ON THE NONLINEAR SCHRÖDINGER EQUATIONS ON SYMMETRIC SPACES AND THEIR GAUGE EQUIVALENT

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Plan of the talk:

1. Introduction
2. FAS and scattering data for the MNLS systems
3. FAS and scattering data for the gauge-equivalent MHF systems
4. Hierarchy of Hamiltonian structures for MNLS type models
5. Dressing Method and Soliton Solutions
6. Example ($\mathfrak{g} \simeq so(5, \mathbb{C})$)
7. Open problems

1. Introduction

- Invariance of the Lax representation w.r. to the group of gauge transformations

$$[L(\lambda), M(\lambda)] = 0 \quad \rightarrow \quad [\tilde{L}(\lambda), \tilde{M}(\lambda)] = 0$$
$$\tilde{L}(\lambda) = g^{-1}L(\lambda)g, \quad \tilde{M}(\lambda) = g^{-1}M(\lambda)g$$

- **Example:** NLS equation and HF equation ($\mathfrak{g} \simeq sl(2)$)

$$iu_t + u_{xx} + 2|u^2|u(x,t) = 0 \quad \textbf{(NLS)}$$

$$iS_t^{(0)} = \frac{1}{2}[S^{(0)}(x,t), S_{xx}^{(0)}] \quad S^{(0)}(x,t) = g^{(0)-1}\sigma_3 g^{(0)}(x,t); \quad (S^{(0)})^2 = \mathbb{1} \quad \textbf{(HF)}$$

[Zakharov,Takhtajan;1979], [Lakshmanan;1977]

- $g^{(0)}$ is determined by $u(x,t)$ through

$$i\frac{dg^{(0)}}{dx} + q^{(0)}(x,t)g^{(0)}(x,t) = 0, \quad q^{(0)} = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \qquad \lim_{x\to\infty} g^{(0)}(x,t) = \mathbb{1}.$$

- Both equations are infinite dimensional completely integrable Hamiltonian systems.

- Generalized Zakharov-Shabat system related to arbitrary simple Lie algebra $\mathfrak{g}$ (of rank $r > 1$):

$$L(\lambda)\psi \equiv \left(i\frac{d}{dx} + q(x,t) - \lambda J \right) \psi(x,t,\lambda) = 0,$$

where $q(x,t), J \in \mathfrak{g}$.

- Fixing the gauge 1: $J \in \mathfrak{h}$- (real) constant, **non**-regular

  $L(\lambda) \to g_0^{-1} L(\lambda) g_0, \quad g_0(x,\lambda) \in \mathfrak{G}$

  $\to \quad \exists \Delta_0 \ni \alpha \, : \, \alpha(J) = 0$: $q(x,t) \in \mathfrak{g} \backslash \mathfrak{g}_0$, $\Delta_0 \subset \Delta$, so

$$q(x,t) = \sum_{\alpha \in \Delta^+ \backslash \Delta_0} (q_\alpha(x,t) E_\alpha + q_{-\alpha}(x,t) E_{-\alpha})$$

  $E_{\pm\alpha}$ - root vectors of $\mathfrak{g}$, Δ_+ - positive roots: $\Delta = \Delta_+ \cup (-\Delta_+)$.

  H_i - Cartan generators, $\{H_i, E_{\pm\alpha}\}$ - Cartan-Weyl basis for $\mathfrak{g}$.

- This choice of J makes more difficult:

  1) the derivation of FAS;

  2) the construction of the related recursion operator;

  3) the application of the gauge transformation.

- MNLS type equations on $\mathfrak{g}$:

$$i\frac{dq}{dt} + 2\mathrm{ad}\,_J^{-1}\frac{d^2q}{dx^2} + [q, \pi_0[q, \mathrm{ad}\,_J^{-1}q]] - 2i(\mathbb{1} - \pi_0)[q, \mathrm{ad}\,_J^{-1}q_x] = 0,$$

$L(\lambda)$ and $M(\lambda)$ - Lax pair for MNLS:

$$M(\lambda) \equiv i\frac{d}{dt} - V_0^{\mathrm{d}} + 2i\mathrm{ad}_J^{-1}q_x + 2\lambda q - 2\lambda^2 J.$$

where $V_0^d = \pi_0\left([q, \mathrm{ad}\,_J^{-1}q_x]\right)$ and π_0 is the projector onto $\mathfrak{g}_J = \{X \in \mathfrak{g}\,|\,[J, X] = 0,\ \forall J \in \mathfrak{h}\}$.

