Proper Sequences of Ordinary Differential Equations

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Contents:

MOTIVATION and DEFINITIONS:

Differential sequence of m ODEs,

$$\{E_1,\, E_2,\, \ldots,\, E_m\},$$

in the form:

$$E_1 := F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0$$
$$E_2 := R^{[k]}[u]\, F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0$$
$$E_3 := (R^{[k]}[u])^2\, F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0$$
$$\vdots$$
$$E_m := (R^{[k]}[u])^{m-1}\, F(u, u_x, u_{xx}, \ldots, u_{nx}) = 0,$$

where $R^{[k]}[u]$ is a kth-order integrodifferential operator of the form

$$R^{[k]}[u] = G_k D_x^k + G_{k-1} D_x^{k-1} + \cdots + G_0 + Q D_x^{-1} \circ J.$$

E_1 **is the <u>seed equation</u> of the differential sequence.**

$Z^i(E_i)$ **the vertical symmetry generator of the equation** E_i **in the sequence, namely**

$$Z^i(E_i) = Q(x, u, u_x, u_{xx}, u_{3x}, \ldots, u_{jx})\partial_u$$

where the necessary and sufficient invariance condition for equation E_i **is**

$$L_{E_i} Q \Big|_{E_i=0} = 0.$$

Definition 1:

The sequence admits a p**-dimensional Lie point symmetry algebra,** $\mathcal{L}$**, spanned by the linearly independent symmetry generators**

$$\{Z_1^i(E_i),\ Z_2^i(E_i),\ \ldots,\ Z_p^i(E_i)\}$$

if each equation in the sequence, $\{E_1,\ E_2, \ldots,\ E_m\}$**, admits a** p**-dimensional Lie point symmetry algebra,** $\mathcal{L}'$**, isomorphic to** $\mathcal{L}$**.**

Definition 2:

$J = J(x, u, u_x, u_{xx}, \ldots)$ **is an <u>integrating factor</u> for the differential sequence if** J **is an integrating factor for each equation in the sequence.**

Definition 3:

$R^{[k]}[u]$ **is defined as a** k**th-order recursion operator of the differential sequence under the following conditions:**

$$\left[L_{E_i}[u],\, R^{[k]}[u]\right] = 0, \qquad i = 1, 2, \ldots, m,$$

$$(R^{[k]})^{*}[u] J_k = \alpha J_l \quad \forall \quad k, l = 1, 2, \ldots, p,$$

where α **is a nonzero constant,** $i = 1, 2, \ldots m$ **and** p **is the total number of integrating factors,** J_l**, valid for all members of the sequence. For some values of** l**,** J_l **may be zero.**

Definition 4:

A proper differential sequence of ODEs is a differential sequence which admits at least one recursion operator.

Definition 5:

An integrable differential sequence is defined as a proper differential sequence of ODEs for which each equation in the sequence is integrable.

Let

$$E_i := u_{qx} - f_i(x, u, u_x, u_{xx}, \dots, u_{(q-1)x}) = 0,$$

where

$$q = n + (m-1)k.$$

We introduce the following total derivative operator

$$D_{E_i} = D_x \Big|_{E_i=0} = \frac{\partial}{\partial x} + \sum_{j=0}^{q-1} u_{jx} \frac{\partial}{\partial u_{(j-1)x}} + f_i \left(x, u, u_x, \dots, u_{(q-1)x} \right) \frac{\partial}{\partial u_{(q-1)x}}.$$

Proposition 1: [*Bluman and Anco*]

J_s **is an integrating factor for the proper differential sequence if and only if the following conditions are satisfied:**

$$L^*_{E_i[u]} J_s(x, u, u_x, \dots) \Big|_{E_i=0} = 0, \qquad i = 1, 2, \dots, m,$$

$$\frac{\partial J_s}{\partial u_{(q-2r)x}} + \sum_{j=1}^{2r-1} (-1)^{j-1} \frac{\partial}{\partial u_{(q-1)x}} \left\{ D^{j-1}_{E_i} \left(\frac{\partial f_i}{\partial u_{(j+q-2r)x}} J_s \right) \right\}$$

$$+ \frac{\partial}{\partial u_{(q-1)x}} \left(D^{2r-1}_{E_i} J_s \right) = 0, \qquad s = 1, 2, \dots, p, \quad r = 1, 2, \dots, \left[\frac{q}{2} \right].$$

