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Spectral Analysis and Zeta Determinant on the Deformed Spheres

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Abstract: We consider a class of singular Riemannian manifolds, the deformed spheres S_k^N, defined as the classical spheres with a one parameter family $g[k]$ of singular Riemannian structures, that reduces for $k = 1$ to the classical metric. After giving explicit formulas for the eigenvalues and eigenfunctions of the metric Laplacian $\Delta_{S_k^N}$, we study the associated zeta functions $\zeta(s, \Delta_{S_k^N})$. We introduce a general method to deal with some classes of simple and double abstract zeta functions, generalizing the ones appearing in $\zeta(s, \Delta_{S_k^N})$. An application of this method allows to obtain the main zeta invariants for these zeta functions in all dimensions, and in particular $\zeta(0, \Delta_{S_k^N})$ and $\zeta'(0, \Delta_{S_k^N})$. We give explicit formulas for the zeta regularized determinant in the low dimensional cases, $N = 2, 3$, thus generalizing a result of Dowker [25], and we compute the first coefficients in the expansion of these determinants in powers of the deformation parameter k.

1. Introduction

In the last decades there has been a (continuously increasing) interest in the problem of obtaining explicit information on the zeta regularized determinant of differential operators [2, 49, 37, 50, 43, 61]. Despite the lack of a general method, a lot of results are available in the literature for various particular cases or by means of some kind of approximation. Moreover, quite complete results have been obtained for the geometric case of the metric Laplacian on a Riemannian compact manifold for some classes of simple spaces: spheres [18, 11], projective spaces [54], balls [5], orbifolded spheres [25], compact (and non-compact) hyperbolic manifolds [20, 9, 10] or in particular cases: Sturm operators on a line segment [8, 45], cone on a circle [56].

In particular, many works in the recent physical literature applied this zeta function regularization process to study the modifications induced at quantum level by some

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kind of deformation of the background space geometry of physical models [40, 24, 22, 52]. In this context, a full class of deformed spaces, called deformed spheres, has been introduced in [52], where the perturbation of the heat kernel expansion has been studied. This is a particularly interesting class of spaces in Einstein theory of gravitation and in cosmology, since the appearance of a non-trivial deformation produces a symmetry breaking of the space. In fact, the deformed sphere may be considered as the Euclidean version of the a deformed de Sitter space, which is particularly relevant in modern cosmology, since it represents the inflationary as well as the recent accelerated phase. It is well known that the quantum effective action is related to the regularized functional determinant of Laplace type operators (see, for example [30] and references therein). As a consequence, an expansion of the functional determinant with respect to the deformation parameter around its spherical symmetric value describes the effects of such geometric symmetry breaking.

It is therefore a natural question to see if the explicit calculation of the zeta determinant for the Laplace operator on this class of spaces is possible. In this work we give a positive answer to this question, establishing a general method that permits to compute the zeta regularized determinant on a deformed sphere of any dimension. Actually, for a particular discrete set of values of the deformation parameter k, the N-dimensional deformed sphere turns out to be isometric to the so-called orbifolded sphere, the quotient space S^N/Γ, of the standard N-sphere by a finite subgroup of the rotation group $O_N(\mathbb{R})$. Determinants on these spaces have been studied by J.S. Dowker in a series of works [25–27], where results are also obtained for different couplings. Under this point of view, the present work is a generalization of the results of Dowker to the continuous range of variation of the deformation parameter k, and in fact the results are consistent (see Sect. 4).

The main motivation of the present work, beside the particular result, is that the method introduced has the advantage of being completely general and not related to this specific problem. In particular, we show how it can be applied to obtain the main zeta invariants of some classes of abstract simple and double zeta functions (Sect. 4.2 and 4.3). In order to give the explicit form for the zeta function on the deformed spheres, we produce an explicit description of the spectrum and of the eigenfunctions of the associated Laplace operator in any dimension (Proposition 3.2). In particular, the 2 dimensional case turns out to be very interesting, from the point of view of geometry: in fact the 2 dimensional deformed sphere is a space with singularities of conical type. This class of singular spaces was introduced and studied by Cheeger [17] and although since then became a subject of deep interest and investigation, there are in fact relatively few occasions where explicit results can be obtained.

2. The Geometry of the Deformed Spheres

In this section we provide the definition of the N dimensional deformed sphere S_k^N, where k is the deformation parameter, and we study its geometry. This produces a particular interesting relation with elliptic function and conical singularity, at least in the 2 dimensional case. The deformed N-sphere is defined as the standard N-sphere with a singular Riemannian structure. When $N = 2$, we have an isometry with the surface immersed in $\mathbb{R}^3$ that can be obtained by rotating around an axis a curve described by an elliptic integral function. The surface obtained presents two singular points of conical type, as considered by Brüning and Seeley in [7] generalizing the definition of metric cone of Cheeger [17]. Thus, the 2 dimensional deformed sphere is a space with singularities of conical type, and due to the great interest in this kind of singular space, both from the point of view of differential geometry and zeta function analysis (see for example [23, 34, 19, 6, 68, 21]), its study is of particular interest (compare also with [56]).

Consider the immersion of the $N+1$ dimensional sphere S^{N+1} in $\mathbb{R}^{N+2}$,

$$\begin{cases} x_0 = \sin\theta_0 \sin\theta_1 \ldots \sin\theta_N, \\ x_1 = \sin\theta_0 \sin\theta_1 \ldots \cos\theta_N, \\ \ldots \\ x_{N+1} = \cos\theta_0, \end{cases}$$

and the induced metric (in local coordinates) $g_{S^{N+1}} = (d\theta_0)^2 + \sin^2\theta_0 g_{S^N}$. We deform this metric as follows. Let k be a real parameter with $0 < k \leq 1$, and consider the family

$$g_{S^{N+1}}[k] = (d\theta_0)^2 + \sin^2\theta_0 g_{S^N}[k],$$

$$g_{S^1}[k] = k^2(d\theta_0)^2.$$

This is a one parameter family of singular Riemannian metric on S^{N+1}. We call the singular Riemannian manifolds $(S^{N+1}, g_{S^{N+1}}[k])$ the deformed spheres of dimension $N+1$ and we use the notation S_k^{N+1}. By direct inspection, we see that the locus of the singular points of the metric in dimension $N+1$ is a sub-manifold isomorphic to two disjoint copies of S^{N-1}. In particular, in the 2 dimensional case we have

$$g_{S^2}[k] = (d\theta_0)^2 + k^2 \sin^2\theta_0 (d\theta_1)^2,$$

that shows that the deformed 2-sphere S_k^2 is a space with singularities of conical type as defined in [7]. Proceeding as in [7] Sect. 7, we will show in the next subsection that the singularity is generated by rotation of a curve in the plane.

Observe that, in a different language, S_k^2 is a periodic *lune*, that is to say it can be pictured by taking a segment of the standard 2-sphere (a lune) and identifying the sides. This situation generalizes to higher dimensions [1], and when the angle of the lune is $\frac{\pi}{n}$, $n \in \mathbb{Z}$, we obtain a spherical orbifold S^N/Γ, as pointed out in the introduction.

Note also that, by direct verification on the local description of the metric $g_{S^N}[k]$, the non-compact Riemannian manifold obtained by subtracting the singular subspace of the metric from S_k^N is a space of constant curvature and locally symmetric. It is not symmetric, as it is clear from the geometry of the low dimensional cases, or observing that it is not simply connected (see Corollary 8.3.13 of [65]). On the other side, the classical sphere S_1^N is a symmetric space; for example, the 2 dimensional one having the maximum number $\frac{N(N+1)}{2}$ of global isometries, namely the 3 spatial rotations. Therefore, the variation of the parameter k away from the trivial value produces a breaking of the global symmetric type of the space. In particular for example on the 2 sphere it breaks two continuous rotations in one discrete symmetry, namely the reflection through the horizontal plane.

We conclude this subsection with the explicit expression for the Laplace operator. With $a = \frac{1}{k}$, the (negative) of the induced Laplace operator on the deformed sphere S_k^{N+1} is

$$\Delta_{S_k^{N+1}} = -d_{\theta_0}^2 - N\frac{\cos\theta_0}{\sin\theta_0} d_{\theta_0} + \frac{1}{\sin^2\theta_0}\Delta_{S_k^N}.$$

2.1. Elliptic integrals and the deformed 2-sphere. The geometry of the 2 dimensional case is particularly interesting and this subsection is dedicated to its study. The ellipse $x^2 + \frac{y^2}{b^2} = 1$ can be given parametrically in the first quadrant by the formula

$$\begin{cases} x = t, \\ y = b\sqrt{1-t^2}, \end{cases}$$

where $0 \leq t \leq 1$. If we assume $b \leq 1$, the arc length is

$$l(t) = \int_0^t \sqrt{\frac{1-k^2s^2}{1-s^2}} ds,$$

where $k = \sqrt{1-b^2}$. With the new variables $t = \sin\theta$, $s = \sin\psi$, we obtain

$$\begin{cases} x = \sin\theta, \\ y = b\cos\theta, \end{cases}$$

with $0 \leq \theta \leq \frac{\pi}{2}$, and the arc length is

$$E(\theta, k) = l(\sin\theta) = \int_0^\theta \sqrt{1-k^2\sin^2\psi} d\psi,$$

that is the elliptic integral of the second kind in Legendre normal form [35] 8.110.2 (see [48] or [64] for elliptic functions and integrals). Note that we cannot find a parameterization of the curve by the arc length reversing the above equation using Jacobi elliptic functions. Consider now the curve $f(\sin\theta) = E(\theta, k)$. This is a smooth curve in the interval $0 \leq t \leq 1$, with $f(0) = 0$ and $f(1) = E(\frac{\pi}{2}, k)$. We can rotate this function around the horizontal axis getting a surface with a geometric singularity at the origin. For further use, it is more convenient to place the surface in the upper half space. Thus, we consider the function

$$f(t) = E(\arccos\frac{t}{k}, k) = \int_0^{\sqrt{1-\frac{t^2}{k^2}}} \sqrt{\frac{1-k^2s^2}{1-s^2}} ds,$$

with $0 \leq t \leq k$, and the curve: $x = t$, $y = f(t)$. We reparametrize this curve by its arc length

$$\theta = l(t) = \int_0^t \sqrt{1+(y'(s))^2} ds = \int_0^t \frac{1}{\sqrt{k^2-s^2}} ds = \arcsin\frac{t}{k},$$

obtaining

$$\begin{cases} x = k\sin\theta, \\ y = f(k\sin\theta) = E(\frac{\pi}{2}-\theta, k) = \int_0^{\frac{\pi}{2}-\theta} \sqrt{1-k^2\sin^2\psi} d\psi, \end{cases}$$

with $0 \leq \theta \leq \frac{\pi}{2}$ (as before θ is the angle from the vertical axis).

