Higher Order Adjoint Symmetries of Evolution Equations and Conservation Laws

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Introduction

Evolution equation

$$E = u_t - F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0$$

Lie-Bäcklund Symmetry Generator:

$$Z = \eta(u, u_x, \ldots, u_{sx}) \frac{\partial}{\partial u}$$

Invariance Condition:

$$Z^{(n)} E \Big|_{E^{(j)}=0} = 0 \quad \textbf{or} \quad L_E[u]\eta \Big|_{E^{(j)}=0} = 0$$

where

$$L_E[u] = \frac{\partial E}{\partial u} + \frac{\partial E}{\partial u_t} D_t + \frac{\partial E}{\partial u_x} D_x + \frac{\partial E}{\partial u_{xx}} D_x^2 + \cdots + \frac{\partial E}{\partial u_{nx}} D_x^n$$

The Infinitesimal symmetry transformation

$$\tilde{u} = u + \varepsilon \eta$$

keeps the equation invariant to order ε.

Symmetry Integrable Evolution Equations:

Equation admits ∞ number of nontrivial commuting Lie-Bäcklund symmetries.

Recursion operators generate Lie-Bäcklund symmetries:

Assume the form

$$R[u] = G_0 + \sum_j G_j D_x^j + \sum_i I_i D_x^{-1} \circ \Lambda_i$$

such that

$$\eta_{j+1} = R^j[u]\eta_1, \qquad j = 1, 2, 3, \ldots$$

where

$$Z = \eta_k \frac{\partial}{\partial u}, \qquad L_E[u]\eta_k \Big|_{E^{(j)}=0} = 0, \quad k = 1, 2, 3, \ldots$$

IFF

$$[L_F[u],\, R[u]]\, \Phi = (D_t R[u])\, \Phi$$

where

$$L_F[u] = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial u_x} D_x + \frac{\partial F}{\partial u_{2x}} D_x^2 + \cdots + \frac{\partial F}{\partial u_{nx}} D_x^n$$

Adjoint symmetries (Cosymmetries): J
IFF

$$L_E^*[u]J\Big|_{E^{(j)}=0} = 0$$

L^* **is the adjoint of the operator** L

Λ **is an integrating factor for the conservation law of** E

$$\Lambda(x,t,u,u_x,u_{xx},\ldots)E = D_t(\Phi^t) + D_x(\Phi^x) = 0$$

IFF

$$L_E^*[u]\Lambda = 0$$

$$L_\Lambda[u]E = L_\Lambda^*[u]E$$

Λ **is a variational expression, i.e self-adjoint**

Note: [*S Anco and G Bluman, J Appl. Math. Vol. 13, 2002*].

Every integrating factor is an Adjoint Symmetry.

Recall the Recursion Operator:

$$R[u] = G_0 + \sum_j G_j D_x^j + \sum_i I_i D_x^{-1} \circ \Lambda_i$$

here

I_i**: Lie Symmetries of** E
Λ_i**: Integrating Factors (Adjoint Symmetries) of** E**.**

$$\Lambda_{j+1} = (R^*[u])^j \Lambda_1$$

$$R^*[u] = G_0 + \sum_j (-1)^j D_x^j \circ G_j - \sum_i \Lambda_i D_x^{-1} \circ I_i$$

Let, for example,

$$\Phi^t = \Phi^t(u, u_x, u_{2x}, u_{3x})$$

then

$$\Lambda = \hat{E}_u \Phi^t$$

$$\Phi^x = -D_x^{-1}\{\Lambda F\} - \frac{\partial \Phi^t}{\partial u_x} F - \frac{\partial \Phi^t}{\partial u_{2x}}(D_x F) - \frac{\partial \Phi^t}{\partial u_{3x}}(D_x^2 F)$$

$$+ F D_x \left(\frac{\partial \Phi^t}{\partial u_{2x}}\right) - F D_x^2 \left(\frac{\partial \Phi^t}{\partial u_{3x}}\right) + (D_x F) D_x \left(\frac{\partial \Phi^t}{\partial u_{3x}}\right)$$