Here $\left[\frac{q}{2}\right]$ **is the largest natural number less than or equal to the number** $\frac{q}{2}$**,** $i = 1, 2, \dots m$**, and** p **is the total number of integrating factors,** J_s**, valid for all members of the sequence, i.e.** $s = 1, 2, \dots, p$.

Example 1: The seed equation

$$u_{xx} + u_x^2 = 0$$

admits the recursion operator

$$R[u] = D_x + u_x.$$

This gives a proper differential sequence

$$E_j := R^{j-1}[u]\left(u_{xx} + u_x^2\right) = 0.$$

That is

$$E_1 := F(u, u_x u_{xx}) = u_{xx} + u_x^2 = 0$$

$$E_2 := R[u]F(u, u_x u_{xx}) = u_{3x} + 3u_x u_{xx} + u_x^3 = 0$$

$$E_3 := R^2[u]F(u, u_x u_{xx}) = u_{4x} + 4u_x u_{3x} + 3u_{xx}^2 + 6u_x^2 u_{xx} + u_x^4 = 0$$

$$\vdots$$

$$E_m := R^{m-1}[u]F(u, u_x u_{xx}) = u_{(m+1)x} + \cdots = 0.$$

with zeroth-order integrating factors

$$J_1(x, u) = e^u, \qquad J_2(x, u) = xe^u.$$

Here

$$R^*[u]e^u = 0, \qquad R^*[u](xe^u) = -e^u.$$

More details: Consider the RO Ansatz

$$R[u] = G_1(u, u_x)D_x + G_0(u, u_x)$$

for

$$E_1 := u_{xx} + u_x^2 = 0$$

where

$$L[u] = \frac{\partial E_1}{\partial u} + \frac{\partial E_1}{\partial u_x} D_x + \frac{\partial E_1}{\partial u_{xx}} D_x^2,$$

under the condition

$$[L[u],\, R[u]] = 0.$$

This leads to

$$R[u] = D_x + u_x + f(\omega), \qquad \omega := u_x\, e^u$$

so that

$$E_2 := R[u]E_1 = u_{3x} + 3u_x u_{xx} + u_x^3 + (u_{xx} + u_x^2) f(\omega) = 0.$$

Note that

$$J_1 = e^u, \qquad J_2 = xe^u$$

are integrating factors of E_1 and E_2. But

$$R^*[u]e^u = e^u\, f(\omega)$$

is NOT an integrating factor of E_2! We find that

$$f(\omega) = 0.$$

and therefore the most general RO for this sequence is

$$R[u] = D_x + u_x.$$

An alternative description: Integration of sequences

Consider a proper differential sequence:

$$\{E_1,\, E_2,\, \ldots,\, E_m\}.$$

Idea:
Construct an alternative sequence (AS),

$$\{\tilde{E}_1,\, \tilde{E}_2,\, \ldots,\, \tilde{E}_m\},$$

such that

- $\tilde{E}_1 \equiv E_1$
- **Order**$(\tilde{E}_j)$ $=$ **Order**$(\tilde{E}_1)$ **for all** j**.**
- **Compatibility (at least one solution) or Complete Compatibility (all solutions)**

Thus:
The alternative sequence define integrals of the proper differential sequence.

