An outlook on two different approaches to the quantum phase problem

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How does the classical notion of phase apply to quantum systems?

Example 1 *The classical decomposition of the electric field into amplitude and phase components follows by simply writing the complex c-number mode expansion coefficient in its polar form:*

$$\vec{E}(\vec{r},t) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 L^3}} \sum_k \left[a_{\vec{k}}\, e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)} + c.c\right] = \sqrt{\frac{2\hbar\omega}{\varepsilon_0 L^3}} \sum_k |a_{\vec{k}}| \, \cos\left(\vec{k}\cdot\vec{r} - \omega t + \varphi_{\vec{k}}\right)$$

.

1. *What is the meaning of the phase of the field at the quantum level?*
2. *Once defined, how it is to be measured?*
3. *How does one write down a quantum mechanical operator corresponding to the phase observable of the field?*

• In a QM-treatment $a_{\vec{k}} \to \hat{a}_{\vec{k}}$ and $a^*_{\vec{k}} \to \hat{a}^\dagger_{\vec{k}}$, with $\left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}\right] = \delta_{\vec{k},\vec{k}'}\hat{1}$; the polar decomposition is no longer straightforward. Decompose each mode operator into Hermitian amplitude and phase components as (Dirac, 1927):

$$\hat{a} = e^{i\hat{\phi}}\, \hat{N}^{\frac{1}{2}} \quad \to \quad e^{i\hat{\phi}} = \hat{a}\, \hat{N}^{-\frac{1}{2}} \qquad , \quad \hat{N} = \left(\hat{a}^\dagger \hat{a}\right)^{1/2} \tag{1}$$

$[e^{i\hat{\phi}}, \hat{N}] = e^{i\hat{\phi}}$ (Lerner criterion) would result into $[\hat{N}, \hat{\phi}] = i$; $\hat{N}$ and $\hat{\phi}$ are canonically conjugate operators and $\Delta\hat{N}\,\Delta\hat{\phi} \geq \frac{1}{2}$. But *upon closer examination*

$$(n-m)\,\langle n|\hat{\phi}|m\rangle \; = i\delta_{n,m} \tag{2}$$

since one has both $\langle n|[\hat{N},\hat{\phi}]|m\rangle = i\delta_{n,m}$ and $\langle n|\hat{N}\hat{\phi} - \hat{\phi}\hat{N}|m\rangle = (n-m)\,\langle n|\hat{\phi}|m\rangle$. The diagonal matrix elements have therefore contradictory properties since we are lead to the equation $0 = i$!

Conclusion 2 *There is something wrong with the Dirac approach to the problem.*

Sources of difficulties

• Pauli's argument: there is no self-adjoint operator canonically conjugate to a Hamiltonian if the Hamiltonian spectrum is bounded from below.

• Within the Dirac framework the phase operator $\hat{\phi}$ cannot be a Hermitian operator. The spectrum of $\hat{N}$ does not extend to negative values and $e^{i\hat{\phi}}$ as defined via (1) *is not unitary* (a property needed to obtain $\hat{a} = e^{i\hat{\phi}}\,\hat{N}^{\frac{1}{2}}$, since $\hat{N} = \hat{a}^\dagger a = R^\dagger \left(e^{-i\hat{\phi}}\right)^\dagger e^{i\hat{\phi}} R$). Besides, is not possible to divide both sides of an operator of the type $\hat{a} = \hat{u}\hat{N}^{1/2}$ by $\hat{N}^{1/2}$ carelessy; we can use $\hat{N}_-^{1/2}$, being

$$\hat{N}_- = \sum_{n=1}^{\infty} n^{-1}|n >< n|$$

the pseudo-inverse of $\hat{N}$. The operator $\hat{u}$ is not defined uniquely.

Outlining ways out

• Introduction of *negative number states*? For instance (Pegg-Barnett)

$$e^{i\hat{\phi}} = \sum_{n=-\infty}^{\infty} |n\rangle \langle n+1| \quad ,$$

Unitarity of $e^{i\hat{\phi}}$, $\left(e^{i\hat{\phi}}\right)^\dagger e^{i\hat{\phi}} = \sum_{n=-\infty}^{\infty} |n+1\rangle \langle n+1| = \hat{1}$, and hermiticity of $\hat{\phi}$, would follow. However, one would be still faced with relevant problems.

• When dealing with angle operators, δ-functions should be actually introduced into the basic commutation relation:

$$[\hat{N}, \hat{\phi}] = i\, - 2\pi i\, \delta\,(\phi - \pi) \qquad , \quad -\pi < \phi \leq \pi \quad ; \qquad (3)$$

$\langle n|[\hat{N},\hat{\phi}]|m\rangle = \langle n|i\, - 2\pi i\, \delta(\phi - \pi)|m\rangle$ provides a mathematically consistent relation ($\langle\phi|n\rangle \propto e^{in\phi}$), and reduces to the canonical form $[\hat{N}, \hat{\phi}] = i$ except at one boundary of the domain of $\hat{\phi}$.

• Working with *periodic functions* of the operator rather than with the operator itself? Proposal by Susskind and Glogower connecting the operators

$$\hat{E} \equiv \left(\hat{N}+1\right)^{-\frac{1}{2}} \hat{a} \qquad , \quad \hat{E}^{\dagger} \equiv \hat{a}^{\dagger} \left(\hat{N}+1\right)^{-\frac{1}{2}} \quad ,$$

analogs of $e^{\pm i\hat{\phi}}$, and the basic field operator $\hat{a}$. Analogs of cosine and sine operators are defined according to

$$\hat{C} = \frac{1}{2}\left(\hat{E}+\hat{E}^{\dagger}\right) \quad , \qquad \hat{S} = \hat{C} = \frac{1}{2i}\left(\hat{E}-\hat{E}^{\dagger}\right)$$

Theory with some remaining formal problems. One may expect problems with the unitarity of the operators. Indeed

$$\hat{E}\hat{E}^{\dagger} = \hat{1} \quad , \qquad \hat{E}^{\dagger}\hat{E} = 1 - |0\rangle\langle 0| \quad ,$$

$$\left[\hat{C}, \hat{N}\right] = i \quad , \quad \hat{S}\left[\hat{S}, \hat{N}\right] = -i\hat{C} \quad , \quad ,$$

$$\left[\hat{C}, \hat{S}\right] = \frac{1}{2}|0\rangle\langle 0| \quad , \quad \hat{C}^2 + \hat{S}^2 = \hat{1} - \frac{1}{2}|0\rangle\langle 0|$$

Remark 3 ***The vacuum-state projector spoils the unitarity of $\hat{E}$.***

The SG operator can be made unitary on a smaller Hilbert space by introducing an upper limit $n_{\max}$ on the number of allowed modes. The dimension of the Hilbert space is allowed to tend to infinity only after $n_{\max}$-phase statistics have been calculated. In states with large average number occupations $<\hat{N}>= \bar{n} \gg 1$, the operator $\hat{E}$ may be treated as approximately unitary.

An overcomplete set of eigenfunctions for $\hat{E}\left|e^{i\phi}\right\rangle = e^{i\phi}\left|e^{i\phi}\right\rangle$ can be found of the form

$$\left|e^{i\phi}\right\rangle = \sum_{n=0}^{\infty} e^{in\phi}\,|n\rangle \quad , \qquad \hat{1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\,\left|e^{i\phi}\right\rangle\left\langle e^{i\phi}\right|$$

Another set of eigenstates (*coherent phase states*): $|\xi\rangle = \sqrt{1-|\xi|^2}\sum_{n=0}^{\infty}\xi^n\,|n\rangle$, $(|\xi| < 1)$.

• Owing to the difficulties in defining self-adjoint phase operator through Poisson brackets quantization, one may wonder if phase (more generally, an angle variable) does correspond to a proper quantum variable. A multiplicity of quantum phase concepts have been proposed and investigated in the years (see e.g. topical reviews by a) Carruthers and Nieto, 1968; b) Bergou and Englert, 1991; c) Lynch, 1995; d) Pegg-Barnett, 1997; and books by i) Perinova, Luks and Perina, 1998; ii) Dubin, Hennings and Smith, 2000). These concepts find important applications in quantum measurements, quantum communication, quantum cryptography, BEC, etc.