- Fixing the gauge 2 (pole gauge): [Zakharov, Mikhailov; 1978-80]

$$\tilde{L}\tilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dx} - \lambda\mathcal{S}(x,t)\right)\tilde{\psi}(x,t,\lambda) = 0,$$

where $\tilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda)$,

$$\mathcal{S}(x,t) = \mathrm{Ad}_g \cdot J \equiv g^{-1}(x,t) J g(x,t).$$

and $g(x,t) = \psi(x,t,0)$- the Jost sol's at $\lambda = 0$.

$$\tilde{M}\tilde{\psi} \equiv \left(i\frac{d}{dt} - 2i\lambda \mathrm{ad}_{\mathcal{S}}^{-1}\mathcal{S}_x - 2\lambda^2 \mathcal{S} \right) \tilde{\psi}(x,t,\lambda) = 0,$$

- $[\tilde{L}(\lambda), \tilde{M}(\lambda)] = 0 \quad \rightarrow$

$$i\frac{d\mathcal{S}}{dt} + 2\frac{d}{dx}\left(\mathrm{ad}_{\mathcal{S}}^{-1} \frac{d\mathcal{S}}{dx} \right) = 0.$$

- $L(\lambda)$ and $\tilde{L}(\lambda)$ have equivalent spectral properties and spectral data

  $\rightarrow$ the classes of NLEE related to $L(\lambda)$ and $\tilde{L}(\lambda)$ are also equivalent.

- **the "squared" solutions:**

$$e_\alpha^{\pm}(x,t,\lambda) = (\mathbb{1} - \pi_0)\left(\chi^{\pm}(x,t,\lambda) E_\alpha \hat{\chi}^{\pm}(x,t,\lambda)\right),$$

where $\chi^{\pm}(x,t,\lambda)$ is the FAS of the Lax operator L (see below)

- their completeness relations [Gerdjikov, Kilish; 1981-6] provide us the spectral decompositions of the so-called **generating (or recursion) operators** $\Lambda_\pm$:

$$\Lambda_+ e^{\pm}_{\pm\alpha} = \lambda e^{\pm}_{\pm\alpha}, \qquad \Lambda_- e^{\pm}_{\mp\alpha} = \lambda e^{\pm}_{\mp\alpha}.$$

$\Lambda_\pm$ play crucial role in deriving the properties of the NLEE.

*) AKNS approach;

**) Gelfand–Dickey approach.

- The interpretation of the ISM as a generalized Fourier transform and the expansions over the "squared solutions" allows one to study all the fundamental properties of the relevant NLEE's. These include:

  1) the description of the whole class NLEE related to the Lax operator $L(\lambda)$ solvable by the ISM;

  2) derivation of the infinite family of integrals of motion

  3) the Hamiltonian formulation of the NLEE's.

2. FAS and scattering data for the MNLS systems

- The direct scattering problem is based on the Jost solutions:

$$\lim_{x\to\infty} \psi(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,\lambda)e^{i\lambda Jx} = \mathbb{1},$$

and the scattering matrix:

$$T_J(\lambda) = (\psi(x,\lambda))^{-1}\phi(x,\lambda).$$

The FAS $\xi^{\pm}(x,\lambda)$ of $L(\lambda)$ are analytic functions of λ for $\lambda \gtrless 0$ and are related to the Jost solutions by

$$\xi^{\pm}(x,\lambda) = \phi(x,\lambda)S_J^{\pm}(\lambda) = \psi(x,\lambda)T_J^{\mp}(\lambda)D_J^{\pm}(\lambda),$$

where $S_J^{\pm}(\lambda)$, $T_J^{\pm}(\lambda)$ and $D_J^{\pm}(\lambda)$ are the factor of the **generalized Gauss decomposition** for $T_J(\lambda)$:

$$T_J(\lambda) = T_J^-(\lambda) D_J^+(\lambda) \hat{S}_J^+(\lambda) = T_J^+(\lambda) D_J^-(\lambda) \hat{S}_J^-(\lambda).$$

