Wobbling kinks in ϕ^4 theory

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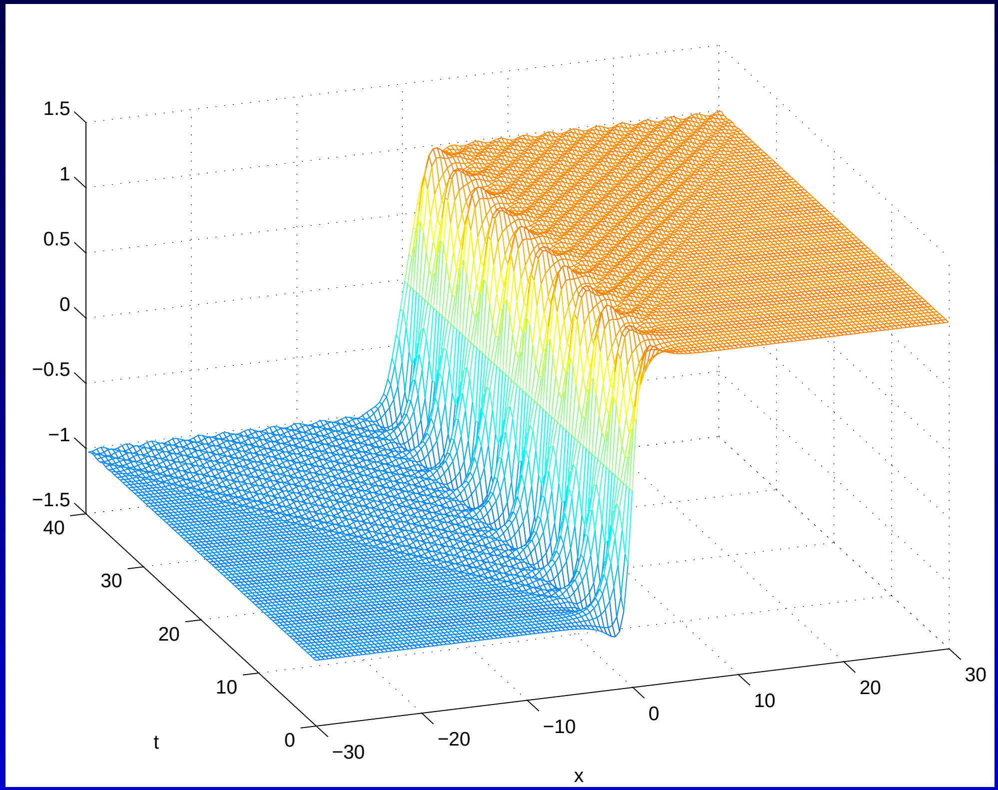
The ϕ^4-theory (since 1960s):

$$\frac{1}{2}\phi_{tt} - \frac{1}{2}\phi_{xx} - \phi + \phi^3 = 0,$$

with its kink solution:

$$\phi(x,t) = \tanh x.$$



When the amplitude of the kink's internal mode becomes large, the wobbling kink has to be regarded as a fundamental nonlinear excitation of the ϕ^4 theory; hence the need for a fully-nonlinear description.

Outline of this talk:

- Freely wobbling kink
- Parametrically driven damped wobbler ($1:1$ resonance)
- Parametrically driven damped wobbler ($2:1$ resonance)
- Directly driven damped wobbler ($1:1$ resonance)
- Directly driven damped wobbler ($1:2$ resonance)

Theoretical work so far

- Numerical simulations: Boris Getmanov, JETP Letters (1975)
- Collective coordinates (free kink): M J Rice and E J Mele, Solid State Commun **35** 487 (1980)
- Regular asymptotic expansion (free kink): H Segur, J Math Phys **24**, 1439 (1983)
- Parametrically driven kink (averaging over large frequency): Yu S Kivshar, A Sánchez, and L Vázquez, Phys Rev A **45** 1207 (1992)
- Damped-driven kink away from resonances: A L Sukstanskii and K I Primak, Phys Rev Lett **75** 3029 (1995)
- Collective coordinates (directly driven kink): N Quintero, A Sánchez, and F Mertens, Phys Rev Lett **84**, 871 (2000)
- Collective coordinates (parametrically driven kink): N Quintero, A Sánchez, and F Mertens, Phys Rev E **64**, 046601 (2001)

