Solutions of Nahm and SG equations and quantization via zeta function

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June 13, 2008

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Motivation

- **Euclidean version:** reduce by deleting dependence on cartesian coordinates $\Rightarrow$ equations, differing by a single sign: interesting in a very different context: intimately related via the ADHMN (Atiyah-Drinfeld-Hitchin-Manin-Nahm) construction to static monopole solutions to Yang-Mills-Higgs theories in four dimensions in the Bogomolnyi-Prasad-Sommerfield limit. The ADHMN construction $\Leftrightarrow$ equivalence between self-dual equations, one - unidimensional, the other in three dimensions (reduced Euclidean four dimensional theory by deleting dependence on a single variable) - E. Corrigan et al (1985).

- in **Minkowski space**: to delete dependence on spatial variables: Baseyan, et al. Non-linear plane waves in the massless Yang- Mills theory. Pis'ma ZETP 29, 641 (1979),

- **Essentially chaotic.** Nikolaevskii, E.S., Schur, L.N.: Non-integrability of the classical Yang-Mills fields. JETP Letts. 36, 218 (1982) Sawiddy, G.K.: The Yang-Mills classical mechanics as a Kolmogorov K-system. Phys. Letts. 130 B, 303 (1983) Chang, S.J.: Classical Yang-Mills solutions and iterative maps. Phys. Rev. D29, 259 (1984)

General purposes

Physics: Quantization of YM field (Yang-Mills particles mass spectrum)

Math.: Development of the zeta-function technique to finite-gap potentials applications. Pre-publication on GL,SG (Leble,Zaitsev arXiv:0804.1255): Letter from M. Pawellek, Quantum mass correction for twisted kink arXiv:0802.0710v1

Introduction. On Nahm models.

Yang-Mills equations in four Euclidean dimensions

$$D_\mu T_{\mu\nu} = 0, \qquad (1)$$

for the gauge fields $T_\mu = T_\mu^+$, where

$$T_{\mu\nu} = T_{\nu,\mu} - T_{\mu,\nu} - \imath[T_\mu, T\nu], \quad D_\mu \Phi = \partial_\mu - i[T_\mu, \Phi].$$

independence on variables x_k, k=1,2,3; setting $x_4 = z$,

$$\frac{d^2 T_k}{dz^2} = [T_j[T_j, T_k]], \quad [T_k, \frac{dT_k}{dz}] = 0. \qquad (2)$$

The self-dual equations,

$$\frac{dT_i}{dz} = \pm\varepsilon_{ijk} T_j T_k, \qquad (3)$$

imply Eqs. (2). Simple solution $T_i = \phi_i(z)\sigma_i$: the Euler system for $\phi_i(z)$ solved in Jacobi functions. Solutions are dressed by the gauge-Darboux transformations (Leble 2005).

The ansatz

$$T_i = \phi(z)\alpha_i, i = l, 2, 3, \tag{4}$$

α_i - constant matrices, yields

$$\begin{gathered} 2\alpha_i = \sum_{j=1}^{3} [\alpha_j [\alpha_j, \alpha_i]] \\ \phi''(z) = 2\phi^3. \end{gathered} \tag{5}$$

In: E. Corrigan et al Comm. Math. Phys. 98, 259-272 (1985) Comments on the Non-Self-Dual Nahm Equations

The equation for $\phi(z)$ enters a specific case (m=0) of Ginzburg-Landau model with $V = \phi^4/2 + c$.

On Feynmann quantization of a classical field.

An attention to Feynmann quantization formalism of a classical field was recently attracted in connection with a link to a SUSY quasiclassic quantization condition. G. Junker, Supersymmetric Methods... 1996.

Most known - the corrections to the kink mass - the separatrix solution of the massive field ϕ^4 (GL) model. (Recent paper M. Bordag - Arxiv 02). Zaitsev,Yurov,Leble, applied Darboux transformations technique (Arxiv 06,08).

V.Konoplich (TMF 1987,89): quantum corrections to kinks by Riemann zeta-function. *The main target of this paper: The formalism extension to periodic solutions quantization problem;* is posed e.g. in the review of Tuszyński, Dixon, Grundland, 1994.

