Gauge-invariant description of some (2+1)-dimensional integrable nonlinear evolution equations

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Contents

Introduction

The fundamental ideas of gauge invariance and gauge transformations are wide spread and in common use in almost every part of physics.

The first applications of such ideas in the theory of integrable nonlinear equations by

- Zakharov and Shabat (1974) [1],
- Kuznetsov and Mikhailov (1977) [2],
- Zakharov and Mikhailov (1978) [3],
- Zakharov and Takhtadzhyan (1979) [4],
- Konopelchenko (1982) [5],
- Konopelchenko and Dubrovsky (1983, 1984) [6, 7]

and others have been made, see also the books [8, 9, 10, 11, 12, 13] and references therein.

Now a lot of gauge-equivalent to each other integrable nonlinear models are well known.

In one-dimensional case the most famous are nonlinear Schrödinger and Heisenberg ferromagnet equations:

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + 2\kappa|\psi|^2\psi, \tag{1}$$

$$\frac{\partial\vec{S}}{\partial t} = \vec{S} \times \frac{\partial^2\vec{S}}{\partial x^2}, \tag{2}$$

KdV and mKdV equations:

$$\frac{\partial u_0}{\partial t} + \frac{\partial^3 u_0}{\partial x^3} + 6u_0\frac{\partial u_0}{\partial x} = 0, \tag{3}$$

$$\frac{\partial u_1}{\partial t} + \frac{\partial^3 u_1}{\partial x^3} + 6u_1^2\frac{\partial u_1}{\partial x} = 0, \tag{4}$$

massive Thirring model and two-dimensional relativistic field model and so on.

In two-dimensional case the most famous are Kadomtsev–Petviashvili (KP) and modified Kadomtsev–Petviashvili (mKP) nonlinear equations:

$$\frac{\partial u_0}{\partial t} + \frac{\partial^3 u_0}{\partial x^3} + 6u_0\frac{\partial u_0}{\partial x} + 3\sigma^2\partial_x^{-1}\frac{\partial^2 u_0}{\partial y^2} = 0, \tag{5}$$

$$\frac{\partial u_1}{\partial t} + \frac{\partial^3 u_1}{\partial x^3} - \frac{3}{2}u_1^2\frac{\partial u_1}{\partial x} + 3\sigma^2\partial_x^{-1}\frac{\partial^2 u_1}{\partial y^2} - 3\sigma\frac{\partial u_1}{\partial x}\partial_x^{-1}\frac{\partial u_1}{\partial y} = 0, \tag{6}$$

Davey–Stewartson

$$p_t - \kappa_1 p_{\xi\xi} + \kappa_2 p_{\eta\eta} - 2\kappa_1 p\partial_\eta^{-1}(pq)_\xi + 2\kappa_2 p\partial_\xi^{-1}(pq)_\eta = 0, \tag{7}$$

$$q_t + \kappa_1 q_{\xi\xi} - \kappa_2 q_{\eta\eta} + 2\kappa_1 q\partial_\eta^{-1}(pq)_\xi - 2\kappa_2 p\partial_\xi^{-1}(pq)_\eta = 0, \tag{8}$$

and Ishimori

$$\vec{S}_t + \frac{1}{2}\vec{S}\times(\vec{S}_{\xi\xi} + \vec{S}_{\eta\eta}) + \frac{1}{2}\varphi_\xi\vec{S}_\xi + \frac{1}{2}\varphi_\eta\vec{S}_\eta = 0, \tag{9}$$

$$\varphi_{\xi\eta} - \vec{S}\cdot[\vec{S}_\xi\times\vec{S}_\eta] = 0, \tag{10}$$

integrable systems of nonlinear equations and so on. See some references in the books [8, 9, 10, 11, 12, 13, 14].

Let us underline the unified role of gauge transformations and gauge-invariance by the simple example of interaction of nonrelativistic spinless charged particle with electromagnetic field.

Let us perform in nonstationary Schrödinger equation for such particle

$$i\hbar\psi_t = \frac{\hat{\vec{p}}^2}{2m}\psi = \hat{H}\psi \tag{11}$$

gauge transformation

$$\psi \to \psi' = g^{-1}\psi, \quad \psi = g\psi' = \exp\left(\frac{i\chi(\vec{r},t)q}{\hbar}\right).\psi' \tag{12}$$

for the wave function. Under substitution (12) into (11) one obtains Schrödinger equation for the transformed wave function ψ'

$$i\hbar\psi'_t = \frac{\left[\hat{\vec{p}} - (-q\vec{\nabla}\chi)\right]^2}{2m}\psi' + q\chi_t\psi'. \tag{13}$$

From another side it is known from the electrodynamics that the vector $\vec{A}$ and scalar ϕ potentials due to gauge freedom are determine nonuniquely

$$\vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}\chi, \quad \phi \to \phi' = \phi + \chi_t, \tag{14}$$

at the same time the electromagnetic fields $\vec{B} = [\vec{\nabla} \times \vec{A}]$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ did not change. One can rewrite the equation (13) due to (14) in the form

$$i\hbar\psi'_t = \frac{\left(\hat{\vec{p}} - q\vec{A}^{(0)}\right)^2}{2m}\psi' + q\phi^{(0)}\psi', \tag{15}$$

where $\vec{A}^{(0)} = -\vec{\nabla}\chi$, $\phi^{(0)} = \chi_t$. It is evident that the fields $\vec{B} = 0$ and $\vec{E} = 0$ as in the case of initial equation (11) and also in the transformed equation (15) are equal to zero:

$$\vec{B}^{(0)} = [\vec{\nabla} \times \vec{A}^{(0)}] = 0, \quad \vec{E}^{(0)} = -\vec{\nabla}\phi^{(0)} - \frac{\partial \vec{A}^{(0)}}{\partial t} = 0. \tag{16}$$

Nevertheless the lesson from such passage is that the equation (15) gives right gauge-invariant form of nonstationary Schrödinger equation also in the case of nontrivial fields $\vec{B} = [\vec{\nabla} \times \vec{A}] \neq 0$, $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \neq 0$. This right form of nonstationary Schrödinger equation due to (15) is the following:

$$i\hbar\psi_t = \frac{(\hat{\vec{p}} - q\vec{A})^2}{2m}\psi + q\phi\psi. \tag{17}$$

Let us consider the equation (17), from IST point of view as auxiliary linear problem, PDE with variable coefficients for the wave function ψ. Under gauge transformation (12) the equation (17) preserves its form if the potentials $\vec{A}$ and ϕ have the following laws of transformations:

$$\vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}\chi, \quad \phi \to \phi' = \phi + \chi_t, \tag{18}$$

in accordance with the rule (14) known from electrodynamics.

