Solutions for real dispersionless Veselov-Novikov hierarchy

Jen-Hsu Chang
(Joint Work with Yu-Tung Chen)

Department of Computer Science
National Defense University, Taiwan

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Outline

1. Introduction
2. The dVN hierarchy
3. Faber Polynomials of 2-dBKP
4. Symmetry constraint of dVN hierarchy and conserved densities
5. Hodograph solutions of dVN hierarchy
6. $2N$-component case

- The Veselov-Novikov equation [Veselov & Novikov 84]

$$u_\tau = (uV)_z + (u\bar{V})_{\bar{z}} + u_{zzz} + u_{\bar{z}\bar{z}\bar{z}}, \quad V_{\bar{z}} = -3u_z, \qquad (1)$$

where $z = x + iy$ and it is a two-dimensional integrable extension of the KdV equation.

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From 2-dBKP hierarchy to dVN hierarchy

The dispersionless Hirota equations of two-component BKP system [Takasaki 06; Tu & Chen 06]

$$\frac{p(\lambda)-p(\mu)}{p(\lambda)+p(\mu)} = \exp(-D(\lambda)S(\mu)), \quad (3)$$

$$\frac{\tilde{p}(\lambda)-\tilde{p}(\mu)}{\tilde{p}(\lambda)+\tilde{p}(\mu)} = \exp(-\tilde{D}(\lambda)\tilde{S}(\mu)), \quad (4)$$

$$\frac{p(\lambda)-\tilde{q}(\mu)}{p(\lambda)+\tilde{q}(\mu)} = \exp(-D(\lambda)\tilde{S}(\mu)), \quad (5)$$

$$\frac{\tilde{p}(\lambda)-q(\mu)}{\tilde{p}(\lambda)+q(\mu)} = \exp(-\tilde{D}(\lambda)S(\mu)), \quad (6)$$

where the quasiclassical vertex operators are

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{2\lambda^{-2n-1}}{2n+1}\partial_{t_{2n+1}}, \quad \tilde{D}(\lambda) = \sum_{n=0}^{\infty} \frac{2\lambda^{-2n-1}}{2n+1}\partial_{\tilde{t}_{2n+1}},$$

and the quasiclassical wave functions are

$$S(\lambda) = \sum_{n=0}^{\infty} t_{2n+1}\lambda^{2n+1} - D(\lambda)\mathcal{F},$$
$$\tilde{S}(\lambda) = \sum_{n=0}^{\infty} \tilde{t}_{2n+1}\lambda^{2n+1} - \tilde{D}(\lambda)\mathcal{F}.$$

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$$p(\lambda) = \partial_{t_1} S(\lambda), \quad q(\lambda) = \partial_{\tilde{t}_1} S(\lambda), \tag{7}$$
$$\tilde{p}(\lambda) = \partial_{\tilde{t}_1} \tilde{S}(\lambda), \quad \tilde{q}(\lambda) = \partial_{t_1} \tilde{S}(\lambda). \tag{8}$$

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$$p(\lambda)q(\lambda) = \tilde{p}(\mu)\tilde{q}(\mu).$$

Letting $\lambda, \mu \to \infty$ one obtains

$$-2\mathcal{F}_{t_1,\tilde{t}_1} = -2\mathcal{F}_{\tilde{t}_1,t_1} = u \tag{9}$$

where $u = u(t_1, t_2, \ldots; \tilde{t}_1, \tilde{t}_2, \ldots)$ is a scalar function and, for arbitrary λ, one has

$$p(\lambda)q(\lambda) = \tilde{p}(\lambda)\tilde{q}(\lambda) = u. \tag{10}$$

dVN hierarchy: Denoting

$$H_{2n+1}=2\frac{\partial^2\mathcal{F}}{\partial t_{2n+1}\partial t_1},\qquad \hat{H}_{2n+1}=2\frac{\partial^2\mathcal{F}}{\partial t_{2n+1}\partial \tilde{t}_1},$$
$$\tilde{H}_{2n+1}=2\frac{\partial^2\mathcal{F}}{\partial \tilde{t}_{2n+1}\partial \tilde{t}_1},\qquad \hat{\tilde{H}}_{2n+1}=2\frac{\partial^2\mathcal{F}}{\partial \tilde{t}_{2n+1}\partial t_1},$$

