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Reductions of multicomponent NLS and mKdV Equations with Nonvanishing Boundary Conditions

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Съдържание

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1 NLS and MKdV eqs. on symmetric spaces

NLS and MKdV

$$iq_t + q_{xx} + 2\epsilon |q|^2 q(x,t) = 0, \qquad \epsilon = \pm 1,$$

$$q_t + q_{xxx} + 6\epsilon q_x q^2(x,t) = 0, \qquad \epsilon = \pm 1,$$

have natural multicomponent generalizations related to the symmetric spaces Fordy, Kulish (1983). Lax representation:

$$L\psi \equiv \left(i\frac{d}{dx} + Q(x,t) - \lambda J \right) \psi(x,t,\lambda) = 0,$$

$$Q(x,t) = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

$$M_{\mathrm{MNLS}}\psi \equiv \left(i\frac{d}{dt} + V_1(x,t) + \lambda V_2(x,t) - 2\lambda^2 J \right) \psi(x,t,\lambda) = \psi(x,t,\lambda)\lambda^2 J,$$

$$M_{\mathrm{MMKdV}}\psi \equiv \left(i\frac{d}{dt} + V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t) - 4\lambda^3 J \right) \psi(x,t,\lambda)$$

$$= \psi(x,t,\lambda)C(\lambda),$$

$$V_2(x,t) = 4Q(x,t), \quad V_1(x,t) = 2iJQ_x + 2JQ^2, \quad V_0(x,t) = -Q_{xx} - 2Q^3.$$

MNLS eqs.:

$$i\frac{\partial Q}{\partial t} + \frac{\partial^2 Q}{\partial x^2} + 2Q^3(x,t) = 0.$$

MMKdV eqs.:

$$\frac{\partial Q}{\partial t} + \frac{\partial^3 Q}{\partial x^3} + 3\left(Q_x Q^2 + Q^2 Q_x\right) = 0.$$

Local coordinates on symmetric spaces:

A.III-type $SU(n+p)/S(U(n)\times U(p))$

$$Q(x,t) = [J, X(x,t)] = \begin{pmatrix} 0 & \mathbf{q} \\ \mathbf{p} & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_p \end{pmatrix}$$

D.III-type $SO(2n)/S(O(n)\times O(n))$

$$Q(x,t) = [J, X(x,t)] = \begin{pmatrix} 0 & \mathbf{q} \\ \mathbf{p} & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}, \qquad Q + S_0 Q^T S_0 = 0$$

BD.I-type $SO(n+2)/(SO(n)\times SO(2))$

$$Q(x,t)=[J,X(x,t)]=\begin{pmatrix}0 & \vec{q}^T & 0\\ \vec{p} & \mathbf{0} & s_0\vec{q}\\ 0 & \vec{p}^T s_0 & 0\end{pmatrix},\quad J=\begin{pmatrix}1 & 0 & 0\\ 0 & \mathbf{0} & 0\\ 0 & 0 & -1\end{pmatrix},\quad Q+S_0Q^TS_0=0$$

$$S_0^{(k)}=\begin{cases}\sum_{s=0}^{k}(-1)^{k+1}E^{(k)}_{s,k+1-s} & \text{for } k=2r+1\\ \sum_{s=0}^{(k-1)/2}(-1)^{k+1}(E^{(k)}_{s,k+1-s}+E^{(k)}_{k+1-s,s}) & \text{for } k=2r\end{cases}$$

C.I-type $SP(2n)/SU(n)$

$$Q(x,t)=[J,X(x,t)]=\begin{pmatrix}0 & \mathbf{q}\\ \mathbf{p} & 0\end{pmatrix},\qquad J=\begin{pmatrix}\mathbb{1}_n & 0\\ 0 & -\mathbb{1}_n\end{pmatrix},\qquad Q+S_1Q^TS_1=0$$

$$S_1^{(k)}=\sum_{s=0}^{(k-1)/2}(-1)^{k+1}(E^{(k)}_{s,k+1-s}-E^{(k)}_{k+1-s,s}).$$

Mikhailov's reduction group G_R (1981)

G_R is a finite group which preserves the Lax representation.

i) $G_R \subset \mathrm{Aut}\mathfrak{g}$ and
ii) $G_R \subset \mathrm{Conf}\,\mathbb{C}$.

$$g_k \Longrightarrow C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda), \quad g_k \Longrightarrow C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda),$$

where $C_k \in \mathrm{Aut}\,\mathfrak{g}$ and $\Gamma_k(\lambda) \in \mathrm{Conf}\,\mathbb{C}$. For each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$. If $N_k = 2$ then $G_R \simeq \mathbb{Z}_2$.

$$[L, M] = 0, \qquad L \equiv i\frac{d}{dx} + U(x,t,\lambda), \quad M \equiv i\frac{d}{dt} + V(x,t,\lambda),$$

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \qquad V(x,t,\lambda) = V_1(x,t) + \lambda V_2(x,t) - 2\lambda^2 J,$$

Reductions:

1)	$C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda),$	$C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda),$
2)	$C_2(U^T(\kappa_2(\lambda))) = -U(\lambda),$	$C_2(V^T(\kappa_2(\lambda))) = -V(\lambda),$
3)	$C_3(U^*(\kappa_1(\lambda))) = -U(\lambda),$	$C_3(V^*(\kappa_1(\lambda))) = -V(\lambda),$
4)	$C_4(U(\kappa_2(\lambda))) = U(\lambda),$	$C_4(V(\kappa_2(\lambda))) = V(\lambda),$

Choices: a) $C_k \in \mathfrak{H}$ - Cartan subgroup; b) $C_k \in \mathfrak{W}$ - Weyl group.
Reductions 1a) - well known, typical.

Reductions must be compatible with the dispersion law:

$$C_1(f(\kappa_1(\lambda))) = f(\lambda)$$

$$f_{\rm MNLS} = -2\lambda^2 J \quad \longleftrightarrow \quad C_k(J) = J, \quad \kappa_k(\lambda) = \lambda^*;$$

$$f_{\rm MMKdV} = -4\lambda^3 J \quad \longleftrightarrow \quad \begin{cases} C_1(J) = J, & \kappa_1(\lambda) = \lambda^* \\ C_2(J) = -J, & \kappa_2(\lambda) = -\lambda \end{cases};$$

Reductions acting on λ trivially: they restrict $U(x,t,\lambda)$ and $V(x,t,\lambda)$ on the subalgebra $\mathfrak{g}_0$ which is stable with respect to C_k.

Example: choose **A.III** of the form $SU(2n)/S(U(n) \times U(n))$ and take reduction 2b) with $C_2 = S_0$. Then we get L and M which are related to **D.III** symmetric space.