Let us now consider the surface Y_k^+ obtained by rotating the above curve along the vertical axis. We have the parameterization

$$Y_k^+ : \begin{cases} x = k \sin\theta \cos\phi, \\ y = k \sin\theta \sin\phi, \\ z = E(\frac{\pi}{2} - \theta, k), \end{cases}$$

where $0 \leq \phi \leq 2\pi$. This is clearly a smooth surface except at the possible singular point $(0, 0, E(\pi/2, k)))$, with the circle C_k: $x^2 + y^2 = k^2$, $z = 0$ of radius k as boundary. Moreover, since the coordinate line tangent vectors on the boundary are $v_\phi = -\sin\phi e_x + \cos\phi e_y$ and $v_\theta = e_z$, the tangent space is vertical and hence we can glue smoothly Y_k^+ with the surface Y_k^- obtained by reflecting through the horizontal plane. We call the surface obtained $Y_k^2 = Y_k^+ \cup Y_k^-$, and the parameter k deformation parameter. The surface X_k obtained from Y_k^2 by removing the poles $(0, 0, \pm k)$ is clearly a smooth (non-compact) surface. The Riemannian metric induced on X_k from the immersion in $\mathbb{R}^3$ is

$$g_{Y_k^2}(\theta, \phi) = (d\theta)^2 + k^2 \sin^2\theta (d\phi)^2.$$

It is clear that the local map $f : (\theta, \phi) \mapsto (\theta, \phi)$, extends to a diffeomorphism $f : Y_k^2 \to S^2$, and since $f^* g_{S^2}[k] = g_{Y_k^2}$, it follows that f is an isometry between $S_k^2 = (S^2, g_{S^2}[k])$ and $(Y_k^2, g_{Y_k^2})$.

3. Spectral Analysis

In this section we give the eigenvalues and eigenfunctions of the Laplace operator on the deformed sphere. As observed in Sect. 2, the two dimensional case is of particular interest, since it represents an instance of a space with singularities of conical type that can be solved explicitly. Therefore we spend a few words to describe the concrete operator appearing in that case, using the language of spectral analysis for spaces with conical singularities [17, 7]. With $a = \frac{1}{k}$, the (negative) of the induced Laplace operator on the deformed sphere S_k^2 is

$$\Delta_{S^2_{1/a}} = -\partial_\theta^2 - \frac{\cos\theta}{\sin\theta}\partial_\theta - \frac{a^2}{\sin^2\theta}\partial_\phi^2,$$

on $L^2(S^2_{1/a})$. With the Liouville transform $u = Ev$, with $E(\theta) = \frac{1}{\sqrt{\sin\theta}}$, we obtain the operator

$$L_a = -\partial_\theta^2 - \frac{a^2}{\sin^2\theta}\partial_\phi^2 - \frac{1}{4}\left(1 + \frac{1}{\sin^2\theta}\right).$$

This is a regular singular operator as defined in [7],

$$L_a = -d_\theta^2 + \frac{1}{\theta^2}A(\theta),$$

where

$$A(\theta) = \frac{\theta^2}{\sin^2\theta}\left(-a^2\partial_\phi^2 - \frac{1}{4}\right) - \frac{1}{4}\theta^2,$$

is a family of operators on the section of the cone, that is the circle of radius 1. It is clear that the operator $-\partial_\phi^2$ has the complete system $\{\mu_m = m^2, \mathrm{e}^{im\phi}\}$, with $m \in \mathbb{Z}$, where all the eigenvalues are double up to the null one that is simple with the unique eigenfunction given by the constant map. Since the problem decomposes spectrally on this system, we reduce to study the family of singular Sturm operators

$$T_{am} = -d_\theta^2 + \frac{a^2m^2 - \frac{1}{4}}{\sin^2\theta} - \frac{1}{4}.$$

In order to define an appropriate self adjoint extension, we introduce the following boundary conditions at the singular points:

$$BC_0 : \qquad \lim_{\theta\to 0} \theta^{am}\left[\left(am + \frac{1}{2}\right)\frac{1}{\theta} f(\theta) - \sqrt{\theta} f'(\theta)\right] = 0,$$

and

$$BC_\pi : \qquad \lim_{\theta\to\pi} (\theta-\pi)^{am}\left[\left(am + \frac{1}{2}\right)\frac{1}{\theta-\pi} f(\theta-\pi) - \sqrt{\theta-\pi} f'(\theta-\pi)\right] = 0.$$

These are the natural generalizations of the classical Dirichlet boundary conditions (compare with [62] 8.4) and were first considered in [7]. In particular, it was proved in [7], Sect. 7, that the self adjoint extension defined by these conditions is the Friedrich extension.

The eigenvalues equation associated with the operators T_{am}, can be more easily studied going back to the original Hilbert space. This equation was in fact already studied by Gromes [36], who found a complete solution. Generalizing the standard approach used for the standard sphere (see for example [39]), we can prove that in fact this solution provide a complete set of eigenvalues and eigenfunctions for the metric Laplacian, as stated in the following lemma.

Lemma 3.1. *The operator $\Delta_{S^2_{1/a}}$, has the complete system:*

$$\lambda_{n,m} = (a|m| + n)(a|m| + n + 1), \ \ n \in \mathbb{N}, m \in \mathbb{Z},$$

where all the eigenvalues with $m \neq 0$ are double with eigenfunctions (where the P_ν^μ are the associated Legendre functions)

$$\mathrm{e}^{iam\phi} P_{am+n}^{-am}, \quad \mathrm{e}^{-iam\phi} P_{am+n}^{-am},$$

while the eigenvalues $n(n+1)$ are simple with eigenfunctions the functions P_n.

Next, we pass to the higher dimensions. The (negative) of the induced Laplace operator on the deformed sphere S_k^{N+1} is

$$\Delta_{S_k^{N+1}} = -d_{\theta_0}^2 - N\frac{\cos\theta_0}{\sin\theta_0} d_{\theta_0} + \frac{1}{\sin^2\theta_0}\Delta_{S_k^N}$$

on $L^2(S_k^{N+1})$. Projecting on the spectrum of $\Delta_{S_k^N}$, we obtain the differential equation

$$\left[-d_{\theta_0}^2 - N\frac{\cos\theta_0}{\sin\theta_0} d_{\theta_0} + \frac{\lambda_{S_k^N}}{\sin^2\theta_0}\right] u = \lambda_{S_k^{N+1}} u.$$

Following [52], we make the substitutions

$$u(\theta_0) = \sin^b \theta_0 v(\theta_0),$$

$$z = \frac{1}{2}(\cos\theta_0 + 1),$$

where $b = \frac{1}{2}\left(1 - N + \sqrt{(N-1)^2 + 4\lambda_{S_k^N}}\right)$. This gives the hypergeometric equation [35] 9.151,

$$z(1-z)v'' + [\gamma - (\alpha + \beta + 1)z]v' - \alpha\beta v = 0,$$

with

$$\alpha = \frac{1}{2}\left(2b + N \mp \sqrt{N^2 + 4\lambda_{S_k^{N+1}}}\right),$$

$$\beta = \frac{1}{2}\left(2b + N \pm \sqrt{N^2 + 4\lambda_{S_k^{N+1}}}\right),$$

$$\gamma = \frac{1}{2}(2b + N + 1).$$

Boundary conditions give the equation

$$2n + 2b + N = \pm\sqrt{N^2 + 4\lambda_{S_k^{N+1}}},$$

where $n \in \mathbb{N}$, that, in turn, gives the recurrence relation

$$\lambda_{S_k^{N+1}} = n^2 + \left(1 + \sqrt{(1-N)^2 + 4\lambda_{S_k^N}}\right) n + \frac{1}{2}\left(1 - N + \sqrt{(1-N)^2 + 4\lambda_{S_k^N}} + 2\lambda_{S_k^N}\right).$$

We can prove that this recurrence relation is satisfied by the numbers

$$\lambda_{S_k^N} = \left(am + n_1 + \cdots + n_{N-1} + \frac{N-1}{2}\right)^2 - \frac{(N-1)^2}{4},$$

where $n_i \in \mathbb{N}$, must be a positive integer. We have obtained

$$b = am + n_1 + \cdots + n_{N-1},$$

$$\alpha = -n_N,$$

$$\beta = 2(am + n_1 + \cdots + n_{N-1}) + n_N + N,$$

$$\gamma = am + n_1 + \cdots + n_{N-1} + \frac{N+1}{2},$$

and the family of solutions for the eigenvalues equation (up to a constant)

$$u_{n_N}(\cos\theta_0) = \sin^{\frac{1-N}{2}} \theta_0 P^{-am-n_1-\cdots-n_{N-1}-\frac{N-1}{2}}_{am+n_1+\cdots+n_{N-1}+\frac{N-1}{2}+n_N}(\cos\theta_0).$$

Using standard argument, we can then prove the following result.