then, from Eq.(9) the evolution of u with respect to t_{2n+1} and $\tilde{t}_{2n+1}$ can be read respectively as

$$\frac{\partial u}{\partial t_{2n+1}} = -(H_{2n+1})_{\tilde{t}_1} = -(\hat{H}_{2n+1})_{t_1}, \tag{11}$$
$$\frac{\partial u}{\partial \tilde{t}_{2n+1}} = -(\tilde{H}_{2n+1})_{t_1} = -(\hat{\tilde{H}}_{2n+1})_{\tilde{t}_1}. \tag{12}$$

Define τ_{2n+1}-flow:

1. $\partial_{\tau_{2n+1}} := \partial_{t_{2n+1}} + \partial_{\tilde{t}_{2n+1}}$
2. $\tilde{t}_{2n+1}$ as the complex conjugate of t_{2n+1}, in particular, $t_1 := z$ and $\tilde{t}_1 := \bar{z}$, where $z = x + iy$.
3. the functions $\tilde{H}_{2n+1} = \bar{H}_{2n+1}$, and $\tilde{\hat{H}}_{2n+1} = \bar{\hat{H}}_{2n+1}$.

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(11) + (12) for $n \geq 1$ together with $n = 0$ in (11) (or (12)) we get the dVN hierarchy

$$u_{\tau_{2n+1}} = -(\hat{H}_{2n+1})_z - (\bar{\hat{H}}_{2n+1})_{\bar{z}}, \qquad u_z = -(H_1)_{\bar{z}}, \tag{13}$$

(For $n = 1 \Longrightarrow$ It's the dVN equation)

The n-th Faber polynomials

Def. $\Phi_n(w)$ is the n-th Faber plynomial of $p(\lambda)$ defined by

$$\log \frac{p(\lambda) - w}{\lambda} = -\sum_{n=1}^{\infty} \frac{\Phi_n(w)}{n} \lambda^{-n}, \tag{14}$$

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Take into account the symmetry conditions [Takasaki 06]

$$p(-\lambda) = -p(\lambda), \qquad \Phi_n(-w) = (-1)^n \Phi_n(w).$$

One can write

$$\log \frac{p(\lambda)-w}{p(\lambda)+w} = -\sum_{n=0}^{\infty} \frac{2\Phi_{2n+1}(w)}{2n+1}\lambda^{-2n-1}, \tag{15}$$

Replace$w$ in (15) by $p(\mu)$ and by $\tilde{q}(\mu)$, respectively. Then Eqs.(3) and (5) reduce to the following system of Hamilton-Jacobi equations

$$\frac{\partial S(\mu)}{\partial t_{2n+1}} = \Phi_{2n+1}(p(\mu)), \qquad \frac{\partial \tilde{S}(\mu)}{\partial t_{2n+1}} = \Phi_{2n+1}(\tilde{q}(\mu)). \tag{16}$$

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Differentiating the above equations with respect to $z, \bar{z}$, we have time evolutions of $p(\mu), q(\mu), \tilde{p}(\mu)$ and $\tilde{q}(\mu)$ in t_{2n+1}-flow:

$$\frac{\partial p(\mu)}{\partial t_{2n+1}} = \partial_z \Phi_{2n+1}(p(\mu)), \quad \frac{\partial q(\mu)}{\partial t_{2n+1}} = \partial_{\bar{z}} \Phi_{2n+1}(p(\mu)), \tag{17}$$

$$\frac{\partial \tilde{p}(\mu)}{\partial t_{2n+1}} = \partial_{\bar{z}} \Phi_{2n+1}(\tilde{q}(\mu)), \quad \frac{\partial \tilde{q}(\mu)}{\partial t_{2n+1}} = \partial_z \Phi_{2n+1}(\tilde{q}(\mu)). \tag{18}$$