Excluding gauge function χ from (18) one obtain the evident but nontrivial consequences

$$[\vec{\nabla} \times \vec{A}'] = [\vec{\nabla} \times \vec{A}], \quad -\vec{\nabla}\phi' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}. \tag{19}$$

This means that the quantities

$$\vec{B} \stackrel{\text{def}}{=} [\vec{\nabla} \times \vec{A}], \quad \vec{E} \stackrel{\text{def}}{=} -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{20}$$

are invariants under gauge transformations (12). Moreover from definitions (20) for invariants $\vec{B}$ and $\vec{E}$ follows famous subsystem

$$\mathrm{div}\vec{B} = \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{21}$$

of fundamental Maxwell equations.

Analogous considerations based on ideas of gauge transformations and gauge-invariance can be applied as well to integrable nonlinear equations. The separation of physical and pure gauge degrees of freedom in the integrable nonlinear equations and their manifestly gauge-invariant formulation may be very useful for the understanding of structure of these equations and the interrelations between different gauge-equivalent to each other equations.

In the present report manifestly gauge-invariant formulation of two-dimensional nonlinear evolution equations integrable by the following two scalar auxiliary linear problems

$$L_1\psi = \left(\partial^2_{\xi\eta} + u_1\partial_\xi + v_1\partial_\eta + u_0\right)\psi = 0, \quad (22)$$

$$L_2\psi = \left(\partial_t + u_3\partial^3_\xi + v_3\partial^3_\eta + u_2\partial^2_\xi + v_2\partial^2_\eta + \tilde{u}_1\partial_\xi + \tilde{v}_1\partial_\eta + v_0\right)\psi = 0 \quad (23)$$

is developed. Here as usual $\xi = x + \sigma y$, $\eta = x - \sigma y$, $\sigma^2 = \pm 1$ and $\partial_\xi = \partial/\partial\xi$, $\partial_\eta = \partial/\partial\eta$, $\partial^2_\xi = \partial^2/\partial\xi^2$, etc.

Two cases of auxiliary linear problems (22), (23) with different second auxiliary linear problem (23) are studied:

- (i) $u_3 = \kappa_1 = \text{const}$, $v_3 = \kappa_2 = \text{const}$, – third-order problem $L_2\psi = 0$, such choice of second auxiliary problem (23) leads to famous Nizhnik–Veselov–Novikov, (1980,1984) (NVN) [15, 16]; modified Nizhnik–Veselov–Novikov, (1990) (mNVN) [17] and other equations;
- (ii) $u_3 = v_3 = 0$, $u_2 = \kappa_1 = \text{const}$, $v_2 = \kappa_2 = \text{const}$, – second-order problem $L_2\psi = 0$, such choice of second auxiliary problem (23) leads to famous two-dimensional generalization of dispersive long-wave equation, (1987) (2DDLW) [18]; Davey–Stewartson (DS) system of equations, (1974) [19] and its reductions and other equations.

All above mentioned famous integrable nonlinear equations via compatibility condition of auxiliary linear problems (22) and (23) in the form of Manakov's triad representation, (1976) [20]

$$[L_1, L_2] = BL_1 \qquad (24)$$

have been previously established [15, 16, 17, 18].

Gauge transformations

$$\psi \to \psi' = g^{-1}\psi \tag{25}$$

with arbitrary gauge function $g(\xi,\eta,t)$ of auxiliary linear problems (22) and (23) are studied. The convenient for gauge-invariant formulation field variables, classical gauge invariants w_2, $\tilde{w}_2$, w_1

$$w_2 \stackrel{\text{def}}{=} u_0 - u_{1\xi} - u_1 v_1 = u_0' - u_{1\xi}' - u_1' v_1', \tag{26}$$

$$\tilde{w}_2 \stackrel{\text{def}}{=} u_0 - v_{1\eta} - u_1 v_1 = u_0' - v_{1\eta}' - u_1' v_1', \tag{27}$$

$$w_1 \stackrel{\text{def}}{=} u_{1\xi} - v_{1\eta} = u_{1\xi}' - v_{1\eta}' \tag{28}$$

and pure gauge variable ρ connected with field variable $u_1(\xi,\eta,t)$ by the formula

$$u_1 \stackrel{\text{def}}{=} (\ln\rho)_\eta \tag{29}$$

are introduced. The variable ρ corresponds to pure gauge degrees of freedom and has under (25) the following simple law of transformation:

$$\rho \to \rho' = g\rho. \tag{30}$$

Gauge-invariant formulation of NVN system

In the case (i) of third order linear auxiliary problem (23) the first invariant w_1 is equal to zero $w_1 \equiv 0$ and the established integrable system of nonlinear equations in terms of ρ, w_2 has the form:

$$\rho_t = -\kappa_1 \rho_{\xi\xi\xi} - \kappa_2 \rho_{\eta\eta\eta} - 3\kappa_1 \rho_\xi \partial_\eta^{-1} w_{2\xi} - 3\kappa_2 \rho_\eta \partial_\xi^{-1} w_{2\eta} + v_0 \rho, \quad (31)$$

$$w_{2t} = -\kappa_1 w_{2\xi\xi\xi} - \kappa_2 w_{2\eta\eta\eta} - 3\kappa_1 \left(w_2 \partial_\eta^{-1} w_{2\xi}\right)_\xi - 3\kappa_2 \left(w_2 \partial_\xi^{-1} w_{2\eta}\right)_\eta. \quad (32)$$

It is remarkable that the gauge-invariant subsystem of the system (31)-(32), the equation (32) for the gauge invariant $w_2 = u_0 - u_{1\xi} - u_1 v_1$, coincides in form with the famous NVN equation [15, 16]:

$$u_t = -\kappa_1 u_{\xi\xi\xi} - \kappa_2 u_{\eta\eta\eta} - 3\kappa_1 \left(u \partial_\eta^{-1} u_\xi\right)_\xi - 3\kappa_2 \left(u \partial_\xi^{-1} u_\eta\right)_\eta. \quad (33)$$

