, we first notice that weak Hausdorff k-spaces are T_1 and hence, G is Hausdorff because topological groups are regular. The assertion is now proved by induction and [15, Proposition A.5.1].

Thus, if G is compact and $(G, \{u_G\})$ is a closed cofibration, the previous observation together with Theorem A.3.4 show that the map $p_G : E_G \longrightarrow B_G$ is a Hurewicz fibration with fibre G.

The last result of this section is about the homotopy type of G.

Proposition A.3.6 *Let G be a compact Hausdorff topological group and suppose that $(G, \{u_G\})$ is a closed cofibration. Then there exists an H-space preserving map*

$$\delta : \Omega B_G \longrightarrow G$$

which is a homotopy equivalence.

[3]Thus (E_G, p_G, B_G) is a universal bundle for numerable principal G-bundles.

Proof – Let L_{p_G} be the pullback of the triad

$$E_G \xrightarrow{p_G} B_G \xleftarrow{\epsilon_1} \mathcal{P}B_G \ .$$

Then, according to [35, Lemma 2.2.2], $G \sim L_{p_G}$. Since E_G is contractible, $L_{p_G} \sim E_G \times \Omega B_G \sim \Omega B_G$. Now use [6, Lemma 3.2]. □

A.4 Inclusions, proclusions and cofibrations of classifying spaces

Theorem A.4.1 *The functors $\mathcal{E}$ and $\mathcal{B}$ preserve closed inclusions.*

Proof – Let $\vartheta : G \longrightarrow H \in TopGr \cap wHk(Top)$ be an inclusion. According to the notation of Section A.3 we write

$$E_{-1} := \emptyset \ , \ D_0 := \{u_G\} \ ,$$

$$\phi_{-1} : \emptyset \times G \longrightarrow \emptyset \ , \ h'_0 : \{u_G\} \times I \longrightarrow \{u_G\} \ ,$$

$$\widehat{E_{-1}} := \emptyset \ , \ \widehat{D_0} := \{u_H\} \ ,$$

$$\widehat{\phi_{-1}} : \emptyset \times H \longrightarrow \emptyset \ , \ \widehat{h'_0} : \{u_H\} \times I \longrightarrow \{u_H\} \ ,$$

$$E_G := \bigcup_{n=-1}^{\infty} E_n = \bigcup_{n=0}^{\infty} D_n \ ,$$

$$E_H := \bigcup_{n=-1}^{\infty} \widehat{E_n} = \bigcup_{n=0}^{\infty} \widehat{D_n} \ ,$$

$$E_{\vartheta,n} : E_n \longrightarrow \widehat{E_n} \ , \ D_{\vartheta,n} : D_n \longrightarrow \widehat{D_n} \ ,$$

and

$$E_\vartheta = \bigcup_{n=-1}^{\infty} E_{\vartheta,n} = \bigcup_{n=0}^{\infty} D_{\vartheta,n} : E_G \longrightarrow E_H \ .$$

The hypotheses of Proposition A.2.1 are satisfied for the map of pairs

$$(D_{\vartheta,0}, E_{\vartheta,-1}) : (D_0, E_{-1}) \longrightarrow (\widehat{D_0}, \widehat{E_{-1}})$$

and those of Proposition A.2.4 hold true for the map of pairs

$$(E_{\vartheta,0}, D_{\vartheta,0}) : (E_0, D_0) \longrightarrow (\widehat{E_0}, \widehat{D_0}) \ ;$$

therefore, $D_{\vartheta,1}$ and $E_{\vartheta,0}$ are closed inclusions; an induction argument plus Propositions A.2.1 and A.2.4 prove that, for every $n \geq 0$, both $D_{\vartheta,n}$ and $E_{\vartheta,n}$ are closed inclusions (this assertion is also true for $E_{\vartheta,-1}$). But the topology of E_G coincides with the final topology of the family $\{\iota_n : E_n \longrightarrow E_G | n \geq -1\}$ (or of the family $\{\iota'_n : D_n \longrightarrow E_G | n \geq 0\}$) as one can see from [15, Lemma A.2.4] (and similarly for E_H). It follows that the family of closed inclusions $\{E_{\vartheta,n} | n \geq -1\}$ (or $\{D_{\vartheta,n} | n \geq 0\}$) induces a closed inclusion $E_\vartheta : E_G \longrightarrow E_H$ (see Lemma 1.1.3). This shows that $\mathcal{E}$ preserves closed inclusions.

Now we study the map $B_\vartheta : B_G \longrightarrow B_H$ induced by E_ϑ. We notice first that the map $E_{\vartheta,0} : E_0 \longrightarrow \widehat{E_0}$ is relation bipreserving in view of Corollary A.2.2; by induction, all maps $E_{\vartheta,n}$ are relation bipreserving, for every $n \geq -1$ and thus, E_ϑ is relation bipreserving. This shows that the induced map B_ϑ is injective.