In the same way, for the Hirota equations (4) and (6), one can derive the corresponding Hamilton-Jacobi equations via the expression of Faber polynomials as

$$\log \frac{\tilde{p}(\lambda) - w}{\tilde{p}(\lambda) + w} = -\sum_{n=0}^{\infty} \frac{2\tilde{\Phi}_{2n+1}(w)}{2n+1} \lambda^{-2n-1}. \qquad (19)$$

With substitutions of $w = \tilde{p}(\mu)$ and $w = q(\mu)$, we have also the following time evolutions of $p(\mu), q(\mu), \tilde{p}(\mu)$ and $\tilde{q}(\mu)$ with respect to $\tilde{t}_{2n+1}$-flow

$$\frac{\partial \tilde{p}(\mu)}{\partial \tilde{t}_{2n+1}} = \partial_{\bar{z}} \tilde{\Phi}_{2n+1}(\tilde{p}(\mu)), \quad \frac{\partial \tilde{q}(\mu)}{\partial \tilde{t}_{2n+1}} = \partial_z \tilde{\Phi}_{2n+1}(\tilde{p}(\mu)), \qquad (20)$$

$$\frac{\partial p(\mu)}{\partial \tilde{t}_{2n+1}} = \partial_z \tilde{\Phi}_{2n+1}(q(\mu)), \quad \frac{\partial q(\mu)}{\partial \tilde{t}_{2n+1}} = \partial_{\bar{z}} \tilde{\Phi}_{2n+1}(q(\mu)). \qquad (21)$$

$H_{2n+1}, \hat{H}_{2n+1}$ generate Faber polynomials

We first differentiate (15) to the both sides with respect to λ and obtain

$$\frac{wp'(\lambda)}{p^2(\lambda)-w^2}=\sum_{n=0}^{\infty}\Phi_{2n+1}(w)\lambda^{-2n-2}.$$

Putting $p(\lambda)=\lambda-\sum_{n=0}^{\infty}\frac{H_{2n+1}}{2n+1}\lambda^{-2n-1}$ into this expression and comparing coefficients of powers of λ on both sides, we solve recursively

$$\begin{aligned}\Phi_1(w) &= w,\\ \Phi_3(w) &= w^3+3H_1w, \qquad (22)\\ \Phi_5(w) &= w^5+5H_1w^3+5(H_1^2+H_3/3)w,\\ \Phi_7(w) &= w^7+7H_1w^5+7(2H_1^2+H_3/3)w^3\\ &\quad +7(H_1^3+(2/3)H_1H_3+H_5/5)w. \qquad (23)\end{aligned}$$

They obey the recursive formula, for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} \Phi_{2n+5}(w) &= w^2\Phi_{2n+3}(w) \\ &\quad -\sum_{m=0}^{n}\sum_{k=0}^{n-m}\frac{H_{2n-2m-2k+1}H_{2k+1}}{(2n-2m-2k+1)(2k+1)}\Phi_{2m+1}(w) \\ &\quad +2\sum_{m=0}^{n+1}\frac{H_{2n-2m+3}}{2n-2m+3}\Phi_{2m+1}+wH_{2n+3}, \end{aligned}$$

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Similarly, the differentiation of (19) with respect to λ obtains

$$\frac{w\tilde{p}'(\lambda)}{\tilde{p}^2(\lambda) - w^2} = \sum_{n=0}^{\infty}\tilde{\Phi}_{2n+1}(w)\lambda^{-2n-2}.$$

The conservation laws

Consider the symmetry constraint for real u and S [Bogdanov, Konopelchenko & Moro,2004]

$$u_x = (S^i)_{z\bar{z}}, \tag{24}$$

- $S^i = S(\mu_i)$ is evaluated at some point μ_i.
- $\partial_z S^i = p(\mu_i) = p^i$ and $\partial_{\bar{z}} S^i = q(\mu_i) = q^i = \bar{p}^i$ obey the algebraic relation $u = p^i q^i = p^i \bar{p}^i$ (by the extra equation (10)).
- Eq.(24) implies $p^i_{\bar{z}} = q^i_z = u_x$.

