Stability and instability dynamics of lattice solitons

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UC Merced

- 10th campus of University of California opened 2005
- 2008–9: 2500 students, @ full build-out: 25,000
- Applied Mathematical Sciences major + graduate

Outline

1. Soliton stability in homogeneous NLS
2. Stability theory for lattice solitons
3. Instability dynamics: asymptotics & computations
4. Conclusions

Collaborators:

- Yonatan Sivan, Tel Aviv University, Israel
- Gadi Fibich, Tel Aviv University, Israel
- Michael I. Weinstein, Columbia University, NY, USA
- Mark J. Ablowitz, University of Colorado at Boulder, CO, USA

Experiments with lattice solitons

- optical media with modulated refractive index – Segev; Christodoulides; Kivshar; Chen; ... – application to optical computing, Photonic Crystal fibers, ...
- Bose-Einstein condensates with optically induced lattices – Cornell; ...
- solitons are localized NL dispersive waves that propagate in lattice-type potentials while maintaining their shape
- observed solitons are usually stable

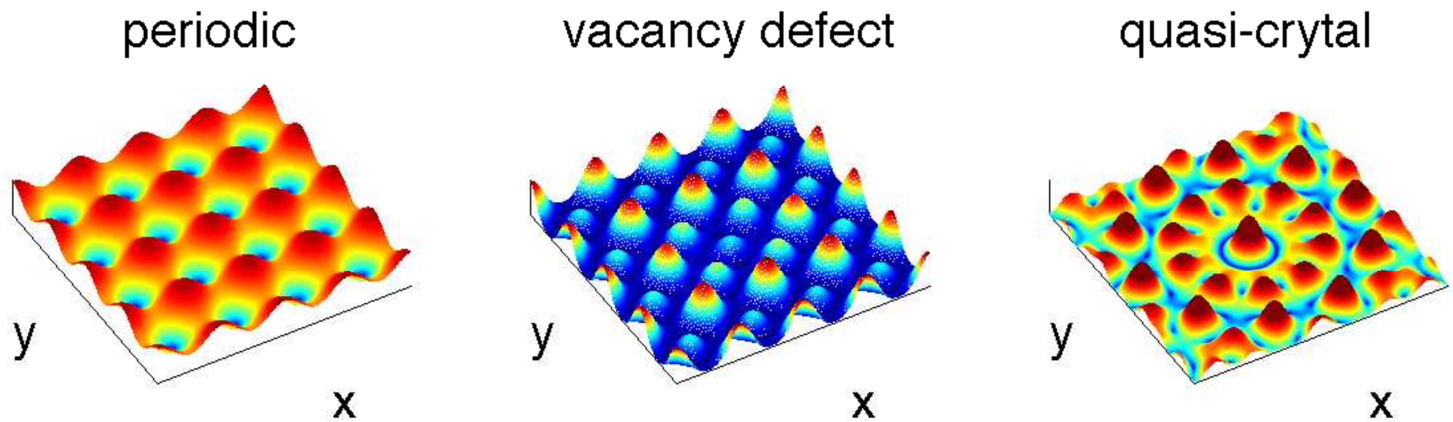
Inhomogeneous NLS model

NL Maxwell + refractive index modulation $\Longrightarrow$ focusing inhomogeneous nonlinear Schrödinger (NLS) model

$$i\psi_t(\mathbf{x},t) + \Delta\psi + |\psi|^{2\sigma}\psi - V(\mathbf{x})\psi = 0\ , \quad \psi(\mathbf{x},0) = \psi_0(\mathbf{x})\ ,$$

- $\psi =$ dimensionless complex electric field
- $t =$ time or propagation direction
- $\mathbf{x} \in \mathbb{R}^d\ ,\ \Delta \equiv \sum_{i=1}^{d} \partial^2_{x_i x_i}$ (diffraction/dispersion)
- $\sigma = 1$: Kerr effect (cubic NLS)
- $V(\mathbf{x}) =$ modulation of linear refractive index
- also saturable NL, NL potentials, ...