2 MNLS eqs on BD.I-symmetric spaces

These symmetric spaces are $SO(n+2)/SO(n)\times SO(2)$.

$$L\psi = \left(i\frac{d\psi}{dx} + U(x,t,\lambda)\right)\psi(x,t,\lambda) = 0,$$

$$U(x,\lambda) = q(x,t) - \lambda J, \qquad q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{aligned} i\partial_t \vec{q} + \partial^2_{xx}\vec{q} + 2(\vec{p},\vec{q})\vec{q} - (\vec{q}s_0\vec{q})\, s_0\vec{p} = 0, \\ i\partial_t \vec{q} - \partial^2_{xx}\vec{q} - 2(\vec{p},\vec{q})\vec{q} + (\vec{q}s_0\vec{q})\, s_0\vec{p} = 0, \end{aligned}$$

Typical reduction for $n=3$ with $K_1 = \mathrm{diag}\,(\epsilon_1,\epsilon_2,1,\epsilon_2,\epsilon_1)$, $\epsilon_{1,2}^2 = 1$;

$$p_2 = \epsilon_1\epsilon_2 q_2^*, \qquad p_3 = \epsilon_1 q_3^*, \qquad p_4 = \epsilon_1\epsilon_2 q_4^*;$$

gives a 3-component system of NLS equation

$$iq_{2,t} \;+\; q_{2,xx} + 2\epsilon_1(\epsilon_2|q_2|^2 + |q_3|^2)q_2 + \epsilon_1\epsilon_2 q_3^2 q_4^* = 0,$$

$$
\begin{aligned}
iq_{3,t} \; + \; q_{3,xx} + 2\epsilon_1 q_2 q_4 q_3^* + \epsilon_1(2\epsilon_2|q_2|^2 + 2\epsilon_2|q_4|^2 + |q_3|^2)q_3 = 0, \\
iq_{4,t} \; + \; q_{4,xx} + 2\epsilon_1(\epsilon_2|q_4|^2 + |q_3|^2)q_4 + \epsilon_1\epsilon_2 q_3^2 q_2^* = 0,
\end{aligned}
$$

Its soliton solution is given by

$$
\begin{aligned}
q_2 &= \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(\epsilon_2 e^{\nu(x-ut)} n_{0,1} n_{0,2}^* + \epsilon_1 e^{-\nu(x-ut)} n_{0,4} n_{0,5}^*\right), \\
q_3 &= \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,3}^* - \epsilon_1 e^{-\nu(x-ut)} n_{0,3} n_{0,5}^*\right), \\
q_4 &= \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(\epsilon_2 e^{\nu(x-ut)} n_{0,1} n_{0,4}^* + \epsilon_1 e^{-\nu(x-ut)} n_{0,2} n_{0,5}^*\right), \\
\Delta &= \epsilon_1 e^{2\nu(x-ut)}|n_{0,1}|^2 + \epsilon_2\left(|n_{0,2}|^2 + |n_{0,4}|^2\right) + |n_{0,3}|^2 + \epsilon_1 e^{-2\nu(x-ut)}|n_{0,5}|^2, \\
v &= \frac{\nu^2 - \mu^2}{\mu}, \qquad u = -2\mu, .
\end{aligned}
$$

A second reduction via a Weyl reflection S_{e_2}:

$$K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad p_2 = q_4^*, \qquad p_3 = -q_3^*, \qquad p_4 = q_2^*.$$

gives rise to another inequivalent system of 3 NLS equations

$$\begin{aligned} iq_{2,t} \;+\; q_{2,xx} + 2(q_2 q_4^* - |q_3|^2) q_2 + q_3^2 q_2^* &= 0, \\ iq_{3,t} \;+\; q_{3,xx} - 2q_2 q_4 q_3^* + (2q_2 q_4^* + 2q_4 q_2^* - |q_3|^2) q_3 &= 0, \\ iq_{4,t} \;+\; q_{4,xx} + 2(q_4 q_2^* - |q_3|^2) q_4 + q_3^2 q_4^* &= 0. \end{aligned}$$

Then we have the following one soliton solution

$$\begin{aligned} q_2 &= \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,4}^* + e^{-\nu(x-ut)} n_{0,4} n_{0,5}^* \right), \\ q_3 &= \frac{2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,3}^* + e^{-\nu(x-ut)} n_{0,3} n_{0,5}^* \right), \end{aligned}$$

$$q_4 = \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,2}^* + e^{-\nu(x-ut)} n_{0,2} n_{0,5}^* \right),$$
$$\Delta = e^{2\nu(x-ut)} |n_{0,1}|^2 + \left(n_{0,2} n_{0,4}^* + n_{0,2}^* n_{0,4} \right) - |n_{0,3}|^2 + e^{-2\nu(x-ut)} |n_{0,5}|^2.$$

Next we consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction, which is a combination of reductions with K_1 and K_2. This is possible only for $\epsilon_1 = -1$. Then

$$p_{2,4} = -\epsilon_2 q_{2,4}^*, \qquad q_2 = -\epsilon_2 q_4, \qquad p_3 = -q_3^*. \tag{1}$$

and we obtain the following system of two equations

$$iq_{2,t} + q_{2,xx} - 2(\epsilon_2 |q_2|^2 + |q_3|^2) q_2 + q_3^2 q_2^* = 0, \tag{2}$$
$$iq_{3,t} + q_{3,xx} - (4\epsilon_2 |q_2|^* + |q_3|^2) q_3 + 2\epsilon_2 (q_2)^2 q_3^* = 0. \tag{3}$$

and its one soliton solution takes the form

$$q_2 = \frac{2i\nu}{\Delta} e^{-i\mu(x-vt)} \epsilon_2 \left(e^{\nu(x-ut)} n_{0,1} n_{0,2}^* + e^{-\nu(x-ut)} n_{0,2} n_{0,5}^* \right), \tag{4}$$
$$q_3 = \frac{2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,3}^* + e^{-\nu(x-ut)} n_{0,3} n_{0,5}^* \right), \tag{5}$$
$$\Delta = e^{2\nu(x-ut)} |n_{0,1}|^2 - 2\epsilon_2 |n_{0,2}|^2 - |n_{0,3}|^2 + e^{-2\nu(x-ut)} |n_{0,5}|^2, \tag{6}$$

$$v = \frac{\nu^2 - \mu^2}{\mu}, \qquad u = -2\mu. \tag{7}$$

3 MMKdV eqs on BD.I-symmetric spaces

The M-operator for the MMKdV equations takes the form

$$\begin{aligned} M\psi(x,t,\lambda) &\equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t) - \lambda^3 J)\psi(x,t,\lambda), \\ V_2(x,t) &= q(x,t), \qquad V_1(x,t) = i\operatorname{ad}_J^{-1}\partial_x q + \frac{1}{2}\left[\operatorname{ad}_J^{-1} q, q(x,t)\right], \\ V_0(x,t) &= -\partial_{xx}^2 q + \frac{1}{2}\left[\operatorname{ad}_J^{-1} q, \left[\operatorname{ad}_J^{-1} q, q(x,t)\right]\right] + i\left[\partial_x q, q\right], \end{aligned} \tag{8}$$

It will be convenient to introduce the following notations for the n-component vectors

$$\vec{q} = (q_1, \ldots, q_n)^T, \qquad \vec{p} = (p_1, \ldots, p_n)^T,$$

and also the matrices S_0 and s_0

$$S_0 = \sum_{k=1}^{n+2} (-1)^{k+1} E_{k,2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (9)$$

The MMKdV equations can be written down in compact form as

$$\partial_t \vec{q} + \partial_{xxx}^3 \vec{q} + 3(\vec{p},\vec{q})\partial_x \vec{q} + 3\left(\partial_x \vec{q}, \vec{p}\right)\vec{q} - 3\left(\partial_x \vec{q} s_0 \vec{q}\right) s_0 \vec{p} = 0, \qquad (10)$$

$$\partial_t \vec{p} + \partial_{xxx}^3 \vec{p} + 3(\vec{p},\vec{q})\partial_x \vec{p} + 3\left(\partial_x \vec{p}, \vec{q}\right)\vec{p} - 3\left(\partial_x \vec{p} s_0 \vec{p}\right) s_0 \vec{q} = 0, \qquad (11)$$

Analogously the MNLS eqs. generalizing the vector NLS are:

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q},\vec{p})\vec{q} - (\vec{q}, s_0\vec{q}) s_0 \vec{p} = 0,$$

$$i\vec{p}_t - \vec{p}_{xx} - 2(\vec{q},\vec{p})\vec{p} + (\vec{p}, s_0\vec{p}) s_0 \vec{q} = 0,$$

Consider a $\mathbb{Z}_2$ reduction of the type

$$U^\dagger(\lambda^*) = U(\lambda), \qquad \Rightarrow \qquad \vec{p} = \vec{q}^*. \qquad (12)$$

Then we obtain the following reduced systems of MMKdV

$$\partial_t \vec{q} + \partial_{xxx}^3 \vec{q} + 3|\vec{q}|^2 \partial_x \vec{q} + 3\left(\partial_x \vec{q}, \vec{q}^*\right)\vec{q} - 3\left(\partial_x \vec{q} s_0 \vec{q}\right) s_0 \vec{q}^* = 0.$$

and MNLS

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^\dagger, \vec{q})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{q}^* = 0,$$

Applications: for $n = 2$ and $n = 3$ describe $F = 1$ and $F = 2$ BEC (Wadati (2006)).

