Integrability in $\mathcal{N} = 4$ SYM
a tool for QCD ?

Matteo Beccaria

Dipartimento di Fisica, Universita' del Salento

with
M. Staudacher, A. Rej, S. Zieme (AEI, Potsdam)
G. Marchesini (MIB), Y. L. Dokshitzer (Paris VI)
V. Forini (Humboldt, Berlin), F. Catino (Lecce)

Gallipoli, June 13, 2008

The central role of $\mathcal{N} = 4$ SYM

From string theory back to strong interactions

$$\underbrace{AdS_5 \times S^5 \longrightarrow}_{\text{I}} \boxed{\begin{matrix}\mathcal{N} = 4 \\ \text{SYM}\end{matrix}} \underbrace{\longrightarrow \begin{matrix}\mathcal{N} = 1,2 \\ \text{SYM}\end{matrix} \longrightarrow \text{QCD}}_{\text{II}}$$

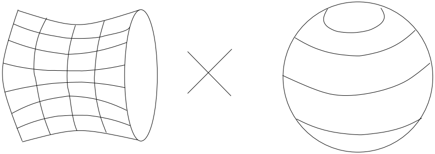
- I AdS/CFT, duality $g \leftrightarrow \frac{1}{g}$
- II QCD-*like* superconformal model, $\gamma(N)$

$\downarrow$

adjoint representation
planar limit, ...

AdS/CFT duality and predictions

- $\boxed{\text{IIB superstring on } AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4 \text{ SYM in } d = 4}$
- **Kinematics ≡ symmetry = OK**
- Isometries of $AdS_5 \times S^5$: $SO(4,2) \times SO(6)$



- **and** in $\mathcal{N} = 4$ SYM

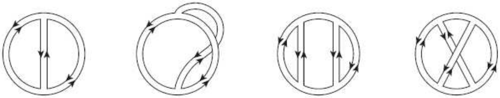
$$SO(4,2) \times SO(6) \overset{bosonic}{\subset} PSU(2,2|4) \qquad !$$

- **What about dynamics ?**

- Gauge/string coupling relations

$$\underbrace{\frac{4\pi\lambda}{N_c}}_{gauge} = \underbrace{g_s}_{string}, \qquad (\lambda = N_c\, g_{\rm YM}^2)$$

- Planar limit $N_c \to \infty \Longrightarrow g_s \to 0$, **free string**



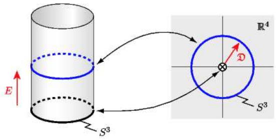
- **Weak - Strong** duality

$$\boxed{\sqrt{\lambda} = \frac{R^2}{\alpha'} \equiv (\text{non-linear } \sigma\text{-model coupling})^{-1}}$$

- Non trivial: Strong coupling $\leftrightarrow$ supergravity limit
  Weak coupling $\leftrightarrow$ strongly coupled σ-model

Holography and inherited integrability

- $\mathcal{N} = 4$ SYM lives on $\partial(AdS_5 \times S^5) = \mathbb{R} \times S^3 \longrightarrow \mathbb{R}^4$
- Time translations $\longrightarrow$ dilatations !



- In a quantum CFT, you get **anomalous dimensions** Δ_{CFT}

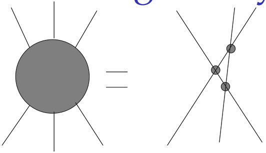
basic prediction: $\quad E_{\text{string}} = \Delta_{CFT}$

- Classical integrability on $AdS_5 \times S^5$ [Bena, Polchinski, Roiban, 03]

$$\Downarrow$$

Integrability properties of Δ_{CFT} ?

Which integrability in Δ_{CFT} ?



it is **not** the factorization of the S matrix !