The approximate quantum corrections to the solutions of the equation KG = SUSY quasiclassic quantization are obtained by Feynmann functional integral (via stationary phase analog); the basic relation

$$\exp[-\frac{S_{qu}}{\hbar}] \simeq \frac{A}{\sqrt{detD}}, \tag{6}$$

S_{qu} denotes quantum action, for a potential $V(\phi)$, A - quantity determined by the vacuum state at $V(\phi) = 0$,

$$D = -\partial_x^2 - \Delta_y + V''(\phi(x)). \tag{7}$$

$y \in R^{d-1}$ stands for the transverse variables.

D appears while the second variational derivative of the quantum action functional (Lagrange density$\mathcal{L} = (\partial\phi)^2/2 - V(\phi)$) is evaluated. Vacuum action: S_{vac} the relation (6) is valid $D \to D_0 = -\partial_x^2 - \Delta_y$. The quantum correction by twice use of (6)

$$\Delta S_{qu} = S_{qu} - S_{vac} = \frac{\hbar}{2} \ln(\frac{\det D}{\det D_0}). \qquad (8)$$

The generalized Riemann zeta-function and Green function of heat equation.

Scheme for $zeta$

Let the set $\{\lambda_n\} = S$ be a spectrum of a linear operator L,

$$\ln(\det L) = \sum_{\lambda_n \in S} \ln \lambda_n, \tag{9}$$

the sum in the r.h.s. is formal one.

The generalized Riemann zeta-function $\zeta_L(s)$ is defined by the equality

$$\zeta_L(s) = \sum_{\lambda_n \in S} \lambda_n^{-s}. \tag{10}$$

This definition should be interpreted as analytic continuation to the complex plane of s from the half plane $Res > \sigma$ in which the sum in (10) converges. Differentiating the relation (10) with respect to s at the point $s = 0$ yields

$$\ln(\det L) = \zeta_L'(0). \tag{11}$$

The generalized zeta-function (72) admits the representation via the diagonal g_L of a Green function of the operator $\partial_t + L$. Such link function (*heat kernel formalism*) is used in quantum theory since works by Fock (Izvestija, 1937).

A representation in terms of the formal sum over the spectrum

$$g_L(t, \mathbf{r}, \mathbf{r_0}) = \sum_{\mathbf{n}} \exp[-\lambda_{\mathbf{n}} \mathbf{t}] \psi_{\mathbf{n}}(\mathbf{r}) \psi^*_{\mathbf{n}}(\mathbf{r_0}), \qquad (12)$$

by the normalized eigenfunctions $\psi_n(\mathbf{r})$ of the operator L.

Def :

$$\gamma_L\,(t) = \int g_L\,(t, x, x)\,dx \qquad (13)$$

Mellin transformation yields zeta of L:

$$\zeta_L(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \gamma_L(t) dt \qquad (14)$$

$$L \to D, D_0, D_\nu$$

$$D = -\Delta + V''(x)$$

$$D_0 = -\Delta = -\frac{d^2}{dx^2} - \Delta_y$$

,

$$D_\nu = -\Delta = -\frac{d^2}{dx^2} - \Delta_y + \nu$$

,

$$\Delta S = \frac{\hbar}{2} ln \left(\det D\right) - \frac{\hbar}{2} ln \left(\det D_\nu\right), \qquad (15)$$

Main problem for a periodic V(x) :

$$\left(\frac{\partial}{\partial t} + D_1\right) g_D\left(t, x, x_0\right) = \delta\left(t\right)\delta\left(x - x_0\right)$$

where $D_1 = -\frac{d^2}{dx^2} + V\left(x\right)$, $V \leq V_m$ $\exists\lambda \in [\lambda_0, +\infty)$.

The Green function expansion for $\forall D_1^+ = D_1$:

$$g_{D_1}\left(t, x, x_0\right) = \sum_k e^{-\lambda_k t}\psi_k\left(x\right)\psi_k^*\left(x_0\right)\Theta\left(t\right)$$

Let t>0,

Def :

$$\gamma_{D_1}(t) = \int g_{D_1}(t,x,x)\,dx$$

$$\gamma_{D_1}(t) = \sum_k e^{-\lambda_k t} \tag{16}$$

Mellin transformation:

$$\zeta_{D_1}(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \gamma_D(t) dt$$

