# Fission of a longitudinal strain solitary wave in a delaminated bar

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# Overview

- Motivation
- Model equation for long nonlinear longitudinal waves
- Problem formulation
- Weakly nonlinear solution
- Fission of an incident strain solitary wave
  - Transmitted wave
  - Reflected wave
  - Higher-order corrections
- Experiments
- Concluding remarks

# 1. Motivation

- Study of longitudinal bulk solitary waves in nonlinearly elastic waveguides (Nariboli and Sedov 1970, Ostrovsky and Sutin 1977, Engelbrecht 1981, Samsonov et al. 1984-present)
- Lattice modelling of nonlinear waves in a bi-layer with delamination (Khusnutdinova and Silbershmidt 2003)
- Longitudinal bulk strain solitary waves can propagate for considerable distances without any significant decay (Samsonov et al. 2006)
- Aim: find experimentally observable nonlinear effects caused by damage/ delamination. Show that splitting of a waveguide leads to fission of a bulk solitary wave.
- Fission of a solitary wave in various problems of fluid mechanics (Tappert and Zabusky 1971, Pelinovsky 1971, Johnson 1972, Djordevic and Redekopp 1978, Grimshaw et al. 2007). No consistent mathematical approach to problems of this type (e.g., no approach to the description of higher-order corrections). No studies in mechanics of elastic solids.

# 2. Model equation for long longitudinal waves

The Doubly Dispersive Equation (DDE) has been derived to describe long nonlinear longitudinal waves in a cylindrical bar of circular cross section by Samsonov (1984). The DDE can also be derived to describe the propagation of a long nonlinear longitudinal bulk wave in an isotropic elastic bar of rectangular cross section $\sigma = \{-a \le y \le a; -b \le z \le b\}$:

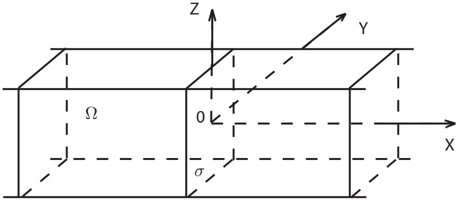


Figure: Elastic bar of rectangular cross section.

# 2. Model equation for long longitudinal waves

Main assumptions:

- Murnaghan's 5-constant model for the density of potential energy Π:

$$\Pi = (\lambda + 2\mu) I_1^2/2 - 2\mu I_2 + (l + 2m) I_1^3/3 - 2m I_1 I_2 + n I_3 + \ldots$$

where $I_k$ are invariants of the deformation tensor
$\mathbf{C} = [\nabla \mathbf{U} + (\nabla \mathbf{U})^T + \nabla \mathbf{U} \cdot (\nabla \mathbf{U})^T]/2$.

- The planar cross-section hypothesis and the approximate relations for the transverse displacements:

$$u = u(x,t) + \ldots, \; v = -y\nu u_x + \ldots, \; w = -z\nu u_x + \ldots,$$

where $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is Poisson's ratio (Love, Volterra).

- The scaling

$$\varepsilon = \frac{Wave\ amplitude}{Wave\ length} = \left(\frac{Bar\ width}{Wave\ length}\right)^2$$

# 2. Model equation for long longitudinal waves

The DDE for long nonlinear longitudinal displacement waves in a bar of rectangular cross section has the form:

$$u_{tt} - c^2 u_{xx} = \frac{\beta}{\rho} u_x u_{xx} + \frac{J\nu^2}{\sigma}(u_{tt} - c_1^2 u_{xx})_{xx}, \qquad (1)$$

where $c = \sqrt{E/\rho}$, $c_1 = \sqrt{\mu/\rho} = c/\sqrt{2(1+\nu)}$, and
$J = \int_\sigma (y^2 + z^2) d\sigma = \frac{4ab}{3}(a^2 + b^2)$.
Differentiating (1) with respect to $x$, we obtain the following equation for the linear strain component $e \equiv u_x$:

$$e_{tt} - c^2 e_{xx} = \frac{\beta}{2\rho}(e^2)_{xx} + \frac{J\nu^2}{\sigma}\left(e_{tt} - c_1^2 e_{xx}\right)_{xx}. \qquad (2)$$