Examples of potentials



- can be extended, a-symmetric, a-periodic, ...
- solitons can be computed
- theory can be challenging

Rigorous theory

Existence of solitons $(V \equiv 0)$

$$i\psi_t(\mathbf{x},t) + \Delta\psi + |\psi|^{2\sigma}\psi = 0$$

Ansatz: $\psi(\mathbf{x},t) = f_E(\mathbf{x})e^{iEt}$

- E = propagation constant
- $f_E(\mathbf{x})$ = real profile, positive bright soliton satisfies bound-state eqn

$$\hookrightarrow \left[-\Delta - (f_E)^{2\sigma}\right] f_E = -E f_E \ , \quad f_E(|\mathbf{x}|) \overset{|\mathbf{x}|\to\infty}{\longrightarrow} 0$$

- ground states: $\exists f_E(\mathbf{x}) = R_{\sigma,d}(|\mathbf{x}|;E) > 0$, e.g.:
  - $\sigma = 1, d = 1$: integrable (1+1)D NLS, $\operatorname{sech}(x)$
  - $\sigma = 1, d = 2$: non-integrable (2+1)D NLS, $R(r)$

Orbital stability $(V \equiv 0)$

$$i\psi_t(\mathbf{x},t) + \Delta\psi + |\psi|^{2\sigma}\psi = 0$$

- Fundamental scale invariants:
  - phase: $\psi \to \psi e^{i\gamma}$
  - translation: $\psi \to \psi(\mathbf{x}+\mathbf{x}_0)$
- f_E is “orbitally stable” if $\psi(\mathbf{x},t)$ remains close to $\psi_0 = f_E + \varepsilon$ for all time modulo phase & translation

$$\rho_E^2[\psi, f_E](t) \equiv \inf_{\mathbf{x}_0\in\mathbb{R}^d} \inf_{\gamma\in[0,2\pi)} \Big[\|\nabla\psi(\mathbf{x}+\mathbf{x}_0,t)e^{i\gamma} - \nabla f_E(\mathbf{x})\|_2^2 + E\|\psi(\mathbf{x}+\mathbf{x}_0,t)e^{i\gamma} - f_E(\mathbf{x})\|_2^2 \Big]$$

- orbital (Lyapunov) stability $\gg$ linear stability

Stability theory ($V \equiv 0$)

A ground state f_E of the homogeneous NLS is orbitally stable iff it satisfies power-slope/V-K condition

$$\frac{dP(E)}{dE} > 0\ , \qquad P(E) \equiv \|f_E\|_2^2 = \int f_E^2\, d\mathbf{x}$$

- Vakhitov & Kolokolov (1973); Weinstein (1985, 1986); Grillakis, Shatah, & Strauss (1987)
- soliton is stable $\Longleftrightarrow \sigma d < 2$
- instability manifested in amplitude, e.g., collapse
- many studies use this criterion

Inhomogeneous NLS

$$
\begin{aligned}
i\psi_t(\mathbf{x},t) + \Delta\psi + |\psi|^{2\sigma}\psi - V(\mathbf{x})\psi &= 0 \\
\hookrightarrow \left[-\Delta + V(\mathbf{x}) - (f_E)^{2\sigma}\right] f_E &= -E f_E
\end{aligned}
$$

- no translation invariance
- orbital stability redefined: ψ remains close to f_E modulo phase (alone)
- most studies rely on power-slope (V-K) condition
- rigorous studies: need *"spectral condition"* as well, e.g., Rose & Weinstein (1988); Oh (1989); Fukuizumi & Ohta (2002); Stuart (2006); Fibich, Sivan, & Weinstein (2006); Rapti *et al.* (2007); Sivan *et al.* (2008)
- no general proof

Stability theorem

Ilan & Weinstein (preprint): *A positive bright soliton, f_E, is orbitally stable if it satisfies two conditions*

1. *power-slope/V-K:* $\frac{dP}{dE} > 0$
2. *spectral condition:*
   $L_+ = -\Delta + E + V(\mathbf{x}) - (2\sigma + 1)(f_E)^{2\sigma}$
   has at most one (simple) negative e-v