It remains to prove that B_ϑ is closed. This is done by induction on the components of B_ϑ. Clearly, $B_{\vartheta,0}$ is closed; assume that $B_{\vartheta,n-1}$ is closed and take the commutative diagram

$$\begin{tikzcd}
E_{n-1} \arrow[rrr, "E_{\vartheta,n-1}"] \arrow[dr, "\iota_{n-1}"] \arrow[ddd, "p_{G,n-1}"'] & & & \widehat{E_{n-1}} \arrow[dl, "\iota'_{n-1}"'] \arrow[ddd, "p_{H,n-1}"] \\
& D_n \arrow[r, "D_{\vartheta,n}"] \arrow[d, "\bar{p}_{G,n-1}"'] & \widehat{D_n} \arrow[d, "\bar{p}_{H,n-1}"] & \\
& B_n \arrow[r, "B_{\vartheta,n}"'] & \widehat{B_n} & \\
B_{n-1} \arrow[ur, "\bar{\iota}_{n-1}"] \arrow[rrr, "B_{\vartheta,n-1}"'] & & & \widehat{B_{n-1}} \arrow[ul, "\bar{\iota}'_{n-1}"']
\end{tikzcd}$$

Let $C \subset B_n$ be closed; then $K := \bar{p}_{G,n-1}^{-1}(C)$ is closed in D_n and, because $D_{\vartheta,n}$ is a closed map, $D_{\vartheta,n}(K)$ is closed in $\widehat{D_n}$. Using [14, VI.6.2] we conclude that

$$B_{\vartheta,n}(C) \subset \widehat{B}_n \text{ closed } \iff p_{H,n-1}(D_{\vartheta,n}(K) \cap \widehat{E_{n-1}}) \subset \widehat{B_{n-1}} \text{ closed} .$$

Since $D_{\vartheta,n}$ is injective and $D_{\vartheta,n}(D_n) \cap \widehat{E_{n-1}} = E_{\vartheta,n-1}(E_{n-1})$ (compare with Proposition A.2.1), it follows that

$$D_{\vartheta,n}(K) \cap \widehat{E_{n-1}} = E_{\vartheta,n-1}(K \cap E_{n-1}) .$$

Therefore,

$$p_{H,n-1}(D_{\vartheta,n}(K) \cap \widehat{E_{n-1}}) = p_{H,n-1} E_{\vartheta,n-1}(K \cap E_{n-1}) = B_{\vartheta,n-1} p_{G,n-1}(K \cap E_{n-1})$$

which is closed in $\widehat{B_{n-1}}$ since $B_{\vartheta,n-1}$ is a closed map, and $p_{G,n-1}(K \cap E_{n-1})$ is closed in B_{n-1}, again by [14, VI.6.2], because $\bar{p}_{G,n-1}(K) = C$ is closed in B_n.

□

Theorem A.4.2 *The functors $\mathcal{E}$ and $\mathcal{B}$ preserve proclusions.*

Proof – Let $\vartheta : G \longrightarrow H$ be a proclusion of two weak Hausdorff k-groups. According to the notation of Lemma A.3.1

$$E_G = \bigcup_{n=-1}^{\infty} E_n = \bigcup_{n=0}^{\infty} D_n ,$$

$$E_H = \bigcup_{n=-1}^{\infty} \widehat{E_n} = \bigcup_{n=0}^{\infty} \widehat{D_n}$$

and

$$E_\vartheta = \bigcup_{n=-1}^{\infty} E_{\vartheta,n} \ : \ E_G \longrightarrow E_H .$$

Induction and Proposition A.2.6 show that the maps $D_{\vartheta,n} : D_n \longrightarrow \widehat{D_n}$ (see notation of Lemma A.3.1) are proclusions; this, coupled with Proposition A.2.5 and induction, shows that the maps $E_{\vartheta,n} : E_n \longrightarrow \widehat{E_n}$ are proclusions. The map E_θ is clearly surjective as a union function of surjective functions, and by Lemma 1.1.3 its image has the final topology. This means that E_ϑ is a proclusion.

To complete the proof, first notice that $B_\vartheta p_G = p_H E_\vartheta$ and that p_G, p_H and E_ϑ are proclusions and then apply [9, 1.2.4, Proposition 7]. $\square$

Theorem A.4.3 *The functors $\mathcal{E}$ and $\mathcal{B}$ preserve closed cofibrations between weak Hausdorff k-groups with non-degenerate unit.*

Proof – Let $G, H \in (TopGr \cap wHk(Top)$ be such that $(G, \{u_G\})$ and $(H, \{u_H\})$ are closed cofibrations (i.e., u_G and u_H are non-degenerate). Suppose that the pair (G, H) is a closed cofibration; we wish to prove that the pairs (E_G, E_H) and (B_G, B_H) are closed cofibrations.

We first recall that under the non-degeneracy hypothesis of u_G all pairs in the infinite sequence of spaces **(ED)** defined during the construction of E_G are closed cofibrations (similarly for E_H). Then, we prove that, for every $n \geq -1$, $(\widehat{E_n}, E_n)$ is a closed cofibration and use the appropriate result analogous to [15, Proposition A.5.5]. The proof follows the same lines as Theorem A.4.1.