The 1-soliton solution of the MMKdV reads

$$q_k = \frac{-i\nu e^{-i\mu(x-ut-\delta_0)}}{\cosh(2\nu(x-vt-\xi_0)) + \mathcal{C}} \left(e^{\nu(x-vt-\xi_0)} c_k^* + (-1)^k e^{-\nu(x-vt-\xi_0)} c_{n+3-k} \right),$$

$$c_k = \frac{n_{0,k}}{\sqrt{|n_{0,1}||n_{0,n+2}|}}, \qquad k = 2, \dots, 2r \qquad \mathcal{C} = \sum_{k=2}^{2r} |n_{0,k}|^2 / 2|n_{0,1}||n_{0,n+2}|,$$

$$v = \nu^2 - 3\mu^2, \qquad u = 3\nu^2 - \mu^2, \qquad \delta_0 = \frac{\arg n_{0,1}}{\mu}, \qquad \xi_0 = \frac{1}{2\nu} \ln \frac{|n_{0,n+2}|}{|n_{0,1}|}.$$

provided we have fixed $\arg n_{0,1} = -\arg n_{0,n+2}$ by using the natural $U(1)$ symmetry of the solution.

Consider $\mathbb{Z}_2$ reduction of MKdV related to $so(5)$ with

$$KU^\dagger(-\lambda^*)K^{-1} = U(\lambda), \qquad \Rightarrow \quad Kq^\dagger K^{-1} = q, \qquad KJK^{-1} = -J$$

$$W_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Rightarrow \qquad K = \begin{pmatrix} 0 & 0 & 0 & 0 & -\epsilon_1 \\ 0 & \epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 & 0 \\ -\epsilon_1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

is the Weyl reflection with respect to the hyperplane orthogonal to e_1. The following interrelations hold true

$$q_3 = -\epsilon_1\epsilon_2 q_1^*, \qquad q_2 = -\epsilon_1 q_2^*,$$
$$p_3 = -\epsilon_1\epsilon_2 p_1^*, \qquad p_2 = -\epsilon_1 p_2^*.$$

As a consequence of the reduction we have

$$\lambda^+ = -(\lambda^-)^*, \qquad |m\rangle = K|n\rangle^*, \tag{13}$$

or

$$(\lambda^\pm)^* = -\lambda^\pm, \qquad |n\rangle = SK|n\rangle^*, \qquad \langle m| = \langle m|^*(SK)^{-1}. \tag{14}$$

Applying another $\mathbb{Z}_2$ reduction of the type

$$U^T(-\lambda) = -U(\lambda), \qquad \Rightarrow \qquad q^T = -q, \tag{15}$$

we obtain that

$$\lambda^+ = -\lambda^-, \qquad |m\rangle = |n\rangle. \tag{16}$$

The corresponding system of MKdV is

$$q_{2,t} + q_{2,xxx} - 3(q_2 q_3)_x q_3 + 3\epsilon_1\epsilon_2 q_3 q_2^* q_{3,x} - 6q_2^2 q_{2,x} = 0,$$
$$q_{3,t} + q_{3,xxx} + 3\epsilon_1\epsilon_2 |q_2|_x^2 q_3 - 3(q_2 q_3)_x q_2 - 3(q_2^* q_3)_x q_2^* - 3q_3^2 q_{3,x} = 0.$$

and is new to the best of our knowledge. Again two types of solitons.
The **doublet soliton** $\rightarrow$ $\lambda^\pm = \pm i\nu$ and $|n\rangle = SK|n\rangle^*$ and is given by

$$q_2 = \frac{i\nu e^{i\delta_0}}{\epsilon_1 \cosh 2\nu(x - vt - \xi_0) + \mathcal{C}} \left(e^{\nu(x-vt-\xi_0)} c_2 + e^{-\nu(x-vt-\xi_0)} c_4 \right),$$
$$q_3 = \frac{2i\nu c_3 e^{i\delta_0} \sinh \nu(x - vt - \xi_0)}{\epsilon_1 \cosh 2\nu(x - vt - \xi_0) + \mathcal{C}}, \qquad \delta_0 = \arg n_{0,1} = \arg n_{0,5} = \frac{l\pi}{2}, \quad l \in \mathbb{Z}$$
$$\mathcal{C} = (2\epsilon_2 \mathrm{Re}\,(n_{0,2} n_{0,4}) + |n_{0,3}|^2)/2|n_{0,1}||n_{0,5}|,$$
$$c_1^* = -\epsilon_1 c_1, \quad c_2^* = -\epsilon_2 c_4, \quad c_3^* = -c_3, \quad c_5^* = -\epsilon_1 c_5, \quad c_k = \frac{n_{0,k}}{\sqrt{|n_{0,1}||n_{0,n+2}|}},$$

q_3 is either real or purely imaginary valued function.

The quadruplet soliton solution:

$$u(x,t,\lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda-\lambda_0} - \frac{KSA^*(x,t)SK}{\lambda+\lambda_0^*} - \frac{SA(x,t)S}{\lambda+\lambda_0} + \frac{KA^*(x,t)K}{\lambda-\lambda_0^*}. \tag{17}$$

$$q(x,t) = [J, A - KSA^*SK - SAS + KA^*K](x,t). \tag{18}$$

Find the matrix $A(x,t) = XF^T$ – algebraic set of equations. Here X and F are rectangular matrices of rank $s \leq r$ and $\lambda_0 = \mu + i\nu$. It can be checked that

$$F(x,t) = e^{i(\lambda_0 x + \lambda_0^3 t)J} F_0, \qquad F_0 = \text{const.}$$

In the simplest $s = 1$ case for the factor X one can obtain the following

$$X = \frac{a^* F + bSKF^* - cKF^*}{|a|^2 + b^2 - c^2},$$

where $\phi_{\rm R}^{\pm} = \phi_{\rm R} \pm \frac{1}{2}\ln\frac{|F_{0,2}|}{|F_{0,4}|}$, $\qquad \phi_{\rm I}^{\pm} = \phi_{\rm I} \pm \arg F_{0,4}$ and

$$a(x,t) = \frac{|F_{0,1}F_{0,5}|}{\lambda_0}(\cosh 2(\phi_{\rm R} - i\phi_{\rm I}) + \mathcal{C}_a), \quad \mathcal{C}_a = \frac{F_{0,2}^2 + F_{0,3}^2 + F_{0,4}^2}{2|F_{0,1}F_{0,5}|},$$

$$b(x,t) = \frac{i|F_{0,1}F_{0,5}|}{\nu}(\cosh 2\phi_{\rm R} + \mathcal{C}_b), \quad \mathcal{C}_b = \frac{2\mathrm{Re}\,(F_{0,2}^* F_{0,4}) + |F_{0,3}|^2}{2|F_{0,1}F_{0,5}|},$$

$$c(x,t) = \frac{|F_{0,1}F_{0,5}|}{\mu}(\cos 2\phi_{\rm I} + \mathcal{C}_c), \quad \mathcal{C}_c = \frac{|F_{0,2}|^2 - |F_{0,3}|^2 + |F_{0,4}|^2}{2|F_{0,1}F_{0,5}|},$$