The one-parameter family of exact solitary wave solutions of (2):

$$e = e_0 \operatorname{sech}^2 \frac{1}{\Lambda}(x - st), \qquad (3)$$

where

$$e_0 = \frac{3\rho(s^2 - c^2)}{\beta}, \quad \Lambda^2 = \frac{2\nu^2 J}{(1+\nu)\sigma}\left[1 + \frac{(1+2\nu)s^2}{s^2 - c^2}\right].$$

# 3. Problem formulation

Consider the propagation of long nonlinear longitudinal bulk waves in a delaminated, symmetric two-layered elastic bar (layers are identical and have the same width $2a$ and the same height $b$). We assume that there is a perfect interface when $x < 0$ and complete debonding when $x > 0$. The material to the left of $x = 0$ may be different from the material to the right, in which case we assume that the cross section $x = 0$ is also a perfect interface.

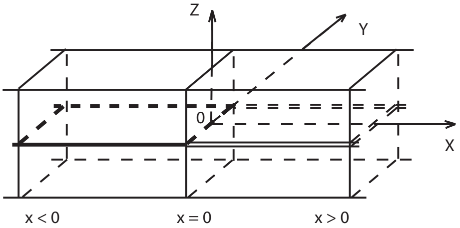


Figure: Delaminated, two-layered elastic bar.

# 3. Problem formulation

Assuming that the delamination area is $\{x > 0, -a < y < a, z = 0\}$, using the planar cross section hypothesis and the symmetry of the structure, we consider the (1+1) - dimensional formulation of the problem:

$$u^-_{tt} - u^-_{xx} = 2\varepsilon[-6u^-_x u^-_{xx} + u^-_{xxxx}], \quad (4)$$
$$u^+_{tt} - c^2 u^+_{xx} = 2\varepsilon[-6\beta u^+_x u^+_{xx} + \gamma u^+_{xxxx}], \quad (5)$$
$$u^-|_{x=0} = u^+|_{x=0}, \quad (6)$$
$$u^-_x + 2\varepsilon[-3(u^-_x)^2 + u^-_{xxx}]|_{x=0} =$$
$$c^2 u^+_x + 2\varepsilon[-3\beta(u^+_x)^2 + \gamma u^+_{xxx}]|_{x=0}. \quad (7)$$

Here, we used asymptotic relations $u^-_{ttxx} = u^-_{xxxx} + O(\varepsilon)$ and $u^+_{ttxx} = c^2 u^+_{xxxx} + O(\varepsilon)$, and introduced dimensionless parameters

$$c^2 = \frac{c_+^2}{c_-^2}, \; \beta = \frac{\beta_+ \rho_-}{\beta_- \rho_+}, \; \gamma = \frac{J_+ \nu_+^2 (c_+^2 - c_{1+}^2)\sigma_-}{J_- \nu_-^2 (c_-^2 - c_{1-}^2)\sigma_+}.$$

The same problem (4) – (7) appears in a continuum approximation for long weakly nonlinear waves in an inhomogeneous FPU chain.

# 4. Weakly nonlinear solution

For $x < 0$, we look for a leading order solution in the form

$$\begin{aligned} u^- &= I(\xi_-, X) + R(\eta_-, X) \\ &+ \varepsilon P(\xi_-, \eta_-, X) + O(\varepsilon^2), \end{aligned} \qquad (8)$$

where $\xi_- = x - t$ and $\eta_- = x + t$ are the fast characteristic variables, and $X = \varepsilon x$ is the slow space variable. This ansatz is similar to that introduced by Miles (1977) in the study of the interaction of solitons, apart from the replacement of the slow time with the slow space variable.

For $x > 0$, we look for a solution in the form

$$u^+ = T(\xi_+, X) + \varepsilon Q(\xi_+, \eta_+, X) + O(\varepsilon^2), \qquad (9)$$

where, again, we introduce the characteristic variables
$\xi_+ = x - ct$, $\eta_+ = x + ct$, and the slow space variable $X = \varepsilon x$.