Equivalently, in terms of variables $\phi = \ln \rho$ and w_2, system of equations (31)-(32) takes the form:

$$\phi_t = -\kappa_1 \phi_{\xi\xi\xi} - \kappa_2 \phi_{\eta\eta\eta} - \kappa_1 (\phi_\xi)^3 - \kappa_2 (\phi_\eta)^3 - \\ - 3\kappa_1 \phi_\xi \phi_{\xi\xi} - 3\kappa_2 \phi_\eta \phi_{\eta\eta} - \\ - 3\kappa_1 \phi_\xi \partial_\eta^{-1} w_{2\xi} - 3\kappa_2 \phi_\eta \partial_\xi^{-1} w_{2\eta} + v_0, \tag{34}$$

$$w_{2t} = -\kappa_1 w_{2\xi\xi\xi} - \kappa_2 w_{2\eta\eta\eta} - \\ - 3\kappa_1 \left(w_2 \partial_\eta^{-1} w_{2\xi}\right)_\xi - 3\kappa_2 \left(w_2 \partial_\xi^{-1} w_{2\eta}\right). \tag{35}$$

Remarkable that the equation (32) (or(35)) for the gauge invariant w_2 of the last systems exactly coincides in form with famous NVN equation [15, 16]. Due to this reason it is worthwhile to name the integrable systems (31)-(32) (or (34)-(35)) as NVN system of equations.

The NVN system of equations (31)-(32) (or (34)-(35)) has gauge-transparent structure. It contains:

- explicity gauge-invariant subsystem – the equation (32) (or (35)) for invariant w_2;
- the equation (31) (or(34)) for pure gauge variable ρ (or ϕ) with some terms containing gauge invariant w_2 and field variable v_0 from second linear auxiliary problem (23).

Manakov's triad representation $[L_1, L_2] = B(W_2)L_1$ (24) for NVN system of equations (31)-(32) (or (34)-(35)) includes the following operators L_1, L_2 of auxiliary linear problems and coefficient $B(w_2)$:

$$L_1 = \partial^2_{\xi\eta} + \frac{\rho_\eta}{\rho}\partial_\xi + \frac{\rho_\xi}{\rho}\partial_\eta + w_2 + \frac{\rho_{\xi\eta}}{\rho}, \qquad (36)$$

$$L_2 = \partial_t + \kappa_1\partial^3_\xi + \kappa_2\partial^3_\eta + 3\kappa_1\frac{\rho_\xi}{\rho}\partial^2_\xi + 3\kappa_2\frac{\rho_\eta}{\rho}\partial^2_\eta + $$
$$+ 3\kappa_1\left(\frac{\rho_{\xi\xi}}{\rho} + \left(\partial^{-1}_\eta w_{2\xi}\right)\right)\partial_\xi + 3\kappa_2\left(\frac{\rho_{\eta\eta}}{\rho} + \left(\partial^{-1}_\xi w_{2\eta}\right)\right)\partial_\eta + v_0, \qquad (37)$$

$$B(w_2) = 3\kappa_1\partial^{-1}_\eta w_{2\xi\xi} + 3\kappa_2\partial^{-1}_\xi w_{2\eta\eta}. \qquad (38)$$

In the case $w_2 = 0$ of zero invariant NVN system of equations (31)-(32) (or (34)-(35)) reduces to linear equation

$$\rho_t = -\kappa_1 \rho_{\xi\xi\xi} - \kappa_2 \rho_{\eta\eta\eta} + v_0 \rho, \tag{39}$$

which is integrable by auxiliary linear problems (22) and (23) with L_1 and L_2 from (36), (37) under $w_2 = 0$. Compatibility condition in this case, due to $B(w_2) = 0$, has Lax form $[L_1, L_2] = 0$. In terms of variable $\phi = \ln \rho$ linear equation (39) looks like Burgers-type equation of third order

$$\phi_t = -\kappa_1 \phi_{\xi\xi\xi} - \kappa_2 \phi_{\eta\eta\eta} - \kappa_1 (\phi_\xi)^3 - \kappa_2 (\phi_\eta)^3 - 3\kappa_1 \phi_\xi \phi_{\xi\xi} - 3\kappa_2 \phi_\eta \phi_{\eta\eta} + v_0, \tag{40}$$

which linearizes by the substitution $\phi = \ln \rho$ and consequently is C-integrable.

Let us denote by $C(\phi, u_0, v_0)$ the gauge which corresponds to nonzero field variables $u_1 = \phi_\eta$, $v_1 = \phi_\xi$, u_0 and v_0 of linear problems (22) and (23) (with operator L_2) and consequently to NVN system (34)-(35) in general position. Under different gauges from NVN system follow different integrable nonlinear equations which are gauge-equivalent to each other. The solutions of these equations by some Miura-type transformation are connected.

For example in the gauge $C(0, u_0, 0)$ the NVN system of equations (34)-(35) reduces to the famous NVN equation [15, 16] for the field variable u_0:

$$u_{0t} = -\kappa_1 u_{0\xi\xi\xi} - \kappa_2 u_{0\eta\eta\eta} - 3\kappa_1 \left(u_0 \partial_\eta^{-1} u_{0\xi}\right)_\xi - 3\kappa_2 \left(u_0 \partial_\xi^{-1} u_{0\eta}\right)_\eta. \quad (41)$$

In another gauge $C(\phi, 0, v_0)$ the NVN system (34)-(35) takes the form:

$$\phi_t = -\kappa_1\phi_{\xi\xi\xi} - \kappa_2\phi_{\eta\eta\eta} - \kappa_1(\phi_\xi)^3 - \kappa_2(\phi_\eta)^3 + \\ + 3\kappa_1\phi_\xi\partial_\eta^{-1}\left(\phi_\xi\phi_\eta\right)_\xi + 3\kappa_2\phi_\eta\partial_\xi^{-1}\left(\phi_\xi\phi_\eta\right)_\eta + v_0, \tag{42}$$

$$\left(\partial^2_{\xi\eta} + \phi_\eta\partial_\xi + \phi_\xi\partial_\eta\right)\phi_t = \left(\partial^2_{\xi\eta} + \phi_\eta\partial_\xi + \phi_\xi\partial_\eta\right) \times \\ \times \Big[-\kappa_1\phi_{\xi\xi\xi} - \kappa_2\phi_{\eta\eta\eta} - \kappa_1(\phi_\xi)^3 - \kappa_2(\phi_\eta)^3 + \\ + 3\kappa_1\phi_\xi\partial_\eta^{-1}\left(\phi_\xi\phi_\eta\right)_\xi + 3\kappa_2\phi_\eta\partial_\xi^{-1}\left(\phi_\xi\phi_\eta\right)_\eta\Big], \tag{43}$$

i. e. due to (42)-(43) NVN system (34)-(35) reduces in the gauge $C(\phi, 0, v_0)$ to the following system of equations:

$$\phi_t = -\kappa_1\phi_{\xi\xi\xi} - \kappa_2\phi_{\eta\eta\eta} - \kappa_1(\phi_\xi)^3 - \kappa_2(\phi_\eta)^3 + \\ + 3\kappa_1\phi_\xi\partial_\eta^{-1}\left(\phi_\xi\phi_\eta\right)_\xi + 3\kappa_1\phi_\eta\partial_\xi^{-1}\left(\phi_\xi\phi_\eta\right)_\eta + v_0, \tag{44}$$

$$\left(\partial^2_{\xi\eta} + \phi_\eta\partial_\xi + \phi_\xi\partial_\eta\right)v_0 = 0. \tag{45}$$