- $V(\mathbf{x}) \in \mathbb{R}$, periodic, defect, quasi-crystal, ...
- f_E can be centered @ any critical point of $V(\mathbf{x})$
- can check both condition numerically

Linearized operators

Linearize NLS eqn around f_E. Get

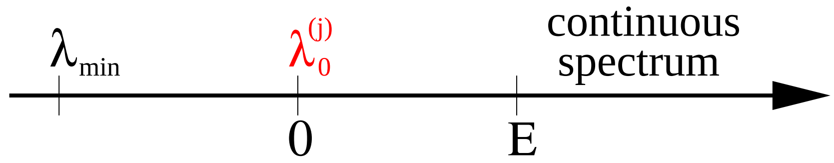
$$L_- \equiv -\Delta + E + V(\mathbf{x}) - (f_E)^{2\sigma}$$

$$L_+ \equiv -\Delta + E + V(\mathbf{x}) - (2\sigma+1)(f_E)^{2\sigma}$$

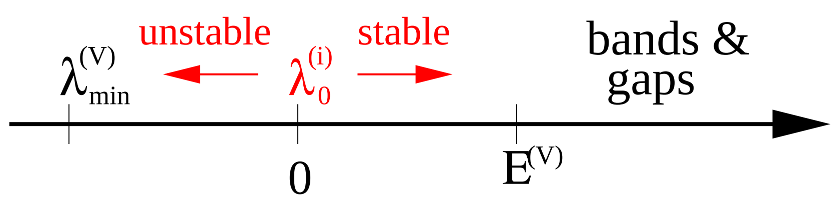
- orbital stability requires bounding $L_\pm$
- $L_- f_E = 0$; 0 is the smallest e-v & is simple
- for L_+:
  - $\lambda_{\min} < 0$
  - no translation invariance $\Longrightarrow \lambda_0^{(i)}$ shifted from 0

Spectrum of L_+

$$V(\mathbf{x}) \equiv 0 \Longrightarrow \lambda_0^{(i)} = 0 \; (i = 1 \ldots d)$$



$$V(\mathbf{x}) \neq 0 \Longrightarrow \lambda_0^{(i)} \neq 0 \; (i = 1 \ldots d)$$



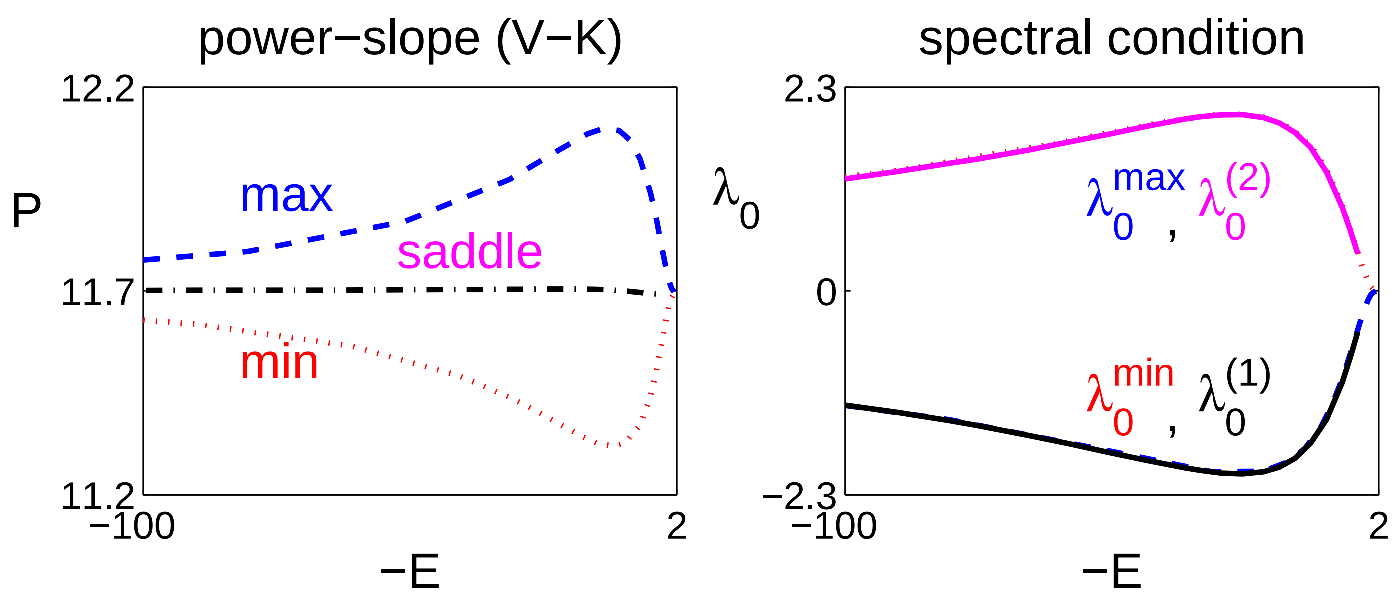
Conclusion: unstable if $\exists \lambda_0^{(i)} < 0$

