Unstable modes of dark photorefractive solitons

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Beam propagation in biased photorefractive media

- Photorefractive (PR) materials are lightly doped electro-optic crystals.
- PR media exhibit a reversible change of refractive index induced by spatial variation of an optical field.
- Photorefractive solitons happen whenever the variations in refractive index produced by a beam is sufficient to compensate for its diffraction.
- Screening solitons are PR solitons on PR media under an external electric field.

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Evolution Equation

Optical field of the polarized beam:

$$\mathbf{E} = \hat{\mathbf{x}}\phi(x,z)\exp[i(kz-\omega t)], \qquad k = \omega n_e/c$$

The slowly varying envelope ϕ propagates according to:

$$i\phi_z + \frac{1}{2k}\phi_{xx} - \frac{k}{2}(n_e^2 r_{\text{eff}} E_{\text{sc}})\phi = 0$$

- Propagation along z,
- Diffraction only allowed in the x direction (optical c axis);

The **space charge field** due to external field and redistribution of charge:

$$E_{\text{sc}} = E_0 \frac{I_\infty + I_d}{I + I_d}, \qquad \text{neglecting charge diffusion}$$

$I_\infty = I(x \to \pm\infty)$, I_d - dark irradiance (thermal ionization) $E_0 = E(x \to \pm\infty)$

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Normalization:

$$q = \left(\frac{n_e}{2\eta_0 I_d}\right)^{1/2}\phi, \quad \tilde{z} = (kn_e^2 r_{\text{eff}}|E_0|/2)z, \quad \tilde{x} = kn_e(r_{\text{eff}}|E_0|/2)^{1/2}x$$

Adimensional evolution equation

$$iq_{\tilde{z}} + q_{\tilde{x}\tilde{x}} - \text{sgn}(E_0)(1+\rho)\frac{q}{1+|q|^2} = 0, \qquad \rho = \frac{I_\infty}{I_d}$$

The above equation admits *bright* and *dark* solitary wave solutions.

Evolution equation

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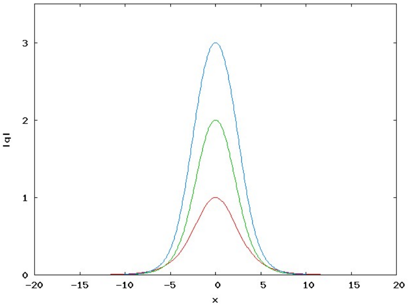
Bright solitons

$I_\infty = 0 \rightarrow \rho = 0$ & $\mathrm{sgn}(E_0) > 0$

Evolution equation for bright solitons

$$iq_z + q_{xx} - \frac{q}{1+|q|^2} = 0 \qquad q \rightarrow 0 \qquad x \rightarrow \pm\infty$$



- Solutions parameterized by the peak value or power.

(Christodoulides and Carvalho 1995)

- Experimentally observed.

(Shih 1995/96)

- Stable for any peak value.

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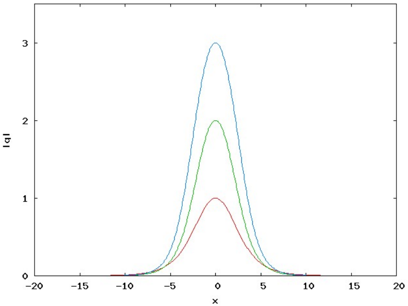

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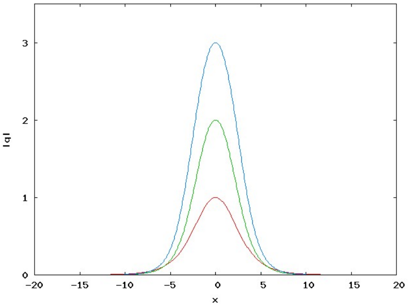

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Dark solitons $I_\infty \neq 0 \rightarrow \rho \neq 0$ & $\text{sgn}(E_0) < 0$