• The *canonical phase distribution* for a quantum state described by the density operator $\hat{\rho}$

$$dP(\varphi) = \frac{d\varphi}{2\pi} \sum_{n,m=0} \rho_{nm} \exp\left[i(n-m)\,\varphi\right] \qquad (4)$$

plays a special role. It has been derived on very different grounds (London, Helstrom, Holevo etc). By using the full statistical content of Born' statistical rule, $dP\left[\hat{\rho}(z)\right] = Tr\ \{\hat{\rho}\, d\hat{\mu}(z)\}$, a *canonical quantum phase* for the quantized harmonic oscillator can be uniquely defined from correspondence principle in the context of elementary quantum mechanics (Paris, 1997).

Remark 4 *Another way to both enlarge the 1-photon Hilbert space and to allow for a a "negative number" of photon states would be possible if we consider the original mode interacting with an appropiate apparatus. Consider the (Shapiro-Wagner) operator* $\hat{Y}_{SW} = \hat{a} + \hat{b}^{\dagger}$*, where* $\hat{a}$ *denotes the annihilation operator for a photon signal mode and* $\hat{b}^{\dagger}$ *by a creation operator of an image mode. Then we have a system composed of two independent and distinguishable subsystems in the extended Hilbert space* $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$*. Signal and image modes can be described by the complete orthogonal discrete bases of Fock states* $|m>_A \otimes |n>_B$*,* $m, n = 0, 1, ...$*, or alternatively by means of the Relative Number State* (RNS) *representation*

$$|n, m >>= \Theta(n)|m+n>_A\ |m>_B + \Theta(-n-1)|m>_A\ |m-n>_B, \qquad (5)$$

where $-\infty < n < \infty$*,* $m \geq 0$*,* $\Theta(n) = 1$ *for* $n \geq 0$ *and* $\Theta(n) = 0$ *for* $n < 0$*, so that* $|n-m, min(m,n) >>= |m>_A\ |n>_B$ *.Basis of the RNS's is complete and orthonormal, and*

$$\hat{N}|n, m >>= n|n, m >> \quad , \qquad \hat{N} = \hat{a}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}. \qquad (6)$$

The spectrum of $\hat{N}$ is unbounded and an unitary operator $\hat{D}$ exists on the Hilbert space $\mathcal{H}$ obeying $\hat{D}\hat{D}^\dagger = \hat{D}^\dagger\hat{D} = \hat{1}$ and $[\hat{D}, \hat{N}] = \hat{D}$. Precisely,

$$\hat{D} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} |n-1, m >>< < m, n| \quad . \tag{7}$$

The commutation rule $[\hat{Y}_{SW}, \hat{Y}_{SW}^\dagger] = 0$ tells us that joint measurement of the real and imaginary part of the operator can be performed, e.g. through an heterodyne apparatus (Shapiro-Wagner).

• A consistent progress in better approaching the phase problem follows by distinguishing two ways to proceed while defining quantum mechanical phase operators: one based on the polar decomposition of the annihilation operator of a photon (*ideal phase*), and the other based on the use of phase-measurement processes (*feasible phase*). Concepts of ideal and feasible phases may be linked by resorting to the definition of *generalized measurements* and introduction of positive operator valued measures (*POVM*'s).

Criterion 5 *One way to generalize the measurement concept beyond the (Von Neumann) QM orthogonal measurements is to suppose that a quantum system A is a part of another system $S = A + B$ so that the Hilbert space associated with A is a part of a larger space that has the structure of a direct sum,*

$$\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^\perp \quad .$$

Observers O_A in $\mathcal{H}_A$ have access only to observables $\hat{M}_A$ with support in $\mathcal{H}_A$, for which

$$\hat{M}_A \left|\psi^\perp\right\rangle = \left(\hat{M}_A \left|\psi^\perp\right\rangle\right)^\dagger = 0 \quad , \qquad \forall \left|\psi^\perp\right\rangle \in \mathcal{H}_A^\perp \, .$$

Orthogonal measurements one might perform in the tensor product will not necessarily be orthogonal measurements in A alone. Observer O_A will see only components of that state in $\mathcal{H}_A$. These are not necessarily orthogonal in $\mathcal{H}_A$ and he concludes that the measurement prepares only a set of non-orthogonal states.

• Quantum observables are generally positive operator valued measures. Phase operators are POVM's that transform covariantly under time translations.

• *A POM is necessarily a partial trace of a PVM coming from a selfadjoint operator defined on a larger Hilbert space.*

Theorem 6 (Naimark) *Let $\mathcal{H}_S$ the Hilbert space of a physical system S and let $d\hat{\mu}(y)$ a POM defined on $\mathcal{H}_S$. Then it is possible to prove the existence of:*

1. *a Hilbert space $\mathcal{H} \supset \mathcal{H}_S$ with $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_S$;*
2. *a selfadjoint operator $\hat{Y}$ in $\mathcal{H}$*

$$\hat{Y} \;\; |y\rangle = y \quad |y\rangle \quad , \qquad |y\rangle \in \mathcal{H} \quad , \quad \langle y|y'\rangle = \langle z|z'\rangle = \delta_{\mathcal{Y}}\,(y - y') \quad ;$$

3. *a density operator $\hat{\rho}_P \in \mathcal{H}_P$ in the complement space such that given the PVM $d\hat{E}(y) = |y\rangle\,\langle y|\;\; dy$ one has*

$$d\hat{\mu}(y) = Tr_P\,\left\{\hat{\rho}_P \otimes \hat{1}_A \;\; d\hat{E}(y)\right\} \quad .$$

Generalized measurements of multiboson linear operators

• Interactions that are linear and bilinear in the field modes play a major role in quantum information, and can be experimentally realized in optical and condensate systems.

• The quantum optical problem of simultaneous detection of quadratures is that of simultaneous measurement of a pair of conjugate observables. The quadratures obtained provide the action and phase-angle variables via a polar transformation.

Problem 7 *Generalized measurements of the multimode operator*

$$\hat{Z}^{(m_1,m_2)} = \sum_{k_1=1}^{m_1} A_{k_1}\,\hat{a}_{k_1} + \sum_{k_2=1}^{m_2} B_{k_1}\hat{a}^{\dagger}_{m_1+k_2} \qquad ,$$

$$\left[\hat{Z}^{(m_1,m_2)}, \hat{Z}^{(m_1,m_2)\,\dagger}\right] = \sum_{k_1=1}^{m_1} |A_{k_1}|^2 - \sum_{k_2=1}^{m_2} |B_{k_1}|^2 \neq 0$$

$\left(\left[\hat{a}_r, \hat{a}_s^\dagger \right] = \delta_{rs} \hat{1},\ r, s = 1, \ldots, m = m_1 + m_2 \right)$

Solution 8 *Introduce proper Naimark modes and device likely experimental setups.*

Define

$$\hat{X}^{(m_1,m_2)} = \frac{(\hat{Z}^{(m_1,m_2)} + h.c)}{2} \quad , \quad \hat{P}^{(m_1,m_2)} = \frac{(\hat{Z}^{(m_1,m_2)} - h.c)}{2i}$$

$$[\hat{X}^{(m_1,m_2)}, \hat{P}^{(m_1,m_2)}] = \frac{i}{2} [\hat{Z}^{(m_1,m_2)}, \hat{Z}^{(m_1,m_2)\,\dagger}] \neq 0$$

and

$$\hat{Z}_N^{(m_1,m_2)} = \hat{Z}^{(m_1,m_2)} + \vec{C}_-\, \hat{a}_0 + C_+ \hat{a}_{m+1}^\dagger \quad , \qquad |C_-|^2 - |C_+|^2 = \sum_{k_2=1}^{m_2} |B_{k_1}|^2 - \sum_{k_1=1}^{m_1} |A_{k_1}|^2$$

$$\hat{X}_N = \frac{(\hat{Z}_N^{(m_1,m_2)} + h.c)}{2} \quad , \quad \hat{P}_N = \frac{(\hat{Z}_N^{(m_1,m_2)} - h.c)}{2i}$$

so that

$$\left[\hat{X}_N^{(m_1,m_2)}, \hat{P}_N^{(m_1,m_2)} \right] = \frac{i}{2} \left[\hat{Z}_N^{(m_1,m_2)}, \hat{Z}_N^{(m_1,m_2)\,\dagger} \right] = 0$$