The evolution of composite operators with the renormalization scale $t = \log\,\mu$ is integrable

$$\frac{\partial}{\partial t}\mathcal{O}_a = \mathfrak{D}_{ab}\,\mathcal{O}_b, \qquad \mathfrak{D} \in \mathfrak{psu}(2,2|4)$$
$$\mathfrak{D}\,\Sigma = \Delta_{\text{CFT}}\,\Sigma, \qquad \Sigma \equiv \text{scaling op.}$$

- As in **old one-loop** QCD ! [Lipatov, 94]

$$\boxed{\begin{matrix}\mathcal{N}=4\\ \text{SYM}\end{matrix}} \longrightarrow \begin{matrix}\mathcal{N}=1,2\\ \text{SYM}\end{matrix} \longrightarrow \text{QCD}$$

$\mathcal{N} = 4$ super Yang-Mills in brief

The maximal case

- Bosonic symmetry algebra $\mathfrak{so}(4,2) \oplus \mathfrak{so}(6)$

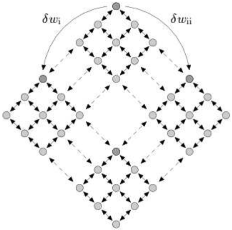
$$\begin{aligned} \mathfrak{so}(4,2) \supset \mathfrak{so}(3,1) &\quad : \quad \text{conformal algebra in } d = 4 \\ \mathfrak{so}(6) \simeq \mathfrak{su}(4) &\quad : \quad \text{internal } R\text{-symmetry} \end{aligned}$$

- Supersimmetries Q^A_α, $\overline{Q}^A_{\dot\alpha}$

φ	$\mathfrak{so}(6)_R$	
A_μ	$\mathbf{1}$	*shared by QCD*
$\lambda^A_\alpha, \quad \overline{\lambda}^A_{\dot\alpha}$	$\mathbf{4} \oplus \overline{\mathbf{4}}$	*SUSY vector multiplet*
Φ^a	$\mathbf{6}$	*$\mathcal{N}$ dependent*

- In the conformal phase, $\mathfrak{psu}(2,2|4)$ symmetric and **UV finite**, $\beta(g) \equiv 0$

- Composite operators build up superconformal multiplets



$$\underbrace{\Delta}_{D}, \underbrace{S_1, S_2}_{\mathfrak{so}(3,1)\simeq\mathfrak{su}(2)\oplus\mathfrak{su}(2)}, \underbrace{J_1, J_2, J_3}_{\mathfrak{so}(6)}.$$

- Composite operators are dual to string states...
- ... and (can) have non trivial anomalous dimensions

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle = \frac{C_{\mathcal{O}}}{(x-y)^{2\,\Delta_{\mathcal{O}}}}, \qquad \Delta_{\mathcal{O}} = \Delta_{\mathcal{O}}(\lambda)$$

$$\Delta_{\mathcal{O}} = \dim\,\mathcal{O} + \boxed{\text{quantum corrections } \mathcal{O}(\hbar)}$$

- Protected operators (conserved currents, BPS, ...)

$$\begin{aligned} \mathcal{O} &= \text{Tr}\left(F_{\mu\lambda}\,F^{\lambda\nu} - \frac{1}{4}\,\delta^\nu_\mu\,F^2 + \text{scalars} + \text{fermions}\right) \\ \Delta &= 4. \end{aligned}$$

- Non degenerate operators without mixing (Konishi)

$$\begin{aligned} \mathcal{O} &= \text{Tr}\,\Phi^a\,\Phi^a \\ \Delta &= 2 + \frac{3\,\lambda}{4\,\pi^2} - \frac{3\,\lambda^2}{16\,\pi^4} + \frac{21\,\lambda^3}{256\,\pi^6} + \cdots \end{aligned}$$

- The calculation of $\Delta(\lambda)$ for unprotected operators $\equiv$ difficult mixing problem (esp. for large charges...)

$$\begin{aligned} \mathcal{O} &= \text{Tr}\left[\Phi^a\,\Phi^a\,\Phi^b\,\Phi^b + \mathcal{O}(\lambda)\,\Phi^a\,\Phi^b\,\Phi^a\,\Phi^b + \cdots\right] \\ \Delta &= 4 + \mathcal{O}(\lambda) \end{aligned}$$

The dilatation operator

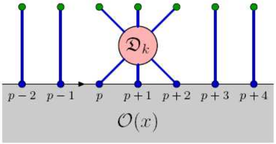
- **The dilatation operator** $\mathfrak{D} \in \mathfrak{psu}(2,2|4)$
  (dual to t-isometry on the string side)

- In the **planar** limit,
  $\mathfrak{D} \longrightarrow$ integrable Hamiltonians