Proposition 2:
Consider a proper differential sequence $\{E_1,\, E_2, \ldots,\, E_m\}$ with recursion operator $R^{[k]}[u]$. An alternative sequence, $\{\tilde{E}_1,\, \tilde{E}_2, \ldots,\, \tilde{E}_m\}$, of the form

$$\tilde{E}_1 := F(u, u_x u_{xx}, \ldots, u_{nx}) = 0$$

$$\tilde{E}_{j+1} := F(u, u_x u_{xx}, \ldots, u_{nx}) = Q_j(x, u, u_x, \ldots, \omega^1, \omega^2, \ldots; c_1, c_2, ...)$$

$$j = 1, 2, \ldots, m-1,$$

is compatible with the proper differential sequence if

$$R^{[k]} Q_1 = 0$$

$$R^{[k]} Q_i = Q_{i-1}, \qquad i = 2, 3, \ldots, m.$$

Here $\omega^1,\ \omega^2 \ldots, \omega^\ell$ are nonlocal coordinates defined by

$$\frac{d\omega^1}{dx} = g_1(u),$$

$$\frac{d\omega^2}{dx} = g_2(\omega^1), \quad \frac{d\omega^3}{dx} = g_3(\omega^2), \ \ldots, \ \frac{d\omega^\ell}{dx} = g_\ell(\omega^{\ell-1})$$

for some differentiable functions g_k.

Example: Consider again the proper differential sequence, already introduced in Example 1, where

$$R[u] = D_x + u_x.$$

That is

$$E_1 := F(u, u_x, u_{xx}) = u_{xx} + u_x^2 = 0$$

$$E_2 := R[u]F(u, u_x, u_{xx}) = u_{3x} + 3u_x u_{xx} + u_x^3 = 0$$

$$E_3 := R^2[u]F(u, u_x, u_{xx}) = u_{4x} + 4u_x u_{3x} + 3u_{xx}^2 + 6u_x^2 u_{xx} + u_x^4 = 0$$

$$\vdots$$

$$E_m := R^{m-1}[u]F(u, u_x, u_{xx}) = u_{(m+1)x} + \cdots = 0.$$

We apply Proposition 2: The second member in the alternative sequence is

$$u_{xx} + u_x^2 = Q_1(x, u, u_x, \ldots)$$

under the condition

$$R[u]Q_1(x, u, u_x, \ldots) = 0.$$

Condition () is of the form

$$D_x(Q_1) = -u_x Q_1$$

with general solution

$$Q_1(u, c_1) = c_1 e^{-u},$$

where c_1 **is an arbitrary constant of integration. The second member in the alternative sequence is**

$$u_{xx} + u_x^2 = c_1 e^{-u}.$$

The third member:

$$u_{xx} + u_x^2 = Q_2(x, u, u_x, \ldots)$$

under the condition

$$R[u]Q_2(x, u; c_1, c_2) = Q_1(u; c_1),$$

which admits the general solution

$$Q_2(x, u; c_1, c_2) = c_1 x e^{-u} + c_2 e^{-u}$$

with c_2 **another constant of integration. The third member in the alternative sequence is**

$$u_{xx} + u_x^2 = e^{-u} \left(c_1 x + c_2\right)$$

which can be presented in the form

$$u_{xx} + u_x^2 = e^{-u} D_x^{-1} c_1.$$

In explicit form the alternative sequence is

$$\tilde{E}_1 := u_{xx} + u_x^2 = 0$$

$$\tilde{E}_2 := u_{xx} + u_x^2 = Q_1 \quad \textbf{with} \quad Q_1 = e^{-u} c_1$$

$$\tilde{E}_3 := u_{xx} + u_x^2 = Q_2 \quad \textbf{with} \quad Q_2 = e^{-u} \left(c_1 x + c_2\right)$$

$$\tilde{E}_4 := u_{xx} + u_x^2 = Q_3, \quad \textbf{with} \quad Q_3 = e^{-u} \left(\frac{1}{2} c_1 x^2 + c_2 x + c_3\right)$$

$$\vdots$$

$$\tilde{E}_m := u_{xx} + u_x^2 = Q_{m-1} \quad \textbf{with} \quad Q_{m-1} = e^{-u} \left(\sum_{j=1}^{m-1} \frac{c_j}{(m-j-1)!} x^{m-j-1}\right)$$

Compatibility or complete compatibility:

- **Compare E_2 and $\tilde{E}_2$:**
  A first integral for E_2 is given by $\tilde{E}_2$, namely

$$c_1 = e^u \left(u_{xx} + u_x^2\right). \tag{9}$$

  Therefore the general solution of $\tilde{E}_2$ gives the general solution of E_2 with c_1 as one of the constants of integration for E_2. Hence the two equations, E_2 and $\tilde{E}_2$, are completely compatible.