Lemma 3.2. *The operator* $\Delta_{S^{N+1}_{1/a}}$, *has the complete system:*

$$\lambda_{m,n_1,\ldots,n_N} = (a|m| + n_1 + \cdots + n_N)(a|m| + n_1 + \cdots + n_N + N), \; n_i \in \mathbb{N}, m \in \mathbb{Z},$$

where all the eigenvalues with $m \neq 0$ *are double with eigenfunctions (up to normalization)*

$$\mathrm{e}^{iam\theta_N} \prod_{j=0}^{N-1} \sin^{\frac{1-N+j}{2}}(\theta_j) P^{-am-n_1-\cdots-n_{N-1-j}-\frac{N-1-j}{2}}_{am+n_1+\cdots+n_{N-j}+\frac{N-1-j}{2}}(\cos\theta_j),$$

$$\mathrm{e}^{-iam\theta_N} \prod_{j=0}^{N-1} \sin^{\frac{1-N+j}{2}}(\theta_j) P^{-am-n_1-\cdots-n_{N-1-j}-\frac{N-1-j}{2}}_{am+n_1+\cdots+n_{N-j}+\frac{N-1-j}{2}}(\cos\theta_j),$$

while the eigenvalues with $m = 0$ *are simple with eigenfunctions*

$$\prod_{j=0}^{N-1} \sin^{\frac{1-N+j}{2}}(\theta_j) P^{-n_1-\cdots-n_{N-1-j}-\frac{N-1-j}{2}}_{n_1+\cdots+n_{N-j}+\frac{N-1-j}{2}}(\cos\theta_j).$$

4. Zeta Regularized Determinants

In this section we study the zeta function associated to the Laplace operator on the deformed sphere S^{N+1}_k. For this, we introduce two quite general classes of zeta functions and we compute the main zeta invariants of them. This allows us to define a general technique to obtain the zeta regularized determinant of the Laplace operator on S^{N+1}_k as a function of the deformation parameter. We apply this technique to the lower cases, $N = 1$ and 2, giving explicit formulas. Our last result is the computation of the coefficients in the expansions of the zeta determinants in powers of the deformation parameter.

By Lemma 3.2, the zeta function on S^{N+1}_k is the function defined by the series

$$\zeta(s, \Delta_{S^{N+1}_{1/a}}) = \sum_{n\in\mathbb{N}^N_0} [n(n+N)]^{-s} + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{N}^N} [(am+n)(am+n+N)]^{-s},$$

when $\mathrm{Re}(s) > N+1$, and by analytic continuation elsewhere. Here n is a positive integer vector $n = (n_1, \ldots, n_N)$, and the notation $\mathbb{N}^N_0$ means $\mathbb{N} \times \cdots \times \mathbb{N} - \{0, \ldots, 0\}$.

Multidimensional gamma and zeta functions, namely zeta functions where the general term is of the form $(n^T an + b^T n + c)^{-s}$, where a is a real symmetric matrix of rank $k \geq 1$, b a vector in $\mathbb{R}^k$, c a real number and n an integer vector in $\mathbb{Z}^k$, were originally introduced by Barnes [3, 4] and Epstein [32, 33] as natural generalizations of the Euler gamma function. Whenever the sum is on the integers (i.e. $n \in \mathbb{Z}^k$), there is a large symmetry that allows one to express the zeta function by a theta series. Multidimensional theta series have been deeply studied in the literature, and by a generalization of the Poisson summation formula (see for example [16] XI.2, 3) it is possible to compute the main zeta invariants for multiple series of this type (see [63, 47, 30, 31 , and 15] and references there in). The main problem in the present case is that the zeta functions are associated to series of Dirichlet type, namely the sums are over $\mathbb{N}^k_0$. We lose then many symmetries and in particular a formula of Poisson type. Consequently, it is more difficult to find general results, and different techniques have been introduced to deal with the specific cases (see for example [12, 13, 18, 29, 54, 55] for simple series or series that can be reduced to simple series or [15, 46] for multiple linear series). Note in particular that the case of a double ($k = 2$) homogeneous quadratic series of Dirichlet type is much harder. The zeta functions of this type (with integer coefficients) appear when dealing with the zeta functions of a narrow ideal class for a real quadratic field as shown by Zagier in [66 and 67], where he also computes the values at non-positive integers (see also [51, 28, 14, 15], and in particular [58] for the derivative). Beside, we can overcome this difficulty in the case under study first by reducing the multi-dimensional zeta functions to a sum of 2 dimensional linear and quadratic zeta functions, and then studying the quadratic one by means of a general method introduced in [59] in order to deal with non homogeneous zeta functions. Note that, for particular values of the deformation parameter, the zeta function can be reduced to a sum of zeta functions of Barnes type [3], and this allows a direct computation of the main zeta invariants [25, 26]. This approach does not work for generic values of the deformation parameter, and therefore the more sophisticated technique introduced here is necessary.

We present in the next subsection some generalizations of some results of [59] necessary in order to treat the present case, and we give in the following subsections some applications to the case of some general classes of abstract simple and double zeta functions. As explained hereafter, by means of these two classes of zeta functions, we can in principle calculate the zeta invariants for the deformed sphere in any dimensions. Eventually in the last subsections we apply the method to obtain the main zeta invariants for the zeta functions on the 2 and 3 dimensional deformed spheres.

By the following lemma (see [60 or 59]), we can reduce $\zeta(s, \Delta_{S_k^{N+1}})$ to a sum of simple and double zeta functions.

Lemma 4.1. *Let $f(z)$ be a regular function of z. Then*

$$\sum_{n\in\mathbb{N}^{N+1}} f(n) = \sum_{n=0}^{\infty} \binom{n+N}{N} f(n), \quad \sum_{n\in\mathbb{N}_0^{N+1}} f(n) = \sum_{n=1}^{\infty} \binom{n+N}{N} f(n).$$

Proposition 4.2. *The zeta function associated with the Laplace operator on the $N + 1$ dimensional deformed sphere is ($N \geq 1$)*

$$\zeta(s, \Delta_{S_{1/a}^{N+1}}) = \sum_{n=1}^{\infty} \binom{n+N-1}{N-1} [n(n+N)]^{-s} + 2 \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \binom{n+N-1}{N-1} [(am+n)(am+n+N)]^{-s}.$$

Since $\binom{n+N-1}{N-1} = P_N(n)$ is a polynomial of order N in n, and since given any polynomial $P_N(n)$ we have a polynomial $Q_N(n + x)$ for any given x, such that $P_N(n) = Q_N(n + x)$ (and we can find explicitly the coefficients of Q as functions on those of P and x), it is sufficient to consider the two classes of zeta functions

$$z(s; \alpha, 2, x, p) = \sum_{n=1}^{\infty} (n + x)^{\alpha} [(n + x)^2 + p]^{-s},$$

and

$$Z(s; \alpha, a, x, p) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{\alpha}[(n + am + x)^2 + p]^{-s}.$$

This will be done in Subsects. 4.2 and 4.3, but first, the next subsection is dedicated to recall and generalize some results on sequences of spectral type and associated zeta functions introduced in [59], necessary in the following.

4.1. Sequences of spectral type and zeta invariants. In this subsection we will use some concepts and results developed in [59], that briefly we recall here. We refer to that work for further details and complete proofs.

Let $T = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers with unique accumulation point at infinite, finite exponent s_0 and genus q. We associate to T, the heat function

$$f(t, T) = 1 + \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t},$$

the logarithmic Fredholm determinant

$$\log F(z, T) = \log \prod_{n=1}^{\infty} \left(1 + \frac{z}{\lambda_n}\right) \mathrm{e}^{\sum_{j=1}^{q} \frac{(-1)^j}{j} \frac{z^j}{\lambda_n^j}},$$

and the zeta function

$$\zeta(s, T) = \sum_{n=1}^{\infty} \lambda_n^{-s}.$$

The sequence T is called of spectral type if there exists an asymptotic expansion of the associated heat function for small t in powers of t and powers of t times positive integer powers of $\log t$. In particular it is said to be a simply regular sequence of spectral type if the associated zeta function has at most simple poles (see [59] pp. 4 and 9). Formulas to deal with the zeta invariants for sequences of spectral type are given in [59]. In particular, there are considered non-homogeneous sequences as well. We generalize the concept of non-homogeneous sequence here, by considering, for any given sequence of spectral type $T_0 = \{\lambda_n\}_{n=1}^{\infty}$, the shifted sequence $T_d = \{\lambda_n + d\}_{n=1}^{\infty}$, where d is a parameter, subject to the unique condition that $\mathrm{Re}(\lambda_n + d)$ is always positive. We can prove the following results for a shifted sequence (see [59] Proposition 2.9 and Corollary 2.10 for details).