[Beisert et al, 03]

$$\mathfrak{D} = \sum_{\ell \geq 1} \lambda^{\ell} \mathcal{H}^{(\ell)}_{\text{integrable}}.$$

$$\mathfrak{D}_k = \sum_{p=1}^{L}$$

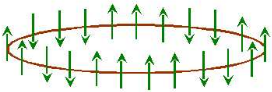


- $\mathcal{H}^{(\ell)}_{\text{integrable}} \longrightarrow$ spin chain with range $\sim \ell \longrightarrow$ **wrapping**

An explicit example: the $\mathfrak{su}(2)$ sector at one loop

In $\mathfrak{su}(2)$, $Z = \varphi_1 + i\,\varphi_2$, $W = \varphi_3 + i\,\varphi_4$,

$$\mathcal{O}_\alpha^{J_1,J_2} = \mathrm{Tr}\left(\underbrace{Z\cdots Z}_{J_1}\,\underbrace{W\cdots W}_{J_2} + \text{permutations} \right).$$



- At **tree level**, $\mathfrak{D}^{(0)} = J_1 + J_2 =$ classical dimension.
- At 1 loop, in the **planar limit** [Minahan, Zarembo, 02]

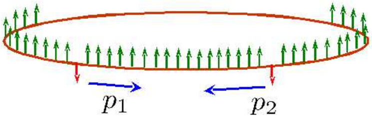
$$\mathfrak{D}^{(1)} = \frac{\lambda}{8\pi^2}\sum_{i=1}^{L}(1 - P_{i,i+1}) = \frac{\lambda}{16\pi^2}\sum_{i=1}^{L}(1 - \vec{\sigma}_i\cdot\vec{\sigma}_{i+1}) = \frac{\lambda}{8\pi^2}\,\mathcal{H}_{\mathrm{XXX}}$$

Bethe Ansatz...

- The Bethe wave-function is determined by $\{p_i\}$

$$\boxed{e^{ip_k L} = \prod_{\substack{i=1 \\ i \neq k}}^{M} S(p_k, p_i).}$$

where



$$S(p_i, p_j) = \frac{\varphi(p_i) - \varphi(p_j) + i}{\varphi(p_i) - \varphi(p_j) - i}, \qquad \varphi(p) = \frac{1}{2}\cot\frac{p}{2}.$$

- Δ is the nothing but the second conserved charge

$$Q_2 = \sum_n (1 - P_{n,n+1}) = \sum_{i=1}^{M} 4\sin^2\frac{p_i}{2}.$$

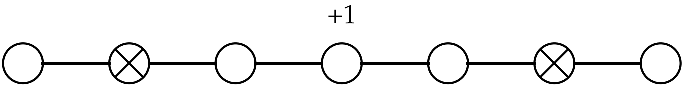
One-loop Bethe Ansatz for $\mathcal{N} = 4$ SYM

The generic $\mathfrak{g}$ case

- **minimal** integrable chain with (super) algebra $\mathfrak{g}$
- rank r, state with $K = K_1 + \cdots + K_r$ Bethe roots u_i, $i = 1, \ldots, K$.
- $k_j = 1, \ldots, r$ labels which simple roots is associated with u_j

Bethe equations [Ogievetsky, Wiegmann, 86]

$$\left(\frac{u_j + \frac{i}{2} V_{k_j}}{u_j - \frac{i}{2} V_{k_j}}\right)^L = \prod_{\substack{\ell=1 \\ \ell \neq j}}^{K} \frac{u_j - u_\ell + \frac{i}{2} M_{k_j,k_\ell}}{u_j - u_\ell - \frac{i}{2} M_{k_j,k_\ell}}.$$



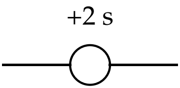
Cyclicity condition and energy

$$1 = \prod_{j=1}^{K} \frac{u_j + \frac{i}{2} V_{k_j}}{u_j - \frac{i}{2} V_{k_j}}, \qquad E = \sum_{j=1}^{K} \left(\frac{i}{u_j + \frac{i}{2} V_{k_j}} - \frac{i}{u_j - \frac{i}{2} V_{k_j}} \right).$$

Example: $\mathfrak{sl}(2)$

rank 1, a single simple root,

$k_j = 1$, $V_{k_j} = V_1 = 2\,s$, $M_{k_j,k_\ell} = 2$.