- **Compare E_3 and $\tilde{E}_3$:**
  A second integral for E_3 is given by $\tilde{E}_3$, namely

$$c_1 x + c_2 = e^u \left(u_{xx} + u_x^2\right). \tag{10}$$

  Therefore the general solution of $\tilde{E}_3$ gives the general solution of E_3 (with c_1 and c_2 as two of the constants of integration for E_3) and the two equations E_3 and $\tilde{E}_3$ are completely compatible. A similar argumant follows for all equations in the proper differential sequence.

Conclusion: The two sequences are completely compatible.

Linearisation:

The proper differential sequence is linearisable by

$$w(X) = u_x e^u, \qquad X = x.$$

Also the alternative sequence is linearisable by

$$w(X) = e^u, \qquad X = x.$$

Conclusion: The proper differential sequence is integrable.

Symmetry properties:

The symmetry characteristic, η_j**, for the symmetry generator**

$$\Gamma_j^s = \eta_j(x, u)\partial_u,$$

of the solution symmetry for E_j **is given by** Q_{j+1} **of the equation** $\tilde{E}_{j+2}$ **in** $\{\tilde{E}_1, \tilde{E}_2, \ldots \tilde{E}_m\}$ **for all** $j = 1, 2, \ldots, m$**.**

Example: $E_1 := u_{xx} + u_x^2 = 0$ **admits the solution symmetry**

$$\Gamma_1^s = Q_2 \partial_u,$$

where $Q_2 = e^{-u}(c_1 x + c_2)$ **corresponds to** $\tilde{E}_3$**.**

Example: Recall the Harry-Dym Equation:

$$u_t = u^3 u_{xxx}$$

with recursion operator

$$R[u] = u^2 D_x^2 - u u_x D_x + u u_{xx} + u^3 u_{xxx} D_x^{-1} \circ u^{-2}.$$

Proper differential sequence of ODEs:

$$E_1 := u^3 u_{3x} = 0$$

$$E_2 := u^5 u_{5x} + 5u^4 u_x u_{4x} + 5u^4 u_{xx} u_{3x} + \frac{5}{2} u^3 u_x^2 u_{3x} = 0$$

$$E_3 := u^7 u_{7x} + 14u^6 u_x u_{6x} + \cdots = 0 \qquad \vdots$$

Alternative sequence:

$$\tilde{E}_1 := u^3 u_{3x} = 0$$
$$\tilde{E}_2 := u^3 u_{3x} = Q_1$$
$$\tilde{E}_3 := u^3 u_{3x} = Q_2$$
$$\vdots$$

where

$$R[u]Q_1 = 0, \quad R[u]Q_2 = Q_1, \quad R[u]Q_3 = Q_2, \quad \ldots$$

We obtain

$$\tilde{E}_2 := u^3 u_{3x} = u^2 \left(\frac{a_0 + a_1 x + a_2 x^2}{u} \right)_x$$

or, after integration,

$$u u_{xx} - \frac{1}{2} u_x^2 = \frac{P_1(x)}{u} + C_1, \qquad P_1(x) = a_0 + a_1 x + a_2 x^2, \tag{11}$$

which is a third integral of E_2. NOTE: With $u = v^2$, (11) is a generalized Ermakov-Pinney equation:

$$v_{xx} = \frac{C_1}{2v^3} + \frac{P_1(x)}{2v^5}.$$

Reference:

N. Euler and P.G.L. Leach, *Aspects of proper differential sequences of ordinary differential equations*, **nlin arXiv:0802.1459 (2008).**