Evolution equation for dark solitons

$$iq_z + \frac{1}{2}q_{xx} + (1+\rho)\frac{q}{1+|q|^2} = 0, \qquad q \rightarrow \sqrt{\rho}e^{i(\theta_0 \pm S/2)} \quad \text{as} \quad x \rightarrow \pm\infty$$

S - phase jump across x

Let $q(z,x) = \sqrt{\rho}y(\eta)e^{i\theta(z,\eta)}$,

where $\eta = x - \omega z + \eta_0$

Phase

$$\theta(z,\eta) = z - \omega \int_0^\eta \frac{d\eta'}{y^2} + \omega\eta + \theta_0.$$

Profile $|y| \leq 1$, $|y| \rightarrow 1$ as $\eta \rightarrow \pm\infty$

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Dark solitons

- Solutions parameterized by ρ and ω, $\omega^2 < \rho/(1+\rho)$
- The minimum of $y_{\min} = \sqrt{m}$ is related with the velocity ω:

$$\omega^2 = \frac{2m}{1-m}\left[\frac{1}{1-m}\frac{1+\rho}{\rho}\ln\left(\frac{1+\rho}{1+\rho m}\right)-1\right].$$

- Their phase is not constant across x - *phase jump*.

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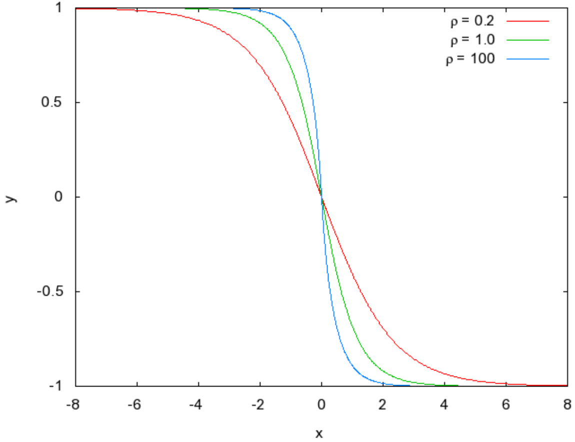
Profile and phase jump of black solitons



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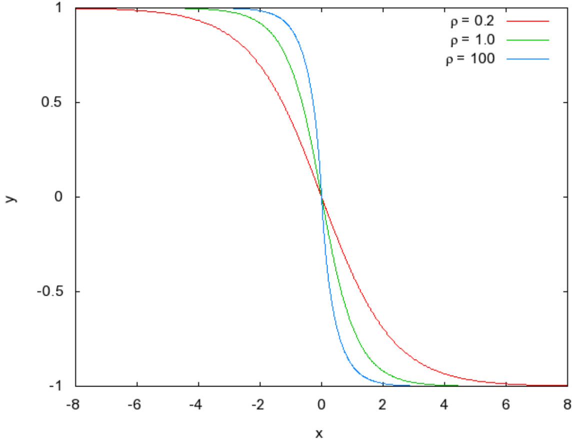

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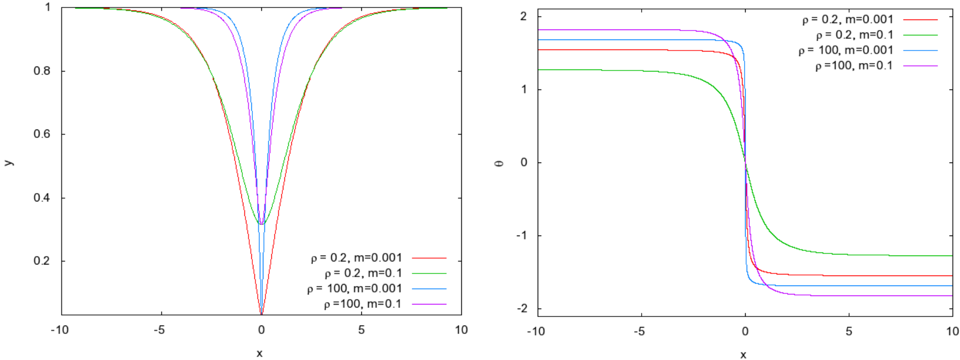
Profile and phase jump of gray solitons