Sketch of proof

- Lyapunov functional $\mathcal{E}[\psi] = H[\psi] - EP[\psi]$, $H \equiv$ Hamiltonian
- bound $\delta^2\mathcal{E}[\psi] \Longrightarrow$ bound $L_\pm$
- upper bounds using Gagliardo-Nirenberg inequalities
- lower bound of L_-: 0 is simple e-v of L_- ala Perron-Frobenius (cf. Reed & Simon IV)
- lower bound of L_+: use Lagrange multipliers. Need both power-slope & spectral condition

Stability conditions

$$V(x, y) = 2.5\left[\cos^2(2\pi x) + \cos^2(2\pi y)\right]$$



- power-slope (V-K) & spectral conditions are independent
- orbital stability requires <u>both</u> conditions

Interim summary

Rigorous theory: soliton is orbitally stable iff

1. power-slope/V-K: $\frac{dP}{dE} > 0$
2. spectral condition: $\forall i, \ \lambda_0^{(i)} > 0$

Questions:

1. What happens if only one condition is satisfied?
2. Qualitative/quantitative measure of instabilities?

Need asymptotic + computational tools!

Instability dynamics

Asymptotics

Sivan, Ilan, & Fibich (2008, preprint):

- $\left|\frac{dP}{d\mu}\right|$ determines strength of amplitude (in)stability
- $|\lambda_0^{(i)}|$ determine strength of drift (in)stability
  - short-time dynamics of center of mass:

$$\frac{d^2 <x_i>}{dt^2} \sim -C^2 \lambda_0^{(i)} <x_i> \ ,$$

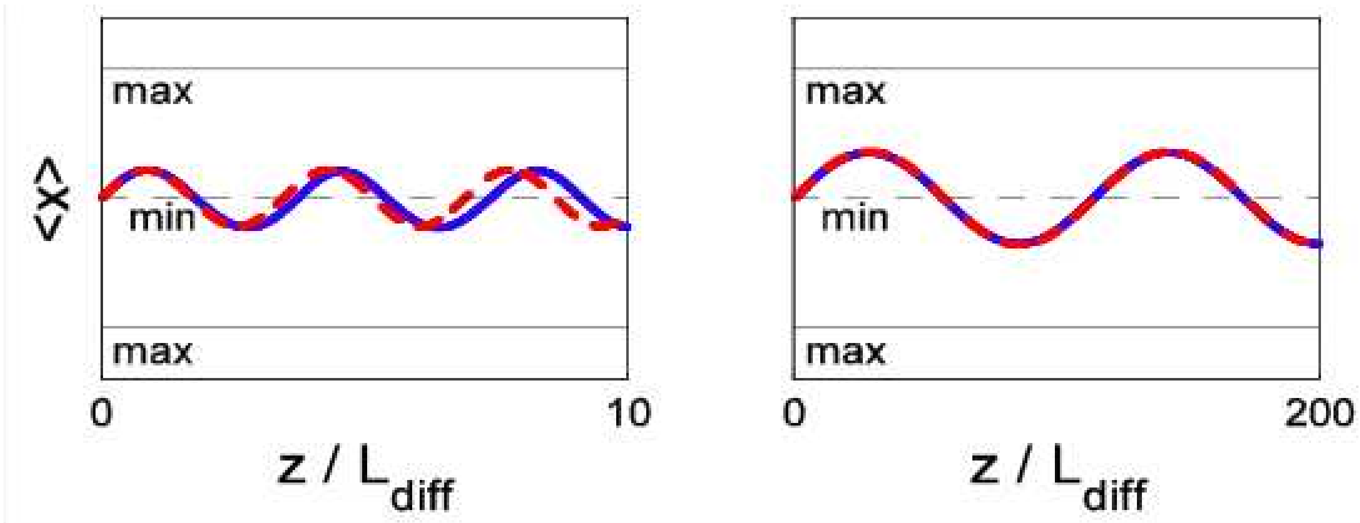
$$C^2 \equiv \frac{<f_E, f_E>}{<f_E, (L_-)^{-1} f_E>} > 0$$