where $\hat{E}$ **is the Euler operator**

$$\hat{E} = \frac{\partial}{\partial u} - D_x \circ \frac{\partial}{\partial u_x} + D_x^2 \circ \frac{\partial}{\partial u_{2x}} - D_x^3 \circ \frac{\partial}{\partial u_{3x}}$$

and recall $E = u_t - F(u, u_x, u_{2x}, \ldots)$**.**

Example: Harry Dym Equation

$$u_t = u^3 u_{3x}$$

Integrating factor: $\Lambda_1 = u^{-1} u_{xx} - \frac{1}{2} u^{-2} u_x^2$**.**

Conserved density: $\Phi^t = \frac{1}{2} u^{-1} u_x^2$

Flux: $\Phi^x = -u^2 u_x u_{3x} - \frac{1}{2} u u_x^2 u_{xx} + \frac{1}{8} u_x^4 + \frac{1}{2} u^2 u_{xx}^2$

Recursion Operator: $R[u] = u^2 D_x^2 - u u_x D_x + u u_{xx} + u^3 u_{3x} D_x^{-1} \circ u^{-2}$

Hierarchy of Symmetry Integrable Evolution Equations:

IFF
$R[u]$ **is Hereditary:**

$[R'[u],\ R[u]]$ **symmetric bilinear operator,**
i.e
$R'[Rv]w - RR'[v]w = R'[Rw]v - RR'[w]v$
(prime is the Fréchet derivative).

Symmetry Integrable Hierarchy E_n**:**

$$E_n = u_{t_n} - R^n[u]F = 0, \qquad n = 0, 1, 2, \ldots$$

Each equation in the hierarchy, $E_0,\ E_1,\ E_2, \ldots$**, admits**

- ∞ **number of commuting Lie-Bäcklund symmetries**
- ∞ **number of higher-order conservation laws**
- **Conserved quantities in involution:**

Transformations between evolution equations:

$$u_t = F(u, u_x, u_{xx}, \ldots)$$

$$\tilde{u}_t = G(\tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}}, \ldots)$$

- **Reciprocal Bäcklund Transformation** *[J.G. Kingston and C Rogers, Phys. Lett. 1982]*:

$$d\tilde{x} = \Phi^t(u, u_x, u_{xx}, \ldots)dx - \Phi^x(u, u_x, u_{xx}, \ldots)dt$$
$$\tilde{u}(\tilde{x}, t) = g(u, u_x, u_{xx}, \ldots)$$

- x**-Generalized Hodograph Transformation** *[N. Euler and M. Euler, JNMP 2001]*:

$$d\tilde{x} = \Phi^t(x, u)dx - \Phi^x(x, u, u_x, u_{xx}, \ldots)dt$$
$$\tilde{u}(\tilde{x}, t) = g(x)$$

Above $D_t\Phi^t + D_x\Phi^x = 0$**.**

Example 1: Harry Dym Equation $u_t = u^3 u_{3x}$

Adjoint Symmetries:

$$J_1 = u^{-2}, \qquad J_2 = u^{-3}, \qquad J_3 = xu^{-3}, \qquad J_4 = x^2 u^{-3}$$
$$J_5 = u^{-1} u_{xx} - \frac{1}{2} u^{-2} u_x^2$$
$$J_6 = u u_{4x} + 2 u_x u_{3x} + \frac{2}{3} u_{xx}^2 - \frac{3}{2} u^{-1} u_x^2 u_{xx} + \frac{3}{8} u^{-2} u_x^4$$

Recursion Operator: $R[u] = u^2 D_x^2 - u u_x D_x + u u_{xx} + u^3 u_{3x} D_x^{-1} \circ u^{-2}$