Now for (B_G, B_H): we already know that all maps $B_{\vartheta,n}$ are closed injections; in order to prove that the pairs $(\widehat{B_n}, B_n)$ are all closed cofibrations we proceed again by induction; this is a routine exercise (refer back to the diagram of Theorem A.4.1). $\square$

A.5 The group structure of E_G

We begin by observing that the results stated in sections 5, 6 and 7 of [42] are still valid if one works withing the category of weak Hausdorff k-spaces. We have already noted that E_G is a contractible space; moreover, as proved by N. Steenrod, E_G is a topological group [42, Theorem 7.6 (e)]. Our immediate objective is to review this group structure; to do this, it is necessary to conduct a refined analysis on the nature of E_G. First recall how to express the elements of E_G (see details in [42]). Let Δ_n be the n-simplex of $\mathbb{R}^n$ defined by the inequalities $0 \leq t_1 \leq \ldots \leq t_n \leq 1$ and let δ_n be its interior; imbed Δ_n in Δ_{n+1} by adding the $(n+1)^{th}$ coordinate $t_{n+1} = 1$. A point of $G^n \times \Delta_n$ is represented by its coordinates in the shuffled form $[g_1, t_1, \ldots, g_n, t_n]$. Imbed

$G^n \times \Delta_n$ in $G^{n+1} \times \Delta_{n+1}$ by adding the coordinates $g_{n+1} = u_G$ and $t_{n+1} = 1$. Theorem 5.1 of [42] shows that, for every $n \geq 0$, there exists a proclusion $k_n : G^n \times \Delta_n \longrightarrow D_n$, whose restriction to $G^{n-1} \times \Delta_{n-1}$ is k_{n-1} and thus, the union map

$$k := \bigcup_{n=0}^{\infty} k_n : \bigcup_{n=0}^{\infty} (G^n \times \Delta_n) \longrightarrow E_G$$

is a well defined proclusion. Thus, for every $x \in E_G := \bigcup_{n=0}^{\infty} D_n$, there exist a non negative integer n and an element $u \in G^n \times \Delta_n$ such that $x \in D_n$ and $k_n(u) = x$. Two elements $u, v \in G^n \times \Delta_n$ are said to be *equivalent* if $k_n(u) = k_n(v)$; the resulting quotient space can be viewed as the space

$$N_n = \bigcup_{j=0}^{n} (G \setminus \{u_G\})^j \times (\delta_j \cup \delta_{j-1})$$

and the restriction of k_n to N_n is a bijection to D_n (see [42, Corollary 5.4]).

At this point consider the free abstract group $\tilde{E}_G$ generated by all pairs $(g,t) \in G \times I$: this is the set $\bigcup_{n=0}^{\infty} (G \times I)^n$ with multiplication $\mu' : \tilde{E}_G \times \tilde{E}_G \longrightarrow \tilde{E}_G$ defined by juxtaposition of monomials (the unit element is the empty monomial $\perp$ corresponding to $n = 0$). Let E'_G be the quotient group obtained from $\tilde{E}_G$ using the following *Fundamental Relations*: for every $g, g' \in G$ and every $t, t' \in I$,

1. $(g, 0) = (u_G, t) = u_{\tilde{E}_G} = \perp$,
2. $(g,t)(g',t) = (gg', t)$,
3. if $0 < t' < t \leq 1$, then $(g,t)(g',t') = (g \cdot g' \cdot g^{-1}, t')(g,t)$.

Notice that if G is abelian, then E'_G is also abelian.

A monomial $(g_1, t_1) \ldots (g_k, t_k)$ is said to be in *normal form* if it is the empty monomial or if $0 < t_1 < \ldots < t_k \leq 1$ and each $g_i \in G \setminus \{u_G\}$. Moreover, each monomial is equivalent to one and only one monomial in normal form; the equivalence is obtained by using the previous Fundamental Relations. Thus E'_G is isomorphic to the abstract subgroup of $\tilde{E}_G$ determined by all the monomials in normal form (for example, $u_{E'_G}$ corresponds to the empty monomial $\perp$). Next, define $f : E'_G \longrightarrow E_G$ as the function which assigns to each element $(g_1, t_1) \ldots (g_m, t_m)$ in normal form the element $k_m([g_1, t_1, \ldots, g_m, t_m])$ of E_G (if $m = 0$, $f(u_{E'_G}) = f(\perp) = u_G \in E_G$). According to [42, Theorem 7.6 (a)], f is a bijection. One should observe that, even if the two sets $\tilde{E}_G$ and $\bigcup_{n=0}^{\infty} G^n \times \Delta_n$ are not isomorphic, the sets E'_G and N_∞ are indeed isomorphic, via the function

$$\epsilon : E'_G \longrightarrow N_\infty \ , \ (g_1, t_1) \dots (g_n, t_n) \longmapsto [g_1, t_1, \dots, g_n, t_n]$$

(in normal form); then $f = k\epsilon$. Finally, define a multiplication $\mu : E_G \times E_G \longrightarrow E_G$ as the composition

$$E_G \times E_G \xrightarrow{f^{-1} \times f^{-1}} E'_G \times E'_G \xrightarrow{\mu'} E'_G \xrightarrow{f} E_G$$

(the continuity of μ is shown in [42, Theorem 7.6 (d)]).