There are solitons whose **phase jump is higher then π.**

Solitons darker then black

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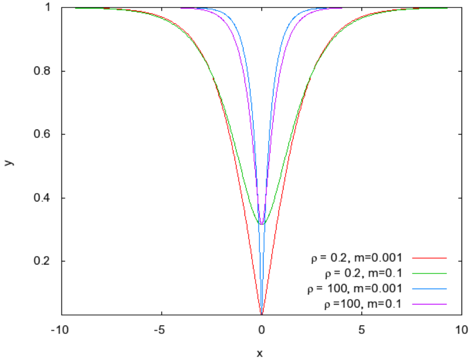

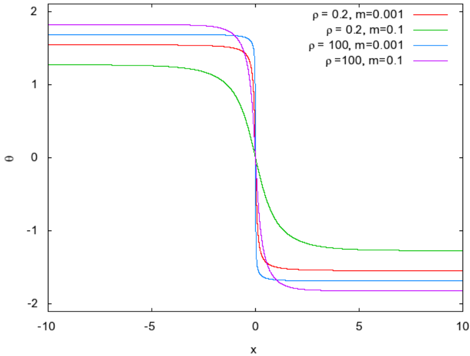

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Solitons darker then black

Linear stability equations

Considering the above **dark solution plus a small perturbation term**:

$$q(z,x) = \sqrt{\rho} e^{i\theta(z,\eta)} \left[y(\eta) + \Delta(z,\eta) \right].$$

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Demanding that Δ and Δ^* have **exponential dependence on** z (study of spectral stability):

$$\Delta(z,\eta) = u(\eta) e^{i\lambda z} + v^*(\eta) e^{-i\lambda^* z}$$

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Stability eigenvalue problem $L\mathbf{w} = \lambda\mathbf{w}$

$$L = \sigma_3\left(\frac{1}{2}\partial_{\eta\eta} + F(\eta) - G(\eta)\right) - i\sigma_2 G(\eta) + iI_2\left(\frac{\omega y'}{y^3} - \frac{\omega}{y^2}\partial_\eta\right)$$

$$\mathbf{w} = \begin{pmatrix} u & v \end{pmatrix}^T$$

Linear stability equations

Where...

- $I_2 \quad \to \quad 2 \times 2$ identity matrix
- σ_2 and $\sigma_3 \quad \to$ Pauli matrices:

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- F and G:

$$F(\eta) = \frac{\omega^2}{2} - 1 - \frac{\omega^2}{2y^4} + \frac{1+\rho}{1+\rho y^2}$$

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Continuous spectrum

The continuous spectrum of L is $\mathbb{R}$

As for L_∞ (L as $\eta \to \infty$):

$$L_\infty = \sigma_3 \left(\frac{1}{2}\partial_{\eta\eta} - \frac{\rho}{1+\rho} \right) - i\sigma_2 \frac{\rho}{1+\rho} - iI_2 \omega \partial_\eta .$$

Discrete eigenvalues

Due to the symmetry of L:

λ is an eigenvalue $\rightarrow -\lambda$, λ^* and $-\lambda^*$ are also eigenvalues

Hence:

The existence of any discrete eigenvalues $\rightarrow$ **instability**

Discrete eigenvalues are searched using the **Evans function method**.

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Stability criterion for dark solitons

Dark solitons are stable for $\omega > \omega_c$ such that

$$\frac{\partial P}{\partial \omega} < 0$$

(Barashenkov 1996, Lin 2002)

Where P is the renormalized momentum

$$P = \frac{i}{2}\int_{-\infty}^{\infty}(q_x^* q - q_x q^*)dx - \rho \text{Arg } q|_{-\infty}^{+\infty}$$
$$= \frac{i}{2}\int_{-\infty}^{\infty}(q_x^* q - q_x q^*)\left(1 - \frac{\rho}{|q|^2}\right)dx$$