where

$$S_J^{\pm}(t,\lambda) = \exp\left(\sum_{\alpha\in\Delta_1^+} s_{J,\alpha}^{\pm}(t,\lambda) E_{\pm\alpha} \right), \quad T_J^{\pm}(t,\lambda) = \exp\left(\sum_{\alpha\in\Delta_1^+} t_{J,\alpha}^{\pm}(t,\lambda) E_{\pm\alpha} \right),$$

$$D_J^{\pm}(\lambda) = \exp(\pm d_1^{\pm}(\lambda) H_1 \pm 2 d_2^{\pm}(\lambda) H_2 + d_{\alpha_1}^{\pm}(\lambda) E_{\alpha_1} + d_{-\alpha_1}^{\pm}(\lambda) E_{-\alpha_1})$$

On the real axis $\xi^+(x,\lambda)$ and $\xi^-(x,\lambda)$ are related by

$$\xi^+(x,\lambda) = \xi^-(x,\lambda) G_{0,J}(\lambda),$$

$$G_{0,J}(\lambda) = \hat{S}_J^-(\lambda) S_J^+(\lambda),$$

and the function $G_{0,J}(\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues [Shabat;1974] [Gerdjikov;1994].

- If $q(x,t)$ evolves according to the MNLS then

$$i\frac{dS_J^\pm}{dt} - 2\lambda^2[J, S_J^\pm(t,\lambda)] = 0, \quad i\frac{dT_J^\pm}{dt} - 2\lambda^2[J, T_J^\pm(t,\lambda)] = 0,$$

while $D_J^\pm(\lambda)$ are time-independent.

$\rightarrow$ the MNLS eq. has four series of integrals of motion.

*)This is due to the special (degenerate) choice of the dispersion law

$$f_{\rm MNLS} = 2\lambda^2 J.$$

**) only two of these four series are in involution, which in turn is related to the non-commutativity of the subalgebra $\mathfrak{g}_J$.

3. FAS and scattering data for the gauge-equivalent MHF systems

- FAS for the gauge equiv. systems:

$$\tilde{\xi}^{\pm}(x,\lambda) = g^{-1}(x,t)\xi^{\pm}(x,\lambda)g_{-}, \quad g_{-} \in \mathfrak{G}_J$$

where $g_{-} = \lim_{x\to-\infty} g(x,t) = \hat{T}_J(0) \in \mathfrak{G}_J$.

$\tilde{\xi}^{\pm}(x,\lambda)$ are analytic w. r. to $\lambda \leftarrow$ the scattering matrix $T_J(0) \in \mathfrak{H}$.

Asymptotics of the FAS for $x \to \pm\infty$:

$$\lim_{x\to-\infty} \tilde{\xi}^{+}(x,\lambda) = T_J(0)S^{+}(\lambda)\hat{T}_J(0)$$

$$\lim_{x\to\infty} \tilde{\xi}^{+}(x,\lambda) = e^{-i\lambda J x} T_J^{-}(\lambda) D_J^{+}(\lambda) \hat{T}_J(0)$$

$$\therefore \quad \tilde{T}_J(\lambda) = T_J(\lambda)\hat{T}_J(0).$$

Obviously $\tilde{T}_J(0) = 1\!\!1$ and

$$\tilde{S}_J^{\pm}(\lambda) = T_J(0) S_J^{\pm}(\lambda)\hat{T}_J(0),$$

$$\tilde{T}_J^{\pm}(\lambda) = T_J^{\pm}(\lambda) \quad \tilde{D}_J^{\pm}(\lambda) = D_J^{\pm}(\lambda)\hat{T}_J(0).$$

On the real axis $\tilde{\xi}^{+}(x,\lambda)$ and $\tilde{\xi}^{-}(x,\lambda)$ are related by:

$$\tilde{\xi}^{+}(x,\lambda) = \tilde{\xi}^{-}(x,\lambda)\tilde{G}_{0,J}(\lambda),$$

$$\tilde{G}_{0,J}(\lambda) = \hat{\tilde{S}}_J^{-}(\lambda)\tilde{S}_J^{+}(\lambda) \quad \tilde{\xi}(x,0) = 1\!\!1.$$

again $\tilde{G}_{0,J}(\lambda)$ can be considered as a minimal set of scattering data.

