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Integrable models for shallow water waves

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1. Governing equations for the inviscid fluid motion

- The motion of inviscid fluid with a constant density ρ is described by the Euler's equations:

  $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \mathbf{g}$,

  $\nabla \cdot \mathbf{v} = 0$,

  where $\mathbf{v}(x, y, z, t)$ is the velocity of the fluid at the point (x, y, z) at the time t, P is the pressure in the fluid, $\mathbf{g} = (0, 0, -g)$ is the constant Earth's gravity acceleration.

- Consider now a motion of a shallow water over a flat bottom, which is located at $z = 0$. We assume that the motion is in the x-direction, and that the physical variables do not depend on y.

- Let h be the mean level of the water and let $\eta(x,t)$ describes the shape of the water surface, i.e. the deviation from the average level. The pressure is

  $$P = P_A + \rho g(h - z) + p(x,z,t),$$

  where P_A is the constant atmospheric pressure, and p is a pressure variable, measuring the deviation from the hydrostatic pressure distribution.

  On the surface $z = h + \eta$, $P = P_A$ and therefore $p = \eta\rho g$. Taking $\mathbf{v} \equiv (u, 0, w)$ we can write the kinematic condition on the surface as (Johnson 1997)

  $$w = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} \qquad \text{on} \qquad z = h + \eta.$$

  Finally, there is no horizontal velocity at the bottom, thus

  $$w = 0 \qquad \text{on} \qquad z = 0.$$

- The equations give the system

$$u_t + uu_x + wu_z = -\frac{1}{\rho}p_x,$$

$$= w_t + uw_x + ww_z = -\frac{1}{\rho}p_z,$$

$$u_x + w_z = 0$$

$$w = \eta_t + u\eta_x, \quad p = \eta\rho g, \qquad \text{on} \qquad z = h + \eta$$

$$w = 0 \qquad \text{on} \qquad z = 0.$$

- Let us introduce now dimensionless parameters

  $\varepsilon = a/h$ and

  $\delta = h/\lambda$,

  where a is the typical amplitude of the wave and λ is the typical wavelength of the wave. Now we can introduce dimensionless quantities, according to the magnitude of the physical quantities, see (Johnson 1997, 2002) for details:

  $x \to \lambda x$

  $z \to zh$,

$$t \to \frac{\lambda}{\sqrt{gh}} t,$$
$$\eta \to a\eta,$$
$$u \to \varepsilon\sqrt{gh}u,$$
$$w \to \varepsilon\delta\sqrt{gh}w,$$
$$p \to \varepsilon\rho gh.$$

This scaling is due to the observation that both w and p are proportional to ε i.e. the wave amplitude, since at undisturbed surface ($\varepsilon = 0$) both $w = 0$ and $p = 0$. The system in the new, dimensionless variables is

$$u_t + \varepsilon(uu_x + wu_z) = -p_x,$$
$$\delta^2(w_t + \varepsilon(uw_x + ww_z)) = -p_z,$$
$$u_x + w_z = 0,$$
$$w = \eta_t + \varepsilon u\eta_x, \quad p = \eta, \qquad \text{on} \qquad z = 1 + \varepsilon\eta,$$
$$w = 0 \qquad \text{on} \qquad z = 0.$$

2. Green-Naghdi Equations

- We present a derivation of the relevant form of the Green -Naghdi (GN) equations (Green and Naghdi 1976), which follows directly from the above system.
- We assume that u is not a function of z. This is not correct at $O(\varepsilon)$, but this approximation is valid for the leading-order problem. This assumption is equivalent to the simplifying approximation used by Green and Naghdi (namely, that w is linear in z in a single-layer model).
- Thus we have $w = -zu_x$, which satisfies $u_x + w_z = 0$ and the bottom condition.
- The second equation gives $p = \eta - \frac{1}{2}\delta^2[(1+\varepsilon\eta)^2 - z^2](u_{xt} + \epsilon u u_x - \epsilon u_x^2)$, which satisfies the pressure condition at the surface.

- This expression for p is now used in the first equation, which is then integrated over all z to give
  $u_t + \varepsilon u u_x + \eta_x = \frac{\delta^2/3}{1+\varepsilon\eta}[(1+\varepsilon\eta)^3(u_{xt} + \varepsilon u u_{xx} - \varepsilon u_x^2)]_x,$
- The first order in the small parameters is
  $u_t - \frac{\delta^2}{3} u_{xxt} + \varepsilon u u_x + \eta_x = 0.$
- The condition on the surface gives
  $\eta_t + [(u(1+\varepsilon\eta)]_x = 0.$

3. Two component Camassa-Holm system

One can demonstrate that the Green-Naghdi system can be related to the following two component Camassa-Holm system in the first order with respect to ε and δ^2:

$$m_t + 2u_x m + u m_x + \rho\rho_x = 0,$$

$$\rho_t + (u\rho)_x = 0,$$

where $m = u - u_{xx}$.