Limitations of the above criterion

- Not applicable to black solitons $\rightarrow$ $\partial P/\partial \omega$ **does not exist.**

$$\int_0^1 \frac{\left(y-\frac{1}{y}\right)^2}{\sqrt{2(y^2-1)+\frac{2(1+\rho)}{\rho}\ln\left(\frac{1+\rho}{1+\rho y^2}\right)}}dy \qquad \text{diverges}$$

(Menza and Gallo 2007 - the sign of limit of the Vakhitov-Kolokolov function at 0 must be determined)

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- **Unstable modes unknown.**

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ODE for the black soliton $y'' - 2y + (1+\rho)\frac{2y}{1+\rho y^2} = 0$

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Continuous spectrum of L_0 is $(-\infty, 0]$

Stability criterion for black solitons - stability EV problem

ODE for the black soliton $y'' - 2y + (1+\rho)\frac{2y}{1+\rho y^2} = 0$

$$L_0 V = 2\lambda U, \qquad L_1 U = 2\lambda V$$

$$L_0 = \partial_{\eta\eta} - 2 + \frac{2(1+\rho)}{1+\rho y^2}$$

$$L_1 = \partial_{\eta\eta} - 2 + \frac{2(1+\rho)}{1+\rho y^2} - \frac{4\rho y^2(1+\rho)}{(1+\rho y^2)^2}$$

Continuous spectrum of L_0 is $(-\infty, 0]$

Continuos spectrum of L_1 is $(-\infty, \frac{4\rho}{1+\rho}]$

Stability criterion for black solitons - VK procedure

Sturm-Liouville theory gives:

- $L_0 y = 0$, y possesses one zero $\rightarrow$ L_0 has one positive eigenvalue, α_0.
- $L_1 y' = 0$, y' has no zero $\rightarrow$ L_1 is non-positive

In the subspace orthogonal to y':

$$\lambda^2 = \frac{1}{4} \frac{\langle V, L_0 V \rangle}{\left\langle V, L_1^{-1} V \right\rangle}$$

- $\left\langle V, L_1^{-1} V \right\rangle$ is negative
- If $\max(\langle V, L_0 V \rangle)$ is positive $\rightarrow$ y_{black} is unstable.
- If $\max(\langle V, L_0 V \rangle)$ is negative $\rightarrow$ y_{black} is stable.

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Stability criterion for black solitons -VK procedure

Following standard Vakhitov-Kolokolov procedure, maximization of $\langle V, L_0 V \rangle$ gives:

$$g(\xi) = \left\langle y', (L_0 - \xi)^{-1} y' \right\rangle = 0$$

where $\xi = \max \langle V, L_0 V \rangle$ and $\xi \in (0, \alpha_0)$.

- $g(\xi)$ has an asymptote to ∞ at $\xi = \alpha_0$
- If $g(\xi) < 0$ as $\xi \to 0$ there is one positive ξ satisfying the above equation $\to y_{\text{black}}$ is **unstable**.
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Range of ρ for stable black solitons

- To find $\psi(\eta;\xi)$ such that $\psi(\eta,\xi) = (L_0 - \xi)^{-1}\, y'$, we solve:

$$\psi'' - (2+\xi)\psi + (1+\rho)\frac{2\psi}{1+\rho y^2} = y'$$

- Then, determining the sign of the limit of the following integral

$$\int_{-\infty}^{\infty} y'(\eta)\psi(\eta;\xi)d\eta \quad \text{as} \quad \xi \to 0$$

we arrive to:

$$y_{\text{black}} \text{ is stable for } \rho \leq 29.3$$

$$y_{\text{black}} \text{ is unstable for } \rho > 29.4$$

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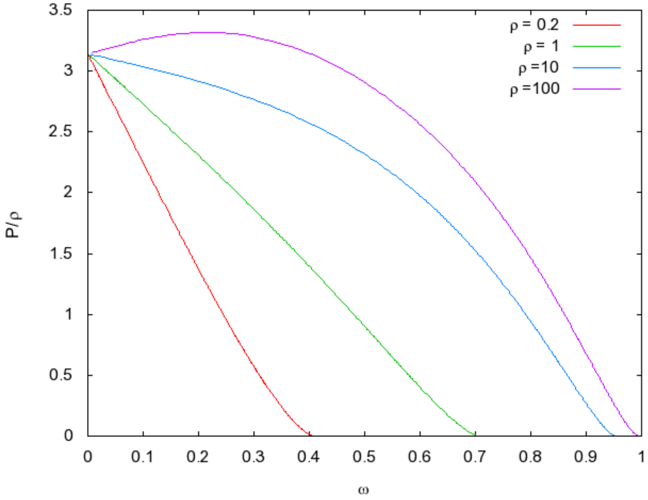
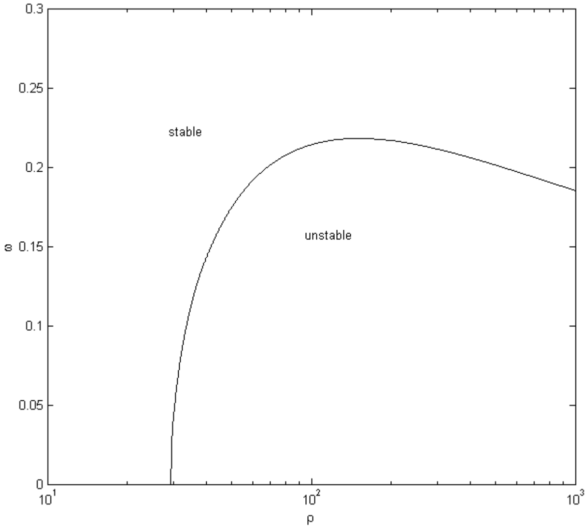
P versus ω



Stability region using the above criterion



Evans function method applied to dark PR solitons

Defining $Y = \begin{pmatrix} u & u_\eta & v & v_\eta \end{pmatrix}^T$, the eigenvalue problem $L\mathbf{w} = \lambda \mathbf{w}$ transforms to:

$$\frac{dY}{d\eta} = A(\eta, \lambda) Y$$

For $\eta \to \pm\infty$:

$$\frac{dY}{d\eta} = A_\infty(\lambda) Y$$

- The asymptotic system is a constant coefficient system of differential equations.
- It has solutions of the form $Y_r^\infty(\eta, \lambda) = y_r(\lambda) \exp[r(\lambda)\eta]$.

$r(\lambda)$ is one of the eigenvalues of $A_\infty(\lambda)$.
$y_r(\lambda)$ is the corresponding eigenvector

- For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, r are such that:

$$\mathrm{Re}(r_{1,2}) > 0 \qquad \mathrm{Re}(r_{3,4}) < 0$$

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The full system $\frac{dY}{d\eta} = A(\eta, \lambda) Y$ has:

- **Bounded solutions as** $\eta \to -\infty$ satisfying

$$Y^{-}_{r_{1,2}}(\eta, \lambda) \sim Y^{\infty}_{r_{1,2}} \qquad \text{as} \qquad \eta \to -\infty,$$

- **Bounded solutions as** $\eta \to +\infty$ satisfying:

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The localized eigenfunction corresponding to discrete eigenvalues should be a linear combination of:

- $Y^-_{r_1}(\eta,\lambda)$ and $Y^-_{r_2}(\eta,\lambda)$ (they span the unstable manifold) and
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Following Alexander *et al* we work on the *exterior space* $\Lambda^2(\mathbb{C}^4)$ where the 2-vectors:

- $U^-(\eta,\lambda) = Y^-_{r_1}(\eta,\lambda) \wedge Y^-_{r_2}(\eta,\lambda)$ represents the unstable manifold
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$$\tilde{D}(\lambda,\eta) = U^{-}(\lambda,\eta) \wedge U^{+}(\lambda,\eta) \quad \text{is independent of} \quad \eta$$

Evans function

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Search for unstable eigenvalues using the argument principle

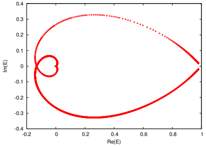
Graphically...

$D(\lambda)$ is analytic and has no zeros on the curve

$$\{\lambda : \lambda = t + i0^-, t \in \mathbb{R}\}$$

$$\Downarrow$$

Number of zeros within that curve (unstable eigenvalues) = number of times the image graph wraps around the origin.