• Minimal Naimark extensions of currents $\hat{Z}^{(m_1,m_2)}$: introduce just one additional mode. That is, take

$$C_- = 0 \quad , \qquad C_+ = \sqrt{\sum_{k_1=1}^{m_1} |A_{k_1}|^2 - \sum_{k_2=1}^{m_2} |B_{k_1}|^2}$$

if $sign([\hat{Z}^{(m_1,m_2)}, \hat{Z}^{(m_1,m_2)\,\dagger}]) > 0$, otherwise

$$C_- = \sqrt{\sum_{k_1=1}^{m_1} |B_{k_1}|^2 - \sum_{k_2=1}^{m_2} |A_{k_1}|^2} \quad , \qquad C_+ = 0$$

Remark 9 *The feasible phase for the operator* $\hat{Z}^{(m_1,m_2)}$ *is*

$$\hat{\theta}_N^{(m_1,m_2)} = \frac{1}{2i} \ln \frac{\hat{Z}_N^{(m_1,m_2)}}{\left(\hat{Z}_N^{(m_1,m_2)}\right)^\dagger} .$$

Cosine and sine quadrature operators

$$\hat{C} = \frac{\exp[i\hat{\theta}_N^{(m_1,m_2)}] + \exp[-i\hat{\theta}_N^{(m_1,m_2)}]}{2} , \quad \hat{S} = \frac{\exp[i\hat{\theta}_N^{(m_1,m_2)}] - \exp[-i\,\hat{\theta}_N^{(m_1,m_2)}]}{2i}$$

obey the correct relation $\hat{C}^2 + \hat{S}^2 = 1$.

• Introduction of the $(m_1 + m_2 + 1)$-mode relative number operator

$$\hat{N} = \sum_{k=1}^{m_2+1} \hat{N}_{m_1+k} - \sum_{k=1}^{m_1} \hat{N}_{m_1+k} , \qquad \hat{N}_k = a_k^\dagger a_k$$

$(k = 1, 2, 3)$, yields to CCR that one expects for genuine phase operators.

• Measurement scheme: modes $\hat{a}_k$ interact each other through an unitary operator $\hat{U}^{(m_1,m_2)}$, which imposes the linear transformation

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \dots \\ \hat{b}_{m_1+m_2+1} \end{pmatrix} = \hat{U}^\dagger_{(m_1,m_2)} \begin{pmatrix} \hat{a}_1 \\ \dots \\ \hat{a}_{m_1+m_2} \\ \hat{a}_N \end{pmatrix} \hat{U}_{(m_1,m_2)} = M^{(m_1,m_2)} \begin{pmatrix} \hat{a}_1 \\ \dots \\ \hat{a}_{m_1+m_2} \\ \hat{a}_N \end{pmatrix} .$$

Quadratures of output modes have to be measured to obtain

$$Tr_{(m_1,m_2)}[R\,\hat{X}^{(m_1,m_2)}] = Tr_{(m_1,m_2,1)}[R \otimes \sigma\,\hat{X}_N^{(m_1,m_2)}]$$
$$Tr_{(m_1,m_2)}[R\,\hat{Y}^{(m_1,m_2)}] = Tr_{(m_1,m_2,1)}[R \otimes \sigma\,\hat{Y}_N^{(m_1,m_2)}]$$

for any state $R \in \mathcal{H}_{(m_1,m_2)}$.

• The rank $m + 1$ transfer matrix M should be implemented in practice, as for example in a quantum optical setting. For modes of the radiation field, the simplest two-mode interaction is the corresponds to a beam splitter.

Problem 10 *Can this kind of extensions be implemented* using only bilinear interactions *among modes followed by measurement of quadratures at the output?*

Solution 11 *Decompose* M *into a set of* $SU(2)$ *transformation (BS's).*

• Consider the case $\left[\hat{Z}^{(m_1,m_2)}, \hat{Z}^{(m_1,m_2)\dagger}\right] > 0$ (i.e.$C_- = 0$). We can pay attention to the case of real positive couplings $A_1, \ldots, A_{m_1}, B_1, \ldots, B_{m_2}$, without loss of generality. Define

$$M_+^{(m_1,m_2)} = \begin{pmatrix} A_1 & \ldots & A_{m_1} & B_1 & \ldots & B_{m_2} & C_+ \\ A_1 & \ldots & A_{m_1} & -B_1 & \ldots & -B_{m_2} & -C_+ \\ M_{31}^{(m_1,m_2)} & \ldots & M_{3,m_1}^{(m_1,m_2)} & M_{3,m_1+1}^{(m_1,m_2)} & \ldots & M_{3,m_1}^{(m_1,m_2)} & M_{3,m+1}^{(m_1,m_2)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ M_{m+1,1}^{(m_1,m_2)} & \ldots & M_{m+1,m_1}^{(m_1,m_2)} & \ldots & \ldots & \ldots & M_{m+1,m+1}^{(m_1,m_2)} \end{pmatrix}$$

where $m = m_1 + m_2$, and $C_+ = \sqrt{\sum_{k_1=1}^{m_1} A_{k_1}^2 - \sum_{k_2=1}^{m_2} B_{k_1}^2})$.

$$M_+^{(m_1,m_2)} M_+^{(m_1,m_2)\,T} = \hat{1}_{m+1,m+1} \quad .$$

From

$$\left(M_+^{(m_1,m_2)} M_+^{(m_1,m_2)\,T}\right)_{nr} = 1 \quad , \qquad n = 1, 2, r$$

one gets

$$\sum_{k_1=1}^{m_1} A_{k_1}^2 = \frac{1}{2} \quad , \quad \sum_{r=1}^{m_1} A_r M_+^{(m_1,m_2)}{}_{k,r} = 0 \quad , \quad \sum_{r=1}^{m+1} M_+^{(m_1,m_2)2}{}_{k,r} = 1$$

and

$$M_+^{(m_1,m_2)}{}_{k,m+1} = -\frac{1}{C_+} \sum_{r=1}^{m_2} B_r M_+^{(m_1,m_2)}{}_{k,m_1+r}$$

Set

$$\mathfrak{M}_+^{(m_1,m_2)} = \frac{M_+^{(m_1,m_2)}}{2\sum_{k_1=1}^{m_1} A_{k_1}^2} = \begin{pmatrix} \alpha_1 & \ldots & \alpha_m & \alpha_{m+1} \\ \alpha_1 & \ldots & -\alpha_m & -\alpha_{m+1} \\ \beta_{31}^{(m_1,m_2)} & \ldots & \ldots & \beta_{3,m+1}^{(m_1,m_2)} \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{m+1,1}^{(m_1,m_2)} & & & \beta_{m+1,m+1}^{(m_1,m_2)} \end{pmatrix}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{m+1}) = \frac{1}{2\sum_{k_1=1}^{m_1} A_{k_1}^2}(A_1, \ldots, A_{m_1}, B_1, \ldots, B_{m_2}, C_+)$ so that $\sum_{k_1}^{m_1} \alpha_{k_1}^2 = \frac{1}{2}$.