4. Hierarchy of Hamiltonian structures for MNLS type models

Both classes of NLEE possess hierarchies of Hamiltonian structures.

- The phase space $\mathcal{M}_{\mathrm{MNLS}}$ of the MNLS type models is the linear space of off-diagonal matrices $q(x,t)$tending to zero fast enough for $|x| \to \infty$

$$\mathcal{M}_J \equiv \{q(x,t), \quad \pi_0 q(x,t) = 0\},$$

and the hierarchy of symplectic structures is given by:

$$\Omega_{\mathrm{q}}^{(k)} = i \int_{-\infty}^{\infty} dx \left\langle \delta q \mathop{\wedge}_{,} \Lambda^k \mathrm{ad}_J^{-1} \delta q(x,t) \right\rangle .$$

- The phase space $\mathcal{M}_{\mathcal{S}}$ of their gauge equivalent equations is the nonlinear manifold of all $\mathcal{S}(x,t)$ satisfying equations of the nonlinear constrains and such that $S(x,t)-J$ are smooth functions tending to zero fast enough for $|x| \to \infty$:

$$\widetilde{\mathcal{M}}_{\mathcal{S}} \equiv \{S(x,t), \quad S(x,t) = g^{-1}Jg(x,t)\}.$$

The family of compatible 2-forms is:

$$\tilde{\Omega}_{\mathcal{S}}^{(k)} = i \int_{-\infty}^{\infty} dx \mathrm{tr} \left(\delta\mathcal{S} \wedge \tilde{\Lambda}^k [\mathcal{S}, \delta\mathcal{S}(x,t)] \right).$$

Here Λ and $\tilde{\Lambda}$ are the recursion operator of the MNLS type equations and its gauge equivalent: $\tilde{\Lambda} = g^{-1}\Lambda g(x,t)$.

5. Dressing Method and Soliton Solutions

- Main goal: starting from a solution $\chi_0^{\pm}(x,t,\lambda)$ of $L_0(\lambda)$ with potential $Q_{(0)}(x,t)$ **to construct** a new singular solution $\chi_1^{\pm}(x,t,\lambda)$ with singularities located at prescribed positions $\lambda_1^{\pm}$;

  the reduction $\mathbf{p} = \mathbf{q}^{\dagger}$ ensures that $\lambda_1^{-} = (\lambda_1^{+})^{*}$.

- The new solutions $\chi_1^{\pm}(x,t,\lambda)$ will correspond to a potential $Q_{(1)}(x,t)$ of $L(\lambda)$ with two additional discrete eigenvalues $\lambda_1^{\pm}$, related to the regular one by

$$\chi_1^{\pm}(x,t,\lambda) = u(x,\lambda)\chi_0^{\pm}(x,t,\lambda)u_-^{-1}(\lambda). \qquad u_-(\lambda) = \lim_{x\to-\infty} u(x,\lambda)$$

Here $u_-(\lambda)$ is a diagonal matrix.

- The dressing factor $u(x,\lambda)$ must satisfy the equation

$$i\frac{du}{dx} + Q_{(1)}(x)u - uQ_{(0)}(x) - \lambda[J, u(x,\lambda)] = 0,$$

and the normalization condition

$$\lim_{\lambda\to\infty} u(x,\lambda) = \mathbb{1}.$$

  – Besides $\chi_i^{\pm}(x,\lambda)$, $i = 0, 1$ and $u(x,\lambda)$ must belong to the corresponding Lie group $\mathfrak{G}$;
  – in addition $u(x,\lambda)$ by construction has poles and/or zeroes at $\lambda_1^{\pm}$.

- The construction of $u(x,\lambda)$ is based on an appropriate anzats specifying explicitly the form of its λ-dependence:

$$u(x,\lambda) = \mathbb{1} + (c_1(\lambda) - 1)\, P_1(x,t) + \left(\frac{1}{c_1(\lambda)} - 1 \right) \overline{P}_1(x,t),$$
$$c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-},$$

where the projectors $P_1(x,t)$ and $\overline{P}_1(x,t)$ are of rank 1 and are related by $\overline{P}_1(x) = SP_1^T(x)S^{-1}$.