Remark. In the context of nonlinear geometry optics, arising from the high-frequency limit of Maxwell equations [Konopelchenko & Moro 04–05], one has

- $u = n^2 = \varepsilon_0 \mu$, n being the refractive index.
- $u = S^i_z S^i_{\bar{z}} \Rightarrow$ standard Eikonal equation in two dimensions.

Under the symmetry reductions, we discuss the relations of conserved densities and the associated Faber polynomials along the following two ways.

(I) Let us take the derivatives of $S(\lambda)$ with respect to $z, \bar{z}, x$ and noticing that $-2\mathcal{F}_{z\bar{z}} = u$, we have

$$\frac{\partial^3 S(\lambda)}{\partial z \partial \bar{z} \partial x} = -D(\lambda)\mathcal{F}_{z\bar{z}x} = \frac{1}{2}D(\lambda)u_x = \frac{1}{2}D(\lambda)S^i_{z\bar{z}}. \qquad (25)$$

Comparing to (3), it follows that

$$(p(\lambda))_x = \left(\frac{1}{2}D(\lambda)S^i\right)_z = -\frac{1}{2}\partial_z\left(\log\frac{p(\lambda)-p^i}{p(\lambda)+p^i}\right), \qquad (26)$$

$$(q(\lambda))_x = \left(\frac{1}{2}D(\lambda)S^i\right)_{\bar{z}} = -\frac{1}{2}\partial_{\bar{z}}\left(\log\frac{p(\lambda)-p^i}{p(\lambda)+p^i}\right). \qquad (27)$$

Using (15) with w replaced by p^i and the expansions of $p(\lambda)$ and $q(\lambda)$, Eqs. (26) and (27) provide that the Hamilton-Jacobi equations (17) can be read respectively by

$$\begin{aligned} \frac{\partial p^i}{\partial t_{2n+1}} &= \partial_z \Phi_{2n+1}(p^i) = -\partial_x H_{2n+1}, \\ \frac{\partial \bar{p}^i}{\partial t_{2n+1}} &= \partial_{\bar{z}} \Phi_{2n+1}(p^i) = -\partial_x \hat{H}_{2n+1}, \end{aligned} \tag{28}$$

where $H_{2n+1} \equiv 2\partial_z \partial_{t_{2n+1}} \mathcal{F}$ and $\hat{H}_{2n+1} \equiv 2\partial_{\bar{z}} \partial_{t_{2n+1}} \mathcal{F}$. Hence, H_{2n+1} and $\hat{H}_{2n+1}$ appear to be the conserved densities characterized by the associated Hamilton-Jabobi equations. They are connected by the compatibility relations

$$\partial_{\bar{z}} H_{2n+1} = \partial_z \hat{H}_{2n+1},$$

and can be obtained by solving (28).

(II) In the similar way, the differentiation of $\tilde{S}(\lambda)$ with respect to $z, \bar{z}, x$ shows that

$$\frac{\partial^3 \tilde{S}(\lambda)}{\partial z \partial \bar{z} \partial x} = -\tilde{D}(\lambda)\mathcal{F}_{z\bar{z}x} = \frac{1}{2}\tilde{D}(\lambda)u_x = \frac{1}{2}\tilde{D}(\lambda)S^i_{z\bar{z}},$$

We finally get the conservation lows

$$\frac{\partial \bar{p}^i}{\partial \bar{t}_{2n+1}} = \partial_{\bar{z}}\tilde{\Phi}_{2n+1}(\bar{p}^i) = -\partial_x \bar{H}_{2n+1},$$
$$\frac{\partial p^i}{\partial \bar{t}_{2n+1}} = \partial_z \tilde{\Phi}_{2n+1}(\bar{p}^i) = -\partial_x \bar{\hat{H}}_{2n+1}. \quad (29)$$

where $\bar{H}_{2n+1} \equiv 2\partial_{\bar{z}}\partial_{\bar{t}_{2n+1}}\mathcal{F}$ and $\bar{\hat{H}}_{2n+1} \equiv 2\partial_z\partial_{\bar{t}_{2n+1}}\mathcal{F}$. They obey