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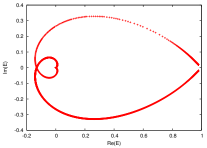

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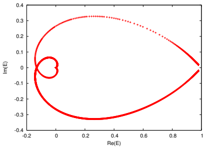
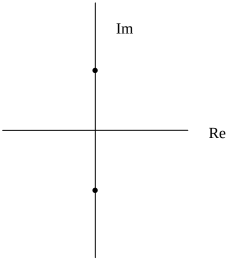


Discrete eigenvalues

- Discrete eigenvalues are found for ρ and ω inside the region of instability.
- In this parameter region, there is only one pair of eigenvalues symmetrically located in the imaginary axis.
- For fixed ρ, they start at some $\pm\lambda_0 i$ and travel toward the origin as ω increases.



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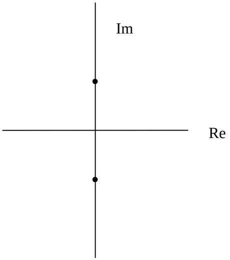

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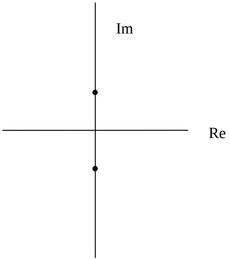

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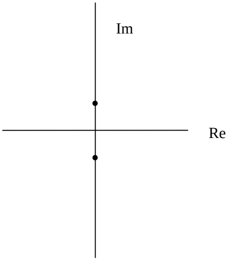

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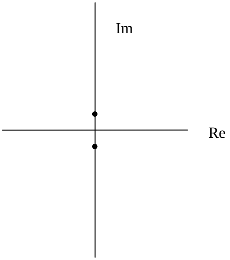

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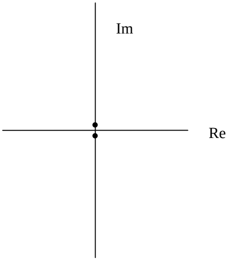
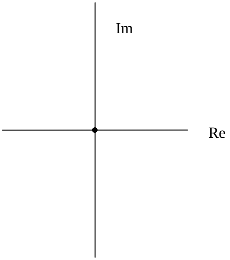
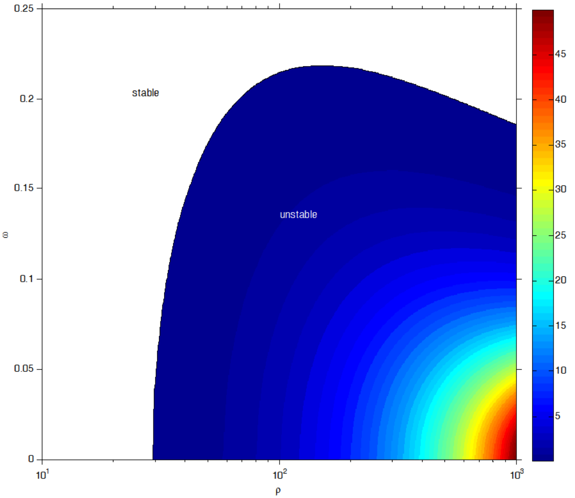


Discrete eigenvalues

- Discrete eigenvalues are found for ρ and ω inside the region of instability.
- In this parameter region, there is only one pair of eigenvalues symmetrically located in the imaginary axis.
- For fixed ρ, they start at some $\pm\lambda_0 i$ and travel toward the origin as ω increases.