- The CH equation can be obtained via the obvious reduction $\rho \equiv 0$.

  The system is integrable, it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ:

$$\Psi_{xx} = \Big(-\zeta^2\rho^2 + \zeta m + \tfrac{1}{4}\Big)\Psi,$$
$$\Psi_t = \Big(\tfrac{1}{2\zeta} - u\Big)\Psi_x + \tfrac{1}{2}u_x\Psi.$$

- The system is also bi-Hamiltonian.
- The first Poisson bracket is
  $$\{A, B\} = -\int \Big[\tfrac{\delta A}{\delta m}(m\partial + \partial m)\tfrac{\delta B}{\delta m} + \tfrac{\delta A}{\delta m}\rho\partial\tfrac{\delta B}{\delta \rho} + \tfrac{\delta A}{\delta \rho}\partial\rho\tfrac{\delta B}{\delta m}\Big]\mathrm{d}x$$
  for the Hamiltonian $H = \frac{1}{2}\int(um + \rho^2)\mathrm{d}x$;
- The second Poisson bracket is
  $$\{A, B\}_2 = -\int \Big[\tfrac{\delta A}{\delta m}(\partial - \partial^3)\tfrac{\delta B}{\delta m} + \tfrac{\delta A}{\delta \rho}\partial\tfrac{\delta B}{\delta \rho}\Big]\mathrm{d}x$$
  for the Hamiltonian $H_2 = \frac{1}{2}\int(u\rho^2 + u^3 + uu_x^2)\mathrm{d}x$.
  It has two Casimirs: $\int \rho\mathrm{d}x$ and $\int m\mathrm{d}x$.

- Let us define $\rho = 1 + \frac{1}{2}\varepsilon\eta - \frac{1}{8}\varepsilon^2(u^2 + \eta^2)$.

  The expansion of ρ^2 in the same order of ε is

  $\rho^2 = 1 + \varepsilon\eta - \frac{1}{4}\varepsilon^2 u^2$.

- With this definition it is straightforward to write in the form

  $$\left(u - \frac{\delta^2}{3}u_{xx}\right)_t + \frac{3}{2}\varepsilon u u_x + \frac{1}{\varepsilon}(\rho^2)_x = 0$$

  or, introducing the variable $m = u - \frac{1}{3}\delta^2 u_{xx}$,

  in the same order (i.e. neglecting terms of order $\varepsilon\delta^2$)

  $m_t + \varepsilon m u_x + \frac{1}{2}\varepsilon u m_x + \frac{1}{\varepsilon}(\rho^2)_x = 0.$

- Next, using the fact that in linear approximation

  $u_t \approx -\eta_x, \qquad \eta_t \approx -u_x,$

  we have $\rho_t = \frac{1}{2}\varepsilon\eta_t + \frac{1}{4}\varepsilon^2(\eta u)_x$.

- With these expressions for ρ and ρ_t the second GN equation can be written as

  $\rho_t + \frac{\varepsilon}{2}(\rho u)_x = 0.$

- The rescaling $u \to \frac{2}{\varepsilon}u$, $x \to \frac{\delta}{\sqrt{3}}\ x$, $t \to \frac{\delta}{\sqrt{3}}t$ in GN equations gives the CH2 system.

- The case with $-\rho\rho_x$ term, which is considered in the most previous works on the system, corresponds to a situation in which the gravity acceleration points upwards.

- Concerning the occurrences of peakons, it was recently established that the only peakons of the CH2 system arise when $\rho \equiv 0$ and $u(x,t) = c\, e^{-|x-ct|}$ for some wave speed $c \neq 0$.

- Wave breaking is the only way that singularities arise in smooth solutions to the system and that for the occurrence of breaking waves it is not necessary to require that $\rho \equiv 0$.

Kaup - Boussinesq system

The Kaup - Boussinesq system is another integrable system matching the GN equation to the first order of the small parameters ε, δ.

- The first GN equation can be written as
  $V_t + \varepsilon V V_x + \eta_x = 0$ where $V = u - \frac{\delta^2}{3} u_{xx}$,
- The second GN equation - first order in ε, δ:
  $\eta_t + V_x + \frac{\delta^3}{3} V_{xxx} + \varepsilon(\eta V)_x = 0$
  rescaling and shift in η leads to the Kaup - Boussinesq system
  $V_t + V V_x + \eta_x = 0$
  $\eta_t + V_{xxx} + (\eta V)_x = 0,$

- which is integrable, with Lax pair

$$\Psi_{xx} = \Big((\zeta - \tfrac{1}{2}V)^2 - \eta\Big)\Psi,$$

$$\Psi_t = -(\zeta + \tfrac{1}{2}V)\Psi_x + \tfrac{1}{4}V_x\Psi.$$

4. Travelling waves

- We are looking for solutions of the form of travelling waves, i.e. solutions that depend on a single variable $\xi = x - ct$ for some constant velocity c.