Lemma 4.3. *Let $T_0 = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of finite exponent s_0 and genus q, then the associated shifted sequence $T_d = \{\lambda_n + d\}_{n=1}^{\infty}$, with d such that* $\mathrm{Re}(\lambda_n + d) > 0$ *for all n, is a sequence of finite exponent s_0 and genus q. Moreover, T_0 is of spectral type if and only if T_d is of spectral type. If T_0 is simply regular, so is T_d.*

Proposition 4.4. *Let $T_0 = \{\lambda_n\}_{n=1}^{\infty}$ be a simply regular sequence of spectral type with finite exponent s_0 and genus q, and $T_d = \{\lambda_n + d\}_{n=1}^{\infty}$, with d such that* $\mathrm{Re}(\lambda_n + d) > 0$ *for all n, an associated shifted sequence. Then,*

$$\zeta(0, T_d) = \zeta(0, T_0) + \sum_{j=1}^{q} \frac{(-1)^j}{j} \mathrm{Res}_1(\zeta(s, T_0), s = j) d^j,$$

$$\zeta'(0, T_d) = \zeta'(0, T_0) - \log F(d, T_0) +$$

$$+ \sum_{j=1}^{q} \frac{(-1)^j}{j} \left[\mathrm{Res}_0(\zeta(s, T_0), s = j) + (\gamma + \psi(j)) \mathrm{Res}_1(\zeta(s, T_0), s = j) \right] d^j.$$

Proposition 4.5. *Let $T_0 = \{\lambda_n\}_{n=1}^{\infty}$ be a simply regular sequence of spectral type with finite exponent s_0 and genus q. Let $L_0 = \{\lambda_n^2\}_{n=1}^{\infty}$, and d such that* $\mathrm{Re}(\lambda_n + d) > 0$ *for all n. Then,*

$$\zeta(0, L_{d^2}) = \frac{1}{2} \left[\zeta(0, T_{id}) + \zeta(0, T_{-id} \right],$$

$$\zeta'(0, L_{d^2}) = \zeta'(0, T_{id}) + \zeta'(0, T_{-id}) - \sum_{j=1}^{\left[\frac{p}{2}\right]} \frac{(-1)^j}{j} \sum_{k=1}^{j} \frac{1}{2k-1} \mathrm{Res}_1(\zeta(s, T_0), s = 2j) d^{2j}.$$

Remark 4.6. Note that the numbers λ_n in the sequence need not to be different, i.e. the cases with multiplicity are covered by Propositions 4.4 and 4.5. In particular, assume the sequence is $T_0 = \{\lambda_n\}_{n=1}^{\infty}$, each λ_n having multiplicity ρ_n (we cover the case of a general abstract multiplicity, given by any positive real number). Then, the unique difficulty can be in defining the exponent of convergence of the sequence. But actually for our purpose it is sufficient to know the genus, and this can be obtained whenever we know the asymptotic of λ_n and ρ_n for large n. In fact, if $\lambda_n \sim n^b$ and $\rho_n \sim n^a$, then the general term of the associated zeta function behaves as n^{a-bs}, and therefore the genus is $q = \left[\frac{a+1}{b}\right]$ (the integer part).

Some more remarks on these results are in order. First, note that the approach of considering some general class of abstract sequences and of studying the analytic properties of the associated spectral functions has been developed by various authors, and in particular instances of Proposition 4.4 can be found in the literature. The original idea is probably due to Voros [61], while a good reference for a rigorous and very general setting is the work of Jorgenson and Lang [41]. However, for our purpose here, the simpler setting of [59] is more convenient. Second, observe that Proposition 4.5 was originally proved by Choi and Quine in [18], and also obtained in [25], Eq. (25). In particular, the reader can see the proof given in [59], as the more rapid route to this result suggested in [25].

4.2. A class of simple zeta functions. We consider the following class of simple zeta functions (compare with [57])

$$z(s; \alpha, \beta, x, p) = \sum_{n=1}^{\infty} (n + x)^{\alpha} [(n + x)^{\beta} + p]^{-s},$$

for $\mathrm{Re}(s) > \frac{1+\alpha}{\beta}$, where α and β are real positive numbers, and x and p are real numbers subject to the conditions that $n + x > 0$ and $(n + x)^\beta + p > 0$ for all n.

Note that different equivalent techniques could be applied to deal with this case; namely one could use the Plana theorem as in [54], a regularized product like in [18], a complex integral representation as in [55], or heat-kernel techniques [30, 31].

Proposition 4.7. *The function $z(s; \alpha, \beta, x, p)$ has a regular analytic continuation in the whole complex s-plane up to simple poles at $s = \frac{1+\alpha}{\beta} - j$, $j = 0, 1, 2, \dots$, whenever these values are not $0, -1, -2, \dots$. The origin is a regular point and if $\frac{1+\alpha}{\beta}$ is not a positive integer*

$$z(0; \alpha, \beta, x, p) = \zeta_H(-\alpha, x + 1),$$

and

$$z'(0; \alpha, \beta, x, p) = \beta\zeta_H'(-\alpha, x + 1) + \sum_{j=1}^{\left[\frac{\alpha+1}{\beta}\right]} \frac{(-1)^j}{j}\zeta_H(\beta j - \alpha, x + 1)p^j + \\ - \log \prod_{n=1}^{\infty} \left(1 + \frac{p}{(n + x)^\beta}\right)^{(n+x)^\alpha} \mathrm{e}^{(n+x)^\alpha \sum_{j=1}^{\left[\frac{\alpha+1}{\beta}\right]} \frac{(-1)^j}{j} \frac{p^j}{(n+x)^{\beta j}}},$$

while if $\frac{1+\alpha}{\beta}$ is a positive integer

$$z(0; \alpha, \beta, x, p) = \zeta_H(-\alpha, x + 1) + \frac{(-1)^{\frac{\alpha+1}{\beta}}}{\alpha + 1} p^{\frac{\alpha+1}{\beta}},$$

and

$$z'(0; \alpha, \beta, x, p) = \beta\zeta_H'(-\alpha, x + 1) + \sum_{j=1}^{\frac{\alpha+1}{\beta}-1} \frac{(-1)^j}{j}\zeta_H(\beta j - \alpha, x + 1)p^j + \\ + \frac{(-1)^{\frac{\alpha+1}{\beta}}}{\frac{\alpha+1}{\beta}} \left[-\Psi(x + 1) + \left(\gamma + \Psi\left(\frac{1 + \alpha}{\beta}\right)\right)\frac{1}{\beta}\right] p^{\frac{\alpha+1}{\beta}} + \\ - \log \prod_{n=1}^{\infty} \left(1 + \frac{p}{(n + x)^\beta}\right)^{(n+x)^\alpha} \mathrm{e}^{(n+x)^\alpha \sum_{j=1}^{\frac{\alpha+1}{\beta}} \frac{(-1)^j}{j} \frac{p^j}{(n+x)^{\beta j}}}.$$

Proof. The result follows applying Proposition 4.4. First, note that the unshifted sequences is $T_0 = \{(n + x)^\beta\}$, with multiplicity $(n + x)^\alpha$. By the Remark 4.6, the sequence has genus $q = \left[\frac{\alpha+1}{\beta}\right]$. The associated zeta function is

$$z(s; \alpha, \beta, x, 0) = \zeta_H(\beta s - \alpha, x + 1),$$

and this clearly shows that T_0 is a simply regular sequence of spectral type, and so is T_p by Lemma 4.3. The unique pole is at $s = \frac{1+\alpha}{\beta}$ and

$$\mathrm{Res}_1\left(z(0;\alpha,\beta,x,0), s=\frac{1+\alpha}{\beta}\right)=\frac{1}{\beta},$$

$$\mathrm{Res}_0\left(z(0;\alpha,\beta,x,0), s=\frac{1+\alpha}{\beta}\right)=-\psi(x+1).$$

The associated Fredholm determinant is

$$F(z,T_0)=\prod_{n=1}^{\infty}\left(1+\frac{z}{(n+x)^\beta}\right)^{(n+x)^\alpha}\mathrm{e}^{(n+x)^\alpha\sum_{j=1}^{q}\frac{(-1)^j}{j}\frac{z^j}{(n+x)^{\beta j}}}.$$

Next, using the expression given in the proof of Proposition 4.4 we have

$$z(s;\alpha,\beta,x,p)=\sum_{j=0}^{\infty}\binom{-s}{j}\zeta_H(\beta(s+j)-\alpha)p^j,$$

thus we have poles when $\beta(s+j)-\alpha=1$, i.e. $s=\frac{1+\alpha}{\beta}-j$, $j=0,1,2,\ldots$, when ever these values are not $0,-1,-2,\ldots$, and the residua are easily computed. To obtain the value at $s=0$, it is useful to distinguish two cases (see [57]). In fact, from the above expression, when $s=0$ the unique term that is singular is the one with $\beta j-\alpha=1$, i.e. $j=\frac{\alpha+1}{\beta}$, that is necessarily a positive integer since $\alpha\geq 0$. Now, if $\frac{\alpha+1}{\beta}$ is not a positive integer, then we have no integer poles, $q=\left[\frac{\alpha+1}{\beta}\right]\neq\frac{\alpha+1}{\beta}$, and hence $z(0;\alpha,\beta,x,p)=z(0;\alpha,\beta,x,0)$, and since

$$\mathrm{Res}_0\left(z(s;\alpha,\beta,x,0), s=j\right)=z(j;\alpha,\beta,x,0)=\zeta_H(\beta j-\alpha,x+1),$$

$$z'(0;\alpha,\beta,x,p)=z'(0;\alpha,\beta,x,0)+\sum_{j=1}^{\left[\frac{\alpha+1}{\beta}\right]}\frac{(-1)^j}{j}\mathrm{Res}_0\left(z(s;\alpha,\beta,x,0), s=j\right)p^j-\log F(p,T_0).$$

If $\frac{\alpha+1}{\beta}$ is a positive integer, we have a pole, $q=\left[\frac{\alpha+1}{\beta}\right]=\frac{\alpha+1}{\beta}$, and we need to take in account also the residuum. As we have seen, since the Hurwitz zeta function has only one pole at $s=1$ with residuum 1, all the terms up to the ones with $j=0$ and the one with $j=\frac{\alpha+1}{\beta}$ have vanishing residuum, and we obtain

$$z(0;\alpha,\beta,x,p)=z(0;\alpha,\beta,x,0)+\frac{(-1)^{\frac{\alpha+1}{\beta}}}{\frac{\alpha+1}{\beta}}\mathrm{Res}_1\left(z(s;\alpha,\beta,x,0), s=\frac{1+\alpha}{\beta}\right)p^{\frac{\alpha+1}{\beta}},$$

and

$$\begin{aligned}z'(0;\alpha,\beta,x,p)&=z'(0;\alpha,\beta,x,0)+\sum_{j=1}^{\frac{\alpha+1}{\beta}-1}\frac{(-1)^j}{j}\mathrm{Res}_0\left(z(s;\alpha,\beta,x,0), s=j\right)p^j+\\&+\frac{(-1)^{\frac{\alpha+1}{\beta}}}{\frac{\alpha+1}{\beta}}\left[\mathrm{Res}_0\left(z(s;\alpha,\beta,x,0), s=\frac{1+\alpha}{\beta}\right)+\right.\\&\left.+\left(\gamma+\Psi\left(\frac{1+\alpha}{\beta}\right)\right)\mathrm{Res}_1\left(z(s;\alpha,\beta,x,0), s=\frac{1+\alpha}{\beta}\right)\right]p^{\frac{\alpha+1}{\beta}}-\log F(p,T_0),\end{aligned}$$

that gives the formula stated in the thesis.