– They must satisfy

$$\overline{P}_1(x,t)P_1(x,t) = P_1(x,t)\overline{P}_1(x,t) = 0.$$

– By S we have denoted the special matrix which enters in the definition of the orthogonal algebra, i.e. $X \in \mathfrak{G}$ if $X + SX^TS^{-1} = 0$.

– In the typical representation of $so(5)$ we have

$$S = \sum_{k=1}^{5} (-1)^{k+1} E_{k,6-k}$$

where $(E_{ij})_{km} = \delta_{ik}\delta_{jm}$.
The explicit construction of $P_1(x,t)$ and $\overline{P}_1(x,t)$ using the 'polarization' vectors is done in [Gerdjikov,Grahovski,Kostov;2005].

– The new potential equals:

$$Q_{(1)}(x,t) - Q_{(0)}(x,t) = (\lambda_1^+ - \lambda_1^-)[J, P_1(x,t) - \overline{P}_1(x,t)].$$

- The λ-dependence of $u(x,\lambda)$ may depend [Gerdjikov,Grahovski,Ivanov,Kostov;2000] on the choice of the representation of $\mathfrak{g}$.

- For the gauge-equivalent MHF systems ($\mathfrak{g} \simeq \mathbf{B}_r, \mathbf{D}_r$):

$$\tilde{u}(x,\lambda) = \openone + \left(\frac{c_1(\lambda)}{c_1(0)} - 1\right)\tilde{P}_1 + \left(\frac{c_1(0)}{c_1(\lambda)} - 1\right)\tilde{P}_{-1},$$

where $\tilde{P}_{\pm 1} = g_{(0)}^{-1} P_{\pm 1} g_{(0)}(x,t)$.

The projectors $\tilde{P}_{\pm 1}$ satisfy the equations:

$$i\frac{d\tilde{P}_1}{dx} + \lambda_1^- \tilde{P}_1 \mathcal{S}_{(0)} - \lambda_1^- \mathcal{S}_{(1)} \tilde{P}_1 = 0,$$

$$i\frac{d\tilde{P}_{-1}}{dx} + \lambda_1^+ \tilde{P}_{-1} \mathcal{S}_{(0)} - \lambda_1^+ \mathcal{S}_{(1)} \tilde{P}_{-1} = 0,$$

and the "dressed" potential can be obtained by:

$$\mathcal{S}_{(1)} = S_{(0)} + i\frac{\lambda_1^+ - \lambda_1^-}{\lambda_1^+ \lambda_1^-}\frac{d}{dx}(-\lambda_1^+ \tilde{P}_1(x) + \lambda_1^- \tilde{P}_{-1}(x)).$$

The dressing factors can be written in the form:

$$\tilde{u}(x,\lambda) = \exp\left[\ln\left(\frac{c_1(\lambda)}{c_1(0)}\right)\tilde{p}(x)\right],$$

where $\tilde{p}(x) = \tilde{P}_1 - \tilde{P}_{-1} \in \mathfrak{g}$ and consequently $\tilde{u}(x,\lambda)$ belongs to the corresponding orthogonal group.

6. Examples

- **Example 1:** $\mathfrak{g} \simeq so(5)$

  $\Delta_+ = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2, \alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = \alpha_1 + 2\alpha_2\}.$

  ▶ Choose $\alpha_1(J) = 0$ $\Delta_1^+ = \{\alpha_2, \alpha_3, \alpha_4\}$ of $so(5)$,

$$q(x,t) \equiv \sum_{\alpha \in \Delta_1^+} (q_\alpha E_\alpha + p_\alpha E_{-\alpha}) = \begin{pmatrix} 0 & 0 & q_{11} & q_{12} & 0 \\ 0 & 0 & q_1 & 0 & q_{12} \\ p_{11} & p_1 & 0 & q_1 & -q_{11} \\ p_{12} & 0 & p_1 & 0 & 0 \\ 0 & p_{12} & -p_{11} & 0 & 0 \end{pmatrix};$$

$$J = \mathrm{diag}\,(a, a, 0, -a, -a).$$

♦ This choice is not related to any symmetric space!!!