<u>**Consider** $J_1 = u^{-2}$</u>

x**-Generalized Hodograph Transformation:**

$$d\tilde{x} = u^{-1} dx - \left(u u_{xx} - \frac{1}{2} u_x^2 \right) dt$$
$$\tilde{u}(\tilde{x}, t) = x$$

leads to the Schwarzian-KdV

$$\tilde{u}_t = \tilde{u}_{3\tilde{x}} - \frac{3}{2} \tilde{u}_{\tilde{x}}^{-1} \tilde{u}_{\tilde{x}\tilde{x}}^2$$

Reciprocal Bäcklund Transformation with $g(u) = u$**:**

$$d\tilde{x} = u^{-1} dx - \left(u u_{xx} - \frac{1}{2} u_x^2 \right) dt$$
$$\tilde{u}(\tilde{x}, t) = u$$

leads to

$$\tilde{u}_t = \tilde{u}_{3\tilde{x}} + \frac{3}{2} \tilde{u}^{-2} \tilde{u}_{\tilde{x}}^3 - 3 \tilde{u}^{-1} \tilde{u}_{\tilde{x}} \tilde{u}_{\tilde{x}\tilde{x}}$$

Reciprocal Bäcklund Transformation with $g(u) = \ln(\alpha u)/\alpha$**:**

$$d\tilde{x} = u^{-1}dx - \left(uu_{xx} - \frac{1}{2}u_x^2\right)dt$$

$$\tilde{u}(\tilde{x}, t) = \frac{1}{\alpha}\ln(\alpha u), \qquad \alpha^2 = -2\lambda_1$$

leads to the potential MKdV equation

$$\tilde{u}_t = \tilde{u}_{3\tilde{x}} + \lambda_1 \alpha u_{\tilde{x}}^3$$

<u>**Consider** $J_2 = u^{-3}$</u>

x**-Generalized Hodograph Transformation:**

$$d\tilde{x} = u^{-2}dx - 2u_{xx}dt$$

$$\tilde{u}(\tilde{x}, t) = x$$

leads to Cavalcante-Tenenblat equation

$$\tilde{u}_t = \tilde{u}_{\tilde{x}}^{-3/2}\tilde{u}_{3\tilde{x}} - \frac{3}{2}\tilde{u}_{\tilde{x}}^{-5/2}\tilde{u}_{\tilde{x}\tilde{x}}^2$$

Reciprocal Bäcklund Transformation with $g(u) = u^{-1}$**:**

$$d\tilde{x} = u^{-2}dx - 2u_{xx}dt$$

$$\tilde{u}(\tilde{x}, t) = u^{-1}$$

leads to a Auto-Bäcklund transformation *(J.G. Kingston, C. Rogers and D. Woodall, J Phys A, 1984)*, **i.e**

$$\tilde{u}_t = \tilde{u}^3\tilde{u}_{3\tilde{x}}$$

Consider $J_5 = u^{-1}u_{xx} - \frac{1}{2}u^{-2}u_x^2$

Reciprocal Bäcklund Transformation with $\Phi^t = u^{-1}u_x^2/2$
$g(u) = u$**:**

$$d\tilde{x} = \frac{1}{2}u^{-1}u_x^2\,dx + (u^2u_xu_{3x} + \frac{1}{2}uu_x^2u_{xx} - \frac{1}{8}u_x^4 - \frac{1}{2}u^2u_{xx}^2)\,dt$$
$$\tilde{u}(\tilde{x},t) = u$$

leads to

$$\tilde{u}_t = 8\tilde{u}^6\tilde{u}_{\tilde{x}}^{-6}\tilde{u}_{3\tilde{x}} - 24\tilde{u}^6\tilde{u}_{\tilde{x}}^{-7}\tilde{u}_{\tilde{x}\tilde{x}}^2 + 24\tilde{u}^5\tilde{u}_{\tilde{x}}^{-5}\tilde{u}_{\tilde{x}\tilde{x}} - 6\tilde{u}^4\tilde{u}_{\tilde{x}}^{-3}$$