4.3. A class of double zeta functions. Consider the following class of double zeta functions

$$Z(s; \alpha, a, x, p) = \sum_{m,n=1}^{\infty} n^{\alpha}[(am + n + x)^2 + p]^{-s},$$

for $\mathrm{Re}(s) > 1 + \alpha$, and where x and p are real constants subject to the conditions that $am + n + x > 0$ and $(am + n + x)^2 + p > 0$ for all n and m, and α is a non-negative integer (the case where α is any real number can be treated by similar methods, but is much more complicated, see [56]).

Remark 4.8. In the more general case

$$Z(s; \alpha, \beta, a, x, p) = \sum_{m,n=1}^{\infty} n^{\alpha}[(am + n + x)^{\beta} + p]^{-s},$$

for $\mathrm{Re}(s) > \frac{2(1+\alpha)}{\beta}$, and where x and p are real constants subject to the conditions that $am + n + x > 0$ and $(am + n + x)^{\beta} + p > 0$ for all n and m, we would have genus $q = \left[\frac{2(1+\alpha)}{\beta}\right]$ by Remark 4.6 since the leading term behaves like $n^{\alpha}n^{-\beta s/2}$, but we would not be able to prove that these are regular sequences of spectral type as in the following proof of Lemma 4.9.

The sequences appearing in these zeta functions are: $S_0 = \{\lambda_{m,n} = (am + n + x)^2\}_{m,n=1}^{\infty}$ and the associated shifted sequence $S_p = \{\lambda_{m,n} + p\}_{m,n=1}^{\infty}$, both with multiplicity n^{α}. These are sequences with finite exponent and genus $q = [1 + \alpha]$ by Remark 4.6. We first show that S_p is a simply regular sequence of spectral type.

Lemma 4.9. *The sequence* $S_p = \{(am + n + x)^2 + p\}_{m,n=1}^{\infty}$ *is a simply regular sequence of spectral type.*

Proof. By Lemma 4.3, we need to show that there exists an expansion of the desired type for the heat function

$$f(t, S_0) = 1 + \sum_{m,n=1}^{\infty} n^{\alpha} \mathrm{e}^{-(am+n+x)^2 t}.$$

Consider the sequence $L = \{am+n+x\}_{m,n=1}^{\infty}$, with multiplicity n^{α}, of finite exponent and genus 2 (since $m^a + n^b \le (mn)^{\frac{ab}{a+b}}$). The associated heat function is

$$f(t, L) = 1 + \sum_{m,n=1}^{\infty} n^{\alpha} \mathrm{e}^{-(am+bn+c)t},$$

and the associated Fredholm determinant is

$$F(z, L) = \prod_{m,n=1}^{\infty} \left(1 + \frac{z}{am + bn + c}\right)^{n^{\alpha}} \mathrm{e}^{\sum_{j=1}^{2} \frac{(-1)^j}{j} \frac{n^{\alpha} z^j}{(am+bn+c)^j}}.$$

Since

$$f(t, L) = 1 + \sum_{m,n=1}^{\infty} n^{\alpha} \mathrm{e}^{-(am+bn+c)t} = 1 + \mathrm{e}^{-ct} \sum_{m=1}^{\infty} \mathrm{e}^{-amt} \sum_{n=1}^{\infty} n^{\alpha} e^{-bnt},$$

and we have an expansion of each factor in powers of t (see [57] Sect. 3.1 for the last sum), it is clear that we have an expansion of the form

$$f(t, L) = \sum_{j=0}^{\infty} e_j t^{\delta_j}.$$

By Lemma 2.5 of [57], L is simply regular, and hence the unique logarithmic terms in the expansion of $F(z, L)$ are of the form $z^k \log z$, with integer $k \leq 2$. Now, consider the product

$$F(iz, L)F(-iz, L) = \prod_{m,n=1}^{\infty} \left(1 + \frac{iz}{am + bn + c}\right)^{n^{\alpha}} \left(1 - \frac{iz}{am + bn + c}\right)^{n^{\alpha}} \times$$

$$\times \mathrm{e}^{\sum_{j=1}^{2} \frac{(-1)^j}{j} \frac{n^{\alpha}(iz)^j}{(am+bn+c)^j}} \mathrm{e}^{\sum_{j=1}^{2} \frac{(-1)^j}{j} \frac{n^{\alpha}(-iz)^j}{(am+bn+c)^j}}.$$

Since $i^j + (-i)^j = 0$ for odd j, and -2 when $j = 2$, this gives

$$F(iz, L)F(-iz, L) = \prod_{m,n=1}^{\infty} \left(1 + \frac{z^2}{(am + bn + c)^2}\right)^{n^{\alpha}} \mathrm{e}^{\frac{1}{2} \frac{n^{\alpha} z^2}{(am+bn+c)^2}} = F(z^2, S_0),$$

and we obtain a decomposition of the Fredholm determinant associated to the sequence S_0. This means that $\log F(z, S_0)$ has an expansion with unique logarithmic terms of the form $z^k \log z$, with integer $k \leq 1$, and therefore S_0 is a simply regular sequence of spectral type by Lemma 2.5 of [59].

Lemma 4.9 shows that the sequence appearing in the definition of the function $Z(s; \alpha, a, x, p) = \zeta(s, S_p)$ are such that we can apply Proposition 4.4 in order to obtain all the desired zeta invariants. For we need explicit knowledge of the zeta invariants of the sequence S_0. This is in the next lemma.

Lemma 4.10. *The function $\chi(s; \alpha, a, x)$ defined for real a and x such that $am+n+x > 0$, for all $m, n \in \mathbb{N}_0$, and α a non-negative integer, by the sum*

$$\chi(s; \alpha, a, x) = \sum_{m,n=1}^{\infty} n^{\alpha} (am + n + x)^{-s},$$

when $\mathrm{Re}(s) > 2(\alpha + 1)$*, can be continued analytically to the whole complex plane up to a finite set of simple poles at $s = 1, 2, \ldots, \alpha + 2$, by means of the following formula:*

$$\chi(s;\alpha,a,x)=\frac{1}{2}a^{-s}\zeta_H(s,(x+1)/a+1)+\frac{a^{1-s}}{s-1}\zeta_H(s-1,(x+1)/a+1)+$$

$$+\sum_{j=1}^{\alpha}\frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{(s-1)(s-2)\dots(s-j-1)}a^{j+1-s}\zeta_H(s-j-1,(x+1)/a+1)+$$

$$+ia^{-s}\int_0^\infty\frac{(1+iy)^\alpha\zeta_H(s,(x+1+iy)/a+1)-(1-iy)^\alpha\zeta_H(s,(x+1-iy)/a+1)}{e^{2\pi y}-1}dy.$$

In particular, this shows that the point $s=0$ is a regular point.

Proof. We apply the Plana theorem as in [54]. Since the general term behaves as $n^\alpha n^{-s/2}$, we assume $\mathrm{Re}(s)>2(\alpha+1)$,

$$\chi(s;\alpha,a,x)=\frac{1}{2}\sum_{m=1}^{\infty}(am+x+1)^{-s}+\sum_{m=1}^{\infty}\int_1^\infty t^\alpha(am+t+x)^{-s}dt+$$

$$+i\sum_{m=1}^{\infty}\int_0^\infty\frac{(1+iy)^\alpha(am+x+1+iy)^{-s}-(1-iy)^\alpha(am+x+1-iy)^{-s}}{e^{2\pi y}-1}dy.$$

Recall that α is a non-negative integer, then we can integrate recursively the middle term obtaining, for $\alpha>0$,

$$\int_1^\infty t^\alpha(am+t+x)^{-s}dt=\sum_{j=0}^{\alpha}\frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{(s-1)(s-2)\dots(s-j-1)}(am+x+1)^{j+1-s};$$

this gives

$$\chi(s;\alpha,a,x)=\frac{1}{2}a^{-s}\sum_{m=1}^{\infty}(m+(x+1)/a)^{-s}+\frac{a^{1-s}}{s-1}(m+(x+1)/a)^{1-s}+$$

$$+\sum_{j=1}^{\alpha}\frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{(s-1)(s-2)\dots(s-j-1)}a^{j+1-s}\sum_{m=1}^{\infty}(m+(x+1)/a)^{j+1-s}+$$

$$+ia^{-s}\sum_{m=1}^{\infty}\int_0^\infty\frac{(1+iy)^\alpha(m+(x+1+iy)/a)^{-s}-(1-iy)^\alpha(m+(x+1-iy)/a)^{-s}}{e^{2\pi y}-1}dy,$$

and, due to uniform convergence of the integral, concludes the proof.