Freely wobbling kink

Transforming to the moving frame,

$$\xi = x - \int_0^t v(t')dt',$$

the equation becomes

$$\frac{1}{2}\phi_{tt} - v\phi_{\xi t} - \frac{v_t}{2}\phi_\xi - \frac{1-v^2}{2}\phi_{\xi\xi} - \phi + \phi^3 = 0.$$

Set $v = \epsilon V$; expand in powers of ϵ:

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots.$$

and introduce a hirerarchy of space and time scales:

$$X_n \equiv \epsilon^n \xi, \quad T_n \equiv \epsilon^n t, \quad n = 0, 1, 2, \ldots.$$

Linear corrections

$$\frac{1}{2}D_0^2\phi_1 + \mathcal{L}\phi_1 = 0,$$

where the Schrödinger operator

$$\mathcal{L} = -\frac{1}{2}\partial_0^2 - 1 + 3\phi_0^2 = -\frac{1}{2}\partial_0^2 + 2 - 3\,\mathrm{sech}^2\,X_0.$$

We keep just the wobbling mode:

$$\phi_1 = Ae^{i\omega_0 T_0}y_1(X_0) + c.c.,$$

where

$$\omega_0 = \sqrt{3}, \quad y_1(X_0) = \mathrm{sech}\,X_0 \tanh X_0.$$

and

$$A = A(X_1, X_2, ..., T_1, T_2, ...).$$

Quadratic corrections

$$\tfrac{1}{2}D_0^2\phi_2 + \mathcal{L}\phi_2 = (\partial_0\partial_1 - D_0D_1)\phi_1 - 3\phi_0\phi_1^2 + VD_0\partial_0\phi_1 + \tfrac{1}{2}D_1V\partial_0\phi_0 - \tfrac{1}{2}V^2\partial_0^2\phi_0,$$

whence $\phi_2 = \varphi_2^{(0)} + \varphi_2^{(1)}e^{i\omega T_0} + c.c. + \varphi_2^{(2)}e^{2i\omega T_0} + c.c.$

$$\varphi_2^{(0)} = 2|A|^2 \operatorname{sech}^2 X_0 \tanh X_0 + \left(\frac{V^2}{2} - 3|A|^2\right) X_0 \operatorname{sech}^2 X_0;$$

$$\varphi_2^{(1)} = -(\partial_1 A + i\omega_0 VA)X_0 \operatorname{sech} X_0 \tanh X_0;$$

$$\varphi_2^{(2)} = A^2 f_1(X_0) = \frac{A^2}{8}\big[6\tanh X_0 \operatorname{sech}^2 X_0 + (3 - \tanh^2 X_0 + ik_0 \tanh X_0)(J_2^* - J_2^\infty)e^{ik_0X_0} + (3 - \tanh^2 X_0 - ik_0 \tanh X_0)J_2e^{-ik_0X_0}$$

$$k_0 = \sqrt{8}, \quad J_2 = \int_{-\infty}^{X_0} e^{ik\xi} \operatorname{sech}^n \xi \, d\xi$$

To avoid infinite velocities, replace A^2 with $B(X_1, X_2, ...; T_1, T_2, ...)$, where $B(0, 0, ..., T_1, T_2, ...) = A^2(T_1, T_2, ...)$.

The amplitude equation

$$\frac{1}{2}D_0^2\phi_3 + \mathcal{L}\phi_3 = (\partial_0\partial_1 - D_0D_1)\phi_2 + (\partial_0\partial_2 - D_0D_2)\phi_1$$
$$+ \frac{1}{2}(\partial_1^2 - D_1^2)\phi_1 - \phi_1^3 - 6\phi_0\phi_1\phi_2 + VD_0\partial_0\phi_2$$
$$+ VD_0\partial_1\phi_1 + VD_1\partial_0\phi_1 + \frac{1}{2}D_2V\partial_0\phi_0 - \frac{1}{2}V^2\partial_0^2\phi_1.$$