For the operator D_0 (Poisson int.):

$$\gamma_{D_0}(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}} d\mathbf{k} \exp(-|\mathbf{k}|^2 t) = (4\pi t)^{-d/2}. \tag{17}$$

For a constant potential ν, for regularization (renormalization)

$$\gamma_{D_\nu}(t) = (4\pi)^{\frac{1-d}{2}} \frac{\Gamma(s + \frac{1-d}{2})}{\Gamma(s)} \nu^{\frac{d-1-2s}{2}}. \tag{18}$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \left(\gamma_{D_\nu}(t) - \gamma_D(t) \right) dt \quad (19)$$

Finally:

$$\Delta S = \frac{\hbar}{2} \zeta'(0) \quad (20)$$

From (13) for $\gamma_D(t)$ follows a property of multiplicity: *if the operator D is a sum of two differential operators $D = D_1 + D_2$, which depend on different variables, the following equality holds*

$$\gamma_D(t) = \gamma_{D_1}(t)\gamma_{D_2}(t). \quad (21)$$

Static solutions of Nahm, Landau-Ginsburg and SG

Integral of (5)

$$(\phi')^2 = 2(\phi^2)^2 - 2w^4. \tag{22}$$

That corresponds to the case m=0 of the GL model.

$$\int_0^{\phi} \frac{d\phi}{\sqrt{\phi^4 - w^4}} = z\sqrt{2}, \tag{23}$$

with invariants

$$\begin{aligned} g_2 &= w^4, \\ g_3 &= 0, \end{aligned} \tag{24}$$

hence, in terms of the Weierstrass function,

$$\phi(z, X) = w + \frac{w^3}{\wp(z; w^4, 0) - w^2/2}. \qquad (25)$$

or, if take into account relations

$$\begin{aligned} &\wp(z; w^4, 0) = e_3 + \tfrac{e_1 - e_3}{sn^2(\sigma z)}, \\ &\sigma = \sqrt{e_1 - e_3}, \quad k = \tfrac{e_2 - e_3}{e_1 - e_3} \end{aligned} \qquad (26)$$

one has for $e_3 = 0, e_1 = -e_2 = w^2/2$, the parameters of the solutions in Jacobi terms $k = i, \sigma = w/\sqrt{2}$

Solution of (5), is a particular but specific case of static GL model (m=0, g=2) A solution of (5) is expressed via Jacobi functions

$$\phi(z) = \sigma \frac{sn[\sigma(z-z_0)]dn[\sigma(z-z_0)]}{cn[\sigma(z-z_0)]}. \tag{27}$$

The potential of heat equation

$$u(x) + \lambda = V''(\phi) = 6\sigma^2 \frac{sn^2[\sigma(z-z_0)]dn^2[\sigma(z-z_0)]}{cn^2[\sigma(z-z_0)]}. \tag{28}$$

Solutions of φ^4 model the potential and the Lagrangian are determined by the formulas

$$V(\varphi) = g\varphi^4/4 - m^2\varphi^2/2,$$

$$L = -(\varphi')^2 - g\varphi^4/4 + m^2\varphi^2/2,$$

the equation of motion has the form

$$\varphi''(x) + m^2\varphi - g\varphi^3 = 0, \qquad (29)$$

that yields

$$(\varphi')^2 = \frac{g}{2}(\varphi^2 - \frac{m^2}{g})^2 + 2W. \qquad (30)$$

Its restricted solutions, as it follows from phase plane analysis, exist if

$$-\frac{m^2}{4g} \leq W \leq 0. \qquad (31)$$

The separatrix (W=0) solution of (30) is the kink/antikink

$$\varphi_0 = \pm\sqrt{\frac{2}{g}} b \tanh(bx)), \qquad b = \frac{m}{\sqrt{2}}. \qquad (32)$$

Inside the interval (31) the equation (30) is expressed in terms of the elliptic Jacobi sinus

$$\varphi = \pm\sqrt{\frac{2}{g}}kbsn(bx;k), \qquad b = \frac{m}{\sqrt{1+k^2}}, \quad 0 < k < 1. \tag{33}$$

The constant W is given by

$$W = -(\frac{1-k^2}{1+k^2})^2\frac{m^4}{4g}. \tag{34}$$

quantization in Pawellek (2008) Leble, Zaitsev (2008)