$$\partial_z \bar{H}_{2n+1} = \partial_{\bar{z}}\bar{\hat{H}}_{2n+1}.$$

Relations of $\hat{H}_{2n+1}$ and H_{2n+1}

Putting $p(\lambda) = \lambda - \sum_{n=0}^{\infty} \frac{\lambda^{-2n-1}}{2n+1} H_{2n+1}$, $q(\lambda) = -\sum_{n=0}^{\infty} \frac{\lambda^{-2n-1}}{2n+1} \hat{H}_{2n+1}$ into $p(\lambda)q(\lambda) = u$, and comparing the coefficients of powers λ, we can solve recursively

$$\hat{H}_1 = -u, \quad \hat{H}_3 = -3uH_1, \quad \hat{H}_5 = -\frac{5}{3}u(3H_1^2 + H_3),$$
$$\hat{H}_7 = -\frac{7}{3}u(3H_1^3 + 2H_1H_3 + \frac{3}{5}H_5), \quad \cdots$$

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$$\hat{H}_1=-u,\quad \hat{H}_3=-3uH_1,\quad \hat{H}_5=-\frac{5}{3}u(3H_1^2+H_3),$$
$$\hat{H}_7=-\frac{7}{3}u(3H_1^3+2H_1H_3+\frac{3}{5}H_5),\quad \cdots$$

Remark. The case of $n=1$ in (13) reduces to the dVN equation

$$u_{\tau_3}=3(uH_1)_z+3(u\bar{H}_1)_{\bar{z}},\quad u_z=-(H_1)_{\bar{z}}, \tag{30}$$

with $\tau:=\tau_3, V:=3H_1$.

How to construct conserved densities?

For simplifying calculations, we use Faber polynomials (22), (23) and the identities

$$\begin{aligned} p^i_z &= p^i_x - u_x, \\ u_z p^i &= u p^i_x - u u_x + u_x (p^i)^2. \end{aligned}$$

Example 1. By (28), for $n = 0$

$$H_{1x} = -\Phi_{1z} = -p^i_z = (u - p^i)_x,$$

After integrating both sides with respect to x one yields

$$H_1 = u - p^i = p^i \bar{p}^i - p^i.$$

Example 2. For $n = 1$ in (28), we use the expressions of Φ_3 and H_1 to get

$$\begin{aligned} H_{3x} &= -\Phi_{3z} = -\left((p^i)^3 + 3(u - p^i)p^i\right)_z, \\ &= \left(3(u - p^i)^2 - (p^i)^3\right)_x = \left(3H_1^2 - (p^i)^3\right)_x, \end{aligned}$$

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After integrating both sides with respect to x, we get

$$H_3 = 3H_1^2 - (p^i)^3.$$

Similar calculations can yield

$$\begin{aligned} H_5 &= -10H_1^3 + \frac{20}{3}H_1H_3 - (p^i)^5, \\ H_7 &= 7\left(5H_1^4 - 5H_1^2H_3 + \frac{6}{5}H_1H_5 + \frac{1}{3}H_3^2 - \frac{(p^i)^7}{7}\right). \end{aligned}$$

Hodograph equations

The conservation laws (28), (29) provide that

$$\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{t_{2n+1}} = \partial_{p^i}\Phi_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_z + \partial_{\bar{p}^i}\Phi_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{\bar{z}},$$

$$\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{\bar{t}_{2n+1}} = \partial_{p^i}\overline{\Phi_{2n+1}}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_z + \partial_{\bar{p}^i}\overline{\Phi_{2n+1}}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{\bar{z}},$$

where we have used the fact that $p^i_{\bar{z}} = \bar{p}^i_z$.

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where we have used the fact that $p^i_{\bar{z}} = \bar{p}^i_z$.
The dVN hierarchy is then governed by

$$\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{\tau_{2n+1}} = \partial_{p^i}M_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_z + \partial_{\bar{p}^i}M_{2n+1}\begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{\bar{z}}, \quad n \geq 1, \quad (31)$$

where $M_{2n+1} = \Phi_{2n+1} + \overline{\Phi_{2n+1}}$.