The solvability condition for the first harmonic

$$\frac{2i\omega_0}{3}D_2A + \zeta|A|^2A - V^2A = 0,$$

where

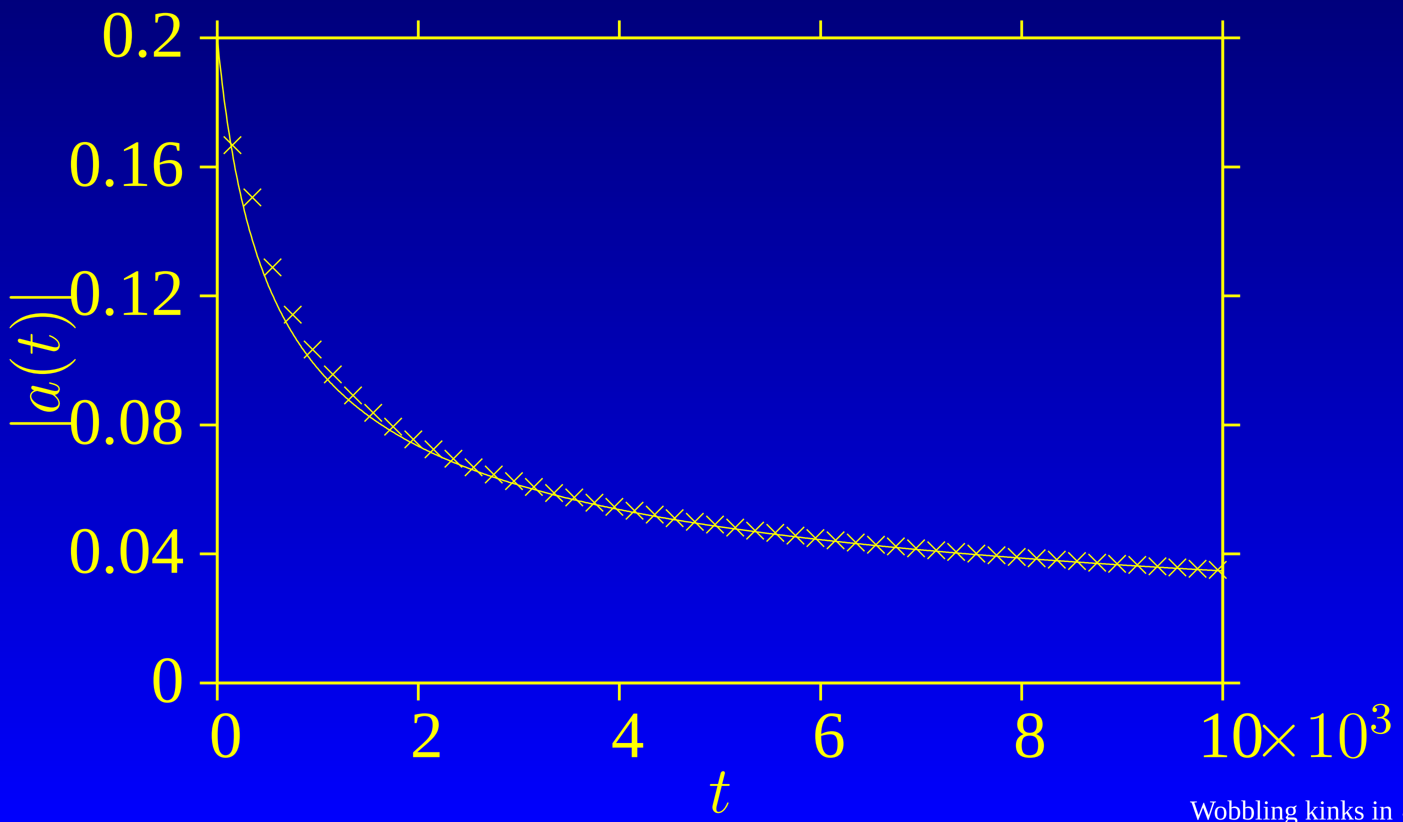
$$\zeta = 6\int_{-\infty}^{\infty} \operatorname{sech}^2 X_0 \tanh^3 X_0 \Big[\frac{5}{2}\operatorname{sech}^2 X_0 \tanh X_0$$
$$- 3X_0 \operatorname{sech}^2 X_0 + f_1(X_0)\Big]\, dX_0$$
$$= -0.8509 + i0.04636$$

All amplitude equations can be assembled into one master equation:

$$ia_t = -\frac{\omega_0}{2}\zeta\,|a|^2 a + \frac{\omega_0}{2} v^2 a + \mathcal{O}\left(|a|^4\right), \qquad a = \epsilon A$$

Decay of the wobbling amplitude:

$$|a(t)|^2 = \frac{|a(0)|^2}{1 + \omega_0\,\mathrm{Im}\,\zeta\,|a(0)|^2 t} = \frac{|a(0)|^2}{1 + 0.08030 \times |a(0)|^2 t}.$$