Let $\vartheta : G \longrightarrow H$ be a homomorphism between two weak Hausdorff k-groups. An analysis of the construction of E_ϑ shows that

$$E_\vartheta((g_1, t_1) \dots (g_n, t_n)) = (\vartheta(g_1), t_1) \dots (\vartheta(g_n), t_n)$$

In particular, we obtain the following result (cf. Theorem A.3.3):

Proposition A.5.1 *$\mathcal{E}$ is a covariant functor of the category of weak Hausdorff k-groups to itself.*

Let E'^0_G be the set of all elements $(g, 1) \in E'_G$; with the identification $(u_G, 1) \equiv \perp$, the elements of E'^0_G are all in normal form and $f : E'^0_G \longrightarrow E_0 \subset E_G$ (see the definition of E_0 in Section A.3). On the other hand, we can identify E'^0_G with the topological group G by the function $i_G : G \longrightarrow E'_G$ which takes an arbitrary $g \in G$ into $(g, 1)$; hence, G can be viewed as a closed topological subgroup of E_G. Furthermore, the multiplication μ restricted to $E_G \times G \subset E_G \times E_G$ coincides with the action $\phi : E_G \times G \longrightarrow E_G$ defined somewhat abstractly in Section A.3; in other words, the following diagram commutes:

$$\begin{tikzcd}
E_G \times G \arrow[r, "\phi"] \arrow[d, "f^{-1} \times i_G"'] & E_G \\
E'_G \times E'_G \arrow[r, "\mu'"'] & E'_G \arrow[u, "f"']
\end{tikzcd}$$

Therefore, ϕ acts on the elements of $E_G \times G$ as follows:

$$\phi(((g_1, t_1) \dots (g_k, t_k)), g) = \begin{cases} f((g_1, t_1) \dots (g_k g, t_k)) & \text{if } t_k = 1 \\ f((g_1, t_1) \dots (g_k, t_k)(g, 1)) & \text{if } t_k < 1 \end{cases}$$

The previous considerations also show that the space $B_G = E_G/G$ of G-orbits in E_G can be identified to the space of right cosets of the topological group E_G with respect to its subgroup G. In particular, if G is normal in E_G, the classifying space B_G is a topological group.

We simplify the notation by omitting f and identifying implicitly the elements of E_G with the corresponding monomials in E'_G in normal form (this can be done as f is bijective and thus, the equivalence classes of E_G are equivalence classes of E'_G).

The orbit of an element $x = (g_1, t_1) \dots (g_n, t_n) \in E_G$ (in normal form) is the set

$$xG = \{(g_1, t_1) \dots (g_n, t_n)(g, 1) \mid g \in G\} \ .$$

Thus, according to the rules established, if $t_n = 1$ the elements of xG have the form $(g_1, t_1) \cdots (g_n g, 1)$ and therefore, we take $(g_1, t_1) \cdots (g_{n-1}, t_{n-1})$ to represent it. This shows that as sets

$$B_G \cong \{(g_1, t_1) \dots (g_n, t_n) \in E_G \mid t_n < 1, n = 0, \cdots, \infty\}$$

(the elements of E_G are always taken in normal form and if $n = 0$, we have the neutral element $\perp$).

If $\vartheta : G \longrightarrow H$ is a morphism between two weak Hausdorff k-groups,

$$B_\vartheta : xG \equiv [x] \longmapsto E_\vartheta(x)H \equiv [E_\vartheta(x)] \ .$$

Proposition A.5.2 *The continuous group multiplication in E_G induces a continuous group multiplication in B_G if, and only if, G is abelian.*

Proof – It is enough to prove that $i_G(G)$ is normal in $E_G \iff G$ is abelian.

Recall that $i_G(G)$ is the set of monoids of the form $(g, 1)$ with $g \in G$. Thus, we should check that $\forall x \in E_G$ and $(g, 1) \in i_G(G)$

$$x(g, 1)x^{-1} \in i_G(G),$$

iff G is abelian.

If G is abelian, the Fundamental Relations 2. and 3. show that, for every monomial $(h, s) \in E_G$ and every $(g, 1) \in i_G(G)$,

$$(g, 1)(h, s) = (ghg^{-1}, s)(g, 1) = (h, s)(g, 1)$$

if $s < 1$, and

$$(g,1)(h,s) = (gh,1) = (hg,1) = (h,1)(g,1)$$

if $s = 1$. Hence, for every $x \in E_G$,

$$x(g,1)x^{-1} = (g,1)xx^{-1} = (g,1) \in i_G(G) \ ,$$

and thus $i_G(G)$ is normal in E_G.