# 4. Weakly nonlinear solution

We assume that the right-propagating incident wave $I(\xi_-, X)$ is known and defined by a solution of the KdV equation:

$$I = \int \tilde{I} d\xi_-, \quad \text{where} \quad \tilde{I}_X - 6\tilde{I}\tilde{I}_{\xi_-} + \tilde{I}_{\xi_-\xi_-\xi_-} = 0. \qquad (10)$$

We need to find the reflected wave

$$R = \int \tilde{R} d\eta_-, \quad \text{where} \quad \tilde{R}_X - 6\tilde{R}\tilde{R}_{\eta_-} + \tilde{R}_{\eta_-\eta_-\eta_-} = 0, \qquad (11)$$

and the higher-order terms:

$$P = 3[RI_{\xi_-} + IR_{\eta_-}] + \phi(\xi_-, X) + \psi(\eta_-, X).$$

Here, we choose the function $\phi(\xi_-, X) = 0$ in accordance with the radiation conditions (there must be no corrections to the given incident wave in the disturbance caused by it), and the function $\psi(\eta_-, X)$ has to be found from the continuity conditions.

# 4. Weakly nonlinear solution

Similarly, in the delaminated area ($x > 0$), we look for the leading order transmitted wave

$$T = \int \tilde{T} d\xi_+, \quad \text{where } \tilde{T}_X - 6\frac{\beta}{c^2}\tilde{T}\tilde{T}_{\xi_+} + \frac{\gamma}{c^2}\tilde{T}_{\xi_+\xi_+\xi_+} = 0, \qquad (12)$$

and the higher-order corrections

$$Q = q(\xi_+, X) + r(\eta_+, X).$$

Here again, $r(\eta_+, X) = 0$ due to the radiation conditions (if the incident wave is coming only from the left, the waves on the right-hand side must be right-going), and $q(\xi_+, X)$ should be found from the continuity conditions.

# 4. Weakly nonlinear solution

Then, from continuity conditions we can find the “initial” conditions for the KdV equations, defining both reflected and transmitted “strain” waves at $x = 0$ in terms of the given incident wave:

$$\tilde{R}|_{x=0} = C_R \tilde{I}|_{x=0} \quad \text{and} \quad \tilde{T}|_{x=0} = C_T \tilde{I}|_{x=0}, \tag{13}$$

where we introduced the reflection coefficient

$$C_R = \frac{c-1}{c+1}, \tag{14}$$

and the transmission coefficient

$$C_T = \frac{2}{c(1+c)}. \tag{15}$$

Note, that if $c = 1$, i.e. if $c_- = c_+$, then the reflection coefficient $C_R = 0$, and there will be no leading order reflected wave.

# 4. Weakly nonlinear solution

We can also restore the dependence of $\psi(\eta_-, X)$and $q(\xi_+, X)$ on their respective characteristic variables

$$\psi(\eta_-, X) = \frac{1}{1+c}\int \left[cf(\eta_-, X) + g(\eta_-, X)\right] d\eta_-,$$
$$q(\xi_+, X) = \frac{1}{c(1+c)} \times$$
$$\int \left[f\left(-\frac{\xi_+}{c}, X\right) - g\left(-\frac{\xi_+}{c}, X\right)\right] d\xi_+ \qquad (16)$$

# 5. Fission of an incident solitary wave

We assume that the leading order right-propagating incident "strain" wave is given by an exact solitary wave solution of the KdV (incident) equation:

$$\tilde{I} = -\frac{v}{2}\, sech^2\, \frac{\sqrt{v}}{2}(\xi_- - vX).$$

We should describe the leading order reflected and transmitted waves, as well as the higher-order corrections. Since experimentally measured quantities are the "strains" $u_x^-$ and $u_x^+$, we aim at finding the explicit leading order asymptotics of these functions for large $t$ and $x$.