$$\phi_{\rm R} = \nu\left(x - vt - \frac{1}{2\nu}\ln\frac{|F_{0,1}|}{|F_{0,5}|}\right), \quad \phi_{\rm I} = \mu\left(x - ut - \frac{\arg F_{0,5}}{\mu}\right),$$

and $\arg F_{0,1} = -\arg F_{0,5}$. Thus for $\epsilon_1 = \epsilon_2 = 1$ one derives

$$\begin{aligned} q_2 \;=\; & \frac{2\sqrt{|F_{0,1}F_{0,2}F_{0,4}F_{0,5}|}}{|a|^2 + b^2 - c^2}\left\{a^* \cosh(\phi_{\rm R}^- - i\phi_{\rm I}^-) - b(\cosh(\phi_{\rm R}^- + i\phi_{\rm I}^+) + \cosh(\phi_{\rm R}^+ - i\phi_{\rm I}^-)) \right. \\ - \; & \left. a\cosh(\phi_{\rm R}^+ + i\phi_{\rm I}^+) + c(\cosh(\phi_{\rm R}^+ + i\phi_{\rm I}^-) - \cosh(\phi_{\rm R}^- - i\phi_{\rm I}^+))\right\}, \\ q_3 \;=\; & \frac{2i\sqrt{|F_{0,1}F_{0,5}|}}{|a|^2 + b^2 - c^2}\mathrm{Im}\,\{(b+c)\sinh(\phi_{\rm R} + i\phi_{\rm I}) - a^*\sinh(\phi_{\rm R} - i\phi_{\rm I})\}F_{0,3}, \end{aligned}$$

4 ISM - Generalized Fourier Transform

Theorem 1 (see VSG (1996)). *The sets* $\{\boldsymbol{\Psi}\}$ *and* $\{\boldsymbol{\Phi}\}$ *form complete sets of functions in* $\mathcal{M}_J$*. The corresponding completeness relation has the form:*

$$\delta(x-y)\Pi_{0J} = \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda (G^{+}(x,y,\lambda) - G^{-}(x,y,\lambda)) - 2i\sum_{j=1}^{N}(G_j^{+}(x,y) + G_j^{-}(x,y)),$$

where

$$\Pi_{0J} = \sum_{i<r}(E_{ir}\otimes E_{ri} - E_{ri}\otimes E_{ir}),$$

$$G^{+}(x,y,\lambda) = \sum_{i<r}\boldsymbol{e}_{ir}^{+}(x,\lambda)\otimes\boldsymbol{e}_{ri}^{+}(y,\lambda), \qquad G^{-}(x,y,\lambda) = \sum_{i<r}\boldsymbol{e}_{ri}^{-}(x,\lambda)\otimes\boldsymbol{e}_{ir}^{-}(y,\lambda),$$

$$G_j^{+}(x,y) = \sum_{i<r}(\boldsymbol{e}_{ir;j}^{+}(x)\otimes\dot{\boldsymbol{e}}_{ri;j}^{+}(y) + \dot{\boldsymbol{e}}_{ir;j}^{+}(x)\otimes\boldsymbol{e}_{ri;j}^{+}(y)),$$

$$G_j^{-}(x,y) = \sum_{i<r}(\dot{\boldsymbol{e}}_{ri;j}^{-}(x)\otimes\boldsymbol{e}_{ir;j}^{-}(y) + \boldsymbol{e}_{ri;j}^{-}(x)\otimes\dot{\boldsymbol{e}}_{ir;j}^{-}(y)),$$

4.1 Expansions of $Q(x)$.

Skipping the calculational details we get the following expansion of $Q(x)$ over the systems $\{\mathbf{\Phi}^{\pm}\}$ and $\{\mathbf{\Psi}^{\pm}\}$:

$$\begin{aligned} Q(x) &= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{i<r} \left(\tau_{ir}^{+}(\lambda)\mathbf{\Phi}_{ir}^{+}(x,\lambda) - \tau_{ri}^{-}(\lambda)\mathbf{\Phi}_{ri}^{-}(x,\lambda)\right) \\ &+ 2\sum_{k=1}^{N}\sum_{i<r}\left(\tau_{ir;j}^{+}\mathbf{\Phi}_{ir;j}^{+}(x) + \tau_{ri;j}^{-}\mathbf{\Phi}_{ri;j}^{-}(x)\right), \end{aligned} \tag{19}$$

$$\begin{aligned} Q(x) &= -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{i<r} \left(\rho_{ri}^{+}(\lambda)\mathbf{\Psi}_{ri}^{+}(x,\lambda) - \rho_{ir}^{-}(\lambda)\mathbf{\Psi}_{ir}^{-}(x,\lambda)\right) \\ &- 2\sum_{k=1}^{N}\sum_{i<r}\left(\rho_{ri;j}^{+}\mathbf{\Psi}_{ri;j}^{+}(x) + \rho_{ir;j}^{-}\mathbf{\Psi}_{ir;j}^{-}(x)\right), \end{aligned} \tag{20}$$

4.2 Expansions of $\mathrm{ad}_J^{-1}\delta Q(x)$.

Next we get the following expansion of $\mathrm{ad}_J^{-1}\delta Q(x)$ over the systems $\{\mathbf{\Phi}^{\pm}\}$ and $\{\mathbf{\Psi}^{\pm}\}$:

$$\begin{aligned} \mathrm{ad}_J^{-1}\delta Q(x) &= \frac{i}{2\pi}\int_{-\infty}^{\infty} d\lambda \sum_{i<r} \left(\delta\tau_{ir}^{+}(\lambda)\mathbf{\Phi}_{ir}^{+}(x,\lambda) + \delta\tau_{ri}^{-}(\lambda)\mathbf{\Phi}_{ri}^{-}(x,\lambda)\right) \\ &+ \sum_{k=1}^{N}\sum_{i<r}\left(\delta' W_{ir;j}^{+}(x) - \delta' W_{ri;j}^{-}(x)\right), \end{aligned} \tag{21}$$

$$\begin{aligned} \mathrm{ad}_J^{-1}\delta Q(x) &= \frac{i}{2\pi}\int_{-\infty}^{\infty} d\lambda \sum_{i<r} \left(\delta\rho_{ri}^{+}(\lambda)\mathbf{\Psi}_{ri}^{+}(x,\lambda) + \delta\rho_{ir}^{-}(\lambda)\mathbf{\Psi}_{ir}^{-}(x,\lambda)\right) \\ &+ \sum_{k=1}^{N}\sum_{i<r}\left(\delta' \tilde{W}_{ir;j}^{+}(x) - \delta' \tilde{W}_{ri;j}^{-}(x)\right), \end{aligned} \tag{22}$$

where

$$\delta' W_{ab;j}^{\pm}(x) = \delta\lambda_j^{\pm}\tau_{ab;j}^{\pm}\dot{\mathbf{\Phi}}_{ab;j}^{\pm}(x) + \delta\tau_{ab;j}^{\pm}\mathbf{\Phi}_{ab;j}^{\pm}(x), \tag{23}$$

$$\delta' \tilde{W}^{\pm}_{ab;j}(x) = \delta\lambda^{\pm}_j \rho^{\pm}_{ab;j} \dot{\mathbf{\Psi}}^{\pm}_{ab;j}(x) + \delta\rho^{\pm}_{ab;j} \mathbf{\Psi}^{\pm}_{ab;j}(x) \qquad (24)$$

These expansions establish the one-to-one correspondence between $Q(x)$ and each of the minimal sets of scattering data $\mathcal{T}_1$ and $\mathcal{T}_2$ as well as one-to-one correspondence between the variation of the potential $\delta Q(x)$ and the variations of the scattering data $\delta\mathcal{T}_1$ and $\delta\mathcal{T}_2$.