Example: Burgers' equation

$$u_{xx} + uu_x = u_t$$

we associate

$$u_{xx} + uu_x = 0$$

which shares the same integrodifferential recursion operator,

$$R[u] = D_x + \tfrac{1}{2}u + \tfrac{1}{2}u_x D_x^{-1} \circ 1.$$

The Burgers Sequence, is

$$E_1 := u_{xx} + uu_x = 0$$
$$E_{j+1} := R^j[u]\,(u_{xx} + uu_x) = 0, \qquad j = 1, 2, \ldots, m.$$

An alternative Burger's Sequence following Proposition 2:
The solution of $R[u]Q_1 = 0$ is

$$Q_1 = \left(-2A \exp\left[-\tfrac{1}{2}\int u\mathrm{d}x\right] + 2B \exp\left[-\tfrac{1}{2}\int u\mathrm{d}x\right] \int \exp\left[\tfrac{1}{2}\int u\mathrm{d}x\right] \mathrm{d}x\right)_x$$

where A and B are constants of integration. Consider

$$w = \int \exp\left[\tfrac{1}{2}\int u\mathrm{d}x\right] \mathrm{d}x$$

so that

$$u_{xx} + uu_x = \left(-2A \exp\left[-\tfrac{1}{2}\int u\mathrm{d}x\right] + 2B \exp\left[-\tfrac{1}{2}\int u\mathrm{d}x\right] \int \exp\left[\tfrac{1}{2}\int u\mathrm{d}x\right]$$

becomes

$$\frac{w_{4x}}{w_x} - \frac{w_{xx}w_{3x}}{{w_x}^2} = \frac{Aw_{xx}}{{w_x}^2} + B\left(1 - \frac{ww_{xx}}{{w_x}^2}\right).$$

In a similar fashion the equation $R[u]Q_2 = Q_1$ has the solution

$$Q_2 = \Bigg\{ 2C \exp\left[-\tfrac{1}{2}\int u\mathrm{d}x\right] \int \exp\left[\tfrac{1}{2}\int u\mathrm{d}x\right]\mathrm{d}x - 2Ax\exp\left[-\tfrac{1}{2}\int u\mathrm{d}x\right]$$

$$+2B\exp\left[-\tfrac{1}{2}\int u\mathrm{d}x\right]\int\left(\int\exp\left[\tfrac{1}{2}\int u\mathrm{d}x\right]\mathrm{d}x\right)\mathrm{d}x\Bigg\}_x,$$

where C is also a constant of integration, and the integrodifferential equation is

$$u_{xx} + uu_x = \Bigg\{ 2C \exp\left[-\tfrac{1}{2}\int u\mathrm{d}x\right] \int \exp\left[\tfrac{1}{2}\int u\mathrm{d}x\right]\mathrm{d}x - 2Ax\exp\Big[-\tfrac{1}{2}$$

$$+2B\exp\left[-\tfrac{1}{2}\int u\mathrm{d}x\right]\int\left(\int\exp\left[\tfrac{1}{2}\int u\mathrm{d}x\right]\mathrm{d}x\right)\mathrm{d}x\Bigg\}_x.$$

The corresponding higher-order ordinary differential equation is

$$\frac{w_{5x}}{w_{xx}} - \frac{w_{3x}w_{4x}}{{w_{xx}}^2} = C\left(1 - \frac{w_x w_{3x}}{{w_{xx}}^2}\right) + A\left(\frac{x w_{3x}}{{w_{xx}}^2} - \frac{1}{w_{xx}}\right) + B\left(\frac{w_x}{w_{xx}} - \frac{w w_{3x}}{{w_{xx}}^2}\right),$$

where now

$$w = \int\left(\int\exp\left[\tfrac{1}{2}\int u\mathrm{d}x\right]\mathrm{d}x\right)\mathrm{d}x$$

or equivalently

$$u = 2\frac{w_{3x}}{w_{xx}}.$$

We thus conclude that the first three terms in the alternative sequence take the following forms