## 5.1 Transmitted wave

The transmitted wave is defined by the spectrum of the Schrödinger equation associated with the (transmitted) KdV equation:

$$\Psi_{\chi\chi} + [\lambda - U(\chi)]\Psi = 0, \qquad (17)$$

where, the potential is given by

$$U(\chi) = -A\, sech^2\, \frac{\chi}{l}, \quad A = \frac{v\beta}{\gamma c(1+c)}, \quad l = \frac{2c}{\sqrt{v}}.$$

# 5.1 Transmitted wave

The number $N$ of secondary solitary waves produced in the delaminated area is defined by the largest integer satisfying the inequality

$$N < \frac{1}{2}\left[\left(1 + \frac{4\alpha^2}{\pi^2}\right)^{1/2} + 1\right],$$

$$\text{where} \quad \alpha = \pi\sqrt{A}l = 2\pi\sqrt{\frac{\beta}{\gamma}\cdot\frac{c}{1+c}}. \qquad (18)$$

Parameters $\beta, \gamma$ and $c$ depend on material properties and geometry of the waveguide, and have been defined in Sec. 3. There is always one solitary wave for small $\alpha$, while more solitons will emerge as $\alpha$ increases. Asymptotically, as $\tau \to +\infty$, the solution evolves into a procession of solitary waves propagating to the right, and some decaying radiation (a dispersive wave train) propagating to the left (e.g., Drazin and Johnson):

$$U \sim -\sum_{n=1}^{N} 2k_n^2 sech^2 k_n(\chi - 4k_n^2\tau - \chi_n) + \text{radiation}.$$

# 5.1 Transmitted wave

Is fission still possible if the waveguide is made of one and the same material? The answer is entirely defined by the geometry of the waveguide. Indeed, in this case, $c = 1$, $\beta = 1$ and $\gamma = \frac{J_+\sigma_-}{J_-\sigma_+} = \frac{4+\kappa^2}{4(1+\kappa^2)}$, where $\kappa = b/a$. This yields

$$N = \left\{ \text{largest integer} < \frac{1}{2}\left[\sqrt{1 + 32\,\frac{1+\kappa^2}{4+\kappa^2}} + 1\right]\right\}.$$

Thus, the number of secondary solitons depends on $\kappa$, and there will be either two (for $\kappa \leq 2\sqrt{2}$) or three (for $\kappa > 2\sqrt{2}$) solitons. For example, for $\kappa = 1$ (i.e., $b = a$) there will be two secondary solitons.

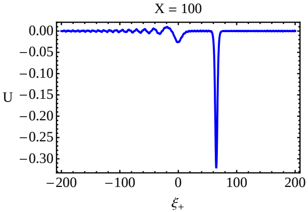


Figure: Two secondary solitons and dispersive radiation in the transmitted wave field at $X = 100$ ($\kappa = 1, v = 0.35$).

# 5.2 Reflected wave

The reflected wave field is defined by the spectrum of the Schrödinger equation, where the potential $U(\chi)$ is given by

$$U(\chi) = -B\ sech^2\ \frac{\chi}{m}, \quad B = \frac{v(c-1)}{2(c+1)}, \quad m = \frac{2}{\sqrt{v}}.$$

Here, the sign of the coefficient $B$ depends on the sign of the reflection coefficient $C_R = \frac{c-1}{c+1}$, and is negative if $c < 1$. In this case the reflected wave field does not contain any solitary waves, and the "initial" pulse degenerates into a dispersive wave train. If $c > 1$, there will be at least one reflected solitary wave accompanied by radiation.

If the structure is made of one and the same material, then the reflection coefficient $C_R = 0$, and there will be no leading order reflected wave.

# 5.3 Higher-order corrections

The expansions of the "strain" fields $u_x^-$ and $u_x^+$ are given by

$$\begin{aligned} u_x^- &= I_x(\xi_-, X) + R_x(\eta_-, X) \\ &+ \varepsilon \Big\{ 3[R\tilde{I}_{\xi_-} + 2\tilde{I}\tilde{R} + I\tilde{R}_{\eta_-}] \\ &+ \frac{1}{1+c} g(\eta_-, X) \Big\}, \end{aligned} \tag{19}$$

and

$$u_x^+ = T_x(\xi_+, X) - \varepsilon \frac{1}{c(1+c)} g(-\frac{\xi_+}{c}, X), \tag{20}$$

respectively. Here, the corrections in (19) describe diffraction in the vicinity of the "jump", and higher-order correction to the reflected wave, while (20) gives higher-order correction to the transmitted wave.