4.3 The generating operators

Standard Fourier transform

$$D_0 = -id/dx, \qquad e^{i\lambda x}, \qquad D_0 e^{i\lambda x} = \lambda e^{i\lambda x}$$

Expansions over the 'squared solutions'. Introduce the generating operators $\Lambda_\pm$ through:

$$\begin{aligned} (\Lambda_+ - \lambda)\mathbf{\Psi}^{+}_{ri}(x,\lambda) &= 0, \qquad (\Lambda_+ - \lambda)\mathbf{\Psi}^{-}_{ir}(x,\lambda) = 0, \\ (\Lambda_- - \lambda)\mathbf{\Phi}^{+}_{ir}(x,\lambda) &= 0, \qquad (\Lambda_- - \lambda)\mathbf{\Phi}^{-}_{ri}(x,\lambda) = 0. \end{aligned} \qquad (25)$$

Their derivation starts by introducing the splitting:

$$e^{\pm}_{ab}(x,\lambda) = e^{\mathrm{d},\pm}_{ab}(x,\lambda) + \boldsymbol{e}^{\pm}_{ab}(x,\lambda), \qquad e^{\mathrm{d},\pm}_{ab}(x,\lambda) = (\mathbb{1} - P_{0J})e^{\pm}_{ab}(x,\lambda), \tag{26}$$

we get

$$i\frac{de^{\mathrm{d},\pm}_{ab}}{dx} + [Q(x), \boldsymbol{e}^{\pm}_{ab}(x,\lambda)] = 0, \tag{27}$$

$$i\frac{d\boldsymbol{e}^{\pm}_{ab}}{dx} + [Q(x), e^{\mathrm{d},\pm}_{ab}(x,\lambda)] = \lambda[J, \boldsymbol{e}^{\pm}_{ab}(x,\lambda)], \tag{28}$$

Eq. (27) can be integrated formally with the result

$$e^{\mathrm{d},\pm}_{ab}(x,\lambda) = C^{\mathrm{d},\pm}_{ab;\epsilon}(\lambda) + i\int_{\epsilon\infty}^{x} dy\,[Q(y), \boldsymbol{e}^{\pm}_{ab}(y,\lambda)], \tag{29}$$

$$C^{\mathrm{d},\pm}_{ab;\epsilon}(\lambda) = \lim_{y\to\epsilon\infty} e^{\mathrm{d},\pm}_{ab}(y,\lambda), \qquad \epsilon = \pm 1. \tag{30}$$

Next insert (29) into (28) and act on both sides by ad_J^{-1}. This gives us:

$$(\Lambda_{\pm} - \lambda)\boldsymbol{e}^{\pm}_{ab}(x,\lambda) = i[C^{\mathrm{d},\pm}_{ab;\epsilon}(\lambda), \mathrm{ad}_J^{-1} Q(x)], \tag{31}$$

where the generating operators $\Lambda_\pm$ are given by:

$$\Lambda_\pm X(x) \equiv \mathrm{ad}_J^{-1} \left(i\frac{dX}{dx} + i\left[Q(x), \int_{\pm\infty}^{x} dy\, [Q(y), X(y)] \right] \right). \tag{32}$$

Thus $\boldsymbol{e}_{ab}^{\pm}(x,\lambda)$ will be an eigenfunction of Λ_+ or Λ_- if only if $C_{ab}^{\mathrm{d},\pm}(y,\lambda) = 0$. Thus we find $(i<r)$:

$$\begin{aligned} (\Lambda_+ - \lambda)\boldsymbol{\Psi}_{ri}^{+}(x,\lambda) &= 0, & (\Lambda_+ - \lambda)\boldsymbol{\Psi}_{ir}^{-}(x,\lambda) &= 0, \\ (\Lambda_+ - \lambda_j^{+})\boldsymbol{\Psi}_{ri;j}^{+}(x) &= 0, & (\Lambda_+ - \lambda_j^{-})\boldsymbol{\Psi}_{ir;j}^{-}(x) &= 0, \end{aligned} \tag{33}$$

$$\begin{aligned} (\Lambda_- - \lambda)\boldsymbol{\Phi}_{ir}^{+}(x,\lambda) &= 0, & (\Lambda_- - \lambda)\boldsymbol{\Phi}_{ri}^{-}(x,\lambda) &= 0, \\ (\Lambda_- - \lambda_j^{+})\boldsymbol{\Phi}_{ir;j}^{+}(x) &= 0, & (\Lambda_- - \lambda_j^{-})\boldsymbol{\Phi}_{ri;j}^{-}(x) &= 0, \end{aligned} \tag{34}$$

$$\begin{aligned} (\Lambda_+ - \lambda_j^{+})\dot{\boldsymbol{\Psi}}_{ri;j}^{+}(x) &= \boldsymbol{\Psi}_{ri;j}^{+}(x), & (\Lambda_+ - \lambda_j^{-})\dot{\boldsymbol{\Psi}}_{ir;j}^{-}(x) &= \boldsymbol{\Psi}_{ir;j}^{-}(x), \\ (\Lambda_- - \lambda_j^{+})\dot{\boldsymbol{\Phi}}_{ir;j}^{+}(x) &= \boldsymbol{\Phi}_{ir;j}^{+}(x), & (\Lambda_- - \lambda_j^{-})\dot{\boldsymbol{\Phi}}_{ri;j}^{-}(x) &= \boldsymbol{\Phi}_{ri;j}^{-}(x), \end{aligned} \tag{35}$$

i.e., $\dot{\boldsymbol{\Psi}}_{ri;j}^{+}(x)$ and $\dot{\boldsymbol{\Phi}}_{ir;j}^{+}(x)$ are **adjoint eigenfunctions** of Λ_+ and Λ_-. Therefore $\lambda_j^{\pm}$, $j=1,\dots,N$ are discrete eigenvalues also of $\Lambda_\pm$ but the corresponding eigenspaces of $\Lambda_\pm$ have dimension 2 since they are spanned by both $\boldsymbol{\Psi}^{\pm}_{ab;j}(x)$ and $\dot{\boldsymbol{\Psi}}^{\pm}_{ab;j}(x)$. Thus the sets $\{\Psi\}$ and $\{\Phi\}$ are the complete sets of eigen- and adjoint functions of Λ_+ and Λ_-.

5 MNLS with Constant Boundary Conditions

Require: i) regular behaviour of the solutions for $t \to \pm\infty$;
ii) require that the spectrum of the two asymptotic operators $L_\pm = id/dx + U_\pm(\lambda)$ have the same spectrum. Here

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \qquad U_\pm(\lambda) \equiv \lim_{x\to\pm\infty} U(x,t,\lambda) = Q_\pm - \lambda J. \quad (36)$$

The first requirement can be satisfied by regularizing the MNLS, i.e. by conveniently adding linear in $\mathbf{q}$ terms. The corresponding regularized MNLS have the form:

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2\mathbf{q}\mathbf{q}^\dagger\mathbf{q} + \mathbf{q}\mu + \bar{\mu}\mathbf{q} = 0, \qquad (37)$$

$$\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{q}_\pm, \qquad \mu = \mathbf{q}_+^\dagger\mathbf{q}_+ = \mathbf{q}_-^\dagger\mathbf{q}_-, \qquad \overline{\mu} = \mathbf{q}_+\mathbf{q}_+^\dagger = \mathbf{q}_-\mathbf{q}_-^\dagger.$$

ii) $\quad Q_+^2 = Q_-^2.$

then $U_+(\lambda)$ and $U_-(\lambda)$ have the same sets of eigenvalues.