The family of the restricted solutions contains also the constant vacuum ones (W=0)

$$\varphi = \pm\frac{m}{\sqrt{g}}. \tag{35}$$

The two-level (kink) potential u(x) :

$$u(x) = -6b^2/ch^2(bx), \tag{36}$$

of one-dimensional Schrødinger equation $-\partial_x^2 + u(x)$ appears, the constant $b = m/\sqrt{2}$. Eigenvalues and the normalized eigenfunctions correspondingly

$$\lambda_1 = -b^2, \qquad \psi_1(x) = \sqrt{3b/2}\,\mathrm{sinh}(bx)/cosh^2(bx);$$

$$\lambda_2 = -4b^2, \qquad \psi_2(x) = \sqrt{3b}/2\,\mathrm{cosh}(bx).$$

SG model: we integrate the KG equation

$$(\varphi')^2 = \frac{4m^4}{3g}(1 + cos(\frac{1}{m}\sqrt{\frac{3g}{2}}\varphi)) + 2W. \qquad (37)$$

The solutions are restricted and hence have the direct physical relevance, if

$$-\frac{4m^4}{3g} \leq W \leq 0, \qquad (38)$$

If $W = 0$ the nontrivial solutions are interpreted as kink and antikink

$$\varphi(x) = \pm\sqrt{\frac{2}{3g}}\operatorname{arcsin}\tanh(mx)(mod\Phi), \qquad \Phi = 2m\pi\sqrt{\frac{2}{3g}}, \qquad (39)$$

while at the interval

$$-\frac{4m^4}{3g} < W < 0,$$

the solution of (37) yields a periodic function expressed via elliptic Jacobi function.

The solution is convenient to found in

$$z = ksn(mx; k), \tag{40}$$

where

$$k = \sqrt{1 + \frac{3gW}{4m^4}} \tag{41}$$

is the module of the elliptic function. Hence

$$W = \frac{4(k^2 - 1)m^4}{3g}. \tag{42}$$

Finally

$$\varphi = \pm 2m\frac{2}{3g} \arcsin ksn(mx; k) \quad (mod\Phi). \tag{43}$$

The class of restricted solutions contains also

$$\varphi = 0(mod\Phi); W = -\frac{3m^4}{3g} \tag{44}$$

and

$$\varphi = \pm\pi m\sqrt{\frac{2}{3g}}(mod\Phi); W = 0. \tag{45}$$

Other static solutions are obtained by shifts

$$x \to x + x_0, \quad \varphi \to \varphi + \Phi,$$

that follows from Klein-Gordon equation invariance and SG equation potential periodicity.

The energy of classic static solutions of SG and ϕ^4 models

Let us evaluate the energy of the nontrivial static solutions of both models via the energy density definitions

$$e(x) = \int_{-\infty}^{\infty} ((\phi')^2/2 + V(\varphi))dx, \tag{46}$$

for kinks and, in a case of periodic solutions,

$$E = 2\int_0^l e(x)dx, \tag{47}$$

the constant "l" is the period of the solution.

For SG kink:

$$E_k = \frac{16m^2}{g}, \tag{48}$$

for a periodic soliton

$$E_p = \frac{8m^2}{g}[(1-k^2)K + 2E], \tag{49}$$

where K(k),E(k) - complete elliptic Legendre integrals.

Equation for Green function diagonal

No necessity to solve the equation for the Green function, it is enough to solve the spectral problem for D_1.

In a spirit of Hermit approach, see, e.g. Adler,Moser, the function $\widehat{g}_L(p,x,x) \equiv G(p,x)$ is a solution of bilinear equation

$$2GG'' - (G')^2 - 4(u(x)-p)G^2 + 1 = 0, \quad (50)$$

in a case of reflectionless and finite-gap solutions is solved more effectively.

Possible to cover all necessary classes of solutions of the models (cases B_0 for Nahm, A,B for ϕ^4 and C,D for SG) via the universal representation by means of polynomials (in p) P, Q

$$G(p,x) = P(p,z)/2\sqrt{Q(p,z)}, \qquad (51)$$

where

$$z = sech^2(bx)$$

for kinks A,C, and

$$z = cn^2(bx;k)$$

for the periodic B_0,B,D.