Conversely, if $i_G(G)$ is normal in E_G, for every $g, h \in G$ and every $s < 1$, $(h,s)(g,1)(h^{-1},s) \in i_G(G)$ and hence,

$$(h,s)(g,1)(h^{-1},s) = (h,s)(gh^{-1}g^{-1},s)(g,1) = (hgh^{-1}g^{-1},s)(g,1) \in i_G(G) \ ;$$

it follows that $hgh^{-1}g^{-1} = u_G$ and thus, G is abelian since g and h were taken arbitrarily. □

Observe that, when G is abelian, the multiplication in B_G is naturally induced from the one of E_G, i.e. the following diagram commutes

$$\begin{tikzcd}
E_Z \times E_Z \arrow[r, "\mu"] \arrow[d, "p_Z \times p_Z"'] & E_Z \arrow[d, "p_Z"] \\
B_Z \times B_Z \arrow[r, "\hat{\mu}"'] & B_Z
\end{tikzcd}$$

Let G and H be two weak Hausdorff k-groups; the function

$$\xi_{G,H} : E_{G\times H} \longrightarrow E_G \times E_H$$

$$\xi_{G,H}\{((g_1,h_1),t_1)\dots((g_n,h_n),t_n)\} = ((g_1,t_1)\dots(g_n,t_n),(h_1,t_1)\dots(h_n,t_n))$$

is a natural homeomorphism of weak Hausdorff k-groups (see of [42, Theorems 6.2 and 7.6 (g)]); in other words, the functor $\mathcal{E}$ preserves finite products.

The natural homeomorphism $\xi_{G,H}$ passes on to the orbit spaces, that is to say, there exists a map

$$\rho_{G,H} : B_{G\times H} \longrightarrow B_G \times B_H$$

$$\rho_{G,H}\{[((g_1,h_1),t_1)\dots((g_n,h_n),t_n)]\} =$$

$$([(g_1,t_1)\dots(g_n,t_n)],[(h_1,t_1)\dots(h_n,t_n)])$$

such that the following diagram commutes

$$\begin{tikzcd}
E_{G\times H} \arrow[r, "\xi_{G,H}"] \arrow[d, "p_{G\times H}"'] & E_G \times E_H \arrow[d, "p_G \times p_H"] \\
B_{G\times H} \arrow[r, "\rho_{G,H}"'] & B_G \times B_H
\end{tikzcd}$$

The map $\rho_{G,H}$ is actually a natural homeomorphism; in other words, the functor $\mathcal{B}$ preserves finite products.

A.6 The functors $\mathcal{E}$ and $\mathcal{B}$: algebraic properties

In this section, we provide the main algebric divices that makes the Milgram-Steenrod construction a powerfull machinary to work with classifying spaces.

A.6.1 Exactness

Theorem A.6.1 *The functors $\mathcal{E}$ and $\mathcal{B}$ are exact.*

Proof – Let

$$A \xrightarrow{\vartheta} G \xrightarrow{\varphi} C$$

be a sequence in $TopGr \cap wHk(Top)$ such that $\operatorname{im}\vartheta = \ker\varphi$; then, the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\vartheta"] \arrow[d, "i_A"'] & G \arrow[r, "\varphi"] \arrow[d, "i_G"] & C \arrow[d, "i_C"] \\
E_A \arrow[r, "E_\vartheta"] \arrow[d, "p_A"'] & E_G \arrow[r, "E_\varphi"] \arrow[d, "p_G"] & E_C \arrow[d, "p_C"] \\
B_A \arrow[r, "B_\vartheta"'] & B_G \arrow[r, "B_\varphi"'] & B_C
\end{tikzcd}$$

Take arbitrarily $z = (a_1, s_1) \dots (a_k, s_k) \in E_A$. Then

$$E_\varphi E_\vartheta(z) = E_{\varphi\vartheta}(z) =$$

$$= (\varphi\vartheta(a_1), s_1) \dots (\varphi\vartheta(a_k), s_k) = (u_C, s_1) \dots (u_C, s_k) = \perp_{E_C} .$$

Hence, $\operatorname{im} E_\vartheta \subseteq \ker E_\varphi$. Now, take $x = (g_1, t_1) \dots (g_n, t_n) \in \ker E_\varphi$ in normal form; then

$$E_\varphi(x) = (\varphi(g_1), t_1) \dots (\varphi(g_n), t_n) = \perp_{E_C}$$

and since $0 < t_i < t_j$ for all $0 \le i < j \le n$, $\varphi(g_j) = u_G$; thus, for every $j \in [0, n]$, there exists an element a_j in A such that $g_j = \vartheta(a_j)$. Therefore, $x = E_\vartheta(z)$ with $z = (a_1, t_1) \dots (a_n, t_n)$, and $\ker E_\varphi \subseteq \operatorname{im} E_\vartheta$.

Now let us look at the sequence of classifying (based) spaces. Let $[z] \in B_A$ be given; then, for every $z \in [z]$ in E_A,

$$B_\varphi B_\vartheta([z]) = [E_\varphi E_\vartheta(z)] = [\perp_{E_C}] ,$$

and thus, $\operatorname{im} B_\vartheta \subseteq \ker B_\varphi$.

Suppose that $[x] \in \ker B_\varphi$ is such that $B_\varphi([x]) = [\perp_{E_C}]$. But $B_\varphi([x]) = [E_\varphi(x)]$ for every representative $x \in [x]$ and thus, $E_\varphi(x) = \perp_{E_C}(c, 1)$, for some $c \in C$. Suppose that $x = (g_1, t_1) \dots (g_n, t_n)$ in $[x]$ with $t_n < 1$[4]; then

[4] This is possible because if $x' = (g'_1, t'_1) \dots (g'_m, t'_m)$ and $t'_m = 1$, it is sufficient to take $x = x'(g'^{-1}_m, t'_m)$.