The M-operators of the MNLS with CBC contains additional terms

$$V_0(x,t) = -[Q, \mathrm{ad}_J^{-1} Q] + 2i\mathrm{ad}_J^{-1} Q_x(x,t) + [Q_\pm, \mathrm{ad}_J^{-1} Q_\pm]. \qquad (38)$$

with $Q_\pm$ which ensure the regular behavior of the solutions for large t.

The Lax operator can be associated with a symmetric spaces if

A.II $\mathfrak{g} \simeq A_{N-1} \equiv sl(N)$, $J = H_{\vec{a}}$, where the vector $\vec{a}$ in the root space $\mathbb{E}^r$ dual to J is given by $\vec{a} = \sum_{k=1}^{s} e_k - \sum_{k=s+1}^{N} e_k$;

In the next two cases $s = r$ and $N = 2r$ is even.

C.II $\mathfrak{g} \simeq C_r \equiv sp(2r)$, $J = H_{\vec{a}}$, where the vector $\vec{a}$ in the root space $\mathbb{E}^r$ dual to J is given by $\vec{a} = \sum_{k=1}^{r} e_k$;

D.III $\mathfrak{g} \simeq D_r \equiv so(2r)$, $J = H_{\vec{a}}$, where the vector $\vec{a}$ in the root space $\mathbb{E}^r$ dual to J is given by $\vec{a} = \sum_{k=1}^{r} e_k$.

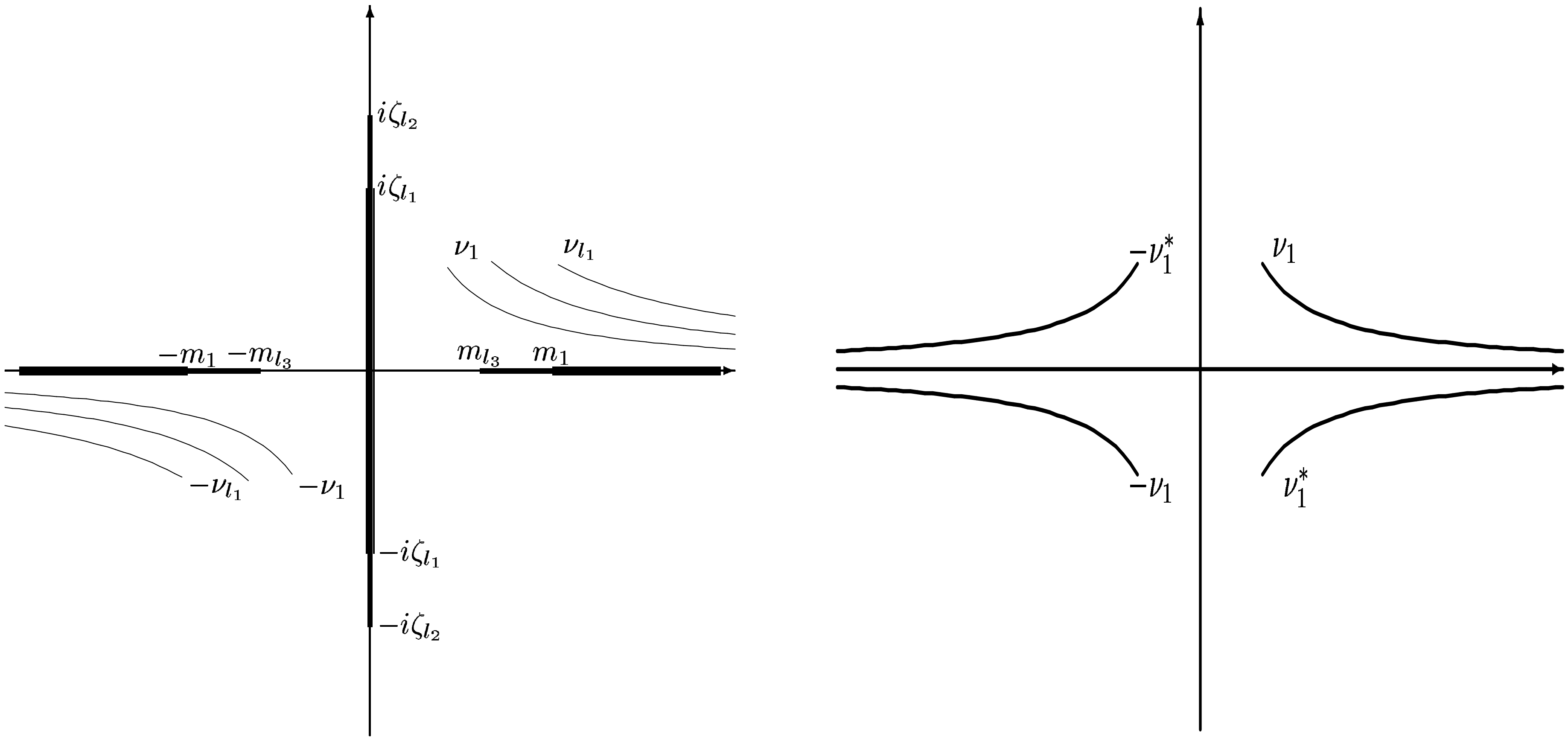
BD.I $\mathfrak{g} \simeq D_r \equiv so(2r)$ for $N = 2r$ and $\mathfrak{g} \simeq B_r \equiv so(2r+1)$ for $N = 2r+1$, $J = H_{e_1}$.

The spectrum of the asymptotic operators $L_\pm$ is purely continuous and is determined by the the eigenvalues of $Q_\pm$ which generically may be arbitrary complex numbers. The spectra of A-type symmetric spaces were described by VSG, Kulish (1983).

a) $\nu_k \neq \pm\nu_k^*$, $k = 1, \dots, l_1$ – two branches of two-fold spectrum filling up the hyperbola's arcs $\mathrm{Re}\,\lambda\,\mathrm{Im}\,\lambda = \mathrm{Re}\,\nu_k \mathrm{Im}\,\nu_k$ on which $|\mathrm{Re}\,\lambda| \geq |\mathrm{Re}\,\nu_k|$;

b) $\nu_{l_1+k} = -\nu_{l_1+k}^* = i\zeta_k$, $k = 1, \dots, l_2$ – two branches of two-fold spectrum filling up the real axis and the segment $|\mathrm{Im}\,\lambda| \leq |\zeta_k|$ of the imaginary axis;

c) $\nu_{l_1+l_2+k} = \nu_{l_1+l_2+k}^* = m_k$, $k = 1, \dots, l_3 = r - l_1 - l_2 + 1$ – two branches of two-fold spectrum filling up the segments $|\mathrm{Re}\,\lambda| \geq |m_k|$ of the real axis;



Фигура 1: Left panel: the continuous spectrum of L, generic case; Right panel: the continuous spectrum of the $sp(4)$ and $so(8)$ MNLS with CBC for $D < 0$; the only difference is that while the multiplicity of the spectra of $sp(4)$ is 2 the one for $so(8)$ is 4.

For $C.II$- and $D.III$-type symmetric spaces the spectra consist of four branches filling up the hyperbola's arcs $\mathrm{Re}\,\lambda\,\mathrm{Im}\,\lambda = \pm\mathrm{Re}\,\nu_1\mathrm{Im}\,\nu_1$ on which $|\mathrm{Re}\,\lambda| \geq |\mathrm{Re}\,\nu_1|$, see the right panel of the figure - VSG 2004.