Wobbling kink

Frequency of the wobbling:

$$\omega = \omega_0 \left[1 - \tfrac{1}{2} v^2 + \tfrac{1}{2} \operatorname{Re} \zeta \, |a|^2 + \mathcal{O}\left(|a|^4 \right) \right]$$

The wobbling kink to the order ϵ^2:

$$\phi(x,t) = \tanh\left(\frac{1 - 3|a|^2}{\sqrt{1-v^2}} \xi \right) + a \operatorname{sech} \xi \tanh \xi e^{i\omega_0 (t - v\xi)} + c.c. + 2|a|^2 \operatorname{sech}^2 \xi \tanh \xi + b f_1(\xi) e^{2i\omega_0 (t - v\xi)} + c.c. + \mathcal{O}\left(|a|^3 \right),$$

where $b = \epsilon^2 B$ and $\xi = x - vt$.

Parametrically driven ϕ^4 (1 : 1 resonance)

$$\frac{1}{2}\phi_{tt} - \frac{1}{2}\phi_{xx} + \gamma\phi_t - [1 + h\cos(\Omega t)]\,\phi + \phi^3 = 0,$$
$$\gamma = \epsilon^2\Gamma, \quad h = \epsilon^3 H, \quad \Omega = \omega_0(1 + \epsilon^2 R).$$

The “free" amplitude equations are replaced with

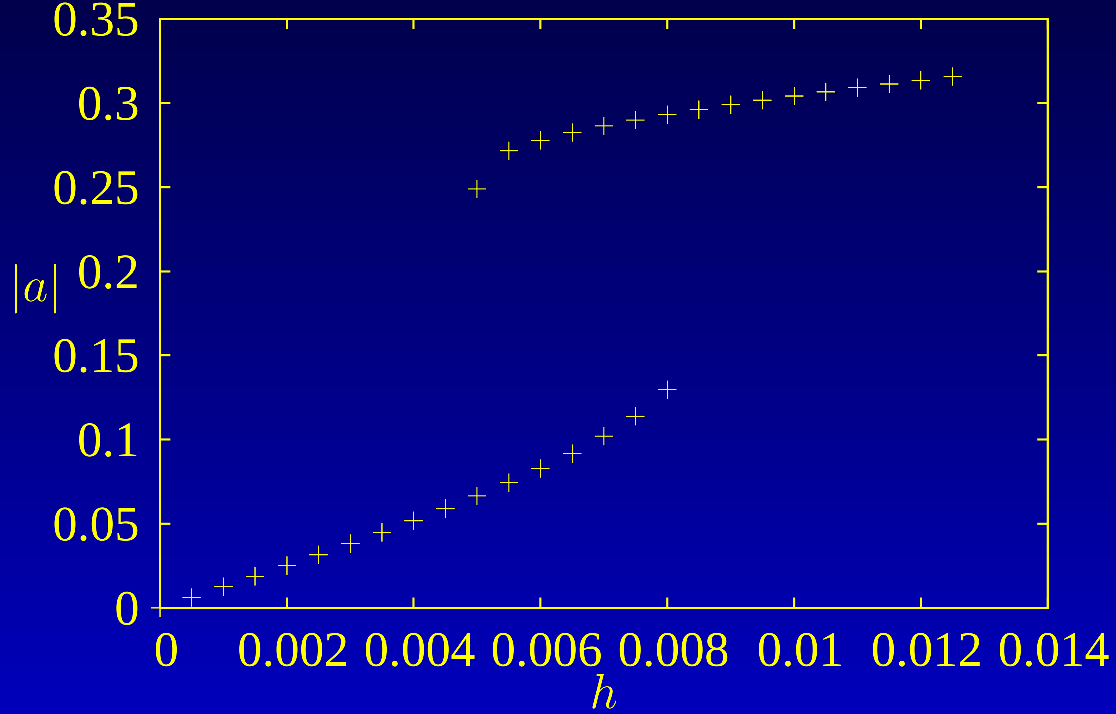
$$D_2 V = -2\Gamma V,$$
$$iD_2 A + \frac{\omega_0}{2}\zeta|A|^2 A - \omega_0(R + \frac{V^2}{2})A = \frac{\pi\omega_0}{8}H - i\Gamma A.$$