Remark 4.11. We could deal with this kind of double zeta function by applying the classical integral formula of Hermite as in the case of the Riemann zeta function. This approach confirms the above results, but it would not give a tractable expression for the singular part.

We can now obtain the zeta invariants of the zeta function $Z(s;\alpha,a,x,p)$ for all the acceptable values of the parameters. This allows us to compute the regularized determinant of the deformed sphere of any dimension, as pointed out at the beginning of this section. Besides, we will give explicit formulas and results for the low dimensional cases in the next subsections.

4.4. Zeta determinant on the deformed 2 sphere. By Proposition 4.2, the zeta function associated to the operator $\Delta_{S^2_{1/a}}$ is the function defined by the series

$$\zeta(s,\Delta_{S^2_{1/a}})=\sum_{n=1}^{\infty}[n(n+1)]^{-s}+2\sum_{m=1,n=0}^{\infty}[(am+n)(am+n+1)]^{-s},$$

when $\mathrm{Re}(s)>2$, and by analytic continuation elsewhere. The aim of this section is to study this zeta function and in particular to obtain a formula for the values of $\zeta(0,\Delta_{S^2_{1/a}})$ and $\zeta'(0,\Delta_{S^2_{1/a}})$. When $a=1$, this reduces to the zeta function on the 2-sphere: $\zeta(s,\Delta_{S^2_{1/a=1}})=\sum_{n=1}^{\infty}(2n+1)(n^2+n)^{-s}$ [18, 54, 55]. The zeta function $\zeta(s,S_{S^2_{1/a}})$ decomposes as

$$\zeta(s,\Delta_{S^2_{1/a}})=z(s;0,2,1/2,-1/4)+2Z(s;0,a,-1/2,-1/4),$$

and we can easily check that the values of the parameters satisfy the condition of definition of these functions. We provide two equivalent formulas for the zeta determinant on the deformed 2-sphere, Theorems 4.15 and 4.16. The first is obtained applying Proposition 4.4, the second applying Proposition 4.5. Computations are given in the proofs of the following lemmas. The first lemma follows by a direct application of Proposition 4.7 and properties of special functions.

Lemma 4.12.

$$z(0;0,2,1/2,-1/4)=-1,$$

$$z'(0;0,2,1/2,-1/4)=-\log 2\pi.$$

Lemma 4.13.

$$Z(0;0,a,-1/2,-1/4)=\frac{a}{12}+\frac{1}{12a},$$

$$Z'(0;0,a,-1/2,-1/4)=\frac{1}{6}\left(\frac{1}{2a}-a\right)\log a+$$

$$+\zeta_H'(0,1/(2a)+1)-2a\zeta_H(-1,1/(2a)+1)-2a\zeta_H'(-1,1/(2a)+1)+$$

$$+2i\int_0^\infty\frac{\zeta_H'(0,(1/2+iy)/a+1)-\zeta_H'(0,(1/2-iy)/a+1)}{e^{2\pi y}-1}dy+$$

$$+\frac{1}{8a^2}\zeta_H(2, 1/(2a)+1) - \frac{1}{4a}\left(\Psi(1/(2a)+1)+1+\log a\right)+$$

$$+\frac{i}{4a^2}\int_0^\infty \frac{\zeta_H(2,(1/2+iy)/a+1)-\zeta_H(2,(1/2-iy)/a+1)}{e^{2\pi y}-1}dy+$$

$$+\prod_{m,n=1}^{\infty}\left(1-\frac{1}{4(am+n-1/2)^2}\right)e^{\frac{1}{4(am+n-1/2)^2}}.$$

Proof. The function $Z(s; 0, a, -1/2, -1/4)$ is the zeta function associated with the sequence $S_{-1/4} = \{(am+n-1/2)^2 - 1/4\}$, all terms with multiplicity 1. By Lemma 4.9, $S_{-1/4}$ is a simply regular sequence of spectral type. In order to apply Proposition 4.4, we need to study the unshifted sequence $S_0 = \{(am+n-1/2)^2\}$. This sequence has genus 1, the associate Fredholm determinant is

$$F(z, S_0) = \prod_{m,n=1}^{\infty}\left(1+\frac{z}{(am+n-1/2)^2}\right)e^{-\frac{z}{(am+n-1/2)^2}},$$

and the associated zeta function is $\zeta(s, S_0) = \chi(2s; 0, a, -1/2)$. By Proposition 4.4 and since the genus is 1, we have that

$$Z(0; 0, a, -1/2, -1/4) = \chi(0; 0, a, -1/2) + \frac{1}{4}\mathrm{Res}_1(\chi(2s; 0, a, -1/2), s=1),$$

and that

$$Z'(0; 0, a, -1/2, -1/4) = \chi'(2s; 0, a, -1/2)|_{s=0} + \frac{1}{4}\mathrm{Res}_0(\chi(2s; 0, a, -1/2), s=1) - \log F(-1/4, S_0),$$

and hence we need to compute the values at $s = 0$ of $\zeta(s, S_0) = \chi(2s; 0, 1, -1/2)$, and the residua at $s = 1$. For, we use the formula provided in Lemma 4.10, namely

$$\chi(2s; 0, a, -1/2) = \frac{1}{2}a^{-2s}\zeta_H(2s, 1/(2a)+1) + \frac{1}{2s-1}a^{1-2s}\zeta_H(2s-1, 1/(2a)+1)+$$

$$+ia^{-2s}\int_0^\infty \frac{\zeta_H(2s,(1/2+iy)/a+1)-\zeta_H(2s,(1/2-iy)/a+1)}{e^{2\pi y}-1}dy. \qquad (1)$$

We obtain

$$\chi(0; 0, a, -1/2) = \frac{1}{2}\zeta_H(0, 1/(2a)+1) - a\zeta_H(-1, 1/(2a)+1)+$$

$$+i\int_0^\infty \frac{\zeta_H(0,(1/2+iy)/a+1)-\zeta_H(0,(1/2-iy)/a+1)}{e^{2\pi y}-1}dy = \frac{a}{12} - \frac{1}{24a},$$

where we have used [35] 9.531 and 9.611.1. Next, we use Eq. (1) to compute the residua at the pole $s = 1$. The unique singular term is the middle one, so we expand the different factors in it near $s = 1$, using [35] 9.533.2,

$$a\frac{a^{-2s}}{2s-1}\zeta_H(2s-1,1+1/(2a)) = \frac{1}{2a}\frac{1}{s-1} - \frac{1}{a}\left(\Psi(1+1/(2a))+1+\log a\right)+O(s-1).$$

This gives

$$\mathrm{Res}_1(\chi(2s;0,a,-1/2),s=1)=\frac{1}{2a},$$

and

$$\mathrm{Res}_0(\chi(2s;0,a,-1/2),s=1)=\frac{1}{2a^2}\zeta_H(2,1/(2a)+1)-\frac{1}{a}(1+\log a)+$$

$$-\frac{1}{a}\Psi(1/(2a)+1)+\frac{i}{a^2}\int_0^\infty\frac{\zeta_H(2,(1/2+iy)/a+1)-\zeta_H(2,(1/2-iy)/a+1)}{e^{2\pi y}-1}dy.$$

Last, we compute the derivative:

$$\chi'(0;0,a,-1/2)=-2\chi(0;0,a,-1/2)\log a+$$

$$+\zeta_H'(0,1/(2a)+1)-2a\zeta_H(-1,1/(2a)+1)-2a\zeta_H'(-1,1/(2a)+1)+$$

$$+2i\int_0^\infty\frac{\zeta_H'(0,(1/2+iy)/a+1)-\zeta_H'(0,(1/2-iy)/a+1)}{e^{2\pi y}-1}dy=$$

$$=\frac{1}{6}\left(\frac{1}{2a}-a\right)\log a+\Gamma(1/(2a)+1)-\frac{1}{2}\log 2\pi+\frac{a}{6}+\frac{1}{4a}+\frac{1}{2}-2a\zeta_H'(-1,1/(2a)+1)+$$

$$+2i\int_0^\infty\log\frac{\Gamma((1/2+iy)/a+1)}{\Gamma((1/2-iy)/a+1)}\frac{dy}{e^{2\pi y}-1}dy.$$

Collecting, we obtain the thesis.

Lemma 4.14.

$$Z'(0;0,a,-1/2,-1/4)=-\left(\frac{a}{6}+\frac{1}{6a}\right)\log a-\frac{1}{2}\log 2\pi+\frac{1}{2}\log\Gamma(1+\frac{1}{a})+$$

$$+\frac{a}{6}+\frac{1}{2}+\frac{3}{4a}-a\zeta_R'(-1)-a\zeta_H'(-1,1+\frac{1}{a})+$$

$$+i\int_0^\infty\log\frac{\Gamma(1+i\frac{y}{a})\Gamma(1+\frac{1}{a}+i\frac{y}{a})}{\Gamma(1-i\frac{y}{a})\Gamma(1+\frac{1}{a}-i\frac{y}{a})}\frac{dy}{e^{2\pi y}-1}.$$

Proof. In the language of Proposition 4.5, we have

$$L_0 = \{(am+n+x)^2\}, \qquad L_{b^2} = \{(am+n+x)^2 + b^2\},$$

$$S_0 = \{am+n+x\}, \qquad S_{ib} = \{am+n+x+ib\},$$

where the genus of S_0 is $p = 2$. Therefore, by Proposition 4.5,

$$Z'(0; 0, a, x, b^2) = \zeta'(0, L_{b^2}) = \zeta'(0, S_{ib}) + \zeta'(0, S_{-ib})) - \mathrm{Res}_1(\zeta(s, S_0), s = 2)b^2.$$