5.1 Spectral properties of $sp(4)$–MNLS with CBC

As mentioned in Section 3, the continuous spectrum of the GZS system is determined by the set of eigenvalues $\{\nu_j, j = 1, 2\}$ of the matrices $q_+r_+ = q_-r_-$. These eigenvalues for $Q_\pm$ with $r = 2$ satisfy the characteristic equation:

$$\nu^2 - K_0\nu + K_1 = 0, \qquad K_0 = \frac{1}{2}\mathrm{tr}\, Q_\pm^2, \qquad K_1 = \det Q_\pm. \tag{39}$$

and determine the end points of the spectrum. If we impose on $Q(x,t)$, and consequently on $Q_\pm$ the involution ($\mathbb{Z}_2$-reduction):

$$B_1^{-1}Q^\dagger B_1 = Q, \qquad B_1 = \mathrm{diag}\,(1, \epsilon, \epsilon, 1), \qquad \epsilon = \pm 1. \tag{40}$$

which in components takes the form:

$$r_1 = \epsilon q_1^*, \qquad r_2 = q_2^*, \qquad r_3 = q_3^*. \tag{41}$$

Then the coefficients K_0 and K_1 equal:

$$K_0 = 2\epsilon|q_1^\pm|^2 + |q_2^\pm|^2 + |q_3^\pm|^2, \qquad K_1 = |(q_1^\pm)^2 + q_2^\pm q_3^\pm|^2 \tag{42}$$

We have three possibilities for the roots ν_1, ν_2 of eq. (39) depending on the sign of the discriminant:

$$D = \frac{1}{4}K_0^2 - 4K_1. \tag{43}$$

a) $D > 0$**,** i.e. the roots $\nu_1 > \nu_2$ are different and real. The continuous spectrum of L fills up two pairs of rays on the real axis $|\mathrm{Re}\,\lambda| > \nu_1$ and $|\mathrm{Re}\,\lambda| > \mathrm{Re}\,\nu_2$;

b) $D = 0$**,** i.e. the roots $\nu_1 = \nu_2$; the two pairs of rays in a) now coincide; the total multiplicity of the spectrum is 4;

c) $D < 0$**,** i.e. the roots ν_j are complex-valued and $\nu_1 = \nu_2^*$; The continuous spectrum of L fils up two branches of two-fold spectrum along the hyperbola's arcs $\mathrm{Re}\,\lambda\,\mathrm{Im}\,\lambda = \mathrm{Re}\,\nu_k \mathrm{Im}\,\nu_k$, see the right panel of fig. 1;

In the generic case there are no apriory limitations as to the positions of the discrete eigenvalues. Such may come up if we consider potentials $Q = -Q^{\dagger}$; then the GZS system become equivalent to a formally self-adjoint linear problem whose spectrum should be confined to the real λ-axis only. The formal self-adjointness takes place for $\epsilon = 1$.

5.2 Spectral properties of $so(8)$-MNLS with CBC

The characteristic equation for $q_{\pm}r_{\pm}$ takes more simple form:

$$\det(q_{\pm}r_{\pm} - \nu) = (\nu^2 - K_0\nu + K_1)^2, \qquad (44)$$

where the coefficients K_j now are given by:

$$K_0 = \frac{1}{2}\mathrm{tr}\,(q_{\pm}r_{\pm}) = \sum_{1\leq i<j\leq 4} q_{ij}^{\pm}r_{ij}^{\pm}, \qquad (45)$$

$$K_1 = (\det(q_{\pm}r_{\pm}))^{1/2} = (q_{13}^{\pm}q_{24}^{\pm} - q_{34}^{\pm}q_{12}^{\pm} - q_{23}^{\pm}q_{14}^{\pm})(r_{13}^{\pm}r_{24}^{\pm} - r_{34}^{\pm}r_{12}^{\pm} - r_{23}^{\pm}r_{14}^{\pm}).$$

An involution of the type (40) gives $r_{ij} = \epsilon_i\epsilon_j q_{ij}^{*}$ with $\epsilon_j = \pm 1$ and makes the coefficients K_0, K_1 real. Besides now each of the eigenvalues

ν_j, $j = 1, 2$ is two-fold. Again we have the three possibilities depending on the value of D; the only difference is that the multiplicity of each of the branches is 4. This imposes certain symmetry on the locations of the eigenvalues of ν_j which in fact determine the end-points of the continuous spectra of L.

5.3 Spectral properties of BD.I-MNLS with CBC

The Jost solutions are determined by their asymptotics for $x \to \pm\infty$ as follows:

$$\psi(x,\lambda) \longrightarrow u_{0,+} e^{i\mu(\lambda)x} \hat{u}_{0,+}, \quad \text{for } x \to \infty;$$

$$\phi(x,\lambda) \longrightarrow u_{0,-} e^{i\mu(\lambda)x} \hat{u}_{0,-}, \quad \text{for } x \to -\infty;$$

$$Q_\pm - \lambda J = u_{0,\pm} \mu(\lambda) \hat{u}_{0,\pm}, \qquad \mu(\lambda) = \operatorname{diag}(\mu_1(\lambda), \dots, \mu_n(\lambda)),$$

$$\mu_{1,n}^2(\lambda) = \frac{\lambda^2 + 2a + \sqrt{\lambda^4 + 4a\lambda^2 + b}}{2}, \quad \mu_{2,n-1}^2(\lambda) = \frac{\lambda^2 + 2a - \sqrt{\lambda^4 + 4a\lambda^2 + b}}{2},$$

$$\mu_{3,4,\dots,n-2} = 0, \quad a = (\vec{r}_\pm, \vec{q}_\pm), \qquad b = 4(\vec{r}_\pm, s_0\vec{r}_\pm)(\vec{q}_\pm, s_0\vec{q}_\pm).$$

The continuous spectrum of L is determined by $\operatorname{Re} \mu_k(\lambda) = 0$. If $b = 4a^2$ this simplifies

$$\mu_{1,n}^2 = \lambda^2 + 2a, \qquad \mu_{2,3,\dots,n-1} = 0.$$

With the reduction $\vec{r} = -\vec{q}^*$ we get that $a = -m_0^2/2 < 0$ and the spectrum fills in the two semiaxis $|\lambda| > m_0$.

6 Hamiltonian properties

The invariants of the transfer matrix $T(\lambda)$ such as, e.g. its principal minors generate integrals of motion, i.e. if all ν_j are different we have only r independent series of conserved quantities. Let us briefly outline the methods of deriving of these integrals as functionals of the potential Q. As starting relation here we consider the Wronskian relation VSG, **nlin.SI/0604005**:

$$\operatorname{tr} \left[i(\chi^+)^{-1} \frac{d\chi^+}{d\lambda} C - \frac{dJ(\lambda)}{d\lambda} xC \right] \Bigg|_{x=-\infty}^{\infty}$$

$$
\begin{aligned}
&= i\sum_{k=1}^{r} \frac{d\delta_k^+}{d\lambda} \operatorname{tr}\left(H_k C\right) + i\operatorname{tr}\left(\hat{\psi}_0(\lambda)\frac{d\psi_0}{d\lambda}C - \hat{\phi}_0(\lambda)\frac{d\phi_0}{d\lambda}C\right) \qquad (46)\\
&= \int_{-\infty}^{\infty} dx \operatorname{tr}\left[\sigma_3 R_C(x,t,\lambda) - \lambda J^{-1}(\lambda)C\right],
\end{aligned}
$$

where C is a constant element of $\mathfrak{h}$ and $R_C(x,t,\lambda) = \chi^+ C(\chi^+)^{-1}(x,t,\lambda)$ is a natural generalization of the diagonal of the resolvent of L. It satisfies the equation:

$$
i\frac{dR_C}{dx} + [Q(x,t) - \lambda J, R_C] = 0, \qquad \lim_{x\to\infty} R_C(x,t,\lambda) = \psi_0(\lambda)C\psi_0^{-1}(\lambda). \tag{47}
$$

Eq. (47) allows one to derive the recurrent relations for evaluating the expansion coefficients

$$
R_C(x,t,\lambda) = C_0 + \sum_{k=1}^{\infty} R_k \lambda^{-k}(x,t), \qquad \psi_0 C\psi_0^{-1}(\lambda) = C_0 + \sum_{k=1}^{\infty} C_k \lambda^{-k}. \tag{48}
$$

The trace identities for the MNLS type equations with CBC can be derived by inserting the asymptotic expansions of $R_C(x,\lambda)$ and $\delta_k^+(\lambda)$:

$$\delta_k^+(\lambda) = \sum_{p=1}^{\infty} I_p^{(k)} \lambda^{-p} \tag{49}$$

in both sides of (46) and equating the corresponding coefficients of λ^{-p}.