In terms of the unscaled variables,

$$\dot{a} = -\gamma a - i\omega_0\rho a + \frac{1}{2}i\omega_0\zeta|a|^2 a - \frac{1}{2}i\omega_0 v^2 a - i\omega_0\frac{\pi}{8}h + \mathcal{O}\left(|a|^5\right),$$
$$\dot{v} = -2\gamma v + \mathcal{O}\left(|a|^5\right).$$

$1:1$ parametric resonance: results

- Wobbling frequency locked to the frequency of the driver
- No threshold for the driving strength to sustain the wobbling
- Bistability and hysteresis for the intermediate driving strengths



- The strongest resonance is at a negative detuning:

$$\frac{\Omega}{\omega_0} - 1 = \frac{\operatorname{Re}\zeta}{2}|a_{\rm res}|^2 < 0, \qquad \frac{64}{\pi^2}|a_{\rm res}|^2 \left(\frac{\gamma}{\omega_0} + \frac{\operatorname{Im}\zeta}{2}|a_{\rm res}|^2\right)^2 = h^2$$

Parametric driving ($2:1$ resonance)

$$\frac{1}{2}\phi_{tt} - \frac{1}{2}\phi_{xx} + \gamma\phi_t - [1 + h\cos(2\Omega t)]\,\phi + \phi^3 = 0,$$

$$\gamma = \epsilon^2\Gamma, \quad h = \epsilon^2 H, \quad \Omega = \omega_0(1+\epsilon^2 R).$$

Master equations:

$$\dot{a} = -\gamma a - i\omega_0\rho a + \frac{1}{2}i\omega_0\zeta|a|^2 a - \frac{1}{2}i\omega_0 v^2 a - \frac{1}{2}i\omega_0\sigma h a^* + \mathcal{O}\left(|a|^5\right),$$

$$\dot{v} = -2\gamma v + \mathcal{O}\left(|a|^5\right)$$

Summary:

- Weaker resonance than in the case of driving near ω_0
- Threshold driving strength:

$$\frac{1}{4}|\sigma|^2 h^2 \geq \frac{\gamma^2}{\omega_0^2} + \left(\frac{\Omega}{\omega_0} - 1\right), \qquad \sigma = 0.5958 + i0.003863$$

- No bistability or hysteresis

Direct driving ($1:2$ resonance)

$$\frac{1}{2}\phi_{tt} - \frac{1}{2}\phi_{xx} + \gamma\phi_t - \phi + \phi^3 = h\cos(\frac{\Omega}{2}t),$$

$$\gamma = \epsilon^2\Gamma, \quad h = \epsilon^2 H, \quad \Omega = \omega_0(1+\epsilon^2 R).$$

The master amplitude equations:

$$\dot{a} = -\gamma a - i\omega_0\rho a + \frac{i}{2}\omega_0\zeta|a|^2 a - \frac{i}{2}\omega_0 v^2 a + \frac{60}{169}i\pi\omega_0 h^2 + \mathcal{O}\left(|a|^5\right),$$

$$\dot{v} = -2\gamma v + \mathcal{O}\left(|a|^5\right).$$

The amplitude equations have the same form as in the case of the parametric driving near the frequency ω_0 where one only needs to replace $h \to h^2$

Direct driving ($1:1$ resonance)

$$\frac{1}{2}\phi_{tt} - \frac{1}{2}\phi_{xx} + \gamma\phi_t - \phi + \phi^3 = h\cos(\Omega t),$$

$$\gamma = \epsilon^2\Gamma, \quad h = \epsilon^2 H, \quad \Omega = \omega_0(1 + \epsilon^2 R).$$

Prior to $\mathcal{O}(\epsilon^2)$, expansion is as in the undamped, undriven case:

$$\phi_0 = \tanh X_0$$

$$\phi_1 = A \operatorname{sech} X_0 \tanh X_0 e^{i\omega T_0} + c.c.$$

At $\mathcal{O}(\epsilon^2)$, we have, as before,

$$\varphi_2^{(0)} = 2|A|^2 \operatorname{sech}^2 X_0 \tanh X_0 + \left(\frac{V^2}{2} - 3|A|^2\right) X_0 \operatorname{sech}^2 X_0,$$

$$\varphi_2^{(2)} = B f_1(X_0), \quad B(0, ..., 0; T_1, ...) = A^2(T_1, T_2, ...),$$

but the first harmonic is altered:

$$\varphi_2^{(1)} = H(1 - 2\operatorname{sech}^2 X_0).$$

At $\mathcal{O}(\epsilon^3)$, the solvability conditions give $D_2 V = -2\Gamma V$ and

$$D_2 A + \Gamma A + i\omega_0 R A - \frac{i}{2}\omega_0 \zeta |A|^2 A + \frac{1}{2} i\omega_0 V^2 A - \frac{3\pi}{4} V H = 0,$$

and the 0th and 1st harmonic components are

$$\varphi_3^{(0)} = -4H(A + A^*)(\operatorname{sech} X_0 + \operatorname{sech}^3 X_0),$$

$$\begin{aligned}\varphi_3^{(1)} = &-\partial_2 A X_0 \operatorname{sech} X_0 \tanh X_0 + |A|^2 A u_1(X_0) \\ &+ \tfrac{1}{2} V^2 A X_0 \operatorname{sech} X_0 (2\operatorname{sech}^2 X_0 - 1) + i\omega_0 V H u_2(X_0),\end{aligned}$$

where $u_1(X_0)$ and $u_2(X_0)$ are the bounded solutions of

$$(\mathcal{L} - 3/2) u_1(X_0) = \frac{3}{2}\zeta \operatorname{sech} X_0 \tanh X_0$$

$$+6 \operatorname{sech} X_0 \tanh^2 X_0 \left[3X_0 \operatorname{sech}^2 X_0 - \frac{5}{2}\operatorname{sech}^2 X_0 \tanh X_0 - f_1(X_0) \right],$$

$$(\mathcal{L} - 3/2) u_2(X_0) = -4\operatorname{sech}^2 X_0 \tanh X_0 + \frac{3\pi}{4}\operatorname{sech} X_0 \tanh X_0.$$

For the second harmonic component we must solve

$$(\mathcal{L}-6)\varphi_3^{(2)} = (\partial_1 B + 2i\omega_0 V B)\,\partial_0 f_1(X_0)$$
$$-2i\omega_0 D_1 B\, f_1(X_0) - 6HA\,\mathrm{sech}\, X_0 \tanh^2 X_0 (1-2\,\mathrm{sech}^2 X_0),$$

where

$$f_1(X_0) \sim \begin{cases} e^{-ik_0 X_0} & \text{for } X_0 \gg 0, \\ e^{ik_0 X_0} & \text{for } X_0 \ll 0. \end{cases}$$

The secular terms will be suppressed if

$$D_1 B = \begin{cases} -\frac{k_0}{\omega_0}(\partial_1 B + 2i\omega_0 V B) & \text{for } X_1 > 0 \\ \frac{k_0}{\omega_0}(\partial_1 B + 2i\omega_0 V B) & \text{for } X_1 < 0, \end{cases}$$

or, writing as a single equation,

$$D_1^2 B = \frac{k_0^2}{4\omega_0^2}\partial_1^2 B + \frac{ik_0^2 V}{\omega_0}\partial_1 B - k_0^2 V^2 B.$$

At $\mathcal{O}\left(\epsilon^5\right)$, the solvability condition is

$$D_4 V = -\frac{3}{2} H\eta|A|^2 A + c.c. + 2\Gamma V^3 + \frac{3\pi}{4} i\omega_0 H\Gamma(A - A^*) + \frac{9\pi}{4} RH(A + A^*) + 2\Gamma RV,$$