# 5.3 Higher-order corrections

There is no leading order reflected wave if $c = 1$. However, a small reflected wave exists in higher-order corrections to the "strain" field $u_x^-$:

$$u_x^- = I_x(\xi_-, X) + \varepsilon r(\eta_-, X), \quad \text{where}$$
$$r(\eta_-, X) = \frac{v^2}{8} \operatorname{sech}^4 \frac{\sqrt{v}}{2}(\eta_- + vX) \times$$
$$\left[1 + 2\gamma - 3\beta + (1-\gamma)\cosh\sqrt{v}(\eta_- + vX)\right].$$

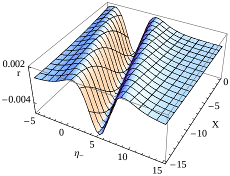


Figure: Higher-order reflected wave $r(\eta_-, X)$ for $c = 1, \beta = 1, \gamma = 5/8$, $v = 0.35$.

There is also a similar correction to the transmitted "strain" wave field.

# 6. Experiments

Experiments in the Ioffe Institute of the Russian Academy of Sciences, St. Petersburg.

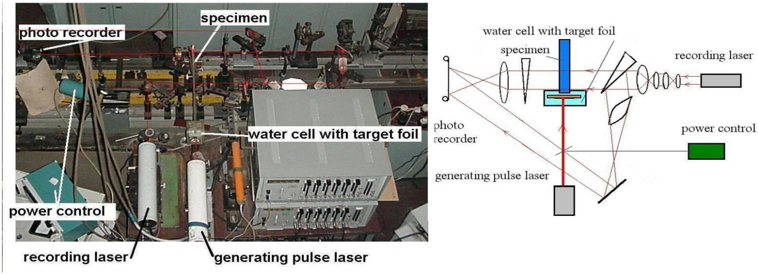


Figure: Experimental set-up.

# 6. Experiments

## Holographic interferograms:

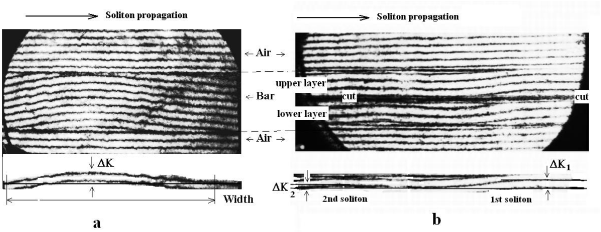


Figure: Experimentally observed secondary solitons in a two-layered PMMA bar.

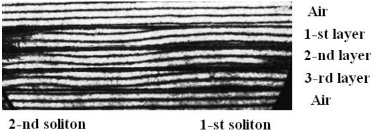


Figure: Experimentally observed secondary solitons in a three-layered PMMA bar.

# 6. Experiments

Theoretical prediction for the ratio of the amplitude of the lead "strain" solitary wave in the delaminated area to the amplitude of the incident "strain" solitary wave in the bonded area:

$$C_T^{A_1} = \frac{\gamma}{4}\left(\sqrt{1+\frac{8}{\gamma}}-1\right)^2, \quad \gamma = \frac{n^2+\kappa^2}{n^2(1+\kappa^2)}$$

Remark: The results have been generalized to the case of a symmetric $n$-layered elastic bar.

The increase of the soliton amplitude is detectable (10-20 %, Dreiden et al., to appear in *Strain*).

An expanded and refined experimental programme is in progress.

# Concluding remarks

Concluding remarks:

- The approach can be applied to other equations with piecewise-constant coefficients.
- The effect of fission of a strain soliton might be useful for the NDT of certain layered structures.

Current work:

- various problems of multiple delamination areas, both for perfect and imperfect interfaces, for other types of incident waves, and for more complicated cases of asymmetric layered structures.

Question: Does anybody know anything about the spectrum of a Schrödinger operator with the potential in the form of an N-soliton solution of the KdV equation multiplied by a constant?

# References and acknowledgements

References:


- K.R. Khusnutdinova, A.M. Samsonov, *Fission of a longitudinal strain solitary wave in a delaminated bar*, Phys. Rev. E. **77**, 066603 (2008).
- G.V. Dreiden, K.R. Khusnutdinova, A.M. Samsonov, I.V. Semenova, *Longitudinal strain solitary wave in a a two-layered polymeric bar*, Strain (2008), *in press*.



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