The above equation has the following implicit form of hodograph equations

$$\begin{aligned} z+\sum_{n=1}^{\infty} f_{2n+1}(p^i,\bar{p}^i)\tau_{2n+1} &= F(p^i,\bar{p}^i),\\ \bar{z}+\sum_{n=1}^{\infty} g_{2n+1}(p^i,\bar{p}^i)\tau_{2n+1} &= G(p^i,\bar{p}^i), \end{aligned} \tag{32}$$

where F and G are the initial data at $\tau_{2n+1}=0$.
$\Rightarrow G$ and F obey the constraint

$$F_{\bar{p}^i} = G_{p^i}, \tag{33}$$

$$p^iF_{p^i} = -\bar{p}^iG_{\bar{p}^i}-(1-p^i-\bar{p}^i)G_{p^i}. \tag{34}$$

- (33) $\Rightarrow$ there exists $\varphi(p^i, \bar{p}^i)$ s.t. $F = \partial_{p^i}\varphi$, $G = \partial_{\bar{p}^i}\varphi$.
- (34) $\Rightarrow p^i\varphi_{p^ip^i} + \bar{p}^i\varphi_{\bar{p}^i\bar{p}^i} + (1 - p^i - \bar{p}^i)\varphi_{p^i\bar{p}^i} = 0$.

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Let $p^i = (\rho_1 - i\rho_2)/2$, $\bar{p}^i = (\rho_1 + i\rho_2)/2$, then

$$\varphi_{\rho_1\rho_1} + 2\rho_2\varphi_{\rho_1\rho_2} + (1 - 2\rho_1)\varphi_{\rho_2\rho_2} = 0. \qquad (35)$$

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In fact, due to the existence of φ,

- $F = \sum_{n\geq 0}\mu_n f_{2n+1}$, $G = \sum_{n\geq 0}\xi_n g_{2n+1}$, where $\mu_n = \xi_n$.

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- $F = \sum_{n\geq 0}\mu_n f_{2n+1}$, $G = \sum_{n\geq 0}\xi_n g_{2n+1}$, where $\mu_n = \xi_n$.
- We deduce that φ has the polynomial-type solution in ρ_1, ρ_2:

$$\varphi = \sum_{n=0}^{\infty}\mu_n M_{2n+1}(\rho_1, \rho_2) \tag{36}$$

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$$\varphi_{\rho_1\rho_1} = 6\rho_1 - 3, \quad \varphi_{\rho_1\rho_2} = 0, \quad \varphi_{\rho_2\rho_2} = 3,$$

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   which satisfy (35).
3. $\varphi = M_5 = \Phi_5 + \overline{\Phi_5} = \rho_1^5 - \frac{20}{3}\rho_1^4 + 10\rho_1^3 + \frac{5}{3}\rho_1^2(\rho_1^2 + \rho_2^2) - \frac{15}{2}\rho_1(\rho_1^2 + \rho_2^2) + \frac{5}{3}(\rho_1^2 + \rho_2^2)^2$, we have the following set of equations satisfied by (35):

$$\begin{aligned}
\varphi_{\rho_1\rho_1} &= 20\rho_1^3 - 40\rho_1^2 + 15\rho_1 + 10\rho_2^2, \\
\varphi_{\rho_1\rho_2} &= 20\rho_1\rho_2 - 15\rho_2, \\
\varphi_{\rho_2\rho_2} &= 10\rho_1^2 - 15\rho_1 + 20\rho_2^2.
\end{aligned}$$

Remark. One has several simple cases of finding solutions of the certain PDEs in Eq. (35):

(a) $\varphi_{\rho_1\rho_1} = 0$ and $2\rho_2\varphi_{\rho_1\rho_2} + (1-2\rho_1)\varphi_{\rho_2\rho_2} = 0$,

(b) $\varphi_{\rho_1\rho_2} = 0$ and $\varphi_{\rho_1\rho_1} + (1-2\rho_1)\varphi_{\rho_2\rho_2} = 0$,

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- Cases (a) and (b) have solutions of polynomial type involved in (36).
- However, case (c) has solutions of the form: $\varphi = c_0 + c_1\rho_1 + c_2\rho_2 + c_3\rho_2\exp(-2\rho_1)$.