$c = u_C$, and $E_\varphi(x) = \perp_{E_C}$. The exactness of the middle row of the previous diagram now implies that there exists z in E_A, such that $x = E_\vartheta(z)$, and hence $[x] = [E_\vartheta(z)] = B_\vartheta([z])$, that is to say, $\ker B_\varphi \subseteq \mathrm{im}\ B_\vartheta$. □

A.6.2 Group theoretical invariants of $\mathcal{E}$

The functor $\mathcal{E}$ behaves well with respect to subgroups, normality and related properties. We start this series of results by observing that if G is a weak Hausdorff k-group and K is a topological subgroup of G, then E_K is a topological subgroup of E_G: in fact, the closed inclusion $\iota : K \longrightarrow G$ gives rise to a closed inclusion $E_\iota : E_K \longrightarrow E_G$ (see Theorem A.4.1) and $E_\iota(E_K)$ is an abstract subgroup of E_G.

Corollary A.6.2 *The functor $\mathcal{E}$ preserves normality of subgroups.*

Proof – Let N be a normal subgroup of G; Theorem A.6.1 applied to the short exact sequence

$$0 \longrightarrow N \xrightarrow{\ i\ } G \xrightarrow{\ p\ } G/N \longrightarrow 0$$

gives rise to a short exact sequence of groups

$$0 \longrightarrow E_N \xrightarrow{\ E_i\ } E_G \xrightarrow{\ E_p\ } E_{G/N} \longrightarrow 0\ .$$

Then, for every $x \in E_G$ and every $y \in E_N$, $xyx^{-1} \in \ker E_p = \mathrm{im}\ E_i$ and thus $xyx^{-1} \in E_N$. □

If A is an abelian subgroup of G, then E_A is an abelian subgroup of E_G; in particular, if ZG is the centre of G, then E_{ZG} is an abelian subgroup of E_G. Actually, we can go one step further:

Lemma A.6.3 *E_{ZG} coincides with the centre ZE_G of E_G.*

Proof – Take two monomials $(g, s) \in E_G$ and $(z, t) \in E_{ZG}$. If $s > t$

$$(g, s)(z, t) = (gzg^{-1}, t)(g, s) = (z, t)(g, s)\ ;$$

if $s < t$

$$(z,t)(g,s) = (zgz^{-1}, s)(z,t) = (g,s)(z,t) \ ;$$

if $t = s$

$$(g,s)(z,t) = (gz, s) = (zg, s) = (z,t)(g,s) \ .$$

For arbitrary elements in normal form, the proof follows after successive applications of the Fundamental Relations. □

More generally,

Corollary A.6.4 *Let Z be a central topological subgroup of $G \in TopGr \cap wHk(Top)$. Then E_Z is a central subgroup of E_G; in short, $\mathcal{E}$ preserves centrality.*

Now we wish to prove that $\mathcal{E}$ preserves quotient groups obtained from normal subgroups: more precisely:

Theorem A.6.5 *Let N be a normal subgroup of the weak Hausdorff k-group G. Then $E_{G/N} \cong E_G/E_N$ in $TopGr \cap wHk(Top)$.*

Proof – Take the short exact sequence of weak Hausdorff k-groups

$$0 \longrightarrow N \xrightarrow{\ \iota\ } G \xrightarrow{\ \pi\ } G/N \longrightarrow 0$$

and notice the following: (i) $E_\iota : E_N \longrightarrow E_G$ is a closed inclusion in $TopGr \cap wHk(Top)$ (Theorem A.4.1); (ii) $E_\pi : E_G \longrightarrow E_{G/N}$ is a proclusion in $TopGr \cap wHk(Top)$ (Theorem A.4.2); (iii) the sequence

$$0 \longrightarrow E_N \xrightarrow{\ E_\iota\ } E_G \xrightarrow{\ E_\pi\ } E_{G/N} \longrightarrow 0$$

is exact (Theorem A.6.1). The last observation shows that as abstract groups, $E_{G/N} \cong E_G/E_N$; the question is now to prove that actually we have a homeomorphism in $TopGr \cap wHk(Top)$. To this end, we refer to [37, Theorem 12, III.19] and notice that we have to adjust Pontrjagin's proof in the sense that in his treatment E_π is an open map, while in our case, E_π is a proclusion. Let $f : E_{G/N} \longrightarrow E_G/E_N$ be the function that maps an element $E_\pi(x) \in E_{G/N}$

into the coset xE_N and consider the diagram

$$\begin{tikzcd} E_G \arrow[d, "E_\pi"'] \arrow[dr, "q"] & \\ E_{G/N} \arrow[r, "f"'] & E_G/E_N \end{tikzcd}$$

in which E_G/E_N has the final topology determined by q. According to [37, Theorem 12, III.19] f is a bijection and its inverse f^{-1} is continuous; the continuity of f follows from the fact that $fp = q$ is continuous. $\square$

A.6.3 Actions of classifying spaces and quotients

There is a similar result for the classifying spaces in the particular case in which N is a central subgroup of the weak Hausdorff k-group G.