  - usually drift-stable when @ minimum of $V(\mathbf{x})$

Computational examples

Solitons on lattice min

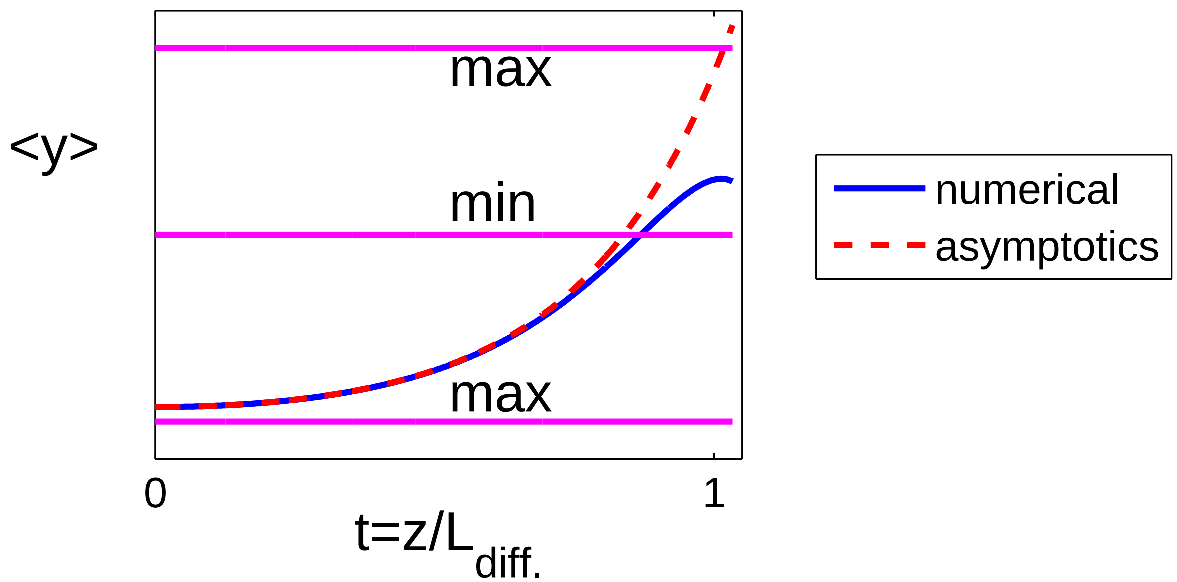
$\psi_0 =$ soliton slightly shifted from lattice min



- solitons centered @ min are drift-stable
- center-of-mass dynamics matches asymptotics