- The second equation gives immediately

  $-c\rho' + (u\rho)' = 0$

  $\rho(\xi) = \frac{\alpha}{u(\xi)-c}$ where α is an integration constant.

  The first equation

  $-cm' + 2u'm + um' + \rho\rho' = 0$, $m = u - u''$ integrated once gives

  $-cm + \frac{3}{2}u^2 - \frac{1}{2}(u')^2 + \frac{1}{2}\rho^2 = \beta = \text{const.}$

  Introducing new variable $z = \frac{u-c}{|c|}$ and using $\rho = \frac{\alpha}{z(\xi)|c|}$ the equation for $z(\xi)$ acquires the form

  $\frac{1}{2}(z')^2 + zz'' = \frac{3}{2}z^2 + 2\frac{c}{|c|}z + \frac{2c^2-\beta}{c^2} + \frac{\alpha^2}{2c^2}z^{-2}$

- integration over z of both sides gives:

  $\int[\frac{1}{2}(z')^2 + zz'']dz = \frac{1}{2}z(z')^2 - \frac{1}{2}\int z2z'dz' + \int zz''dz =$

  $= \frac{1}{2}z(z')^2 - \int zz'\frac{dz'}{d\xi}\frac{d\xi}{dz}dz + \int zz''dz = \frac{1}{2}z(z')^2$

- Finally

  $(zz')^2 = z^4 + 2\frac{c}{|c|}z^3 + \gamma z^2 + \mu z - \frac{\alpha}{c^4}$

  where γ and μ are new letters for the integration constants.

- $z^4 + 2\frac{c}{|c|}z^3 + \gamma z^2 + \mu z - \frac{\alpha}{c^4} = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$

  $\xi = \int \frac{zdz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}} \Rightarrow z(\xi)$

5. Multicomponent CH generalizations

- In order to obtain multi-component generalizations, we consider a more general Lax pair, leading to a hierarchy of Camassa-Holm type:

  $\Psi_{xx} = Q(x, \lambda)\Psi,$
  $\Psi_t = -U(x, \lambda)\Psi_x + \frac{1}{2}U_x(x, \lambda)\Psi,$

  where

  $Q(x, \lambda) = \lambda^n q_n(x) + \lambda^{n-1} q_{n-1}(x) + \ldots + \lambda q_1(x) + \frac{1}{4},$
  $U(x, \lambda) = u_0(x) + \frac{u_1(x)}{\lambda} + \ldots \frac{u_k(x)}{\lambda^k}.$

- The compatibility condition of these gives the equation

  $Q_t = \frac{1}{2}U_{xxx} - 2U_x Q - UQ_x,$

  which, is equivalent to a chain of n evolution equations with $k+1$ differential constraints for the $n+k+1$ variables

  $q_1, q_2, \ldots, q_n, u_0, u_1, \ldots, u_k$

(n and k are arbitrary natural numbers, i.e. positive integers):

$$q_{n-r,t} = -\sum_{s=\max(0,r-k)}^{r} (2u_{r-s,x} q_{n-s} + u_{r-s} q_{n-s,x}),$$
$$r = 0, 1, \dots, n-1,$$
$$0 = \tfrac{1}{2}(u_{r,xxx} - u_{r,x}) - \sum_{s=1}^{\min(n,k-r)} (2u_{r+s,x} q_s + u_{r+s} q_{s,x})$$
$$r = 0, 1, \dots, k-1,$$
$$0 = \tfrac{1}{2}(u_{k,xxx} - u_{k,x}).$$

- *Example 1*: $k = n = 2$.

  The choice $u_2 = -1/2$ solves automatically one of the constraints. The other two differential constraints can be easily integrated, giving

$$q_1 = u_1 - u_{1,xx} + \omega_1,$$
$$q_2 = u_0 - u_{0,xx} + 3u_1^2 - u_{1,x}^2 - 2u_1 u_{1,xx} + 4\omega_1 u_1 + \omega_2,$$

where $\omega_{1,2}$ are arbitrary constants.

The system of equations for u_0, u_1 is

$$q_{2,t} + 2u_{0,x}q_2 + u_0 q_{2,x} = 0,$$
$$q_{1,t} + 2u_{0,x}q_1 + u_0 q_{1,x} + 2u_{1,x}q_2 + u_1 q_{2,x} = 0.$$

- *Example 2*: $k = 1$, $n = 2$.