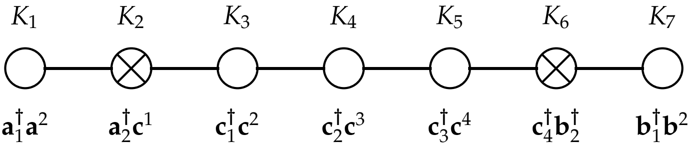


XXX_{-s} Bethe equations

$$\boxed{\left(\frac{u_j + i\,s}{u_j - i\,s} \right)^L = \prod_{\substack{\ell=1 \\ \ell \neq j}}^{K} \frac{u_j - u_\ell + i}{u_j - u_\ell - i}}$$

Application: The $\mathfrak{psu}(2,2|4)$ particular case

- Favourite Dynkin diagram for $\mathcal{N} = 4$ SYM



- Cartan matrix and **singleton** representation on $D^n\,(\varphi, \lambda, A)$
- For any particular (highest weight) state

$$w = [\lambda_1, \lambda_2, \lambda_3]^{\Delta_0}_{(j,\bar{j})}.$$

- **Forget** multi-loop **Feynman diagrams !**

We compute the excitations $K_1, \ldots, K_7$ over the BPS vacuum
and solve (numerically) **the Bethe equations !**

Higher order integrability ? The $\mathfrak{su}(2)$ sector

- Loop expansion of $\mathfrak{D}$

$$\mathfrak{D} = \sum_{\ell=1}^{L} \Big(1 + g^2 H_1 + g^4 H_2 + g^6 H_3 + \cdots\Big)$$

- H_i are **integrable** spin chains with increasing range (hopping expansion of Hubbard model ?)

$$\begin{aligned}
H_1 &= \frac{1}{2}(1 - \sigma_\ell \cdot \sigma_{\ell+1}) \\
H_2 &= -(1 - \sigma_\ell \cdot \sigma_{\ell+1}) + \frac{1}{4}(1 - \sigma_\ell \cdot \sigma_{\ell+2}) \\
H_3 &= \frac{15}{4}(1 - \sigma_\ell \cdot \sigma_{\ell+1}) - \frac{3}{2}(1 - \sigma_\ell \cdot \sigma_{\ell+2}) + \frac{1}{4}(1 - \sigma_\ell \cdot \sigma_{\ell+3}) + \\
&\quad -\frac{1}{8}(1 - \sigma_\ell \cdot \sigma_{\ell+3})(1 - \sigma_{\ell+1} \cdot \sigma_{\ell+2}) + \\
&\quad +\frac{1}{8}(1 - \sigma_\ell \cdot \sigma_{\ell+2})(1 - \sigma_{\ell+1} \cdot \sigma_{\ell+3})
\end{aligned}$$

Long-Range Bethe equations for $\mathfrak{psu}(2,2|4)$

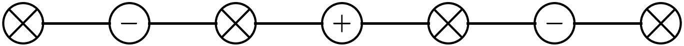
- **Question:** In the full $\mathfrak{psu}(2,2|4)$ theory, can we **encode** the various H's in a single integrable S-matrix with factorized scattering ?

- **Answer:** deformation of the one loop [Beisert, Staudacher, 05]
Bethe equations at **all orders** in the coupling g !

- Deformed spectral variables

$$x(u) = \frac{u}{2}\left(1 + \sqrt{1 - \frac{2\,g^2}{u^2}}\right) \qquad \leftrightarrow \qquad u = x + \frac{g^2}{2\,x},$$

$$x^{\pm}(u) = x\left(u \pm \frac{i}{2}\right) = u \pm \frac{i}{2} + \mathcal{O}(g^2)$$