where

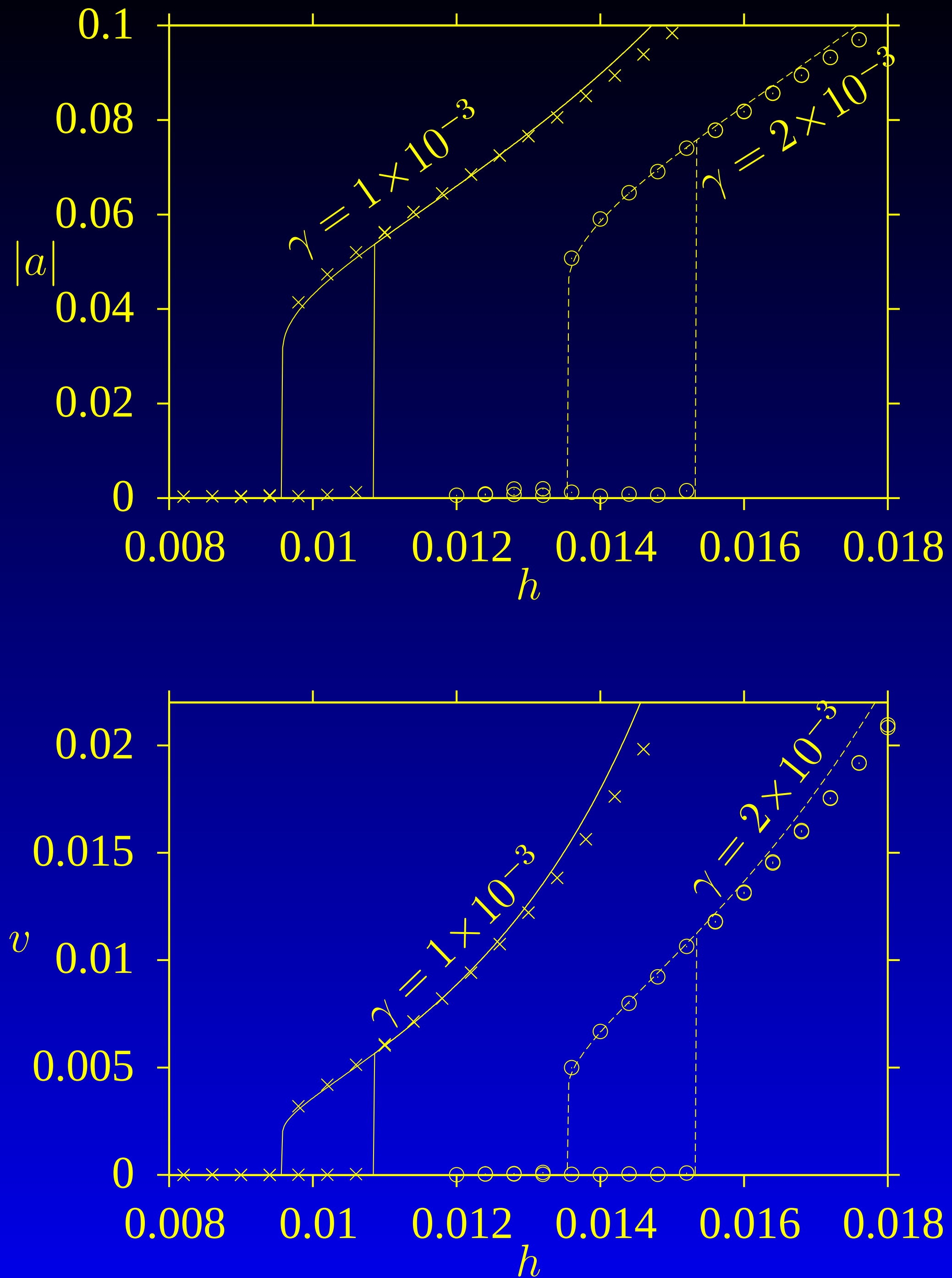
$$\eta = -2.005 - 0.3823i.$$

The master amplitude equations:

$$\dot{a} = -\gamma a - i\omega_0\rho a + \frac{i}{2}\omega_0\zeta|a|^2 a - \frac{i}{2}\omega_0 v^2 a + \frac{3\pi}{4} hv - \frac{3}{2} i\nu h^2 a - i\frac{87}{35} h^2 a^* + \mathcal{O}\Big(|a|^5$$

$$\dot{v} = -2\gamma v + \frac{3\pi}{4} i\omega_0 h\gamma(a - a^*) + \left(\frac{9\pi}{4}\rho - \frac{3}{2}\delta h^2 - \frac{3}{2}\eta h|a|^2\right)(a + a^*) + \mathcal{O}\left(|a|^6\right)$$

Hysteresis



0.1
0.08
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γ = 1×10^−3
γ = 2×10^−3
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0.005
0
v
0.008
0.01
0.012
0.014
0.016
0.018
h
γ = 1×10^−3
γ = 2×10^−3

Conclusions

- Free wobbling kink a long-lived object: the wobbling amplitude decays only as $t^{-1/2}$
- Both direct and parametric driving can sustain the wobbling. The wobbling locks on to a harmonic of the driving frequency
- Parametrically driven wobbling kink: $1:1$ is the strongest resonance
- $1:1$ direct driving: wobbling sustained only if the kink performs translation motion. Radiation important
- Complicated dynamics (bistability and hysteresis) but no chaos