Also, we have that

$$\zeta(s, S_{ib}) = \sum_{m,n=1}^{\infty} (am+n+x+ib)^{-s} = \chi(s; 0, a, x+ib),$$

and therefore, we need information on χ. Use Lemma 4.10. We have, with $z = x \pm ib$,

$$\chi(s; 0, a, z) = \frac{1}{2} a^{-s} \zeta_H(s, \frac{z+1}{a} + 1) + \frac{a^{1-s}}{s-1} \zeta(s-1, \frac{z+1}{a} + 1) +$$

$$+ia^{-s} \int_0^\infty \frac{\zeta_H(s, \frac{z+1+iy}{a} + 1) - \zeta_H(s, \frac{z+1-iy}{a} + 1)}{e^{2\pi y} - 1} dy.$$

This gives

$$\mathrm{Res}_1(\chi(s; 0, a, z), s = 2) = \frac{1}{a},$$

$$\chi(0; 0, a, z) = \frac{1}{4} + \frac{a}{12} + \frac{1}{12a} + \frac{z^2}{2a} + \frac{z}{2a} + \frac{z}{2},$$

$$\chi'(0; 0, a, z) = -\chi(0; 0, a, z) \log a + \frac{1}{2} \zeta'_H(0, \frac{z+1}{a} + 1) - a\zeta_H(-1, \frac{z+1}{a} + 1) +$$

$$-a\zeta'_H(-1, \frac{z+1}{a} + 1) + i \int_0^\infty \log \frac{\Gamma(1 + \frac{z+iy+1}{a})}{\Gamma(1 + \frac{z-iy+1}{a})} \frac{dy}{e^{2\pi y} - 1}.$$

Using the decomposition at the beginning of this subsection and the results in Lemmas 4.12, 4.13 and 4.14 respectively, we can prove the following theorems.

Theorem 4.15.

$$\zeta(0, \Delta_{S^2_{1/a}}) = -1 + \frac{a}{6} + \frac{1}{6a},$$

$$\zeta'(0, \Delta_{S^2_{1/a}}) = -2 \log 2\pi + 1 + \frac{a}{3} - \frac{1}{3} \left(a + \frac{1}{a} \right) \log a + 2 \log \Gamma(1/(2a) + 1) +$$

$$+\frac{1}{4a^2}\zeta_H(2,1/(2a)+1)-\frac{1}{2a}\Psi(1/(2a)+1)-4a\zeta_H'(-1,1/(2a)+1)+$$

$$+4i\int_0^\infty \log\frac{\Gamma((1/2+iy)/a+1)}{\Gamma((1/2-iy)/a+1)}\frac{dy}{e^{2\pi y}-1}+$$

$$+\frac{i}{2a^2}\int_0^\infty \frac{\zeta_H(2,(1/2+iy)/a+1)-\zeta_H(2,(1/2-iy)/a+1)}{e^{2\pi y}-1}dy+$$

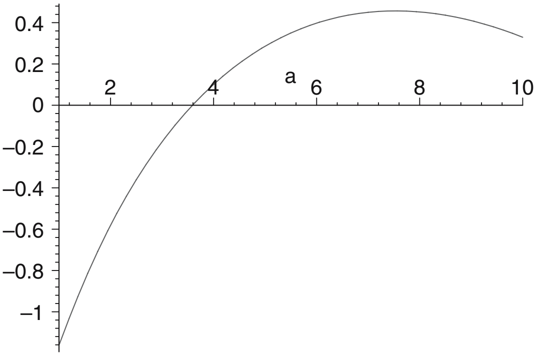
$$-2\log\prod_{m,n=1}^\infty\left(1-\frac{1}{4(am+n-1/2)^2}\right)e^{\frac{1}{4(am+n-1/2)^2}}.$$

Theorem 4.16.

$$\zeta'(0,\Delta_{S^2_{1/a}})=-\left(\frac{a}{3}+\frac{1}{3a}\right)\log a-2\log 2\pi+\frac{a}{3}+1+\frac{3}{2a}+\log\Gamma(1+\frac{1}{a})+$$

$$-2a\zeta_R'(-1)-2a\zeta_H'(-1,1+\frac{1}{a})+2i\int_0^\infty\log\frac{\Gamma(1+i\frac{y}{a})\Gamma(1+\frac{1}{a}+i\frac{y}{a})}{\Gamma(1-i\frac{y}{a})\Gamma(1+\frac{1}{a}-i\frac{y}{a})}\frac{dy}{e^{2\pi y}-1}.$$

Observe that, although the formula given in Theorem 4.16 looks nicer, it is in fact less useful than the one given in Theorem 4.15, since convergence of the integral is much lower than convergence of the infinite product. Note also that the analytic formulas obtained in the previous theorems, provide a rigorous answer to the problem studied in [27], where an attempt to obtain such formulas was performed. In particular, we can compare the graphs given in [27] Sect. XI (where observe the opposite sign), with the following one, where $\zeta'(0,\Delta_{S^2_{1/a}})$ is plotted using the formula given in Theorem 4.15, and the relation with the lune angle ω is $a=\frac{\pi}{\omega}$.



4.5. The zeta determinant on the deformed 3 sphere. On the deformed 3-sphere we have $N = 2$ and

$$\zeta(s, \Delta_{S^3_{1/a}}) = \sum_{n=1}^{\infty}(n+1)[n(n+2)]^{-s} + 2\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(n+1)[(am+n)(am+n+2)]^{-s}.$$

We can check that this reduces to the usual zeta function on the 3-sphere $\zeta(s, \Delta_{S^3_1}) = \sum_{n=1}^{\infty}(n+1)^2[n(n+2)]^{-s}$ [54, 18], and we can decompose it as follows

$$\zeta(s, \Delta_{S^3_{1/a}}) = z(s; 1, 2, 1, -1) + 2Z(s; 1, a, 0, -1).$$

As in the previous subsection, we apply Propositions 4.4 and 4.5 and properties of special functions to prove the following lemmas. Observe that, in this case, an application of Proposition 4.5 gives a simpler formula for $z'(0; 1, 2, 1, -1)$, we thank the referee for pointing out this fact.

Lemma 4.17.

$$z(0; 1, 2, 1, -1) = -1,$$

$$z'(0; 1, 2, 1, -1) = \gamma - 1 + 2\zeta'_H(-1, 2) - \log\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right)^n e^{\frac{1}{n}} = 2\zeta'_H(-1, 2) + \log 2 - 1.$$

Remark 4.18. The above result allows to obtain the following interesting formulas for the Barnes G-function $G(z)$ and the double sine function $S(z)$ (see [3, 53 or 59] for the definition of the G-function, and [44 or 59] for the multiple sine function):

$$\lim_{z\to 1}\frac{G(1-z)}{1-z} = \frac{4}{e},$$

$$\lim_{z\to 1}\frac{S(\pi(1-z))}{1-z} = \frac{\pi}{e}.$$

The proofs of the next lemmas are the same as for Lemmas 4.13 and 4.14. Besides the increasing difficulty of the calculation and the fact that now the multiplicity is not trivial ($\alpha = 1$), the main difference is that a new singular term appears in the unshifted zeta function, namely applying Lemma 4.10, we obtain the expression

$$\chi(2s; 1, a, 0) = \frac{a^{-2s}}{2}\zeta_H(2s, 1/a+1) + \frac{a^{1-2s}}{2s-1}\zeta_H(2s-1, 1/a+1) +$$

$$+\frac{a^{2-2s}}{(2s-1)(2s-2)}\zeta_H(2s-2, 1/a+1) +$$

$$+ia^{-2s}\int_0^{\infty}\frac{(1+iy)\zeta_H(2s, (1+iy)/a+1) - (1-iy)\zeta_H(2s, (1-iy)/a+1)}{e^{2\pi y}-1}dy$$

instead of formula (1).

Lemma 4.19.

$$Z(0;1,a,0,-1) = -\frac{5}{24},$$

$$Z'(0;1,a,0,-1) = \frac{3}{4} - \frac{a}{12} + \frac{1}{2a} + \frac{5}{12}\log a - \frac{11}{12}\log 2\pi + 2\log\Gamma(1/a+1)+$$

$$+\frac{1}{2a^2}\zeta_H(2,1/a+1) - \frac{1}{a}\Psi(1+1/a) - 2a\zeta'_H(-1,1/a+1) + a^2\zeta'_H(-2,1/a+1)+$$

$$+2i\int_0^\infty \log\frac{\Gamma((1+iy)/a+1)}{\Gamma((1-iy)/a+1)}\frac{dy}{e^{2\pi y}-1} - 2\int_0^\infty y\log|\Gamma((1+iy)/a+1)|^2\frac{dy}{e^{2\pi y}-1}+$$

$$+\frac{i}{a^2}\int_0^\infty \frac{\zeta_H(2,(1+iy)/a+1) - \zeta_H(2,(1-iy)/a+1)}{e^{2\pi y}-1}dy+$$

$$-\frac{1}{a^2}\int_0^\infty y\frac{\zeta_H(2,(1+iy)/a+1) + \zeta_H(2,(1-iy)/a+1)}{e^{2\pi y}-1}dy+$$

$$-\log\prod_{m,n=1}^\infty\left(1-\frac{1}{(am+n)^2}\right)^n e^{\frac{n}{(am+n)^2}}.$$

Lemma 4.20.

$$Z'(0;1,a,0,-1) = \frac{5}{12}\log a - 1 - \frac{a}{12} - \frac{5}{12}\log 2\pi + \frac{1}{2}\log\Gamma(\frac{2}{a}+1)+$$

$$-a\left(\zeta'_R(-1) + \zeta'_H(-1,\frac{2}{a}+1)\right) + \frac{a^2}{2}\left(\zeta'_R(-2) + \zeta'_H(-2,\frac{2}{a}+1)\right)+$$

$$+i\int_0^\infty \log\frac{\Gamma(1+i\frac{y}{a})\Gamma(1+\frac{2+iy}{a})}{\Gamma(1-i\frac{y}{a})\Gamma(1+\frac{2-iy}{a})}\frac{dy}{e^{2\pi y}-1}+$$

$$-\int_0^\infty y\log\frac{\pi y}{a\operatorname{sh}\frac{\pi y}{a}}\Gamma(1+\frac{2+iy}{a})\Gamma(1+\frac{2-iy}{a})\frac{dy}{e^{2\pi y}-1}.$$

Using the decomposition at the beginning of this subsection and the results in Lemmas 4.17, 4.19 and 4.20 we can prove the following theorems.