For instance:

- $m_1 = 4, m_2 = 0$; $(\alpha_5 = \xi_4 = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2} = \frac{1}{\sqrt{2}})$

$$\mathfrak{M}^{(4,0)} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \alpha_1 & \alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 \\ -\frac{\xi_2^+}{\xi_1^+}. & \frac{\alpha_1\alpha_2}{\xi_1^+\xi_2^+} & \frac{\alpha_1\alpha_3}{\xi_1^+\xi_2^+} & \frac{\alpha_1\alpha_4}{\xi_1^+\xi_4^+} & 0 \\ 0 & -\frac{\xi_3^+}{\xi_2^+} & \frac{\alpha_2\alpha_3}{\xi_2^+\xi_3^+} & \frac{\alpha_2\alpha_4}{\xi_2^+\xi_3^+} & 0 \\ 0 & 0 & -\frac{\xi_4^+}{\xi_3^+} & \frac{\alpha_3\alpha_4}{\xi_3^+\xi_4^+} & 0 \end{pmatrix}$$

- $m_1 = 3, m_2 = 1$; $(\alpha_5 = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2} = \sqrt{\frac{1}{2} - \alpha_4^2})$.

$$\mathfrak{M}^{(3,1)} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \alpha_1 & \alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 \\ -\frac{\xi_2^+}{\xi_1^+} & \frac{\alpha_1\alpha_2}{\xi_1^+\xi_2^+} & \frac{\alpha_1\alpha_3}{\xi_1^+\xi_2^+} & 0 & 0 \\ 0 & -\frac{\xi_3^2}{\xi_2^+} & \frac{\alpha_2\alpha_4}{\xi_2^+\xi_3^+} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\xi_4}{\xi_3} & \frac{\alpha_4\alpha_5}{\xi_3\xi_4} \end{pmatrix} ,$$

$(\xi_k^+ = \sqrt{\sum_{\substack{s=k \\ e_s=1}} e_s\alpha_s^2},\ e_1 = e_2 = e_3 = -e_4 = 1)$

- $m_1 = 2, m_2 = 2$; $(\alpha_5 = \sqrt{\frac{1}{2} - \alpha_3^2 - \alpha_4^2})$

$$\mathfrak{M}^{(2,2)} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \alpha_1 & \alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 \\ -\frac{\xi_2^+}{\xi_1^+} & \frac{\alpha_1\alpha_2}{\xi_1^+\xi_2^+} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\xi_3}{\xi_2} & \frac{\alpha_3\alpha_4}{\xi_2\xi_3} & \frac{\alpha_3\alpha_5}{\xi_2\xi_3} \\ 0 & 0 & 0 & -\frac{\xi_4}{\xi_3} & \frac{\alpha_4\alpha_5}{\xi_3\xi_4} \end{pmatrix} ,$$

$(\xi_k^+ = \sqrt{\sum_{\substack{s=k \\ e_s=1}} e_s\alpha_s^2},\ e_1 = e_2 = -e_3 = -e_4 = 1)$

Example 12 *Consider the operator*

$$\hat{Z}^{(1,m_2)} = \alpha_1\hat{a}_1 + \alpha_2\hat{a}_2^\dagger + \cdots + \alpha_{m_2+1}\hat{a}_{m_2+1}^\dagger$$

with $\alpha_1^2 - (\alpha_2^2 + \cdots + \alpha_{m_2+1}^2) > 0,\ \alpha_k \in \mathbb{R}_+$. *Hence*

$$\hat{Z}_N^{(1,m_2)} = \alpha_1\hat{a}_1 + \alpha_2\hat{a}_2^\dagger + \cdots + \alpha_{m_2+1}\hat{a}_{m_2+1}^\dagger + C_+\hat{a}_N^\dagger$$

and

$$\mathfrak{M}_+^{(1,m_2)} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_{m_2+1} & \alpha_{m_2+2} \\ \alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & \dots & -\alpha_{m_2+1} & -\alpha_{m_2+2} \\ 0 & -\frac{\xi_2}{\xi_1} & \frac{\alpha_2\alpha_3}{\xi_1\xi_2} & \frac{\alpha_2\alpha_4}{\xi_1\xi_2} & \dots & \frac{\alpha_2\alpha_{m_2+1}}{\xi_1\xi_2} & \frac{\alpha_2\alpha_{m_2+2}}{\xi_1\xi_2} \\ 0 & 0 & -\frac{\xi_3}{\xi_2} & \frac{\alpha_2\alpha_3}{\xi_2\xi_3} & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -\frac{\xi_m}{\xi_{m-1}} & \frac{\xi_m}{\xi_{m-1}} \end{pmatrix},$$

with $\xi_1 = \alpha_1 = \frac{1}{\sqrt{2}},\ \xi_s = \sqrt{\alpha_1^2 - \sum_{k=2}^{s} \alpha_k^2},\ (s = 2, \dots, m_2 + 1)$. *It turns out that*

$$\hat{U}_{(m_1,m_2)} = \hat{\mathfrak{B}}_2\hat{\mathfrak{B}}_{m+1,m}\hat{\mathfrak{B}}_{m+1,m-1}\dots\hat{\mathfrak{B}}_{m+1,1}\hat{\mathfrak{B}}_{m,m-1}\hat{\mathfrak{B}}_{m,m-2}\dots\hat{\mathfrak{B}}_{2,1}$$

where

$$\hat{\mathfrak{B}}_2 = \left[\mathbb{I}_1 \otimes \hat{B}_2(\pi) \otimes \mathbb{I}_3 \otimes \dots \mathbb{I}_{m+1}\right], \qquad \hat{\mathfrak{B}}_{j,k} = \left[\hat{B}_{j,k}(\theta_{j,k}) \prod_{s\neq j,k} \otimes \mathbb{I}_s\right]$$

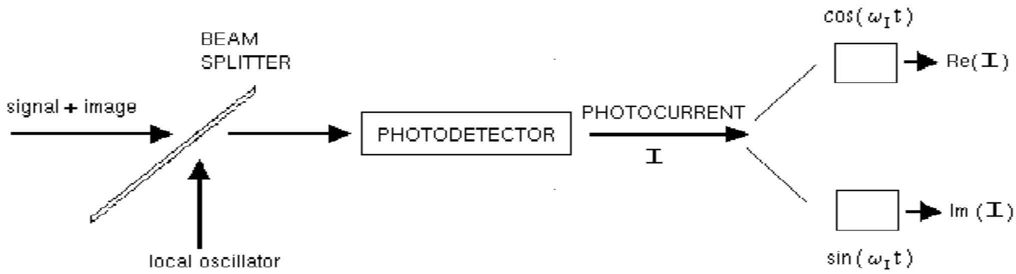
with

$$\hat{B}_2(\pi) = \exp\left(i\pi a_2^\dagger a_2\right), \quad \hat{B}_{jk}(\theta_{jk}) = \exp\left\{-i\theta_{jk}\left(\hat{a}_j\hat{a}_k^\dagger + \hat{a}_k\hat{a}_j^\dagger\right)\right\},$$

The needed beam splitters are identified via

$$\sin\theta_{k1} = \frac{\alpha_k}{\sqrt{\sum_{s=1}^{k} \alpha_s^2}}, \quad \sin\theta_{rj} = \frac{\alpha_r\alpha_j}{\sqrt{\sum_{s=j-1}^{r-1} \alpha_s^2}\sqrt{\sum_{s=j}^{r} \alpha_s^2}},$$

$$(2 < k \leq m+1,\ 2 \leq j \leq m, j < r \leq m).$$



Revisiting heterodyne detection

Heterodyne detection allows to perform the joint measurement of two conjugate quadratures of according to the scheme depicted above (Shapiro and Wagner, 1984). A single-mode signal field E_1 of nominal frequency ω_1 is mixed through a beam-splitter with a local oscillator field E_L whose frequency ω_L is slightly offset by an amount $\omega_I \ll \omega_1$ from that of the input signal, i.e. $\omega_1 = \omega_L + \omega_I$. A photodetector is placed right after the beam-splitter The output photocurrent, which generally depends on fields parameters and on specific assumptions on the apparatus, is filtered at the *intermediate frequency* ω_I. In standard optical heterodyne detection (Shapiro and Wagner), measuring the filtered photocurrent corresponds to realize the quantum measurement of the normal operator $\hat{Y}_{SW} = a_1 + a_2^\dagger$, where $\hat{a}_1$ (res. $\hat{a}_2^\dagger$) denotes the photon annihilator (resp. creation) operator for the input (resp. image) signal. Measuring the real and imaginary parts of the (actually rescaled) output photocurrent thus provides the simultaneous measurement of both input field quadratures.

Remark 13 *Whenever one is not restricted to an input field frequency in the optical regime, but, rather, one is concerned with microwave (or radio) heterodyning, then the interaction of the input signal field with the apparatus (approximatively) results in a different measurement operator. Quite a different situation arises when heterodyne detector is not of the SW type, but rather a power-detector for which the measurement operator takes the form (Caves,1984)*

$$\hat{Y}_C = \sqrt{(1+\frac{\omega_I}{\omega_1})}a_1 + \sqrt{(1-\frac{\omega_I}{\omega_1})}a_2^\dagger \quad , \quad [\hat{Y}_C, \hat{Y}_C^\dagger] = 2\frac{\omega_I}{\omega_1} \neq 0 \qquad (8)$$

The standard heterodyne detection cannot achieve the simultaneous measurements of signal quadratures of $\hat{Y}_C$.