Example of hodograph solution

Example 3. The (2+1)-dimensional solutions in $(z, \bar{z}, \tau)$: Now $\Phi_3 = (p^i)^3 + 3H_1 p^i$ with $H_1 = u - p^i$, we expand (32) up to $\tau_3 = \tau$

$$F(p^i, \bar{p}^i) = z + f_3\,\tau = z + \Big(3(p^i + \bar{p}^i)^2 - 6p^i\Big)\tau,$$
$$G(p^i, \bar{p}^i) = \bar{z} + g_3\,\tau = \bar{z} + \Big(3(p^i + \bar{p}^i)^2 - 6\bar{p}^i\Big)\tau. \quad (37)$$

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$$\begin{aligned} F(p^i, \bar{p}^i) &= z + f_3\tau = z + \Big(3(p^i + \bar{p}^i)^2 - 6p^i\Big)\tau, \\ G(p^i, \bar{p}^i) &= \bar{z} + g_3\tau = \bar{z} + \Big(3(p^i + \bar{p}^i)^2 - 6\bar{p}^i\Big)\tau. \end{aligned} \tag{37}$$

Choosing, for example, $F = f_3, G = g_3$ for $\mu_1 = \xi_1 = 1$, we solve p^i by

$$\begin{aligned} p^i \;\; &= \;\; \frac{1}{12(\tau - 1)}\Big(3(\tau - 1) + (z - \bar{z}) \\ &\qquad \pm \sqrt{9(\tau - 1)^2 - 6(\tau - 1)(z + \bar{z})}\Big), \end{aligned}$$

Therefore, u is read as

$$u = p^i\bar{p}^i = \frac{1}{144(\tau-1)^2}\Big(18(\tau-1)^2 - 6(\tau-1)(z+\bar{z}) - (z-\bar{z})^2 \pm 6(\tau-1)\sqrt{9(\tau-1)^2 - 6(\tau-1)(z+\bar{z})}\Big). \quad (38)$$

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One can verify that (38) satisfies the dVN equation (30). Consider $\partial S^i/\partial z = p^i$ and $\partial S^i/\partial \bar{z} = \bar{p}^i$, then S^i is solved by

$$S^i(z,\bar{z},\tau) = \frac{1}{72(\tau-1)^2}\Big(3(\tau-1)(z-\bar{z})^2 + 18(\tau-1)^2(z+\bar{z}+4C) - 2\sqrt{3}\Big(3(\tau-1)^2 - 2(\tau-1)(z+\bar{z})\Big)^{3/2}\Big),$$

where C is an arbitrary constant.

$2N$-component case

Let us consider a more general symmetry constraint of the form [Bogdanov, Konopelchenko & Moro, 2004]

$$u_x = \sum_{i=1}^{N} \epsilon_i S^i_{z\bar{z}}. \quad \text{Ansatz:} \quad \begin{cases} u = p^i \bar{p}^i, \forall\, i = 1, \ldots, N, \\ \sum_{i=1}^{N} \epsilon_i = 1. \end{cases}$$

we have the following relations between conserved densities and the associated Faber polynomials:

$$(H_{2n+1})_x = -\sum_{i=1}^{N} \epsilon_i \partial_z \Phi_{2n+1}(p^i),$$

$$(\hat{H}_{2n+1})_x = -\sum_{i=1}^{N} \epsilon_i \partial_{\bar{z}} \Phi_{2n+1}(p^i),$$

where the Faber polynomials $\Phi_{2n+1}(p^i)$ are defined as before.