Theorem 4.21.

$$\zeta(0, \Delta_{S^3_{1/a}}) = -1,$$

$$\zeta'(0, \Delta_{S^3_{1/a}}) = \gamma - 1 + 2\zeta'_H(-1, 2) - \log \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)^n e^{\frac{1}{n}} +$$

$$+\frac{3}{2} - \frac{a}{6} + \frac{1}{a} + \frac{5}{6} \log a - \frac{11}{6} \log 2\pi + 4 \log \Gamma(1/a + 1)+$$

$$+\frac{1}{a^2}\zeta_H(2, 1/a + 1) - \frac{2}{a}\Psi(1 + 1/a) - 4a\zeta'_H(-1, 1/a + 1) + 2a^2\zeta'_H(-2, 1/a + 1)+$$

$$+4i \int_0^\infty \frac{(1 + iy) \log \Gamma((1 + iy)/a + 1) - (1 - iy) \log \Gamma((1 - iy)/a + 1)}{e^{2\pi y} - 1} dy+$$

$$+\frac{2i}{a^2} \int_0^\infty \frac{(1 + iy)\zeta_H(2, (1 + iy)/a + 1) - (1 - iy)\zeta_H(2, (1 - iy)/a + 1)}{e^{2\pi y} - 1} dy+$$

$$-2 \log \prod_{m,n=1}^{\infty} \left(1 - \frac{1}{(am + n)^2}\right)^n e^{\frac{n}{(am+n)^2}}.$$

Theorem 4.22.

$$\zeta'(0, \Delta_{S^3_{1/a}}) = \log 2 + 2\zeta'_R(-1) - 1 + \frac{5}{6} \log a - 2 - \frac{a}{6} - \frac{5}{6} \log 2\pi + \log \Gamma(\frac{2}{a} + 1)+$$

$$-2a \left(\zeta'_R(-1) + \zeta'_H(-1, \frac{2}{a} + 1)\right) + a^2 \left(\zeta'_R(-2) + \zeta'_H(-2, \frac{2}{a} + 1)\right) +$$

$$+2i \int_0^\infty \log \frac{\Gamma(1+i\frac{y}{a})\Gamma(1+\frac{2+iy}{a})}{\Gamma(1-i\frac{y}{a})\Gamma(1+\frac{2-iy}{a})} \frac{dy}{e^{2\pi y}-1} - 2 \int_0^\infty y \log \frac{\pi y \Gamma(1+\frac{2+iy}{a})\Gamma(1+\frac{2-iy}{a})}{a \operatorname{sh}\frac{\pi y}{a}} \frac{dy}{e^{2\pi y}-1}.$$

4.6. Expansions. In this subsection we give explicit formulas and numerical values of the first coefficients appearing in the expansions of the determinants of the Laplace operator on the 2 and 3 dimensional deformed sphere S^N_k for small deformations of the parameter $k = 1 - \delta$, with small positive δ. We first state a lemma that allows to deal with the expansion of the values of the zeta function, and thus justify the formal series expansion of all the functions appearing in Theorems 4.15 and 4.21 up to the infinite products, but the last can be treated directly. The proof of Lemma 4.23 follows by the same argument as the one used in the proof of Proposition 4.4.

Lemma 4.23. *Let* x, q *and* δ *be real with* $0 \le \delta \le 1$*, then for all* $\text{Re}(s) > -2$ *we have the expansion*

$$\zeta_H(s, 1 + x + q\delta)) = \zeta_H(s, 1 + x) - s\zeta_H(s + 1, 1 + x)q\delta + \frac{s(s + 1)}{2}\zeta_H(s + 2, 1 + x)q^2\delta^2 + O(\delta^3),$$

and

$$\zeta_H'(s,1+x+q\delta)) = \zeta_H'(s,1+x) - \big(\zeta_H(s+1,1+x) + s\zeta_H'(s+1,1+x)\big)\, q\delta +$$

$$+\left(\left(s+\frac{1}{2}\right)\zeta_H(s+2,1+x) + \frac{s(s+1)}{2}\zeta_H'(s+2,1+x)\right) q^2\delta^2 + O(\delta^3),$$

where note that the coefficients of the second and third term in the second formula are defined as limits.

Proposition 4.24. *For $a = 1+\delta+O(\delta^2)$,*

$$\zeta'(0,\Delta_{S^2_{1-\delta}}) = \zeta'(0,\Delta_{S^2_1}) + Z_2\delta + O(\delta^2),$$

where

$$\zeta'(0,\Delta_{S^2_1}) = 4\zeta_H'(-1) - \frac{1}{2} =$$

$$= -\log 2\pi - \frac{2}{3} + \frac{\pi}{8} + \frac{\gamma}{2} - 4\zeta_H'(-1,1/2) + \frac{i}{2}\int_0^\infty \frac{\Psi'(3/2+iy) - \Psi'(3/2-iy)}{e^{2\pi y}-1}dy +$$

$$+4i\int_0^\infty \log\frac{\Gamma(3/2+iy)}{\Gamma(3/2-iy)}\frac{dy}{e^{2\pi y}-1} - 2\log\prod_{m,n=1}^\infty\left(1-\frac{1}{4(m+n+1/2)^2}\right)e^{\frac{1}{4(m+n+1/2)^2}}$$

$$= -1.161684575,$$

$$Z_2 = -\frac{1}{3} + \frac{\gamma}{2} - \frac{\pi^2}{8} + \frac{7}{4}\zeta_R(3) - 4\zeta_H'(-1,1/2) + 2\pi\int_0^\infty \frac{\tanh \pi y}{e^{2\pi y}-1}dy +$$

$$+4\int_0^\infty y\frac{\Psi(1/2+iy)+\Psi(1/2-iy)}{e^{2\pi y}-1}dy + \frac{1}{2}\int_0^\infty y\frac{\Psi''(3/2+iy)+\Psi'(3/2-iy)}{e^{2\pi y}-1}dy +$$

$$-\frac{i}{4}\int_0^\infty \frac{\Psi''(3/2+iy)-\Psi'(3/2-iy)}{e^{2\pi y}-1}dy - 4\sum_{j=2}^\infty\frac{1}{4^j}\sum_{m,n=1}^\infty\frac{m}{(m+n+1/2)^{2j+1}} =$$

$$= 0.7116523492.$$

Corollary 4.25.

$$\det\Delta_{S^2_{1-\delta}} = \det\Delta_{S^2_1} - Z_2\det\Delta_{S^2_1}\delta + O(\delta^2) = 3.195311305 - 2.273950797\delta + O(\delta^2).$$

Proposition 4.26. *For* $a = 1 + \delta + O(\delta^2)$,

$$\zeta'(0, \Delta_{S^3_{1-\delta}}) = \zeta'(0, \Delta_{S^3_1}) + Z_3\delta + O(\delta^2),$$

where

$$\zeta'(0, \Delta_{S^3_1}) = 2\zeta'_R(-2) + 2\zeta'_R(0) + \log 2 =$$

$$= 3\gamma - \frac{5}{3} - 2\zeta'_R(-1) - \frac{11}{6}\log(2\pi) + \frac{\pi^2}{6} + 2\zeta'_R(-2) - \log\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right)^n e^{\frac{1}{n}} +$$

$$+4i\int_0^\infty \log\frac{\Gamma(2+iy)}{\Gamma(2-iy)}\frac{dy}{e^{2\pi y}-1} - 4\int_0^\infty y\log|\Gamma(2+iy)|^2\frac{dy}{e^{2\pi y}-1} +$$

$$+2i\int_0^\infty \frac{\zeta_H(2,2+iy) - \zeta_H(2,2-iy)}{e^{2\pi y}-1}dy - 2\int_0^\infty y\frac{\zeta_H(2,2+iy)+\zeta_H(2,2-iy)}{e^{2\pi y}-1}dy +$$

$$= -2\log\prod_{m,n=1}^{\infty}\left(1 - \frac{1}{(m+n)^2}\right)^n e^{\frac{n}{(m+n)^2}} = -1.205626800,$$

$$Z_3 = -\frac{1}{2} + 2\gamma + 2\zeta_R(3) - 8\zeta'_R(-1) - 2\log(2\pi) + 4\zeta'_R(-2) +$$

$$-4i\int_0^\infty \frac{(1+iy)^2\Psi(2+iy) - (1-iy)^2\Psi(2-iy)}{e^{2\pi y}-1}dy +$$

$$-4i\int_0^\infty \frac{(1+iy)\Psi'(2+iy) - (1-iy)\Psi'(2-iy)}{e^{2\pi y}-1}dy +$$

$$-2i\int_0^\infty \frac{(1+iy)^2\Psi''(2+iy) - (2-iy)^2\Psi''(2-iy)}{e^{2\pi y}-1}dy + \frac{3}{2} - \frac{\pi^2}{9} = 0.6666666661 = \frac{2}{3}.$$

Corollary 4.27.

$$\det\Delta_{S^3_{1-\delta}} = \det\Delta_{S^3_1} - Z_3\det\Delta_{S^3_1}\delta + O(\delta^2) = 3.338845845 - 2.2258972228\delta + O(\delta^2).$$

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