Soliton on lattice max

$\psi_0 =$ soliton slightly shifted from (2+1)D lattice max

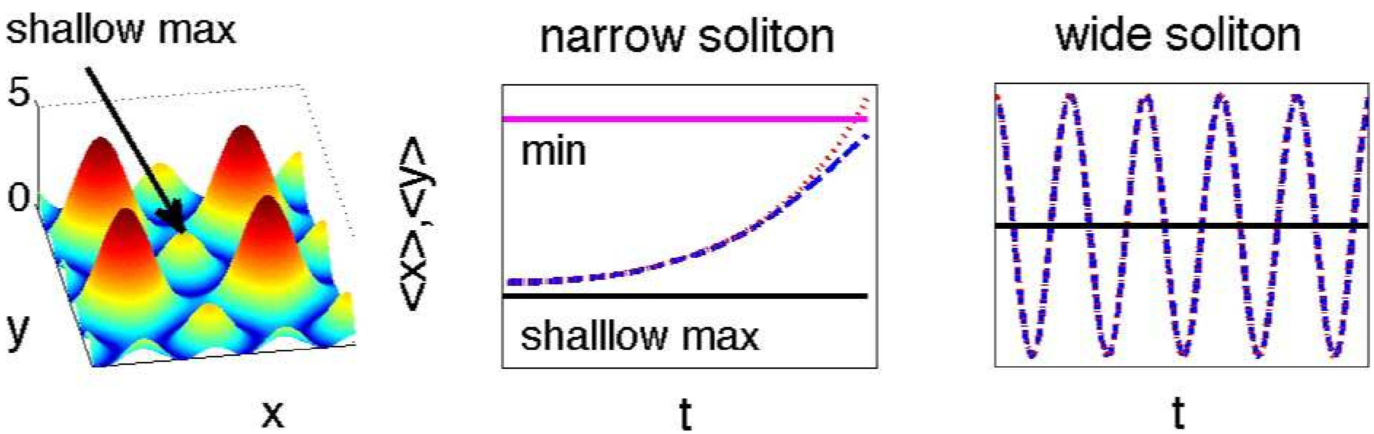


- solitons centered @ max are drift-unstable
- center-of-mass dynamics matches asymptotics

Soliton on shallow max

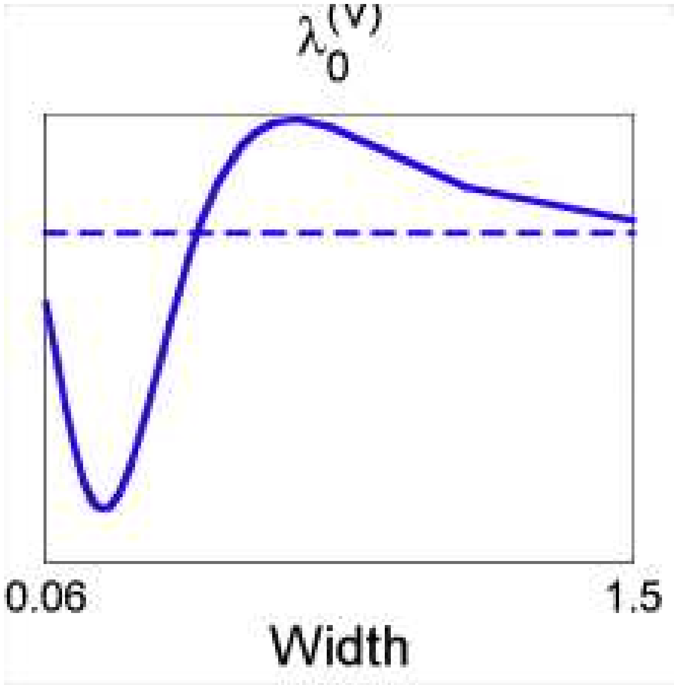
(2+1)D square lattice with shallow max

$$V(x, y) = 2.5\left[1 + \cos(2\pi x) + \cos(2\pi y)\right]^2$$



- narrow soliton @ max $\Longrightarrow$ drift unstable
- wide soliton effectively @ min $\Longrightarrow$ drift stable
- $\lambda_0^{(i)}$ predict drift (in)stability

Shallow max – cont.



- $\lambda_0^{(i)}$ changes sign as soliton becomes wider

Conclusions

- Stability theory:
  - $V(\mathbf{x})$ can be extended, a-symmetric, any dimension
  - need power-slope & spectral conditions
- Instability dynamics:
  - Violation of power-slope condition $\Longrightarrow$ amplitude instability
  - Violation of spectral condition $\Longrightarrow$ drift instability
    - analytic formulae for drift/oscillations
  - $|\frac{dP}{dE}|, |\lambda_0^{(i)}|$ determine strength of (in)stabilities

Thank you for your attention!