Plugging (51) yields

$$b^2(\rho(z)(2PP''-(P')^2)+\rho'(z)PP')-(p+u(z))P^2+Q=0, \tag{52}$$

the primes denote derivatives with respect to z, while

$$\rho = \begin{cases} z^2(1-z), cases \quad A, C; \\ z(1-z)(1-k^2+k^2z), cases \quad B_0, B, D. \end{cases} \tag{53}$$

$$u(z) = \begin{cases} b^2(1-2z), case \quad (A) \\ -\lambda+6\sigma^2\frac{(1-z)(\sqrt{1-k^2}+k^2z)}{z} \quad (B_0) \\ b^2(2k^2-1-2k^2z), case \quad (B) \\ 2b^2(2-3z), case \quad (C) \\ b^2(5k^2-6k^2z), case \quad (D) \end{cases} \tag{54}$$

case A (kink). Polynomial is determined from well-known facts of the reflectionless potentials theory.

$$Q = p^2(p + b^2). \tag{55}$$

The case (B)

$$P = p + P_1(z), \quad Q = p^3 + q_2 p^2 + q_1 p + q_0 m \tag{56}$$

includes (A) in a sense that for the case $q_2 = b^2, \quad q_1 = 0, \quad q_0 = 0$.

Substituting (56) into (52) gives for each power of $p = 0, 1, 2$

$$\begin{gathered} -2P_1(z) - u(z) + q_2 = 0, \\ b^2(2\rho(z)P_1'' + \rho'(z)P_1') - P_1^2 - 2u(z)P_1 + q_1. \\ b^2(\rho(z)(2P_1P_1'' - P_1^2) + \rho'(z)P_1P_1' \\ -u(z)P_1^2 + q_0 = 0. \end{gathered} \tag{57}$$

respectively.

In the case (A) it gives

$$P_1 = b^2 z. \tag{58}$$

For the case (B) we begin from

$$P_1 = k^2 b^2 z, \tag{59}$$

that is the result of substitution of P_1 from the first equation of (57) into the second one. Next, from the third equation yields

$$Q = p(p^2 + (1 - k^2)b^2)(p - k^2 b^2), \tag{60}$$

with the simple roots p_i.

Going to the cases (C,D), generally

$$\begin{array}{c} P = p^2 + P_1(z)p + P_2(z), \\ Q = p^5 + q_4 p^4 + q_3 p^3 + q_2 p^2 + q_1 p + q_0, \end{array} \qquad (61)$$

while in the particular case of (C)

$$q_0 = 0, \quad q_1 = 0, q_2 = 36b^6, \quad q_3 = 33b^4, q_5 = 10b^2.$$

A substitution of (61) into the Hermit equation splits in the system

$$\begin{array}{c} -2P_1 - u + q_2 = 0, \\ -2P_2 - P_1^2 - 2uP_1 + b^2(2\rho P_1'' + \rho' P_1') + q_3, \\ b^2(\rho(2P_2'' + 2P_1 P_1'' - (P_1')^2), \\ +\rho'(P_2' + P_1 P_1')) - 2P_1 P_2) - u(2P_2 + P_1^2) + q_2 = 0, \\ b^2(2\rho(2P_1'' P_2 + P_1' P_2' + P_1 P_2'') \\ +\rho'(P_1 P_2') + P_1' P_2)) - P_2^2 - 2uP_1 P_2 + q_1, \\ b^2(\rho(2P_2 P_2'' - P_2'^2)) + \rho' P_2^2 + q_0. \end{array} \qquad (62)$$

The arguments in (62) are omitted.

Solving the system yields for the case D

$$P_1(z) = \alpha z + \beta, \qquad P_2(z) = a_1 z^2 + b_1 z + c_1, \tag{63}$$

where

$$\begin{gathered}\alpha = 3k^2b^2,\\ \beta = \frac{(1-5k^2)b^2 - q_4}{2},\\ a_1 = 9k^4b^4,\\ b_1 = \frac{3k^2b^2(q_4 + (1-11k^2)b^2)}{2},\\ c_1 = \frac{4q_3 - q_4^2 + 2(1-5k^2)b^2q_4 + 3(1-6k^2+21k^4)b^4}{8},\end{gathered} \tag{64}$$

are functions only of k, b, where

$$\begin{gathered}q_0 = 0, q_1 = -27k^2(1-k^2)b^8,\\ q_2 = -9(1-3k^2-3k^4-k^6)b^6, q_3 = 3(1+k^2+k^4)b^4,\\ q_4 = 5(1+k^2)b^2.\end{gathered} \tag{65}$$