First, let A be an abelian subgroup of $H \in TopGr_* \cap wHk(Top_*)$ and let $r : G \times A \longrightarrow G$ be an action that commutes with the translations of G. Then the group E_A is abelian and $B_A := E_A/A$ is an abelian group with the natural quotient structure (see Section A.5). Because $\mathcal{E}$ and $\mathcal{B}$ preserve finite products (see the end of Section A.5) we can build up the commutative diagram in $wHk(Top_*)$

$$\begin{tikzcd} G \times A \arrow[r, "1"] \arrow[d, "i_G \times i_A"'] & G \times A \arrow[r, "r"] \arrow[d, "i_G \times i_A"] & G \arrow[d, "i_G"] \\ E_G \times E_A \arrow[r, "\xi^{-1}_{G,A}"] \arrow[d, "p_G \times p_A"'] & E_{G\times A} \arrow[r, "E_r"] \arrow[d, "p_{G\times A}"] & E_G \arrow[d, "p_G"] \\ B_G \times B_A \arrow[r, "\rho^{-1}_{G,A}"'] & B_{G\times A} \arrow[r, "B_r"'] & B_G \end{tikzcd}$$

where $\xi^{-1}_{G,A}$ and $\rho^{-1}_{G,A}$ are natural homeomorphisms in $TopGr \cap wHk(Top)$ and $wHk(Top)$, respectively.

Theorem A.6.6 *If $r : G \times A \longrightarrow G$ is a right action which commutes with the translations of G, then the maps*

$$E(r) := E_r \xi^{-1}_{G,A} \text{ and } B(r) := B_r \rho^{-1}_{G,A}$$

are right actions.

The theorem says that, modulo certain natural homeomorphisms, the functors $\mathcal{E}$ and $\mathcal{B}$ preserve the right actions of abelian groups which commute with translations.

Proof – Let $x = (g_1, s_1) \dots (g_n, s_n) \in E_G$ be in normal form. Then

$$E(r)(x, \perp_{E_A}) = E_r((g_1, u_A), s_1) \dots ((g_n, u_A), s_n)) =$$

$$= (r(g_1, u_A), s_1) \dots (r(g_n, u_A), s_n) = x \ ;$$

furthermore, for any representative $z \in [\perp_{E_A}]$,

$$B(r)([x], [\perp_{E_A}]) = B_r([\xi^{-1}_{G,A}(x, z)]) = B_r([x]) = [E_r(x)] = [x] \ .$$

The proof of the associativity is based on the fact that in the present case r and the multiplication m of E_A are homomorphisms. Therefore, we can construct the commutative diagram

$$\begin{tikzcd}
E_G \times E_A \times E_A \arrow[r, "\cong"] \arrow[d, "\cong"'] \arrow[dr, "\cong"] & E_{G\times A} \times E_A \arrow[r, "E_r \times 1"] \arrow[d, "\cong"] & E_G \times E_A \arrow[d, "\cong"] \\
E_G \times E_{A\times A} \arrow[r, "\cong"'] \arrow[d, "1 \times E_m"'] & E_{G\times A\times A} \arrow[r, "E_{r\times 1}"] \arrow[d, "E_1 \times m"'] \arrow[dr, "E_{r(r\times 1)}", "E_{r(1\times m)}"'] & E_{G\times A} \arrow[d, "E_r"] \\
E_G \times E_A \arrow[r, "\cong"'] & E_{G\times A} \arrow[r, "E_r"'] & E_G
\end{tikzcd}$$

where the isomorphisms are given by the maps ξ^{-1} defined at the end of Section A.5. The associativity of $B(r)$ then comes by projection of the diagram onto the corresponding diagram of classifying spaces. □

Notice that, since $E(r)$ is a group homomorphism, it commutes with the multiplication in E_G (see Lemma 1.2.2). Furthermore, by straightforward computations, we can verify the following:

Theorem A.6.7 *Both functors $\mathcal{E}$ and $\mathcal{B}$ preserve freedom of actions.*

Now, consider tha case when the group A acting on G is one of its subgroups, and the action is the natural one given by restriction of multiplication of G; this action is an homomorphism iff A is central in G. Then, if Z is a central subgroup of G, there are two actions of E_Z over E_G, the one defined in Theorem and the one given by restriction of the multiplication, but it is easy to show that actually they coincide, i.e. the following diagram commutes:

$$\begin{tikzcd}
G \times Z \arrow[r, "1"] \arrow[d, "i_G \times i_Z"'] & G \times Z \arrow[r, "r"] \arrow[d, "i_{G\times Z}"] & G \arrow[d, "i_G"] \\
E_G \times E_Z \arrow[r, "\xi^{-1}"] \arrow[d, "1"] & E_{G\times Z} \arrow[r, "E_r"] & E_G \arrow[d, "1"] \\
E_G \times E_Z \arrow[rr, "\alpha"'] & & E_G
\end{tikzcd}$$

For classifying spaces, where we have not a group moltiplication, a weaker result is still valid. In fact, in this case B_Z is a subspace of B_G, and by restriction of the action r in the diagram in the proof of Theorem A.6.6 to $\{u_G\} \times Z$, we can show that the restriction of the action $B(r)$ to $\{b_0\} \times B_Z$ coincides with the inclusion of B_Z in B_G (here b_0 is the preferred point of B_G, i.e. the image of G under p_G).