For $v_0 = 0$ system of equations (44)-(45) reduces to the famous modified Nizhnik-Veselov-Novikov equation

$$\phi_t = -\kappa_1 \phi_{\xi\xi\xi} - \kappa_2 \phi_{\eta\eta\eta} - \kappa_1 (\phi_\xi)^3 - \kappa_2 (\phi_\eta)^3 + \\ + 3\kappa_1 \phi_\xi \partial_\eta^{-1} \left(\phi_\xi \phi_\eta\right)_\xi + 3\kappa_1 \phi_\eta \partial_\xi^{-1} \left(\phi_\xi \phi_\eta\right)_\eta \qquad (46)$$

which at first in the paper [17] of Konopelchenko (1990), in different context was discovered. Let us mention that considered version (46) of mNVN equation derived in the present paper in the framework of manifestly gauge-invariant description is different from mNVN equation discovered in the paper of Bogdanov (1987) [24].

The new system of equations (44)-(45) can be named as modified NVN (mNVN) system of equations. This system due to (36)-(38) and to the choice of the gauge $C(\phi, 0, v_0)$ has following Manakov triad representation (24) with (L_1, L_2, B):

$$L_1 = \partial^2_{\xi\eta} + \phi_\eta \partial_\xi + \phi_\xi \partial_\eta, \qquad (47)$$

$$L_2 = \partial_t + \kappa_1 \partial^3_\xi + \kappa_2 \partial^3_\eta + 3\kappa_1 \phi_\xi \partial^2_\xi + 3\kappa_2 \phi_\eta \partial^2_\eta +$$
$$+ 3\kappa_1 \left(\phi^2_\xi - \partial^{-1}_\eta (\phi_\xi \phi_\eta)_\xi\right)\partial_\xi + 3\kappa_2 \left(\phi^2_\eta - \partial^{-1}_\xi (\phi_\xi \phi_\eta)_\eta\right)\partial_\eta + v_0, \qquad (48)$$

$$B(w_2) = -3\kappa_1 \phi_{\xi\xi\xi} - 3\kappa_2 \phi_{\eta\eta\eta} -$$
$$- 3\kappa_1 \partial^{-1}_\eta (\phi_\xi \phi_\eta)_{\xi\xi} - 3\kappa_2 \partial^{-1}_\xi (\phi_\xi \phi_\eta)_{\eta\eta}. \qquad (49)$$

The mNVN equation (46) has triad representation (47)-(49) with $v_0 = 0$.

It is evident that the solutions u_0 and ϕ of NVN (41) and mNVN (46) equations via invariant $w_2 = u_0 = -\phi_{\xi\eta} - \phi_\xi\phi_\eta$ (calculated in different gauges $C(0, u_0, 0)$ and $C(\phi, 0, 0)$) by Miura-type transformation

$$u_0 = -\phi_{\xi\eta} - \phi_\xi\phi_\eta \tag{50}$$

are connected. In one-dimensional limit, under $\partial_\xi = \partial_\eta$, the mNVN equation (46) reduces to the mKdV equation in potential form:

$$\phi_t = -\kappa\,\phi_{\xi\xi\xi} + 2\kappa(\phi_\xi)^3, \tag{51}$$

where $\kappa = \kappa_1 + \kappa_2$. In terms of variable $v_1 = \phi_\xi$ this is mKdV equation:

$$v_{1t} = -\kappa\,v_{1\xi\xi\xi} + 6\kappa\,v_1^2 v_{1\xi}. \tag{52}$$

Gauge-invariant formulation of 2DGDLW system

In the case (ii) of second-order linear auxiliary problem (23) the established integrable system of nonlinear equations in terms of ρ, w_1 and w_2 has the form:

$$\rho_t = -\kappa_1 \rho_{\xi\xi} - \kappa_2 \rho_{\eta\eta} - 2\kappa_1 \rho \partial_\eta^{-1} w_{2\xi} + 2\kappa_2 \rho_\eta \partial_\xi^{-1} w_1 + v_0 \rho, \qquad (53)$$

$$w_{1t} = -\kappa_1 w_{1\xi\xi} + \kappa_2 w_{1\eta\eta} - 2\kappa_1 w_{2\xi\xi} + 2\kappa_2 w_{2\eta\eta} - - 2\kappa_1 \left(w_1 \partial_\eta^{-1} w_1\right)_\xi + 2\kappa_2 \left(w_1 \partial_\xi^{-1} w_1\right)_\eta, \qquad (54)$$

$$w_{2t} = \kappa_1 w_{2\xi\xi} - \kappa_2 w_{2\eta\eta} - 2\kappa_1 \left(w_2 \partial_\eta^{-1} w_1\right)_\xi + 2\kappa_2 \left(w_2 \partial_\xi^{-1} w_1\right)_\eta. \qquad (55)$$

The gauge-invariant subsystem of the system (53)-(55), the system of equations (54)-(55) for invariants $w_1 = u_{1\xi} - v_{1\eta}$ and $w_2 = u_0 - u_{1\xi} - u_1 v_1$, for the choice $u_1 = 0$, $v_1 = v$, $u_0 = u$ for which $w_1 = -v_\eta$, $w_2 = u$, leads to the well known system of equations, Konopelchenko (1988) [22]:

$$v_t = -\kappa_1 v_{\xi\xi} + \kappa_2 v_{\eta\eta} + 2\kappa_1 \partial_\eta^{-1} u_{\xi\xi} - 2\kappa_2 u_\eta + 2\kappa_1 v v_\xi - 2\kappa_2 v_\eta \partial_\xi^{-1} v_\eta, \quad (56)$$

$$u_t = \kappa_1 u_{\xi\xi} - \kappa_2 u_{\eta\eta} + 2\kappa_1 \big(uv\big)_\xi - 2\kappa_2 \big(u\partial_\xi^{-1} v_\eta\big)_\eta. \quad (57)$$