Finally,

$$Q = \prod_{i=1}^{i=5} (p - p_i), \tag{66}$$

where the polynomial Q the simple roots p_i are ordered so that (0<k<1)

$$\begin{aligned} -(2\sqrt{1-k^2+k^4}+1+k^2)b^2 < -3b^2 < -3k^2b^2 < 0 \\ < -(2\sqrt{1-k^2+k^4}-1-k^2)b^2. \end{aligned} \tag{67}$$

Pick up the expressions determining $\widehat{\gamma}(p)$:

$$\widehat{\gamma}(p) = \int \frac{P(z)}{3\sqrt{Q}} dx =$$
$$\frac{p^2}{2\sqrt{Q}} \int dx + \frac{p}{2\sqrt{Q}} \int (\alpha z + \beta) dx + \frac{1}{2\sqrt{Q}} \int (a_1 z^2 + b_2 z + c_3) dx. \tag{68}$$

We need three integrals over the period K.

$$\int_0^K dx, \int_0^K z dx, \int_0^K z^2 dx. \tag{69}$$

Let us go to the variable z, $dz = d(cn^2(x)) = -2cn(x)cn(x)dn(x)dx = 2sn(x)$, a bit more convenient to put $y = 1 - z, dy = -dz$

$$\int_0^K dx = \int_0^1 \frac{dy}{y(1-y)(1-k^2y)} = 2K(k),$$
$$\int_0^K sn^2(x;k)dx =$$
$$\frac{1}{2}\int_0^1 \frac{ydy}{y(1-y)(1-k^2y)} = \frac{K(k)-E(k)}{k^2}, \tag{70}$$
$$\int_0^K sn^4(x;k)dx = \int_0^1 \frac{z^2dz}{z(1-z)(1-k^2z)} =$$
$$\frac{1}{3k^4}((2+k^2)K(k) - 2(1+k^2)E(k)).$$

hence

$$\hat{\gamma}(p) = 2K(k)\frac{p^2}{2\sqrt{Q}}+$$
$$\alpha(-\frac{K(k)-E(k)}{k^2} + (\beta-\alpha)2K(k)\frac{p}{2\sqrt{Q}}+$$
$$(a_1\frac{(2+k^2)K(k)-2(1+k^2)E(k)}{3k^4}+$$
$$(b_1-2a_1)\frac{K(k)-E(k)}{k^2} + (a_1-b_1+c_1)2K(k))\frac{1}{2\sqrt{Q}}. \tag{71}$$

or, finally

$$\zeta(s) = M^{2s}\int_0^\infty (\int_l \hat{\gamma}(p)e^{pt}dp)t^{s-1}dt/\Gamma(s); \tag{72}$$

which is expressed via integrals

$$\begin{gathered}\int_l \frac{e^{pt}}{\sqrt{Q}}dp \\ \int_l \frac{pe^{pt}}{\sqrt{Q}}dp \\ \int_l \frac{p^2e^{pt}}{\sqrt{Q}}dp\end{gathered} \tag{73}$$

by a contour that contains all branch points of the integrands.

The generalized zeta-function via Baker-Achiezer functions.

The Green function is defined in the Sec.1

The one-dimensional version of

$$\left(\frac{\partial}{\partial t}-\frac{d^2}{dx^2}+V''(x)\right)G\left(t,x,x_0\right)=\delta\left(t\right)\delta\left(x-x_0\right) \tag{74}$$

with a periodic potential $V''(x)$.

Let us integrate the equation (74) by x over $[x_0-\epsilon, x_0+\epsilon]$, hence

$$\lim_{\epsilon\to 0}\int_{x_0-\epsilon,}^{x_0+\epsilon}\left(\frac{\partial}{\partial t}-\frac{d^2}{dx^2}+V''(x)\right)G(t,x,x_0)dx = \lim_{\epsilon\to 0}[-\frac{dG(t,x_0+\epsilon,x_0,x_0)}{dx}+\frac{dG(t,x_0-\epsilon,x_0)}{dx}] = \lim_{\epsilon\to 0}\int_{x_0-\epsilon,}^{x_0+\epsilon}\delta(x-x_0)dx = 1. \tag{75}$$

The function $G(t,x,x_0)$ is supposed to be continuous at t$>$0.