$$
\begin{aligned}
\tilde{E}_1(w) &:= \frac{w_{5x}}{w_{xx}} - \frac{w_{3x}w_{4x}}{{w_{xx}}^2} = 0 \\
&\Leftrightarrow \left(\frac{w_{4x}}{w_{xx}}\right)_x = 0 \\
&\Leftrightarrow \quad w_{4x} = k_1 w_{2x} \\
\tilde{E}_2(w) &:= \frac{w_{5x}}{w_{xx}} - \frac{w_{3x}w_{4x}}{{w_{xx}}^2} = \frac{Aw_{3x}}{{w_{xx}}^2} + B\left(1 - \frac{w_x w_{3x}}{{w_{xx}}^2}\right) \\
&\Leftrightarrow \left(\frac{w_{4x}}{w_{xx}}\right)_x = -\left(\frac{A}{w_{xx}}\right)_x + B\left(\frac{w_x}{w_{xx}}\right)_x \\
&\Leftrightarrow w_{4x} = a_1 w_{xx} + Bw_x - A \\
\tilde{E}_3(w) &:= \frac{w_{5x}}{w_{xx}} - \frac{w_{3x}w_{4x}}{{w_{xx}}^2} = C\left(1 - \frac{w_x w_{3x}}{{w_{xx}}^2}\right) + A\left(\frac{xw_{3x}}{{w_{xx}}^2} - \frac{1}{w_{xx}}\right) \\
&\qquad\qquad\qquad\qquad + B\left(\frac{w_x}{w_{xx}} - \frac{ww_{3x}}{{w_{xx}}^2}\right) \\
&\Leftrightarrow \left(\frac{w_{4x}}{w_{xx}}\right)_x = C\left(\frac{w_x}{w_{xx}}\right)_x - A\left(\frac{x}{w_{xx}}\right)_x + B\left(\frac{w}{w_{xx}}\right)_x \\
&\Leftrightarrow w_{4x} = a_2 w_{xx} + Cw_x + Bw - Ax.
\end{aligned}
$$

The proper differential sequence in the same variable w:

$$E_1(w) := \frac{w_{5x}}{w_{xx}} - \frac{w_{3x}w_{4x}}{{w_{xx}}^2} = 0 \;\Leftrightarrow\; \left(\frac{w_{4x}}{w_{xx}}\right)_x = 0 \quad\Leftrightarrow\quad w_{3x} = k_1 w_x + k_1$$

$$E_2(w) := \frac{w_{6x}}{w_{xx}} - \frac{w_{3x}w_{5x}}{{w_{xx}}^2} = 0 \;\Leftrightarrow\; \left(\frac{w_{5x}}{w_{xx}}\right)_x = 0 \quad\Leftrightarrow\quad w_{4x} = k_2 w_x + k_2$$

$$E_3(w) := \frac{w_{7x}}{w_{xx}} - \frac{w_{3x}w_{6x}}{{w_{xx}}^2} = 0 \;\Leftrightarrow\; \left(\frac{w_{6x}}{w_{xx}}\right)_x = 0 \quad\Leftrightarrow\quad w_{5x} = k_3 w_x + k_3$$

Compatible but not completely compatible.

Linealisation:

The n**th element of the** **Burgers Differential Sequence**

$$R^{n-1}[u]\,(u_{xx}+uu_x)=0,$$

where

$$R[u]=D_x+\tfrac{1}{2}u+\tfrac{1}{2}u_xD_x^{-1},$$

is linearised to

$$v_{(n+1)}=\Omega_n^{n+1}v,$$

where $u=2v_x/v$ **and** Ω **are arbitrary constants.**

The n**th element of the alternative Burgers Differential Sequence written in the integrodifferential form**

$$u_{xx}+uu_x=\exp\left[-\frac{1}{2}\int u\mathrm{d}x\right]\left(\sum_{i=1}^{n-1}B_iD_x^{-i}\exp\left[\frac{1}{2}\int u\mathrm{d}x\right]\right)$$

is linearised to

$$W_{(n+1)x}=B_{n-2}+B_{n-1}W,$$

where $W=D_x^{-(n-1)}\exp\left[\frac{1}{2}\int u\mathrm{d}x\right]$**.**

Conclusion:

The alternative sequence is in general only a compatible and not completely compatible sequence.

All integrals do not in general follow from the alternative sequence, even for integrable proper differential sequences.