Reciprocal Bäcklund Transformation with $\Phi^t = 2u_{xx} - u^{-1}u_x^2$**:**
$g(u,\Phi^t) = \alpha u\Phi^t$:

$$\begin{cases} d\tilde{x} = \alpha\left(-u^{-1}u_x^2 + 2u_{xx}\right)\,dx \\ \qquad + \alpha\left(-uu_{xx}u_x^2 + \frac{1}{4}u_x^4 + u^2u_{xx}^2 + 4u^2u_xu_{xxx} + 2u^3u_{xxxx}\right)\,dt \\ \tilde{u}(\tilde{x},t) = \alpha\left(-u_x^2 + 2uu_{xx}\right), \end{cases}$$

leads to

$$\tilde{u}_t = \tilde{u}^3\tilde{u}_{3\tilde{x}} + 3\tilde{u}^2\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{x}\tilde{x}} + \frac{3}{4\alpha}\tilde{u}^2\tilde{u}_{\tilde{x}}$$

Example 2: Schwarzian KdV Equation $u_t = u_{3x} - \frac{3}{2}\frac{u_{xx}^2}{u_x}$

Adjoint Symmetries:

$$J_1 = \frac{u^2 u_{xx}}{u_x^3} - \frac{2u}{u_x}, \qquad J_2 = \frac{u u_{xx}}{u_x^3} - \frac{1}{u_x}, \qquad J_3 = \frac{u_{xx}}{u_x^3}, \qquad \ldots$$

Consider: $J_1 = \frac{u^2 u_{xx}}{u_x^3} - \frac{2u}{u_x}$

We obtain

$$\Phi^t(u, u_x) = \frac{1}{3}\frac{u^2}{u_x}.$$

Reciprocal transformation with $g(u) = u^3$**:**

$$d\tilde{x} = \frac{1}{3}\frac{u^2}{u_x}\,dx + \left(-\frac{1}{3}\frac{u^2 u_{3x}}{u_x^2} + \frac{1}{6}\frac{u^2 u_{xx}^2}{u_x^3} + \frac{4}{3}\frac{u u_{xx}}{u_x} - \frac{4}{3}u_x\right)\,dt$$

$$\tilde{u}(\tilde{x}, t) = u^3$$

leads to

$$\tilde{u}_t = \frac{\tilde{u}^2 \tilde{u}_{3\tilde{x}}}{\tilde{u}_{\tilde{x}}^{3/2}} - \frac{3}{2}\frac{\tilde{u}^2 \tilde{u}_{\tilde{x}\tilde{x}}^2}{\tilde{u}_{\tilde{x}}^{5/2}} + \frac{4}{9}\tilde{u}_{\tilde{x}}^{3/2}$$

On the other hand:

$$\Phi^t(u, u_x, u_{xx}) = \frac{2}{27}\frac{u^3 u_{xx}}{u_x^3}$$

with $g(u, u_x) = u_x^{-2}$ **leads to**

$$\tilde{u}_t = \frac{\tilde{u}_{\tilde{x}}^4 \tilde{u}_{3\tilde{x}}}{\tilde{u}^{3/2}\tilde{u}_{\tilde{x}\tilde{x}}^3} + \frac{3}{2}\frac{\tilde{u}_{\tilde{x}}^5}{\tilde{u}^{5/2}\tilde{u}_{\tilde{x}\tilde{x}}^2} - 3\frac{\tilde{u}_{\tilde{x}}^3}{\tilde{u}^{3/2}\tilde{u}_{\tilde{x}\tilde{x}}} + \frac{4}{9}\frac{\tilde{u}_{\tilde{x}}}{\tilde{u}^{1/2}}$$

Extended Schwarzian KdV Equations:

$$u_t = u_{xxx} - \frac{3}{2} u_x^{-1} u_{xx}^2 + \lambda_1 u_x^{-1} + \lambda_2 u_x^3 + \lambda_3 u_x + \lambda_0,$$

$$u_t = u_{xxx} - \frac{3}{2} \left(\frac{u_x}{u_x^2 - c} \right) u_{xx}^2 + \lambda_1 (u_x^2 - c)^{3/2} + \lambda_2 u_x^3 + \lambda_3 u_x + \lambda_0$$

$$u_t = u_{xxx} - \frac{3}{4} u_x^{-1} u_{xx}^2 + \lambda_1 u_x^{3/2} + \lambda_2 u_x^2 + \lambda_3 u_x + \lambda_0,$$

None of the above have zero-order integrating factors.