- 6-component MNLS:

$$i\frac{\partial q_{12}}{\partial t} + \frac{1}{2a}\frac{\partial^2 q_{12}}{\partial x^2} + \frac{1}{a}q_{12}(q_1p_1 + q_{11}p_{11} + q_{12}p_{12}) + \frac{i}{a}q_1q_{11,x} - \frac{i}{a}q_{11}q_{1,x} = 0,$$

$$i\frac{\partial q_{11}}{\partial t} + \frac{1}{a}\frac{\partial^2 q_{11}}{\partial x^2} + \frac{1}{a}q_{11}(q_1p_1 + q_{11}p_{11} + \frac{1}{2}q_{12}p_{12}) + \frac{i}{a}q_{12}p_{1,x} + \frac{i}{2a}q_{12,x}p_1 = 0,$$

$$i\frac{\partial q_{1}}{\partial t} + \frac{1}{a}\frac{\partial^2 q_{1}}{\partial x^2} + \frac{1}{a}q_{1}(q_1p_1 + q_{11}p_{11} + \frac{1}{2}q_{12}p_{12}) - \frac{i}{a}q_{12}p_{11,x} - \frac{i}{2a}q_{12,x}p_{11} = 0,$$

$$i\frac{\partial p_{1}}{\partial t} - \frac{1}{a}\frac{\partial^2 p_{1}}{\partial x^2} - \frac{1}{a}p_{1}(q_1p_1 + q_{11}p_{11} + \frac{1}{2}q_{12}p_{12}) - \frac{i}{a}p_{12}q_{11,x} - \frac{i}{2a}p_{12,x}q_{11} = 0,$$

$$i\frac{\partial p_{11}}{\partial t} - \frac{1}{a}\frac{\partial^2 p_{11}}{\partial x^2} - \frac{1}{a}p_{11}(q_1p_1 + q_{11}p_{11} + \frac{1}{2}q_{12}p_{12}) + \frac{i}{a}p_{12}q_{1,x} + \frac{i}{2a}p_{12,x}q_1 = 0,$$

$$i\frac{\partial p_{12}}{\partial t} - \frac{1}{2a}\frac{\partial^2 p_{12}}{\partial x^2} - \frac{1}{a}p_{12}(q_1p_1 + q_{11}p_{11} + q_{12}p_{12}) + \frac{i}{a}p_1p_{11,x} - \frac{i}{a}p_{11}p_{1,x} = 0.$$

- The corresponding MHF system takes the form:

$$iS_t - \frac{5}{4a^2}[S, S_{xx}] + \frac{1}{4a^4}\left((\mathrm{ad}\, _S)^3 S_x\right)_x = 0,$$

where $\mathcal{S}$ is constrained by $\mathcal{S}(\mathcal{S}^2 - a^2)^2 = 0$.

- Impose the "canonical" reduction → 3-component MNLS

$$i\frac{dq_{12}}{dt} + \frac{1}{2a}\frac{d^2 q_{12}}{dx^2} - \frac{1}{a}q_{12}(|q_1|^2 + |q_{11}|^2 + |q_{12}|^2) + \frac{i}{a}q_1 q_{11,x} - \frac{i}{a}q_{11}q_{1,x} = 0$$

$$i\frac{dq_{11}}{dt} + \frac{1}{a}\frac{d^2 q_{11}}{dx^2} - \frac{1}{a}q_{11}(|q_1|^2 + |q_{11}|^2 + \frac{1}{2}|q_{12}|^2) + \frac{i}{a}q_{12}q_{1,x}^* + \frac{i}{2a}q_{12,x}q_1^* = 0$$

$$i\frac{dq_1}{dt} + \frac{1}{a}\frac{d^2 q_1}{dx^2} - \frac{1}{a}q_1(|q_1|^2 + |q_{11}|^2 + \frac{1}{2}|q_{12}|^2) - \frac{i}{a}q_{12}q_{11,x}^* - \frac{i}{2a}q_{12,x}q_{11}^* = 0,$$

For the gauge-equivalent system: $\mathcal{S}^\dagger = \mathcal{S}$.

7. Open problems

- to study reductions of the MNLS and their gauge equiv. systems;
- to study the internal structure of the soliton solutions and soliton interactions (for both types of systems);
- to study the spectral decompositions for the recursion operators (for both types of models).

Thank you!

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