Here we write down the first three of the local integrals of motion coming from the principal series with $C = J$:

$$\begin{aligned} \boldsymbol{H}_k &= \int_{-\infty}^{\infty} dx \, \mathrm{tr}\, \left(\sigma_3 R_{k+1}(x,t) - C_{k+1}\right), \qquad \boldsymbol{H}_1 = \frac{1}{2} \int_{-\infty}^{\infty} dx \, \mathrm{tr}\, (q q^\dagger(x,t) - \bar{\mu}), \\ \boldsymbol{H}_2 &= \frac{i}{4} \int_{-\infty}^{\infty} dx \, \mathrm{tr}\, (q_x q^\dagger - q q_x^\dagger), \qquad \bar{\mu} = q_+ q_+^\dagger, \\ \boldsymbol{H}_3 &= \frac{3}{8} \int_{-\infty}^{\infty} dx \, \mathrm{tr}\, \left[q_x q_x^\dagger + (q q^\dagger(x,t))^2 - \bar{\mu}^2)\right]. \end{aligned} \tag{50}$$

The correct use of the Wronskian relation (46) allowed us to derive renormalized integrals of motion, i.e. ones that converge for $Q(x,t) \in \mathcal{M}$.

However among the integrals in this series one can not find the Hamiltonian of the MNLS (37). In order to obtain the Hamiltonian we need to regularize these integrals. By regularized integral we mean one whose gradient $\delta \boldsymbol{H}_k / \delta Q^T(x,t)$ vanishes for both $x \to \infty$ and $x \to -\infty$. This can be done by considering additional series of integrals, which generically have non-local densities. Fortunately among the simplests of them one may find local ones. For example, the first integral from the series with C choosen to be $C^{(l)} = \sum_{k=1}^{r} m_k^{2l} H_k$ with $1 \leq l \leq r$, is local and has the form:

$$\tilde{\boldsymbol{H}}_1^{(l)} = \frac{1}{4} \int_{-\infty}^{\infty} dx \, \mathrm{tr} \, \left[q q^\dagger(x,t) \bar{\mu}^l + q^\dagger q(x,t) \mu^l - 2\mu^{l+1} \right], \qquad \mu = q_+^\dagger q_+. \tag{51}$$

Note, that $\tilde{\boldsymbol{H}}_1^{(l)}$ is nontrivial, i.e. does not reduce to $\boldsymbol{H}_1$ only if $s \geq 2$, $\nu_1 \neq \nu_2 \neq \ldots$. Using it we can check the validity of

$$\boldsymbol{H}_{\mathrm{MNLS}} = \frac{8}{3} \boldsymbol{H}_3 - 4 \tilde{\boldsymbol{H}}_1^{(1)} = \int_{-\infty}^{\infty} dx \, \mathrm{tr} \, \left[q_x q_x^\dagger + (q q^\dagger(x,t) - \bar{\mu})^2 \right]. \tag{52}$$

In analyzing the Hamiltonian properties of the MNLS with CBC we will make use also of the classical r-matrix approach. It allows one to write down in compact form the Poisson brackets of the transfer (monodromy) matrix. Since our problem is ultra-local in the terminology of Faddeev then the definition of r is independent on the boundary conditions. Using Fordy, Kulish (1983) we find

$$r(\lambda,\mu) = \frac{1}{\lambda-\mu}\left(\sum_{\alpha\in\Delta_0^+\cup\Delta_1^+} (E_\alpha \otimes E_{-\alpha} + E_{-\alpha}\otimes E_\alpha) + \sum_{j=1}^{r} H_j \otimes H_j \right), \tag{53}$$

where E_α and H_j are the Cartan-Weyl generators of the corresponding simple Lie algebra $\mathfrak{g}$. In order to derive the Poisson brackets for the MNLS on the whole axis with CBC we need to take into account the corresponding oscillations coming from the Jost solutions.

Skipping the details we just write down the expressions for the Poisson brackets between the matrix elements of $T(\lambda)$:

$$\left\{ T(\lambda) \underset{,}{\otimes} T(\mu) \right\} = r_+(\lambda,\mu) T(\lambda)\otimes T(\mu) - T(\lambda)\otimes T(\mu) r_-(\lambda,\mu), \tag{54}$$

$$r_\pm(\lambda,\mu) = \lim_{x\to\pm\infty} \tau_\pm^{-1}(x,\lambda,\mu) r(\lambda,\mu) \tau_\pm(x,\lambda,\mu),$$

$$\tau_+(x,\lambda,\mu) = \psi_0(\lambda)e^{-iJ(\lambda)x} \otimes \psi_0(\mu)e^{-iJ(\mu)x},$$
$$\tau_-(x,\lambda,\mu) = \phi_0(\lambda)e^{-iJ(\lambda)x} \otimes \phi_0(\mu)e^{-iJ(\mu)x},$$

where $\{T(\lambda) \underset{,}{\otimes} T(\mu)\}_{ij,kl} \equiv \{T_{ij}(\lambda), T_{kl}(\mu)\}$.

An important and difficult problem here is to take correctly into account the the threshold singularities of $T_{kl}(\lambda)$ of the form $j_k^{-1}(\lambda)$ at the end points of the continuous spectrum.

As a consequence of (54) we get the involution properties of $\delta_k^+(\lambda)$:

$$\{\delta_k^+(\lambda), \delta_j^+(\mu)\} = 0, \qquad (55)$$

for all values of $1 \leq i,j \leq r$ and λ and μ taking values on the continuos spectrum of L. From (55) there follows that $\{I_p^{(k)}, I_s^{(l)}\} = 0$ for all positive values of p and s, and for all $1 \leq k,l \leq r$. A consequence of eq. (55) is the involutivity of the integrals of the principal series $\{\boldsymbol{H}_k, \boldsymbol{H}_p\} = 0$. This is a necessary condition in proving the complete integrability of the MNLS equations with CBC.

Of course the rigorous proof of the complete integrability and the derivation of the basic properties of the MNLS equations must be based on the completeness relation of the relevant 'squared solutions' of L. For the single component NLS such relation has been proposed by Konotop, (1994); for the multicomponent systems this is still open question.

7 Discussion

Possible applications: recently it was discovered by Wadati, that the $so(5)$ and $so(7)$ MNLS with vanishing BC have important applications to BEC with spin $F = 1$ and $F = 2$ respectively. This enhances the interest to the MNLS type models.

Soliton solutions: In particular it will be important to work out the dressing Zakharov-Shabat procedure also for non-vanishing BC. This problem have attracted attention recently and will be discussed in the reports of B. Prinari and T. Tsuchida who have worked out the technique for some special choices of the boundary constants $Q_{\pm}$. In the generic case however, the Riemannian surface related to the Lax operator does not allow uniformization which makes the problem difficult.

Some steps in this directions have been reported in VSG, Kulish

(1983) for the $su(n+m)/s(u(n)\otimes su(m))$ case. Deriving the dressing factors for the MNLS for the symmetric spaces of types $C.II$- and $D.III$- requires substantial changes even for vanishing BC; doing the same for CBC is still a bigger chalenge.

Expansions over 'squared solutions' - to be done.

Thank you	Grazie
for your	per
attention!	attenzione!