Problem 14 *Accomplish simultaneous phase and amplitude measurements for the non-normal operator*

$$\hat{Z}_\gamma = a_1 + \gamma\, a_2^\dagger \quad , \quad \gamma = \sqrt{\frac{\omega_1 - \omega_I}{\omega_1 + \omega_I}} < 1 \qquad , \quad \left[Z_\gamma, Z_\gamma^\dagger\right] = 1 - \gamma^2 \tag{9}$$

Solution 15 *Exploit previous results to realize a suitable linear amplifier.*

Let

$$X_\gamma = \frac{1}{2}\left(Z_\gamma + Z_\gamma^\dagger\right) = \frac{1}{\sqrt{2}}\left(q_1 + \gamma q_2\right) \quad , \quad Y_\gamma = \frac{1}{2}\left(Z_\gamma + Z_\gamma^\dagger\right) = \frac{1}{\sqrt{2}}\left(p_1 - \gamma p_2\right) \quad ,$$

$$q_k = \frac{1}{\sqrt{2}}\left(a_k^\dagger + a_k\right) \quad p_k = \frac{i}{\sqrt{2}}\left(a_k^\dagger - a_k\right) \quad [q_j, p_k] = i\delta_{jk}$$

($k = 1, 2$). Since $[X_\gamma, Y_\gamma] = \frac{i}{2}(1 - \gamma^2)$, then these sum- and difference-quadratures of the two modes can be jointly measured only when a generalized measurement is be devised. Indeed, eigenstates of Z_γ for $\gamma \neq 1$

$$\begin{aligned} &|z\rangle\rangle_\gamma = D(z) \otimes \mathbb{I}\, |\gamma\rangle\rangle .. \quad , \\ &D(z) = \exp\{z a_1^\dagger - z^* a_1\} \quad , \quad |\gamma\rangle\rangle = \sqrt{1-\gamma^2} \sum_n \gamma^n |n\rangle \otimes |n\rangle \end{aligned}$$

do not provide a resolution of the identity,

$$\int \frac{d^2 z_\gamma}{\pi} : |z\rangle\rangle_{\gamma\gamma}\langle\langle z| = (1-\gamma^2)\, \gamma^{2a^\dagger a}$$

The operator Z_γ is defined on the Hilbert-Fock space $\mathcal{H}_{12}$ of two harmonic oscillators. A Naimark extension for the operator Z_γ is a triplet $\left(\mathcal{H}_a, \hat{Z}_N, \sigma\right)$, where $\hat{Z}_N$ is an operator defined on an extended Hilbert space $\mathcal{H}_{12} \otimes \mathcal{H}_a$ and σ is a state (density operator) in $\mathcal{H}_a$, such that for any state $R \in \mathcal{H}_{12}$ we have

$$\begin{aligned} \mathrm{Tr}_{12}\left[R : X_\gamma\right] &= \mathrm{Tr}_{12a}\left[R \otimes \sigma \mathrm{Re}\ \hat{Z}_N\right] \\ \mathrm{Tr}_{12}\left[R : Y_\gamma\right] &= \mathrm{Tr}_{12a}\left[R \otimes \sigma \mathrm{Im}\ \hat{Z}_N\right] \end{aligned} \tag{10}$$

Equations (10) ($\hat{Z}_N$ *traces* the operator $\hat{Z}_\gamma$) do not hold for higher moments:

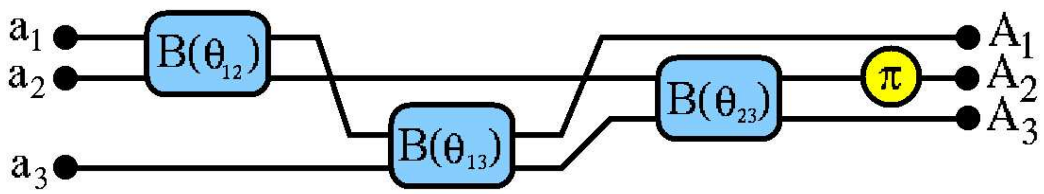
Remark 16 *The generalized measurement of $\hat{Z}_\gamma$ unavoidably introduces some noise of purely quantum origin. In general we have*

$$Tr_{12}\left[R\,X_\gamma^n\right] \neq Tr_{12a}\left[R\otimes\sigma\,(Re\;\hat{Z}_N)^n\right] \quad n\geq 2$$
$$Tr_{12}\left[R\,Y_\gamma^n\right] \neq Tr_{12a}\left[R\otimes\sigma\,(Im\;\hat{Z}_N)^n\right] \quad n\geq 2 \tag{11}$$

Minimal Naimark extension involves just a single additional bosonic mode a_3, $\hat{Z}_N = \hat{Z}_\gamma + \kappa\hat{a}_3^\dagger$. The trace condition of Eqs. (10) and the normality constraint for $\hat{Z}_N$ require

$$\mathrm{Tr}_a\left[\sigma\,a_3^\dagger\right]=0 \quad , \qquad \kappa^2 = 1-\gamma^2 \quad .$$

Naimark extensions have been implemented using only bilinear interactions among modes followed by measurement of quadratures at the output Precisely, in the case under consideration the $SU(2)$ transformations $\hat{B}_{2,1}$, $\hat{B}_{3,1}$, $\hat{B}_{3,2}$, followed by a π-rotation ($R_2 = \exp\{i\pi a_2^\dagger a_2\}$) are needed. For $\gamma\to 1$ the mode a_3 decouples from the other two modes and the scheme reduces to the standard joint measurement of quadratures for the normal operator $Z_1 = a_1 + a_2^\dagger$.



Each outcome from the joint measurement of the quadratures Q_1 and P_2 corresponds to a complex number $\tau = Q_1 + iP_2$ that represents a realization of the observable $\hat{Z}_\gamma$. The probability density of the outcomes $K_\gamma(\tau)$ for a given initial preparation $R\otimes\sigma$ is obtained as the Fourier transform of the moment generating function $\Xi(\lambda)$,

$$K_\gamma(\tau) = \int \frac{d^2\lambda}{\pi^2} e^{\bar{\lambda}\tau-\lambda\bar{\tau}}\,\Xi(\lambda), \qquad \Xi(\lambda) = \mathrm{Tr}\left[R\otimes\sigma\,\exp\left(\lambda\hat{Z}_N{}^\dagger - \bar{\lambda}\hat{Z}_N\right)\right]$$

$$\exp\{\lambda\hat{Z}_N{}^\dagger - \bar{\lambda}\hat{Z}_N\} = D_1(\lambda)\otimes D_2(-\lambda\gamma)\otimes D_3(-\lambda\kappa)$$

where $D_j(z)$ is the displacement operator for the mode a_j. Therefore, the moment generating function rewrites as

$$\Xi_\gamma(\lambda) = \chi_{12}(\lambda)\ \chi_3(-\lambda\kappa) \quad , \quad \chi_{12}(\lambda) = \mathrm{tr}\,[R\, D_1(\lambda) \otimes D_2(-\lambda\gamma)]$$

and $\chi_3(z) = \mathrm{Tr}[\sigma\, D_3(z)]$ is the characteristic function of the mode a_3. The probability density of the outcomes is given by the convolution

$$K_\gamma(\tau) = \frac{1}{\kappa^2}\, H_\gamma(\tau) \star W_3(-\tau/\kappa) \quad , \tag{12}$$

$W_3(z)$ being the Wigner function of the mode a_3, $\star$ the convolution product, and $H_\gamma(z)$ the density obtained by the Fourier transform of $\chi_{12}(\lambda)$.