  In the notations $u_0 \equiv u$, $q_1 \equiv q$ and $q_2 \equiv \rho^2$, and with the choice $u_1 = -1/2$, the system can be written in the form

  $$q_t = uq_x + 2qu_x - \rho\rho_x = 0,$$
  $$\rho_t + (u\rho)_x = 0,$$

  where $q = u - u_{xx} + \omega$ and ω is an arbitrary constant.

- *Example 3*: CH equation

  Taking $u_1 = \omega_1 = 0$ gives $q_1 = 0$, $q_2 = u_0 - u_{0,xx} + \omega_2$ and we obtain exactly the CH equation with $u \equiv u_0$ and $\omega \equiv \omega_2$.

  CH can also be obtained as a reduction from Example 2 by setting $\rho = 0$.

2+1 dimensional generalization

The system

$$m_t + 2U_{xy}m + (U_y + \gamma)m_x + \rho\rho_y = 0$$

$$\rho_t + [(U_y + \gamma)\rho]_x = 0$$

with $m = U_x - U_{xxx} + const$

is integrable, and reduces to CH2 if $x = y$ and $u = U_x$.

It can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ:

$$\Psi_{xx} = \Big(-\zeta^2\rho^2 + \zeta m + \tfrac{1}{4}\Big)\Psi,$$

$$\Psi_t = \tfrac{1}{2\zeta}\Psi_y + (U_y + \gamma)\Psi_x + \tfrac{1}{2}U_{xy}\Psi.$$

6. Peakons of the system in the short-wave limit

- No peakons among the solitary wave solutions.
- However, there are peakon solutions of the 'short wave limit' equation $\sigma_1 = 0$.
- The peakon solutions have the form

$m(x,t) = \sum_{k=1}^{N} m_k(t)\delta(x - x_k(t))$
$u(x,t) = -\frac{1}{2}\sum_{k=1}^{N} m_k(t)|x - x_k(t)|,$
$\rho(x,t) = \sum_{k=1}^{N} \rho_k(t)\theta(x - x_k(t)),$

where θ is the Heaviside unit step function. The asymptotic behaviour $\rho(x,t) \to 0$ for $x \to \infty$ and $\int m \, \mathrm{d}x = 0$ lead to
$\sum_{l=1}^{N} m_l = \sum_{l=1}^{N} \rho_l = 0$, or
$\sum_{l=1}^{N} \mu_l = 0$
in terms of the new complex variable $\mu_k \equiv m_k + i\rho_k$.

The substitution of the above Ansatz into the equations under the assumption that $x_1(t) < x_2(t) < \ldots < x_N(t)$ for all t, (a condition holding for the peakons of HS equation gives the following dynamical system for the time-dependent variables:

$\frac{\mathrm{d}x_k}{\mathrm{d}t} = -\frac{1}{2}\sum_{l=1}^{N} m_l |x_l - x_k|,$

$\frac{\mathrm{d}\mu_k}{\mathrm{d}t} = \frac{\mu_k}{2}\sum_{l=1}^{N} \mu_l \,\mathrm{sgn}(k-l)$

with the convention $\mathrm{sgn}(0) = 0$.

The integrals for this system can be obtained from the integrals of the original system by substituting the expressions . It is convenient to write the system in terms of the new independent variables $\Delta_k \equiv x_{k+1} - x_k,$

$M_k \equiv \mu_1 + \ldots + \mu_k,$

with $k = 1, 2, \ldots, N-1$.

- The Hamiltonian of the new system is $H = \frac{1}{2}\sum_{l=1}^{N-1} |M_k|^2 \Delta_k,$

the equations

$\frac{\mathrm{d}\Delta_k}{\mathrm{d}t} = -\mathrm{Re}(M_k)\Delta_k,$

$\frac{\mathrm{d}M_k}{\mathrm{d}t} = \frac{1}{2}M_k^2$

being Hamiltonian with respect to the bracket

$\{\Delta_k, M_l\} = -\frac{M_k}{\bar{M}_k}\delta_{lk},$

in which the bar stands for complex conjugation. These equations integrate immediately:

$M_k(t) = -\frac{1}{t/2+c_k},$

$\Delta_k(t) = \Delta_k(0)\frac{(t/2+c_{k,1})^2+c_{k,2}^2}{c_{k,1}^2+c_{k,2}^2},$

where $c_k \equiv c_{k,1} + ic_{k,2} = -M_k^{-1}(0)$ is a complex constant with real and imaginary parts $c_{k,1}$ and $c_{k,2}$ respectively. Notice that the large time asymptotics $M_k \sim t^{-1}, \quad \Delta_k \sim t^2$, are the same as those for the peakons of the Hunter-Saxton equation when $\rho_k \equiv 0$.