Reference:

A.V. Mikhailov, A.B. Shabat and V.V. Sokolov, "What is Integrability", 1991
M. Euler and N. Euler, JNMP vol. 14, 2007

Example 3: Krichever-Novikov Equation

$$u_t = u_{3x} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x}, \qquad P^{(V)} = 0.$$

2nd-order integrating factor:

$$J_1 = P^{1/2}\frac{u_{xx}}{u_x^3} - \frac{1}{2}P^{-1/2}P' u_x^{-1}$$

with

$$P(u) = k_3(u^2 + k_1 u + k_2)^2.$$

4th-order integrating factor:

$$J_2 == \frac{u_{4x}}{u_x^2} - 4\frac{u_{2x}u_{3x}}{u_x^3} + 3\frac{u_{2x}^3}{u_x^4} - \frac{2P}{u_x^4}u_{2x} + \frac{P'}{u_x^2}$$

with $P^{(V)} = 0$**.**

Consider

$$P(u) = k_3(u^2 + k_1 u + k_2)^2$$

An Adjoint Symmetry is:

$$J_1 = P^{1/2}\frac{u_{xx}}{u_x^3} - \frac{1}{2}P^{-1/2}P' u_x^{-1}$$

With the assumption $\Phi^t(u, u_x)$ **we obtain**

$$\Phi^t = \frac{P(u)^{1/2}}{u_x}$$

and for $g(u) = u$ **the transformed equation**

$$\tilde{u}_t = P^{3/4}\frac{\tilde{u}_{3\tilde{x}}}{\tilde{u}_{\tilde{x}}^3/2} - \frac{3}{2}P^{3/4}\frac{\tilde{u}_{\tilde{x}\tilde{x}}}{\tilde{u}_{\tilde{x}}^{5/2}} + \frac{4}{3}P^{3/4}\frac{1}{\tilde{u}_{\tilde{x}}^{1/2}} - \frac{1}{8}\left(\frac{9(P')^2 - 12PP''}{P^{5/4}}\right)\tilde{u}_{\tilde{x}}^{3/2}$$

Example 4:

$$u_t = u_{3x} - \frac{3}{2}\left(\frac{u_x}{u_x^2 - c}\right) u_{2x}^2 - \frac{3}{2c}\left(u_x^2 - c\right) u_x P(u),$$

where c **is an arbitrary but nonzero constant and** P **satisfies the equation**

$$(P')^2 = \frac{4}{c} P^3 + a_1 P + a_2$$

with a_1 **and** a_2 **arbitrary constants.**

2nd-order integrating factor:

$$J_1 = \frac{f(u) u_{2x}}{(u_x^2 - c)^{3/2}} - \frac{f'(u)}{(u_x^2 - c)^{1/2}}$$

IFF $f(u)$ **and** $P(u)$ **satisfy**

$$2cf''' - 3fP' - 6f'P = 0.$$

4th-order integrating factor:

$$J_2 = -\frac{u_{4x}}{u_x^2 - c} + \frac{4u_x u_{2x} u_{3x}}{(u_x^2 - c)^2} - \frac{(3u_x^2 + c) u_{2x}^3}{(u_x^2 - c)^3} + \frac{3u_{2x}}{c} P + \frac{3u_x^2 + c}{2c} P'$$

IFF $P(u)$ **satisfy**

$$(P')^2 = \frac{4}{c} P^3 + a_1 P + a_2$$