Remark 17 *For factorized preparations $R = \varrho_1 \otimes \varrho_2$ the moment generating function $\chi_{12}(\lambda) = \chi_1(\lambda)\, \chi_2(-\lambda)$ factorizes into the product of the characteristic functions of ϱ_1 and ϱ_2 respectively, and the density $H_\gamma(\tau)$ reduces to the convolution of the Wigner functions of the two input signals*

$$H_\gamma(\tau) = \frac{1}{\gamma^2}\, W_1(\tau) \star W_2(-\tau/\gamma)$$

Variances of the measured quantities Q_1 and P_2 are related to the variances of the quadratures of interest. We have

$$\Delta Q_1^2 = \Delta X_\gamma^2 + \frac{1}{2}(1-\gamma^2)\Delta q_3^2 \qquad , \qquad \Delta P_2^2 = \Delta Y_\gamma^2 + \frac{1}{2}(1-\gamma^2)\Delta p_3^2\,, \tag{13}$$

where $\Delta q_3^2 = Tr[\sigma\, q_3^2]$ and analogously $\Delta p_3^2 = Tr[\sigma\, p_3^2]$ (remind that Eq. (10) implies $Tr[\sigma\, q_3] = Tr[\sigma\, p_3] = 0$). Notice that the added noise in Eq. (13) is the minimum noise according to generalized uncertainty relations for joint measurement of non commuting observables (Yuen, 1982). On the other hand, the covariance between the measured quadratures i.e. *the quantity*

$$\Sigma_{Q_1P_2} = \frac{1}{2}\, Tr_{12a}\,[R \otimes \sigma\,(Q_1P_2 + P_2Q_1)] - Tr_{12a}\,[R \otimes \sigma\, Q_1]\; Tr_{12a}\,[R \otimes \sigma\, P_2]\ :,$$

may be written as

$$\Sigma_{Q_1P_2} = \Sigma_{X_\gamma Y_\gamma} - \frac{1}{2}(1-\gamma^2)\, Tr_a\left[\frac{1}{2}\sigma\,(p_3q_3 + q_3p_3)\right] \tag{14}$$

where $\Sigma_{X_\gamma Y_\gamma} = \frac{1}{2}\, Tr_{12}\,[R\,(X_\gamma Y_\gamma + Y_\gamma X_\gamma)] - Tr_{12}\,[R\, X_\gamma]\; Tr_{12}\,[R\, Y_\gamma]$ is the covariance of the desired quadratures.

Remark 18 *Notice that the added noise to the covariance, Eq. (14), may vanish for some preparation of the state σ whereas the added noise to the variances, Eq. (13), cannot vanish for any physical preparation σ. This raises the question of the consequences of different field states on the statistics of the measurement and, in turn, of the role played by preparations of states in concrete experiments.*

Within experimental frameworks, one may take full advantage of possible freedom in preparing some of the modes. This is definitively the case of the Naimark mode a_3, even though its preparation needs to be compatible with the prescription (10) for the expectation values of position and momentum operators. In particular, a valid Naimark extension can be obtained by preparing the mode a_3 in the vacuum state $\sigma = |0\rangle\langle 0|$ to let its contribution to the noise in formula (14) to vanish, since $\mathrm{Tr}_a[\sigma(q_3p_3 + p_3q_3)] = 0$, and to minimize Δq_3^2 and Δp_3^2 in (13), since both the terms would be equal to one half. Each of the other two fields may be, for instance, in one among the most meaningful types of states, such as number states, coherent states, thermal states or phase states (eigenstates of the operator $\hat{C} + i\hat{S}$) or prepared in an entangled states.

Example 19 *Consider the fully separable state described by the density operator $\varrho = R \otimes \sigma = \varrho_1 \otimes \varrho_2 \otimes \sigma$, where ϱ_k, with $k = 1, 2$, denotes the preparation for the k-th bosonic field in the arbitrarily mixed state*

$$\varrho_k = \sum_{m=0}^{\infty} p_m^{(k)} |m\rangle\langle m|$$

on the Hilbert space $\mathcal{H}_k$, then the system moment generating function is

$$Tr_k\left[\varrho_k D_k(\alpha_k)\right] = e^{-\frac{|\alpha_k|^2}{2}} \sum_{m=0}^{\infty} p_m^{(k)} L_m\left(|\alpha_k|^2\right) \quad ,$$

where the L_n's are Laguerre polynomials. For instance, for coherent and phase states it should be used with

$$p_m^{(k)} = e^{-|\alpha|^2} \frac{|\alpha|^{2m}}{m!} \qquad \text{and} \qquad p_m^{(k)} = (1 - |z|^2)|z|^{2m}$$

respectively (phase state formulae can be used even when dealing with thermal states upon the identification $z = \exp[-\frac{1}{2}\beta\hbar\omega]$, β being the inverse of temperature).

Suppose no specific conditions do constraint, in principle, the preparation for the mode a_2. Once again a vacuum choice may be advantageous. Let us therefore focus on the specific case of the measurement of $\hat{Z}_\gamma$ on the class of factorized signals described by $R = \varrho_1 \otimes |0\rangle\langle 0|$ where ϱ_1 is a generic preparation of the mode a_1 while $|0\rangle$ is the ground state of the mode a_2. In this case

$$\varrho = \varrho_1 \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| \quad ,$$

Eq. (12) becomes a Gaussian convolution and the moment generating function becomes independent of the parameter γ

$$\Xi(\lambda) = \chi_1(\lambda)\, \exp\left(-\frac{1}{2}|\lambda|^2\right) . \tag{15}$$

The measured variances are thus given by

$$\Delta Q_1^2 = \frac{1}{2}\left(\Delta q_1^2 + 1\right) \quad , \qquad \Delta P_2^2 = \frac{1}{2}\left(\Delta p_1^2 + 1\right) . \tag{16}$$

Equations (15)-(16) contain a *remarkable result* that may be expressed as follows (Paris,GL and Soliani, 2007).

Conclusion 20 *The measurement of $\hat{Z}_\gamma$ on the class of states $R = \varrho_1 \otimes |0\rangle\langle 0|$* does not *lead to added noise with respect to the measurement of the normal operator $\hat{a}_1 + \hat{a}_2$.*

We thus learn how the simultaneous measurement of the field quadratures for a quasi-monochromatic signal can be realized in the case the heterodyne apparatus yields a non-normal measurement operator. Moreover, a suitable preparation enables one to avoid additional noise with respect to that resulting in the measurement of signal field quadratures within the framework of the standard optical heterodyne detection.

• A *feasible phase* can be naturally defined within the Caves description of heterodyning at the cost of introducing of a Naimark mode

$$\hat{\theta}_N = \frac{1}{2i} \ln \frac{\hat{Z}_N}{\hat{Z}_N^\dagger} \qquad , \qquad \left[\hat{Z}_N, \hat{Z}_N^\dagger\right] = 0$$

• Cosine and sine quadrature operators

$$\hat{C} = \frac{\exp\left(i\hat{\theta}_N\right) + \exp\left(-i\theta_T\right)}{2} \quad , \qquad \hat{S} = \frac{\exp\left(i\hat{\theta}_N\right) + \exp\left(-i\theta_T\right)}{2i} \quad ,$$

obey the correct relation $\hat{C}^2 + \hat{S}^2 = 1$.

• Introduction of the 3-mode relative number operator $\hat{N} = \hat{N}_1 - (\hat{N}_2 + \hat{N}_3)$, where $\hat{N}_k = \hat{a}_k^\dagger \hat{a}_k$ $(k = 1, 2, 3)$, yields to what one expects for genuine phase operatorsThe commutator $[\hat{\theta}_N, \hat{N}]$ can then be interpreted as the canonical conjugation of the feasible phase for Caves heterodyne measurement operator with respect to the operator mode number difference $\hat{N}$.

Fortcoming steps in this activity

- Better (more economical) setups for generalized measurements of phase in the case of linear amplifiers?

- Detection schemes to perform phase generalized measurements for non-linear operators of physical interest (quadratic, cubic,...).

Classical action-angle variables for the time-dependent oscillator

• Classical one-dimensional standard harmonic oscillator with constant mass and frequency.

$$H_0 = \frac{p^2}{2m_0} + \frac{m_0\omega_0^2 q^2}{2} \ , \qquad \theta_0 = \tan^{-1}\left(\frac{p}{m_0\omega_0 q}\right) \ , \tag{17}$$

Transformation $(q,p) \to (\theta_0, -\omega_0^{-1}H)$ is canonical, $\{\theta_0, -\omega_0^{-1}H_0\}_{q,p} = 1$.