In terms of variables

$$v = -\frac{q}{2}, \qquad u = \frac{1}{4}(1 + r - q_\eta) \quad (58)$$

integrable system of nonlinear equations (56)-(57) takes the form:

$$q_t = -\kappa_1 \partial_\eta^{-1} r_{\xi\xi} + \kappa_2 r_\eta - \frac{\kappa_1}{2}\big(q^2\big)_\xi + \kappa_2 q_\eta \partial_\xi^{-1} q_\eta, \quad (59)$$

$$r_t = -\kappa_1 q_\xi + \kappa_2 \partial_\xi^{-1} q_{\eta\eta} - \kappa_1 q_{\eta\xi\xi} + \kappa_2 q_{\eta\eta\eta} - \kappa_1 \big(rq\big)_\xi + \kappa_2 \big(r\partial_\xi^{-1} q_\eta\big)_\eta. \quad (60)$$

For the particular value $\kappa_2 = 0$ system of equations (59)-(60) reduces to famous integrable two-dimensional generalization of dispersive long-wave system of equations, Boiti, Leon, Pempinelli (1987) [18]:

$$q_{t\eta} = -\kappa_1 r_{\xi\xi} - \frac{\kappa_1}{2}\left(q^2\right)_{\xi\eta}, \tag{61}$$

$$r_{t\xi} = -\kappa_1\left(qr + q + q_{\xi\eta}\right)_{\xi\xi}. \tag{62}$$

In one-dimensional limit $\xi = \eta$ both systems (59)-(60) with $\kappa_1 - \kappa_2 = 1$ and (61)-(62) with $\kappa_1 = 1$ reduce to the famous dispersive long-wave equation (see e. g. Broer, (1975) [23]). It is worthwhile by this reason to name the system (53)-(55) as two-dimensional generalized dispersive long-wave (2DGDLW) system of equations.

In terms of variables $\phi = \ln \rho$, w_1 and w_2 the integrable system (53)-(55) takes the form:

$$\begin{aligned}\phi_t = -\kappa_1 \phi_{\xi\xi} - \kappa_2 \phi_{\eta\eta} - \kappa_1 (\phi_\xi)^2 - \kappa_2 (\phi_\eta)^2 - \\ - 2\kappa_1 \partial_\eta^{-1} w_{2\xi} + 2\kappa_2 \phi_\eta \partial_\xi^{-1} w_1 + v_0, \qquad (63)\end{aligned}$$

$$\begin{aligned}w_{1t} = -\kappa_1 w_{1\xi\xi} + \kappa_2 w_{1\eta\eta} - 2\kappa_1 w_{2\xi\xi} + 2\kappa_2 w_{2\eta\eta} - \\ - 2\kappa_1 \left(w_1 \partial_\eta^{-1} w_1\right)_\xi + 2\kappa_2 \left(w_1 \partial_\xi^{-1} w_1\right)_\eta, \qquad (64)\end{aligned}$$

$$w_{2t} = \kappa_1 w_{2\xi\xi} - \kappa_2 w_{2\eta\eta} - 2\kappa_1 \left(w_2 \partial_\eta^{-1} w_1\right)_\xi + 2\kappa_2 \left(w_2 \partial_\xi^{-1} w_1\right)_\eta. \qquad (65)$$

In terms of variables $\phi = \ln\rho$, w_2 and $\widetilde{w}_2 = w_2 + w_1$ the integrable system (53)-(55) converts to more symmetrical form:

$$\phi_t = -\kappa_1\phi_{\xi\xi} - \kappa_2\phi_{\eta\eta} - \kappa_1(\phi_\xi)^2 - \kappa_2(\phi_\eta)^2 - \\ - 2\kappa_1\partial_\eta^{-1} w_{2\xi} + 2\kappa_2\phi_\eta\partial_\xi^{-1} w_1 + v_0, \tag{66}$$

$$w_{2t} = \kappa_1 w_{2\xi\xi} - \kappa_2 w_{2\eta\eta} - \\ - 2\kappa_1\left(w_2\partial_\eta^{-1}(\widetilde{w}_2 - w_2)\right)_\xi + 2\kappa_2\left(w_2\partial_\xi^{-1}(\widetilde{w}_2 - w_2)\right)_\eta, \tag{67}$$

$$\widetilde{w}_{2t} = -\kappa_1\widetilde{w}_{2\xi\xi} + \kappa_2\widetilde{w}_{2\eta\eta} - \\ - 2\kappa_1\left(\widetilde{w}_2\partial_\eta^{-1}(\widetilde{w}_2 - w_2)\right)_\xi + 2\kappa_2\left(\widetilde{w}_2\partial_\xi^{-1}(\widetilde{w}_2 - w_2)\right)_\eta. \tag{68}$$

Remember for convenience that the variables $\phi = \ln \rho$, w_1, w_2 and $\widetilde{w}_2$ connected with the field variables u_1, v_1, u_0 of corresponding auxiliary linear problem (22) by the formulae:

$$u_1 = \frac{\rho_\eta}{\rho} = \phi_\eta, \quad v_1 = \frac{\rho_\xi}{\rho} - \partial_\eta^{-1} w_1 = \phi_\xi - \partial_\eta^{-1} w_1, \tag{69}$$

$$w_1 = u_{1\xi} - v_{1\eta}, \tag{70}$$

$$w_2 = u_0 - u_{1\xi} - u_1 v_1 = u_0 - \phi_{\xi\eta} - \phi_\eta \phi_\xi + \phi_\eta \partial_\eta^{-1} w_1 = $$

$$= u_0 - \frac{\rho_{\xi\eta}}{\rho} + \frac{\rho_\eta}{\rho} \partial_\eta^{-1} w_1, \tag{71}$$

$$\widetilde{w}_2 = w_2 + w_1 = u_0 - v_{1\eta} - u_1 v_1. \tag{72}$$

Let us mention also that the invariants w_2 and $\widetilde{w}_2$ of gauge transformations (25) are nothing but the famous classical Laplace invariants

$$h \overset{\text{def}}{=} w_2, \qquad k \overset{\text{def}}{=} \widetilde{w}_2 \tag{73}$$

connected with the first auxiliary problem (22).