- Remarkable 1-4-7 coupling

$$1 = \prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}\eta_1}{u_{1,k} - u_{2,j} - \frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{1 - g^2/2x_{1,k}x_{4,j}^{+\eta_1}}{1 - g^2/2x_{1,k}x_{4,j}^{-\eta_1}},$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i\eta_1}{u_{2,k} - u_{2,j} + i\eta_1} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}\eta_1}{u_{2,k} - u_{3,j} - \frac{i}{2}\eta_1} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}\eta_1}{u_{2,k} - u_{1,j} - \frac{i}{2}\eta_1},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}\eta_1}{u_{3,k} - u_{2,j} - \frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^{+\eta_1}}{x_{3,k} - x_{4,j}^{-\eta_1}},$$

$$1 = \left(\frac{x_{4,k}^-}{x_{4,k}^+}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{x_{4,k}^{+\eta_1} - x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2} - x_{4,j}^{+\eta_2}} \frac{1 - g^2/2x_{4,k}^+ x_{4,j}^-}{1 - g^2/2x_{4,k}^- x_{4,j}^+} \sigma^2(x_{4,k}, x_{4,j})\right)$$
$$\times \prod_{j=1}^{K_1} \frac{1 - g^2/2x_{4,k}^{-\eta_1} x_{1,j}}{1 - g^2/2x_{4,k}^{+\eta_1} x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^{-\eta_1} - x_{3,j}}{x_{4,k}^{+\eta_1} - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^{-\eta_2} - x_{5,j}}{x_{4,k}^{+\eta_2} - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - g^2/2x_{4,k}^{-\eta_2} x_{7,j}}{1 - g^2/2x_{4,k}^{+\eta_2} x_{7,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}\eta_2}{u_{5,k} - u_{6,j} - \frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^{+\eta_2}}{x_{5,k} - x_{4,j}^{-\eta_2}},$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k} - u_{6,j} - i\eta_2}{u_{6,k} - u_{6,j} + i\eta_2} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}\eta_2}{u_{6,k} - u_{5,j} - \frac{i}{2}\eta_2} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + \frac{i}{2}\eta_2}{u_{6,k} - u_{7,j} - \frac{i}{2}\eta_2},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + \frac{i}{2}\eta_2}{u_{7,k} - u_{6,j} - \frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{1 - g^2/2x_{7,k}x_{4,j}^{+\eta_2}}{1 - g^2/2x_{7,k}x_{4,j}^{-\eta_2}},$$

An all-order rank-1 $\mathfrak{sl}(2)$ subsector

- Twist-2 operators $\supset \mathfrak{sl}(2)$

$$\mathbb{O} \quad = \quad \text{Tr}\left(D_+^{n_1}\varphi\, D_+^{n_2}\varphi\right), \qquad n_1 + n_2 = N,$$

- All-order **rank-1** Bethe Ansatz equations

$$u \pm \frac{i}{2} = x^{\pm} + \frac{g^2}{2\,x^{\pm}}$$

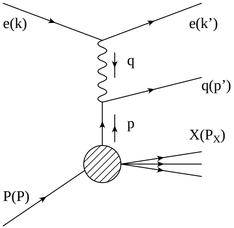
$$\left(\frac{x_k^+}{x_k^-}\right)^2 = \prod_{j\neq k}^{N} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1 - g^2/2\,x_k^+\,x_j^-}{1 - g^2/2\,x_k^-\,x_j^+}$$

$$\gamma(N) \sim \sum_{k=1}^{N}\left(\frac{i}{x_k^+} - \frac{i}{x_k^-}\right).$$

- **Can you deform the XXX_{-s} one-loop solution ???**

$\mathcal{N} = 4$ SYM: a *toy model* for perturbative QCD ?

- Integrability in $\mathcal{N} = 4 \Longrightarrow$ **γ at many loops.** **So What ?!**
- Consider deep inelastic scattering $eP \to eX$ in QCD



- γ of twist-2, spin N ops. $\leftrightarrow$ **splitting functions $P(x)$**

$$\int_0^1 dx\, x^{N-1}\, P(x) = -2\,\pi\, \boxed{\gamma_{\mathcal{O}}(N)}$$

- **Study $P(x)$ at many loops (≥ 3) !** Plenty of Physics...