The result does not hold anymore once a generalized oscillator having mass $m = m(t)$ and frequency $\omega = \omega(t)$ arbitrarily depending on time is considered. The Hamiltonian

$$H = \frac{p^2}{2\,m(t)} + \frac{m(t)\,\omega^2(t)\,q^2}{2} \tag{18}$$

can no longer be an action variable. The naive time-invariance is lost and more complicated symmetry group and conservation law have to be considered.

Angle-action variables for the time-dependent oscillator (18) can be easily obtained. One has a basic quadratic invariant, the Ermakov invariant,

$$I = \kappa\frac{m}{\sigma^2}q^2 + \left[\frac{\sigma}{\sqrt{m}}p - \sqrt{m}\sigma q\frac{d}{dt}\left(\ln\frac{\sigma}{\sqrt{m}}\right)\right]^2 \ , \tag{19}$$

$$q = \ell\cos(\theta+\delta) = C\frac{\sigma}{\sqrt{m}}\cos(\theta+\delta) \tag{20}$$

$$\ddot{\sigma} + \left[\omega^2 - \frac{\dot{M}}{2} - \frac{M^2}{4}\right]\sigma = \frac{\kappa}{\sigma^3} \ , \qquad M = \frac{\dot{m}}{m} \ , \ \kappa \in \mathbb{R}^+ \tag{21}$$

generated by a vector field of the type

$$V = \frac{\sigma^2}{\sqrt{\kappa}}\left\{\partial_t + \left[\frac{d}{dt}\left(\ln\frac{\sigma}{\sqrt{m}}\right)\right]q\partial_q\right\} \ . \tag{22}$$

The quantity $J = \frac{I}{2\sqrt{\kappa}}$ can be therefore as the natural action variable for the time-dependent oscillator. The associated angle variable is

$$\theta(p,q) = \tan^{-1}\left\{\frac{\sigma^2}{\sqrt{\kappa}}\left[\frac{p}{mq} - \frac{d}{dt}\ln\frac{\sigma}{\sqrt{m}}\right]\right\} = \tan^{-1}\left\{\frac{\ell^2}{C^2\sqrt{\kappa}}\left[\frac{p}{q} - m\frac{d}{dt}\ln\ell\right]\right\} \ . \tag{23}$$

(Standard harmonic oscillator: $\sigma^2 \to \frac{\sqrt{\kappa}}{\omega_0}$)

Quantum angle-action operators for the time-dependent oscillator

The quantum phase problem for the standard harmonic oscillator is effectively attacked by resorting to number shifts operators. Another way to proceed is based on the use of *phase-space distributions* and quantization the classical phase variable by means of ordering rules for the momentum and position operators. Each Hermitian phase operator $\hat{\phi}$ such that the phase distribution

$$P(\varphi) = \mathrm{tr}[\delta(\hat{\phi} - \varphi)\hat{\rho}] \tag{24}$$

attributes the correct sharp phase to any large amplitude localized state $\hat{\rho}$ is expressible as the operator obtained from (17) by direct quantization of phase-space variables and introduction of an ordering rule. The relevance of the angle variable $\theta(p, q)$ given above relies on the possibility to adopt a similar strategy in the case of the time-dependent oscillator as well.

A likely way to tackle the problem of the definition of the quantum phase operator for the time-dependent oscillator by taking account of (23), is therefore based on the introduction of an operator of the type

$$\theta(p, q) \to \left\{\hat{\theta}(\hat{p}, \hat{q})\right\}_{\Omega} \tag{25}$$

where Ω means an operator ordering (*e.g.* the Weyl ordering).

Remark 21 *The appearance of* $\frac{d}{dt}\ln \ell = \frac{d}{dt}\ln\frac{\sigma}{\sqrt{m}} = \left(\frac{\dot{\sigma}}{\sigma} - \frac{\dot{m}}{2m}\right)$ *in the classical angle variable, and thus in quantum phase operator (25), for a time-dependent oscillator is meaningful from the physical point of view. At the quantum level the quantity measures in fact the departure from the minimum uncertainty of states that at an initial time are coherent but during their time evolution under the time-dependent oscillator dynamics generally become squeezed:*

$$\Delta_\alpha \hat{q} \Delta_\alpha \hat{p} \geq \frac{\hbar}{2}\sqrt{1 + \frac{\sigma^2}{\kappa}\left(\dot{\sigma} - \frac{\dot{m}}{2m}\sigma\right)^2}\ , \tag{26}$$

Minimum uncertainty is preserved during the time-evolution whenever the amplitude ℓ *is constant, say* $\sigma = c\sqrt{m}$, *or equivalently when* $m\omega = \frac{1}{2c^2}$ *being* *c a constant. In such a case, it would therefore result*

$$\hat{\theta}_{MU} = \tan^{-1}\left[\frac{c^2}{\sqrt{\kappa}}\frac{\hat{p}}{\hat{q}}\right]_\Omega \quad , \qquad \hat{J}_{0,MU} = \frac{1}{2}\left(\frac{\sqrt{k}}{c^2}\hat{q}^2 + \frac{c^2}{\sqrt{k}}\hat{p}^2\right) \tag{27}$$

while the Hamiltonian would read $\hat{H}_{MU} = \frac{1}{2m}\left(\hat{p}^2 + \frac{\hat{q}^2}{4c^2}\right)$*, thus showing an explicit time-dependence.*

Weyl-ordered polynomial quantum form for the TDO angle variable

An explicit formal representation of the quantum phase operator for the time-dependent oscillator in terms of the Weyl ordered operators

$$\hat{T}_{i,j}(\hat{q},\hat{p}) = \frac{1}{2^i}\sum_{s=0}^{\infty}\begin{pmatrix} i \\ s \end{pmatrix}\hat{p}^s\hat{q}^j\hat{p}^{i-s} = \frac{1}{2^j}\sum_{s=0}^{\infty}\begin{pmatrix} j \\ s \end{pmatrix}\hat{q}^s\hat{p}^i\hat{q}^{j-s} \tag{28}$$

can be found in a way similar to that considered for the harmonic oscillator (Bender and Dunne, 1989)

- Let

$$\hat{J} = J(\hat{q},\hat{p},t) = J_{pp}(t)\ \hat{p}^2 + J_{qq}(t)\ \hat{q}^2 + J_{pq}(t)\ \ (\hat{p}\hat{q} + \hat{q}\hat{p}) \tag{29}$$

be an action operator. A solution for the associated angle operator can be found of the form (GL, 2008)

$$\hat{\theta}_W(\hat{q},\hat{p}) = \sum_{k=0}^{\infty}\alpha_{-k,k}(t)\ \hat{T}_{-k,k}(\hat{q},\hat{p}) \quad , \qquad k = 0,1,2,\ldots$$

with

$$\alpha_{0,0} = \frac{\Delta}{J_{pp}} \tan^{-1}\left[\tilde{J}_{qp}^{-1}\right] \quad , \tag{30}$$

$$\alpha_{-1-2k,1+2k} = \frac{(-1)^k}{(1+2k)} \frac{\Delta}{J_{pp}} \left[-\frac{\tilde{J}_{qq}\Delta}{(1+\tilde{J}_{qp}^2)}\right]^{1+2k} {}_2F_1\left(-\frac{1}{2}-k, -k, \frac{1}{2}, -\tilde{J}_{qp}^2\right) \tag{31}$$

$$\alpha_{-2-2k,2+2k} = (-1)^{k+1} \frac{\tilde{J}_{qp}\Delta}{J_{pp}} \left[-\frac{\tilde{J}_{qq}\Delta}{(1+\tilde{J}_{qp}^2)}\right]^{2+2k} {}_2F_1\left(-\frac{1}{2}-k, -k, \frac{3}{2}, -\tilde{J}_{qp}^2\right) \tag{32}$$

$$\Delta = \frac{\Upsilon(t)}{2\sqrt{\tilde{J}_{qq} - \tilde{J}_{qp}^2}} = \frac{\sqrt{\kappa}}{2\sigma^2\sqrt{\tilde{J}_{qq} - \tilde{J}_{qp}^2}} \quad , \quad \tilde{J}_{qp} = \frac{J_{qp}}{J_{pp}} \quad , \quad \tilde{J}_{qq} = \frac{J_{qq}}{J_{pp}} \quad .$$