All considered equivalent to each other 2DGDLW integrable systems of nonlinear equations (53)-(55), (63)-(65) and (66)-(68) have common gauge-transparent structure:

- they contain explicitly gauge-invariant subsystems (54)-(55), (64)-(65) of nonlinear equations for gauge invariants w_1 and w_2 (or equivalently subsystem (67)-(68) for gauge invariants w_2 and $\widetilde{w}_2$);
- they include the equation (53) for pure gauge variable ρ (or equation (63) for variable $\phi = \ln \rho$) (with simple rule of gauge transformation $\rho \to \rho' = g\rho$) with additional terms containing gauge invariants and field variable v_0.

Such structure of 2DGDLW systems reflects existing gauge freedom in auxiliary linear problems (22) and (23).

2DGDLW system (53)-(55) has triad representation $[L_1, L_2] = B(w_1)L_1$ with operators L_1, L_2 and coefficient $B(w_1)$ of the following forms:

$$L_1 = \partial^2_{\xi\eta} + \frac{\rho_\eta}{\rho}\partial_\xi + \left(\frac{\rho_\xi}{\rho} - \left(\partial_\eta^{-1} w_1\right)\right)\partial_\eta + w_2 + \frac{\rho_{\xi\eta}}{\rho} - \frac{\rho_\eta}{\rho}\partial_\eta^{-1} w_1, \quad (74)$$

$$L_2 = \partial_t + \kappa_1\partial^2_\xi + \kappa_2\partial^2_\eta + 2\kappa_1\frac{\rho_\xi}{\rho}\partial_\xi + 2\kappa_2\left(\frac{\rho_\eta}{\rho} - \left(\partial_\xi^{-1} w_1\right)\right)\partial_\eta + v_0, \quad (75)$$

$$B(w_1) = 2\kappa_1\partial_\eta^{-1} w_{1\xi} - 2\kappa_2\partial_\xi^{-1} w_{1\eta}. \quad (76)$$

Let us consider some particular gauges of established 2DGDLW systems of equations (53)-(55), (63)-(65) and (66)-(68). It is convenient to denote the gauge in general position by the symbol $C(u_1, v_1, u_0)$.

In the gauge $C\,(u_1 = \phi_\eta, v_1 = \phi_\xi, u_0 = \phi_{\xi\eta} + \phi_\xi \phi_\eta)$ which due to (26)-(28) corresponds to zero values of invariants w_1 and w_2

$$w_1 = u_{1\xi} - v_{1\eta} = 0, \quad w_2 = u_0 - u_{1\xi} - u_1 v_1 = 0, \quad \widetilde{w}_2 = 0 \tag{77}$$

2DGDLW system of equations (66)-(68) reduces to two-dimensional Burgers equation in potential form

$$\phi_t = -\kappa_1 \phi_{\xi\xi} - \kappa_2 \phi_{\eta\eta} - \kappa_1 (\phi_\xi)^2 - \kappa_2 (\phi_\eta)^2 + v_0, \tag{78}$$

or in terms of variable ρ connected with ϕ by Hopfe-Cole transformation $\phi = \ln \rho$, to linear diffusion equation:

$$\rho_t = -\kappa_1 \rho_{\xi\xi} - \kappa_2 \rho_{\eta\eta} + v_0 \rho. \tag{79}$$

The equation (78) (or (79)) due to our construction is compatibility condition in Lax form

$$[L_1, L_2] = B(w_1) L_1 \equiv 0 \tag{80}$$

of linear problems (22) and (23) with operators L_1, L_2 given by (74), (75) under substitution $w_1 = w_2 = 0$.

In another simple gauge $C\,(u_1 = \phi_\eta, v_1 = 0, u_0 = 0)$ corresponding due to (70)-(72) to the invariants

$$w_1 = \phi_{\xi\eta}, \quad w_2 = -\phi_{\xi\eta}, \quad \widetilde{w}_2 = 0, \tag{81}$$

2DGDLW system of equations (66)-(68) for the choice $v_0 = 0$ again reduces to the single equation of Burgers type in potential form

$$\phi_t = \kappa_1 \phi_{\xi\xi} - \kappa_2 \phi_{\eta\eta} - \kappa_1 (\phi_\xi)^2 + \kappa_2 (\phi_\eta)^2. \tag{82}$$

This equation linearizes by Hopfe-Cole transformation $\phi = -\ln\rho$ to corresponding linear equation

$$\rho_t = \kappa_1 \rho_{\xi\xi} - \kappa_2 \rho_{\eta\eta}. \tag{83}$$

In the less trivial gauge $C\,(u_1=0, v_1=-q_\xi/q, u_0=p\,q)$ the invariants w_1, w_2 and $\widetilde{w}_2$ due to (70)-(72) are given by the following expressions

$$w_1=\left(\ln q\right)_{\xi\eta},\quad w_2=u_0=p\,q,\quad \widetilde{w}_2=p\,q+\left(\ln q\right)_{\xi\eta}, \tag{84}$$

the variable ρ due to (70) has constant value, consequently the variable $\phi=0$. In this case due to (66)

$$v_0=2\kappa_1\partial_\eta^{-1}w_{2\xi}=2\kappa_1\partial_\eta^{-1}\left(p\,q\right)_\xi. \tag{85}$$

and from the 2DGDLW system of equations (66)-(68) one obtains after some calculations the famous DS system of equations [19] for the field variables p and q:

$$p_t=\kappa_1 p_{\xi\xi}-\kappa_2 p_{\eta\eta}+2\kappa_1 p\,\partial_\eta^{-1}\left(p\,q\right)_\xi-2\kappa_2 p\,\partial_\xi^{-1}\left(p\,q\right)_\eta, \tag{86}$$

$$q_t=-\kappa_1 q_{\xi\xi}+\kappa_2 q_{\eta\eta}-2\kappa_1 q\,\partial_\eta^{-1}\left(p\,q\right)_\xi+2\kappa_2 q\,\partial_\xi^{-1}\left(p\,q\right)_\eta. \tag{87}$$

One can consider also the gauge $C\,(u_1 = p_\eta, v_1 = q_\xi, u_0 = p_\eta q_\xi)$ in which due to (70)-(72) the invariants have the following expressions through q and p:

$$w_1 = p_{\xi\eta} - q_{\xi\eta}, \quad w_2 = -p_{\xi\eta}, \quad \widetilde{w}_2 = -q_{\xi\eta}. \tag{88}$$