In similarly lengthy calculations, some of H_{2n+1} for the $2N$-reduction system are given by

$$H_1 = u - \sum_{i=1}^{N} \epsilon_i p^i,$$

$$H_3 = 3H_1^2 - \sum_{i=1}^{N} \epsilon_i (p^i)^3,$$

$$H_5 = -10H_1^3 + \frac{20}{3} H_1 H_3 - \sum_{i=1}^{N} \epsilon_i (p^i)^5,$$

$$H_7 = 35H_1^4 - 35H_1^2 H_3 + \frac{42}{5} H_1 H_5 + \frac{7}{3} H_3^2 - \sum_{i=1}^{N} \epsilon_i (p^i)^7.$$

Under the symmetry constraint, the Hamilton-Jacobi equations can now be written in the following way:

$$\frac{\partial p^k}{\partial t_{2n+1}} = \partial_z \Phi_{2n+1}(p^k; p^1, \ldots, p^N, \bar{p}^1, \ldots, \bar{p}^N),$$
$$\frac{\partial \bar{p}^k}{\partial t_{2n+1}} = \partial_{\bar{z}} \Phi_{2n+1}(p^k; p^1, \ldots, p^N, \bar{p}^1, \ldots, \bar{p}^N),$$
$$\frac{\partial p^k}{\partial \bar{t}_{2n+1}} = \partial_z \bar{\Phi}_{2n+1}(\bar{p}^k; p^1, \ldots, p^N, \bar{p}^1, \ldots, \bar{p}^N),$$
$$\frac{\partial \bar{p}^k}{\partial \bar{t}_{2n+1}} = \partial_{\bar{z}} \bar{\Phi}_{2n+1}(\bar{p}^k; p^1, \ldots, p^N, \bar{p}^1, \ldots, \bar{p}^N),$$

where $k = 1, \ldots, N$.

After incorporating the above evolution equations to the τ_{2n+1}-flow of dVN hierarchy and noting that $p^i_{\bar{z}} = \bar{p}^i_z$ for $i = 1, \ldots, N$, we get

$$\begin{pmatrix} p^k \\ \bar{p}^k \end{pmatrix}_{\tau_{2n+1}} = \sum_{i=1}^{N} f^{(i)}_{2n+1} \begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_z + \sum_{i=1}^{N} g^{(i)}_{2n+1} \begin{pmatrix} p^i \\ \bar{p}^i \end{pmatrix}_{\bar{z}},$$

where $k = 1, \ldots, N$, and

$$\begin{aligned} f^{(i)}_{2n+1}(p^k, \bar{p}^k) &= \partial_{p^i}(\Phi_{2n+1}(p^k) + \bar{\Phi}_{2n+1}(\bar{p}^k)), \\ g^{(i)}_{2n+1}(p^k, \bar{p}^k) &= \partial_{\bar{p}^i}(\Phi_{2n+1}(p^k) + \bar{\Phi}_{2n+1}(\bar{p}^k)). \end{aligned}$$

For example, in the case of $N = 2$ we have

$$\begin{pmatrix} p^1 \\ \bar{p}^1 \\ p^2 \\ \bar{p}^2 \end{pmatrix}_{\tau_{2n+1}} = \begin{pmatrix} f^{(1)}_{1,1} I_2 + g^{(1)}_{1,1}\mathbb{A}(p^1) & f^{(2)}_{1,1} I_2 + g^{(2)}_{1,1}\mathbb{A}(p^2) \\ f^{(1)}_{2,2} I_2 + g^{(1)}_{2,2}\mathbb{A}(p^1) & f^{(2)}_{2,2} I_2 + g^{(2)}_{2,2}\mathbb{A}(p^2) \end{pmatrix} \begin{pmatrix} p^1 \\ \bar{p}^1 \\ p^2 \\ \bar{p}^2 \end{pmatrix}_z ,$$

where we denote $f^{(j)}_{k,k} = f^{(j)}_{2n+1}(p^k, \bar{p}^k)$, $g^{(j)}_{k,k} = g^{(j)}_{2n+1}(p^k, \bar{p}^k)$, and I_2 is the 2×2 identity matrix and

$$\mathbb{A}(p^i) = \begin{pmatrix} 0 & 1 \\ -\frac{\bar{p}^i}{p^i} & \frac{1 - p^i - \bar{p}^i}{p^i} \end{pmatrix}, \qquad i = 1, 2.$$

In principle, the last evolution equation can be solved by hodograph method. But exact solutions are not found.