Physical properties of the splitting functions

Soft gluon emission

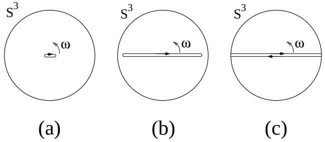
- In the **quasi-elastic limit** $x \to 1$ the most singular piece of P is **universal**, *i.e.* dominated by soft emission

$$P_{qq}(x) = \frac{2\,\Gamma_{\text{cusp}}(g)}{1-x} + \cdots, \qquad P_{gg}(x) = \frac{C_A}{C_F}\,\frac{2\,\Gamma_{\text{cusp}}(g)}{1-x} + \cdots$$

- **Prediction** (in $\mathcal{N} = 4$ SYM)

$$\boxed{\gamma_{ab}(N) = 2\,\delta_{ab}\,\Gamma_{\text{cusp}}(\alpha)\,\boxed{\log\, N} + \cdots}$$

- geometrical interpretation on $AdS_5 \times S^5$



Physical properties of the splitting functions

Gribov-Lipatov reciprocity

- Crossing relation between DIS and e^+e^- annihilation
- **Prediction** RR kernel $\mathcal{P}$ [Marchesini, Dokshizter, Salam, 05]

[Marchesini, Dokshizter, Beccaria, 07]

$$\gamma(N) = \mathcal{P}(N + \gamma(N)) \quad \longrightarrow \quad \boxed{\mathcal{P}(x) = -x\,\mathcal{P}\left(\frac{1}{x}\right)}$$

- In Mellin space, conditions on the large spin expansion

$$\boxed{\mathcal{P} = \underbrace{2\,\Gamma_{\text{cusp}}(\alpha)}_{\alpha_{\text{phys}}} \log J^2 + \sum_{n,m} \log^m(J^2)\,(J^2)^{-n},\ J^2 = N(N+1)}$$

- **or** MVV relations for the singular expansion $P(x) = \frac{A\,x}{(1-x)_+} + B\,\delta(1-x) + C\,\ln(1-x) + D + \cdots$ [Moch, Vermaseren, Vogt, 04]

Integrability at work: Three loop DIS from BA !

- Twist-2 operators $\supset \mathfrak{sl}(2)$

$$\mathbb{O} \;=\; \mathrm{Tr}\left(D_+^{n_1}\varphi\, D_+^{n_2}\varphi\right), \qquad n_1+n_2=N,$$

- Remark: We need $\gamma(N)$ in closed form !?!?!
- For each N we get a **rational** perturbative series

$$\gamma(N) \;=\; \sum_{n\geq 0}\gamma^{(n)}(N)\, g^{2\,n}$$

- The numerology is not very clear...

$$\begin{aligned}
\gamma^{(1)}(N) &= 6, \frac{25}{3}, \frac{49}{5}, \frac{761}{70}, \frac{7381}{630}, \frac{86021}{6930}, \frac{1171733}{90090}, \frac{2436559}{180180}, \dots \\
\gamma^{(2)}(N) &= -12, -\frac{925}{54}, -\frac{45619}{2250}, -\frac{138989861}{6174000}, -\frac{12120281899}{500094000}, -\frac{17061829801679}{665625114000}, \dots \\
\gamma^{(3)}(N) &= \dots .
\end{aligned}$$

Finding a closed formula is a **badly ill-posed** task

QCD-inspired closed expressions save the day

- **KLOV maximal transcendentality principle** [KLOV, 04]

$$\begin{aligned}
\gamma^{(1)}(N) &= \sum_{|X|=1} c_X\, S_X(N) = c\, S_1(N), \\
\gamma^{(2)}(N) &= \sum_{|X|=3} c_X\, S_X(N), \\
\gamma^{(3)}(N) &= \sum_{|X|=5} c_X\, S_X(N),
\end{aligned}$$

$$S_a(N) = \sum_{n=1}^{N} \frac{(\mathrm{sign}(a))^n}{n^{|a|}}, \qquad S_{a,\mathbf{b}}(N) = \sum_{n=1}^{N} \frac{(\mathrm{sign}(a))^n}{n^{|a|}}\, S_{\mathbf{b}}(n)$$

- **Can we prove this from the Bethe equations ?** [Catino's poster]

One-loop is completely solvable

- The XXX_{-s} chain has $\mathfrak{sl}(2)$ symmetry. site $\sim [s]$

- The Bethe roots are encoded in the **Baxter polynomial**

$$Q(u) = \prod_{k=1}^{N}(u - u_k).$$

obeying the **Baxter equation**

$$(u + i\,s)^L\,Q(u+i) + (u - i\,s)^L\,Q(u-i) = t(u)\,Q(u).$$

where

$$t(u) \quad = \quad 2\,u^L + q_2\,u^{L-2} + q_3\,u^{L-3} + \cdots + q_L$$

- The energy is **simply computed** from $Q(u)$

$$E = i\ \left[(\log\ Q(u))'\right]_{-is}^{+is}, \qquad e^{iP} = \frac{Q(+is)}{Q(-is)}.$$