Problem 22 *The concrete action of the operator $\hat{\theta}_W$ should be elucidated. This implies the investigation of phase states associated with it as well as the analysis of the way they are affected by the mechanism of loss of coherence.*

• Weyl ordered $\hat{T}_{-n,n}$-expansions for quantum canonical pairs are known for the:

1) free particle

$$\hat{H} = \frac{\hat{p}^2}{2m}\,, \qquad \hat{T}_{A-B} = -m\hat{T}_{-1,1}(\hat{q},\hat{p})$$

2) harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2}\,, \qquad \hat{\theta} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(1+2k)!}\,\hat{T}_{-2n-1,2n+1}$$

3) time-dependend oscillator:

$$I = \kappa\frac{m}{\sigma^2}\hat{q}^2 + \left[\frac{\sigma}{\sqrt{m}}\hat{p} - \sqrt{m}\sigma\hat{q}\frac{d}{dt}\left(\ln\frac{\sigma}{\sqrt{m}}\right)\right]^2, \qquad \hat{\theta}_W$$

Problem 23 *Study of the operator*

$$\hat{T}_{-n,n} = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}\,\hat{q}^j\,\hat{p}^{-n}\,\hat{q}^{n-j} \quad .$$

This is concerned with the question of the definition of the operator $\hat{p}^{-1}$.

Operator $\hat{p}^{-1}$ is defined by means of

$$\left\langle \phi_k \right| \hat{p}^{-1} \left| \psi \right\rangle = \frac{1}{p} \tilde{\psi} \quad , \qquad \left\langle q \right| \hat{p}^{-1} \left| \psi \right\rangle = \sum_{k \neq 0} \left\langle q \,|\, \hat{p}^{-1} | \phi_k \right\rangle \left\langle \phi_k | \psi \right\rangle$$

where $\phi_k = \phi_k^0 = \frac{1}{\sqrt{2}} e^{ik\pi\, q}$ are the standard momentum eigenfunctions. Hence

$$\langle q' |\, \hat{T}_{-n,n} \, | \psi \rangle = \sum_{k \neq 0} \langle q \,|\, \hat{T}_{-n,n} \, | \phi_k \rangle \, \langle \phi_k | \psi \rangle$$

Coordinate representation of $\hat{T}_{-n,n}$

$$\langle q' |\, \hat{T}_{-n,n} \, | \psi \rangle = \int_{\mathcal{D}(q)} T_{-n,n} \, (q' |\, q) \; \psi(q) \, dq$$

Evaluation of kernels $T_{-n,n}(q' |\, q)$ provides

$$T_{-n,n}(q' |\, q) = \frac{[i\pi(q+q')]^n}{n!} \, B_n \left(\frac{q-q'}{2} \right) \; \Theta(q'-q) + \\ + \frac{[-i\pi(q+q')]^n}{n!} \, B_n \left(\frac{q'-q}{2} \right) \; \Theta(q-q')$$

where the $B_n(x)$'s denote Bernoulli polynomials of n-th order and $\Theta(x)$ is the Heaviside step function.

The action of an arbitrary operator $\hat{O} = \sum_{n=0}^{\infty} c_n \hat{T}_{-n,n}$ on the eigenstate $|m>$ of the HO is

$$\langle q' | \hat{O} | m \rangle = \sum_{n=0}^{\infty} c_n \frac{(i\pi)^n}{n!} \left(\int_{q \leq q'} + (-1)^n \int_{q \geq q'} \right) \left[(q+q')^n B_n (\frac{q'-q}{2}) H_m(q) e^{-q^2} \, dq \right]$$

In particular

$$\langle q' | \sum_{n=0}^{\infty} c_{2n} \hat{T}_{-2n,2n} | m \rangle = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \sum_{\alpha=0}^{2n-k} c_{2n} \left(\frac{i\pi}{2} \right)^{2n} g_{k,\alpha} \; (q'-1)^{2n-\alpha}$$

The problem can be also approached with the assumption that particles is essentially confined, say $q \in [-\ell, \ell]$. Define position and momentum operators as (Galapon et al, 2005)

$$(\hat{q}\psi)\,(q) = q\,\psi(q) \quad \forall\psi \in \mathcal{D}(\hat{q}) = L^2[-\ell,\ell] = \mathcal{H}$$
$$(\hat{p}_\gamma\phi)\,(q) = -i\frac{d\phi}{dq} \quad \forall\phi \in \mathcal{D}(\hat{p}_\gamma) = \left\{ \begin{array}{c} \phi \in \mathcal{H} : \frac{d\phi}{dq} \in \mathcal{H}, \\ \phi(-\ell) = e^{-2i\gamma}\,\phi(\ell),\ \gamma \in [0,1) \end{array} \right\}$$

$\hat{q}$ and $\hat{p}$ are self-adjoint operators canonically conjugate in a dense subspace of $\mathcal{D}(\hat{p}_\gamma)$. The eigenfunctions of the momentum spectral problem $\hat{p}_\gamma \phi_k^\gamma = p_{\gamma,k}\,\phi_k^\gamma$ are

$$\phi_k^\gamma = \frac{1}{\sqrt{2}} e^{i\frac{p_{\gamma,k}}{\hbar}q} \quad , \quad p_{\gamma,k} = \hbar\,(\gamma + k\,\pi) \quad , \qquad k = 0, \pm1, \pm2, \ldots \quad .$$

The inverse momentum operator exists, it is bounded and self-adjoint:

$$\left\langle \phi_k | \hat{p}_\gamma{}^{-1} | \psi \right\rangle = \frac{1}{p_{\gamma,k}}\,\tilde{\psi} \quad , \qquad \left\langle q |\, \hat{p}_\gamma{}^{-1} | \psi \right\rangle = \sum_k \left\langle q\,| \hat{p}_\gamma{}^{-1} | \phi_k \right\rangle\, \left\langle \phi_k | \psi \right\rangle$$

and $\hat{T}^\gamma_{-n,n}\,|\psi>= \ \sum_{k\neq0} \hat{T}^\gamma_{-n,n}\,|\phi_k>\ \langle\phi_k|\psi\rangle$. Seeking for coordinate representations of $\hat{T}^\gamma_{-n,n}$ would lead to the Fredholm integral operator representation

$$\left\langle q' |\, \hat{T}^\gamma_{-n,n}\,|\psi \right\rangle = \int_{-\ell}^{\ell} \hat{T}\,(q'|\,q)\ \psi(q)\,dq$$

with the kernels $T^\gamma_{-m,m}(q'|\,q)$

$$T^\gamma_{-n,n}(q'|q) \propto\ (q+q')^n\, e^{i\gamma(q'-q)}\ \left\{ \left[\lim_{z\to-\frac{\gamma}{\pi}} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{e^{iz\pi|q'-q|}}{e^{2i\pi z}-1} \right) \right] \Theta(q'-q) + \right.$$
$$\left. + \left[\lim_{z\to\frac{\gamma}{\pi}} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{e^{-iz\pi|q'-q|}}{1-e^{-2i\pi z}} \right) \right]\ \Theta(q-q') \right\}$$

(symmetric, $T^\gamma_{-n,n}(q'|q) = T^{\gamma\,*}_{-n,n}(q|q')$, and bounded, $\int_{-l}^{l} dq \int_{-l}^{l} dq' |T^\gamma_{-n,n}(q'|q)| < \infty$, hence self-adjoint). Being compact, kernels have a complete set of eigenfunctions and discrete spectrum.

$$
\begin{aligned}
T_{-2,2}(q'|\,q) &= \frac{\pi^2}{24}[2+6(q-q')+3(q-q')^2] \\
\Psi_{-2,2}(q) &= \sum_{r=0}^{4} \xi_r q^r \\
T^{\gamma}_{-2,2}(q'|\,q) &= \frac{1}{4}(i\cot\gamma-1)(q+q')^2|q-q'| \\
\Psi^{\gamma}_{-2,2}(q) &= \sum_{r=0}^{4} \xi^{\gamma}_r F^{\gamma}_r\left(-\frac{q^4}{16\lambda^{\gamma}_{-2,2}}\right)
\end{aligned}
$$

where the F's are functions of the hypergeometric type.