Finally, the analogous of Theorem A.6.5 for classifying spaces is the following:

Theorem A.6.8 *For any $G \in TopGr \cap wHk(Top)$ and any central subgroup $Z \subset G$, $B_{G/Z} \cong B_G/B_Z$.*

Proof – The sequences

$$0 \longrightarrow E_Z \xrightarrow{E_\iota} E_G \xrightarrow{E_\pi} E_{G/Z} \longrightarrow 0$$

and

$$0 \longrightarrow B_Z \xrightarrow{B_\iota} B_G \xrightarrow{B_\pi} B_{G/Z} \longrightarrow 0$$

are exact in the respective categories by Theorem A.6.1 and $E_{G/Z} \cong E_G/E_Z$ because of Theorem A.6.5. From [10, 3.2.8, Proposition 22] we conclude that

$$B_G = E_G/G \cong (E_G/Z)/(G/Z) \ .$$

This, together with the fact that B_Z acts on B_G (see Theorem A.6.6) and Lemma 1.2.10 imply the following sequence of homeomorphisms:

$$B_G/B_Z \cong ((E_G/Z)/(G/Z))/(E_Z/Z) =$$

$$= ((E_G/Z)/(E_Z/Z))/(G/Z) = (E_G/E_Z)/(G/Z) =$$

$$= E_{G/Z}/(G/Z) = B_{G/Z} \ .$$

□

A.7 The functors $\mathcal{E}$, $\mathcal{B}$ and principal bundles

In this section we seek the pairs of weak Hausdorff k-groups that give rise to (locally trivial) principal bundles by application of $\mathcal{E}$ and $\mathcal{B}$. We begin with the functor $\mathcal{E}$.

Proposition A.7.1 *Let G be a weak Hausdorff k-group with non-degenerate unit and N be a normal subgroup of G such that $i : N \longrightarrow G$ is an N-equivariant closed cofibration. Let $\pi : G \longrightarrow G/N$ be the quotient map. Then $E_i : E_N \longrightarrow E_G$ is an E_N-equivariant closed cofibration and*

$$(E_G, E_\pi, E_{G/N}, E_N)$$

is a principal E_N-bundle.

Proof – E_G is a weak Hausdorff k-group (Proposition A.5.1) with non-degenerate unit (Theorem A.4.3); furthermore, E_N is a normal subgroup of E_G (Corollary A.6.2).

The identity element $u_G \in G$ is a non-degenerate element of N and the class N is non-degenerate in G/N. We also know (from Theorem A.4.3) that $E_i : E_N \longrightarrow E_G$ is a closed cofibration; however, this is not enough. Take the commutative diagram

$$\begin{tikzcd} E_N \arrow[r, "E_i"] \arrow[d, "q"'] & E_G \arrow[d, "q"] \\ E_N/E_N \arrow[r, "\widehat{E_i}"'] & E_G/E_N \end{tikzcd}$$

and observe that $\widehat{E_i} : E_N/E_N \longrightarrow E_G/E_N$ is a closed cofibration; it follows that $E_i : E_N \longrightarrow E_G$ is an E_N-equivariant closed cofibration in view of Lemma 1.2.8.

The second part of the proposition follows from Proposition 1.2.9. $\square$

Now let us go to the functor $\mathcal{B}$.

Theorem A.7.2 *Let G be a weak Hausdorff k-group with non-degenerate unit. Let Z be a central subgroup of G such that the inclusion $Z \subset G$ is a Z-equivariant closed cofibration; finally, let $\pi : G \longrightarrow G/Z$ be the quotient map. Then $(B_G, B_\pi, B_{G/Z}, B_Z)$ is a principal B_Z-bundle.*[5]

Proof – Let us take E_G, G and E_Z for X, G and H, respectively, of Proposition 1.2.13. Take the map $E_p : E_G \longrightarrow E_G/E_Z$ and form the commutative diagram

$$\begin{tikzcd} E_G \arrow[r, "\Delta"] \arrow[d, "E_p"'] & E_G \times E_G \arrow[d, "E_p \times E_p"] \\ E_G/E_Z \arrow[r, "\Delta"'] & E_G/E_Z \times E_G/E_Z \end{tikzcd}$$

[5]In general we do not know if this bundle – and that of Proposition A.7.1 – is numerable.

Then observe that the space E_G^* coincides with $(E_p \times E_p)^{-1}\Delta(E_G/E_Z)$ and thus E_G^* is closed in $E_G \times E_G$ because $E_G/E_Z \cong E_{G/Z}$ is weak Hausdorff. Then, according to Proposition 1.2.13 and Theorem 1.3.4,

$$(B_G/E_Z, q, E_G/G = B_G, E_Z, E_Z)$$

is a principal E_Z-bundle. We conclude the proof using the second part of Lemma 1.2.11 and observing that Z acts trivially on B_G. $\square$

Remark A.7.3 If G is compact, then $B_{G/Z}$ is paracompact (see Remark A.3.5) and so, $(B_G, B_q, B_{G/Z}, B_Z)$ is a numerable bundle and moreover, is a Hurewicz fibration.

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