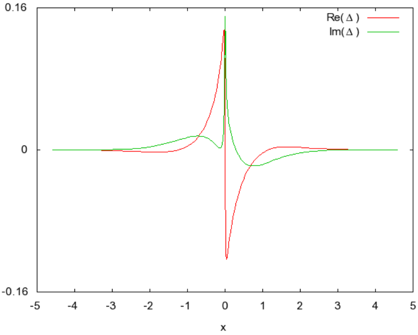


Strength of the instability $|\lambda|$

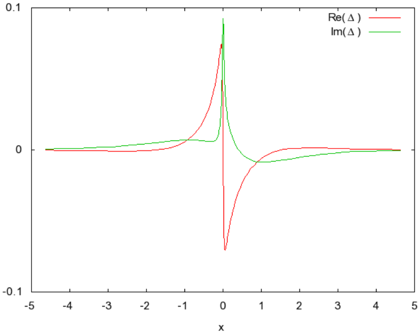


Unstable eigenmodes

Once the eigenvalues are known, the eigenmodes may be determined.

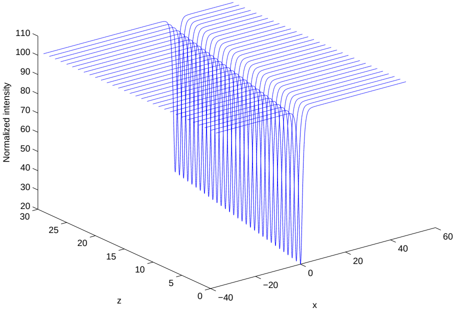


$\rho = 100 \quad \omega = 0.0845$



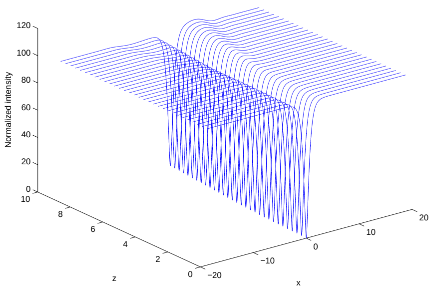
$\rho = 100 \quad \omega = 0.1432$

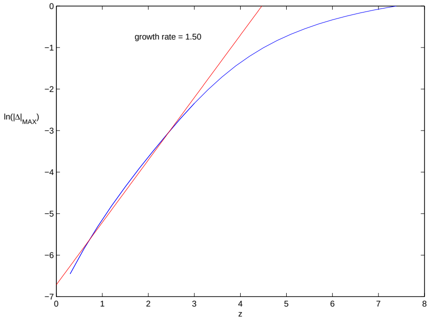
PDE simulations - Stable soliton



$\rho = 100 \qquad \omega = 0.701$

PDE simulations - Unstable soliton

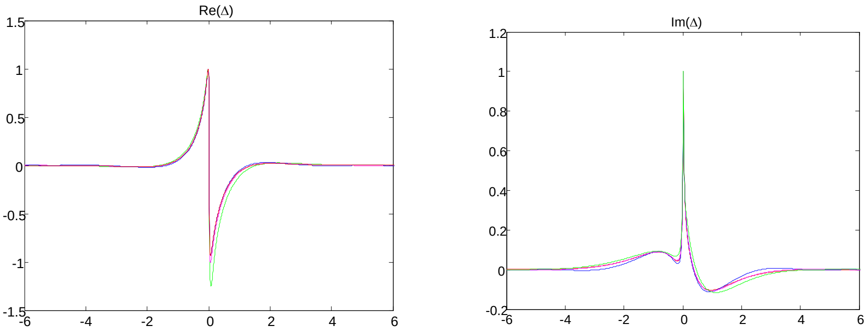




$$\rho = 100 \qquad \omega = 0.1182$$

Growth rate of perturbation agrees reasonably with the estimated $|\lambda_{\text{unstable}}| = 1.35$

Comparation between the growing perturbation and eigenmode



$\rho = 80 \qquad \omega = 0.1034$

Conclusions

- We have determined parameter region for stability of PR solitons.
- For small ρ ($\rho < 29.3$), all the dark solitons are stable. Note that in the limit of small ρ the model resembles the defocusing NLS for which all the dark solitons are stable.
- For $\rho > 29.4$, there are always stable solitons for $\omega > \omega_c$.
- Using Evans function method, we have determined unstable eigenvalues and eigenmodes of the linear stability eigenvalue problem.
- The absolute value of the unstable eigenvalue (strength of the instability) decreases with ω for fixed ρ.
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Thank you