Substitution of w_1, w_2 and $\widetilde{w}_2$ from (88) into the system (66)-(68) leads to the following three equations for p and q. From equation (66) for $\phi \equiv p$ one obtains

$$p_t = \kappa_1 p_{\xi\xi} - \kappa_2 p_{\eta\eta} - \kappa_1 (p_\xi)^2 + \kappa_2 (p_\eta)^2 - 2\kappa_2 p_\eta q_\eta + v_0. \tag{89}$$

Equations (67) and (68) for w_2 and $\widetilde{w}_2$ in terms of variables p, q take the forms

$$\begin{aligned} p_t = \kappa_1 p_{\xi\xi} - \kappa_2 p_{\eta\eta} - \kappa_1 (p_\xi)^2 + \kappa_2 (p_\eta)^2 + \\ + 2\kappa_1 \partial_\eta^{-1}(p_{\xi\eta} q_\xi) - 2\kappa_2 \partial_\xi^{-1}(p_{\xi\eta} q_\eta), \end{aligned} \tag{90}$$

$$\begin{aligned} q_t = -\kappa_1 q_{\xi\xi} + \kappa_2 q_{\eta\eta} + \kappa_1 (q_\xi)^2 - \kappa_2 (q_\eta)^2 - \\ - 2\kappa_1 \partial_\eta^{-1}(q_{\xi\eta} p_\xi) + 2\kappa_2 \partial_\xi^{-1}(q_{\xi\eta} p_\eta). \end{aligned} \tag{91}$$

The equations (89) and (90) are compatible for the choice of v_0 given by the formula

$$v_0 = 2\kappa_1 \partial_\eta^{-1}(p_{\xi\eta} q_\xi) + 2\kappa_2 \partial_\xi^{-1}(q_{\xi\eta} p_\eta), \tag{92}$$

and the system of three equations (89)-(91) reduces to system of two equations (90)-(91) containing in nonlocal terms derivatives $p_{\xi\eta} q_\xi$, $p_{\xi\eta} q_\eta$, etc.

Analogously in the gauge $C\,(u_1 = p_\eta, v_1 = q_\xi, u_0 = 0)$ it follows for w_1, w_2 and $\widetilde{w}_2$ due to (70)-(72)

$$w_1 = p_{\xi\eta} - q_{\xi\eta}, \quad w_2 = -p_{\xi\eta} - p_\eta q_\xi, \quad \widetilde{w}_2 = -q_{\xi\eta} - p_\eta q_\xi. \tag{93}$$

The equation (63) for $\phi \equiv p$ via (93) takes the form

$$\begin{aligned} p_t = \kappa_1 p_{\xi\xi} - \kappa_2 p_{\eta\eta} - \kappa_1 (p_\xi)^2 + \kappa_2 (p_\eta)^2 - \\ - 2\kappa_2 p_\eta q_\eta + 2\kappa_1 \partial_\eta^{-1}(p_\eta q_\xi)_\xi + v_0. \end{aligned} \tag{94}$$

Equation (64) via substitutions from (93) transforms to the form

$$\begin{aligned} p_t - q_t = \kappa_1 (p+q)_{\xi\xi} - \kappa_2 (p+q)_{\eta\eta} - \kappa_1 (p_\xi - q_\xi)^2 + \\ + \kappa_2 (p_\eta - q_\eta)^2 + 2\kappa_1 \partial_\eta^{-1}(p_\eta q_\xi)_\xi - 2\kappa_2 \partial_\xi^{-1}(p_\eta q_\xi)_\eta. \end{aligned} \tag{95}$$

By substraction of equation (95) from equation (94) one obtains the evolution equation for q:

$$q_t = -\kappa_1 q_{\xi\xi} + \kappa_2 q_{\eta\eta} + \kappa_1 (q_\xi)^2 - \kappa_2 (q_\eta)^2 - \\ - 2\kappa_1 p_\xi q_\xi + 2\kappa_2 \partial_\xi^{-1} (p_\eta q_\xi)_\eta + v_0. \qquad (96)$$

The equation (65) for the invariant w_2 due to (93) in terms of variables p, q is

$$(p_{\xi\eta} + p_\eta q_\xi)_t = \kappa_1 (p_{\xi\eta} + p_\eta q_\xi)_{\xi\xi} - \kappa_2 (p_{\xi\eta} + p_\eta q_\xi)_{\eta\eta} - \\ - 2\kappa_1 ((p_{\xi\eta} + p_\eta q_\xi)(p_\xi - q_\xi))_\xi + 2\kappa_2 ((p_{\xi\eta} + p_\eta q_\xi)(p_\eta - q_\eta))_\eta. \qquad (97)$$

The equations (94), (96) and (97) are compatible with each other if the field variable v_0 satisfies to the equation

$$v_{0\xi\eta} + p_\eta v_{0\xi} + q_\xi v_{0\eta} = 0. \qquad (98)$$

For the simple choice $v_0 \equiv 0$ one obtains from the system of three equations (95), (96) and (97) the following equivalent system of two equations:

$$p_t = \kappa_1 p_{\xi\xi} - \kappa_2 p_{\eta\eta} - \kappa_1 \left(p_\xi\right)^2 + \kappa_2 \left(p_\eta\right)^2 - \\ - 2\kappa_2 p_\eta q_\eta + 2\kappa_1 \partial_\eta^{-1} \left(p_\eta q_\xi\right)_\xi, \tag{99}$$

$$q_t = -\kappa_1 q_{\xi\xi} + \kappa_2 q_{\eta\eta} + \kappa_1 \left(q_\xi\right)^2 - \kappa_2 \left(q_\eta\right)^2 - \\ - 2\kappa_1 p_\xi q_\xi + 2\kappa_2 \partial_\xi^{-1} \left(p_\eta q_\xi\right)_\eta. \tag{100}$$

At first this system of equations has been derived in another context in the paper [22] of Konopelchenko, 1988.

In conclusion let us derive Miura-type transformations between different systems of DS-type equations of second order obtained in this section in different gauges. For convenience let us denote by capital letters $P \equiv p$, $Q \equiv q$ the solutions of DS famous system (86)-(87) of equations. By the use of invariants w_1 and w_2 one obtains the following relations between variables ($P \equiv p$, $Q \equiv q$) of DS system (86)-(87) and variables p, q of the system (90)-(91):

$$w_1 = \left(\ln Q\right)_{\xi\eta} = p_{\xi\eta} - q_{\xi\eta}, \quad w_2 = PQ = -p_{\xi\eta}. \tag{101}$$

One derives from (101):

$$Q = e^{p-q}, \quad P = -p_{\xi\eta}\, e^{q-p}. \tag{102}$$

Quite analogously for the pair of DS systems (86)-(87) and (99)-(100) one has

$$w_1 = (\ln Q)_{\xi\eta} = p_{\xi\eta} - q_{\xi\eta}, \quad w_2 = PQ = -p_{\xi\eta} - p_\eta q_\xi. \tag{103}$$

One obtains from (103):

$$Q = e^{p-q}, \quad P = -(p_{\xi\eta} + p_\eta q_\xi)e^{q-p}. \tag{104}$$

Transformations (102) and (104) allow to obtain solutions of famous DS system of equations (87)-(86) from the systems of equations (90)-(91) and (99)-(100), these transformations are Miura-type transformations between gauge-equivalent to each other DS-type systems of equations of second order.