- For twist-2

$$[s] \otimes [s] = \bigoplus_{N=0}^{\infty} [2s + N]$$

and highest weights are labeled by the Lorentz spin N

- The Baxter polynomial with degree N (even or odd) is

$$\boxed{Q(u) = {}_3F_2\left(\begin{array}{ccc|c} -N & N+4s-1 & s-iu & \\ & 2s\quad 2s & & 1 \end{array}\right)}$$

- This **proves** KLOV at one-loop

$$E = i\ \left[(\log\ Q(u))'\right]_{-is}^{+is} = 4\ \left[\psi(N+2s) - \psi(2s)\right]$$

Beyond one-loop...

- Don't know precisely why, **but KLOV works !**

$$
\begin{aligned}
\gamma_{2,s}^{(1)} &= 4\,S_1\,,\\
\gamma_{2,s}^{(2)} &= -4\left(S_3 + S_{-3} - 2\,S_{-2,1} + 2\,S_1\left(S_2 + S_{-2}\right)\right),\\
\gamma_{2,s}^{(3)} &= -8\Big(2\,S_{-3}\,S_2 - S_5 - 2\,S_{-2}\,S_3 - 3\,S_{-5} + 24\,S_{-2,1,1,1}\\
&\quad +6\left(S_{-4,1} + S_{-3,2} + S_{-2,3}\right) - 12\left(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\right)\\
&\quad -\left(S_2 + 2\,S_1^2\right)\left(3\,S_{-3} + S_3 - 2\,S_{-2,1}\right) - S_1\left(8\,S_{-4} + S_{-2}^2\right.\\
&\quad \left.+4\,S_2\,S_{-2} + 2\,S_2^2 + 3\,S_4 - 12\,S_{-3,1} - 10\,S_{-2,2} + 16\,S_{-2,1,1}\right)\Big)
\end{aligned}
$$

- GL reciprocity **OK** $\surd$ [Beccaria, Marchesini, Dokshizter, Basso, Korchemsky, 07]

  cusp anomaly **OK** $\surd$

  nice multi-loop structures **OK** $\surd$

Many other successful applications !

- Extension of KLOV at twist-3, **4 loops** [Beccaria, 07]

  [Beccaria, Marchesini, Dokshitzer, 07]

$$\mathbb{O} = \mathrm{Tr}\,(D_+^{n_1}\varphi\, D_+^{n_2}\varphi\, D_+^{n_3}\varphi)\,, \qquad n_1+n_2+n_3 = N$$

- SUSY universality in the $\lambda\lambda\lambda$ $\mathfrak{sl}(2|1) \supset \mathfrak{sl}(2)$ sector [Beccaria, 07]

  Theorem: $\gamma^{\lambda\lambda\lambda}(N) = \gamma_{\mathrm{twist}\,2}(N+2)$

- Extension of KLOV to 3-gluon operators $\mathbb{O} = \mathrm{Tr}\,D^N(A^3)$
- Higher order gluon condensates $\mathrm{Tr}\,\mathcal{F}^L$ [Beccaria, Forini, 07]
- Hypermagnets $\mathbb{O} = \mathrm{Tr}\,\{\varphi\,\varphi\,D^n\,\overline{D}^m\,\varphi\}$ [Beccaria, Staudacher, Rej, Zieme, 08]
- Sum rules for higher twists in $\mathfrak{sl}(2)$ [Beccaria, Catino, 08]

Conclusions and (selected) open questions

Impressive impact of integrability on AdS/CFT

CFT

- Long-range Bethe equations $\Longrightarrow$ multi-loop calculations
- *QCD-inspired* closed formulae and hyperintegrability
  Why ?
  Can we prove them from the long-range Bethe Ansatz ?
- Physical checks: GL reciprocity, BFKL singularities,
  Non-trivial in the BA !

AdS

- Integrability $\longrightarrow$ flow of anomalous dimensions to strong coupling, **BES and all that...**