Gauge-invariant formulation of KP–mKP and SK–KK systems

Let us consider briefly the cases of KP–mKP and SK–KK systems of integrable nonlinear evolution equations.

For the KP–mKP system of equations, Konopelchenko (1982), Konopelchenko & Dubrovsky (1984) starting with auxiliary linear problems

$$L_1\psi = \left(\sigma\partial_y + \partial_x^2 + u_1\partial_x + u_0\right)\psi = 0, \quad (105)$$

$$L_2\psi = \left(\partial_t + 4\partial_x^3 + v_2\partial_x^2 + v_1\partial_x + v_0\right)\psi = 0, \quad (106)$$

obtains via compatibility condition $[L_1, L_2] = 0$ in terms of pure gauge variable $u_1 = 2\frac{\rho_x}{\rho}$ and gauge invariant $w_0 = u_0 - \frac{1}{2}u_{1x} - \frac{1}{4}u_1^2 - \frac{\sigma}{2}\partial_x^{-1}u_{1y}$ the following system of integrable nonlinear equations:

$$\rho_t + 4\rho_{xxx} + 6\rho_x w_0 + 3\rho w_{0x} - 3\sigma\rho\partial_x^{-1}w_{0y} - \rho v_0 = 0, \quad (107)$$

$$w_{0t} + w_{0xxx} + 6w_0 w_{0x} + 3\sigma^2\partial_x^{-1}w_{0yy} = 0. \quad (108)$$

In the case of SK–KK (Sawada–Kotera and Kaup–Kupershmidt) system of equations, Konopelchenko & Dubrovsky (1984), Dubrovsky & Gramolin (2008) starting with auxiliary linear problems

$$L_1\psi = \left(\sigma\partial_y + \partial_x^3 + u_2\partial_x^2 + u_1\partial_x + u_0\right)\psi = 0, \tag{109}$$

$$L_2\psi = \left(\partial_t + \kappa\partial_x^5 + v_4\partial_x^4 + v_3\partial_x^3 + v_2\partial_x^2 + v_1\partial_x + v_0\right)\psi = 0, \tag{110}$$

obtains in terms of pure gauge variable ρ

$$u_1 = 3\frac{\rho_x}{\rho} \tag{111}$$

and gauge invariants

$$w_1 = u_1 - u_{2x} - \frac{1}{3}u_2^2, \tag{112}$$

$$w_0 = u_0 - \frac{1}{3}u_1 u_2 - \frac{1}{3}u_{2xx} + \frac{2}{27}u_2^3 - \frac{\sigma}{3}\partial_x^{-1}u_{2y}, \tag{113}$$

the following system of integrable nonlinear equations:

$$\rho_t + \kappa\rho_{xxxxx} - \rho v_0 + \frac{5}{9}\kappa(\rho w_1)_{xxx} - \frac{5}{9}\kappa(\rho w_{1xx})_x + \frac{5}{3}\kappa(\rho_x w_0)_x +$$
$$+\frac{5}{9}\kappa\rho_x w_1^2 - \frac{5}{9}\kappa\sigma\rho_x\partial_x^{-1}w_{1y} + \frac{10}{9}\kappa\rho(w_{0xx} + w_0 w_1 - \frac{\sigma}{9}\partial_x^{-1}w_{0y}) = 0,$$
$$w_{1t} - \frac{1}{9}\kappa w_{1xxxxx} - \frac{5}{9}\kappa\left(w_1 w_{1xx}\right)_x - \frac{5}{3}\kappa\left(w_0 w_{1x}\right)_x -$$
$$-\frac{5}{9}\kappa w_1^2 w_{1x} + \frac{10}{3}\kappa w_0 w_{0x} - \frac{5}{9}\kappa\sigma w_{1xxy} - \frac{5}{9}\kappa\sigma w_1 w_{1y} +$$
$$+\frac{5}{9}\kappa\sigma^2\partial_x^{-1}w_{1yy} - \frac{5}{9}\kappa\sigma w_{1x}\partial_x^{-1}w_{1y} = 0,$$
$$w_{0t} - \frac{1}{9}\kappa w_{0xxxxx} - \frac{5}{9}\kappa\left(w_0 w_1\right)_{xxx} - \frac{5}{9}\kappa\left(w_0 w_{1xx}\right)_x +$$
$$+\frac{5}{3}\kappa\left(w_0 w_{0x}\right)_x - \frac{5}{9}\kappa\left(w_0 w_1^2\right)_x - \frac{5}{9}\kappa\sigma w_{0xxy} - \frac{10}{9}\kappa\sigma w_0 w_{1y} -$$
$$-\frac{5}{9}\kappa\sigma w_1 w_{0y} + \frac{5}{9}\kappa\sigma^2\partial_x^{-1}w_{0yy} - \frac{5}{9}\kappa\sigma w_{0x}\partial_x^{-1}w_{1y} = 0.$$

Conclusion

In conclusion let us underline once again that ideas of gauge invariance now are in common use in the theory of integrable nonlinear equations.

There are known attempts to develop invariant description of some nonlinear integrable equations considered in the present paper by the use of matrix linear auxiliary problems. This was done for example in the paper of Yilmaz & Athorne (2002) [26] for the Nizhnik–Veselov–Novikov and Davey–Stewartson equations in the framework of classical invariant theory of second order linear partial differential equations.

Matrix linear auxiliary problems have a bigger number degrees of freedom then the scalar, the performance of reductions from general position to integrable nonlinear equations is more difficult; all this leads to the need of consideration gauge transformations under some restrictions, manifestly gauge-invariant description of integrable nonlinear equations in this case is far from completion and requires additional research work.

The end

See more details of this research in arXiv:0802.2334 and our forthcoming article in J. Phys. A: Math. Theor.

Thank you very much for your attention!

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