Solutions of (74) with the zero r.h.s. is expressed in terms of the Baker-Achiezer functions $\psi_s(x,t;P)$ by the polynomial $q(s) = sx + s^2 t$ of local parameter s.

$$\psi_s(x,t;P) = \exp(\int_{P_0}^{P} \Omega) \ln \frac{\Theta(A(P) - A(D) + U - K)}{\Theta(A(P) - A(D) + U - K)}, \tag{76}$$

where Ω_q is the differential of the second kind on the curve Γ with the principal part dq, normalized as

$$\int_{a_k} \Omega = 0 \tag{77}$$

that define the periods vector

$$U_k = \int_{b_k} \Omega. \tag{78}$$

The non-special divisor is denoted as D, define the general Riemann Θ.

The parametrization by the polynomial $q(s)$ yields

$$\psi_s(x,t;P) = \exp(\int_{P_0}^{P} [x\Omega^{(1)} + t\Omega^{(2)}] \ln \frac{\Theta(xU + tV - A(D) - K)}{\Theta(xU + tV - A(D) - K)}, \tag{79}$$

where the second kind differentials $\Omega^{(}i)$ have the only pole at infinity as

$$\Omega^{(1)} = ds + ..., \quad \Omega^{(2)} = ds^2 + ... \tag{80}$$

The vectors of b-periods

$$U_i = \int_{b_i} \Omega^{(1)} \quad V_i = \int_{b_i} \Omega^{(2)} \tag{81}$$

define the argument of the Θ-function. The link between the potential $u(x)$ and ψ is given by

$$u(x) = 2\frac{\partial}{\partial x} \ln \Theta(xU + tV - A(D) - K) \tag{82}$$

which is recognized as general Matveev-Its formula.

Parameters (w,..) of the potential and the Green function arise from (81) as the position of the cycles b_i depends on parameters of the potential via the curve that determine the Riemann surface. One builds the Green function via the linear independent solution $\psi_{-s}(x,t;P)$ as:

$$G_0(t,x,x_0) = \frac{1}{W}\psi_s(x,t;P)\psi_{-s}(x_0,t;P) \qquad (83)$$

that account the unit jump of the first derivative with respect to x at $x = x_0$ and the Wronskian $W = \psi_s\psi'_{-s} - \psi_{-s}\psi'_s$ is determined by a normalization at the borders of the period.

Condition at $t = 0$ is satisfied if integrate (83) with respect to k along a mentioned contour

$$\int_C ds \frac{1}{W} \psi_s(x, t; P) \psi_{-s}(x_0, t; P) \tag{84}$$

the zero value at $t < 0$ guaranteed by Laplace transform property.

The diagonal values of the Green function are given by

$$\begin{gathered} G(t,x,x) = \int_C dk \tfrac{1}{W} \psi_s(x,t;P) \psi_{-s}(x,t;P) = \\ \int dk \exp\left(\int_{P_0}^{P} \Omega + \int_{P_0}^{P_-} \Omega\right) \ln \frac{\Theta(A(P_)-A(D)+U-K)}{\Theta(A(P)-A(D)+U-K)} \cdot \\ \ln \frac{\Theta(A(P_-)-A(D)+U-K)}{\Theta(A(P_-)-A(D)+U-K)} \end{gathered} \tag{85}$$

P_- corresponds to $q_{-s} = -sx + s^2 t$

BA function and "twisted BPS monopoles" The case of Nahm equation belongs to the elliptic case (genus $g = 1$),let us recall that the change of variables $x = \frac{\alpha z+\beta}{\gamma z+\delta}$ reduce a general polynomial $a_0 + ... + a_4 x^4$ the Weiersrass curve $w^2 = P_3(z) = 4(z - z_1)(z - z_2)(z - z_3)$.

In the case of elliptic potential the Riemann Θ-function is the Jacobi theta

$$\theta(z) = \sum_{n \in Z} \exp[Bn^2/2 + nz], \tag{86}$$

where

$$B = \frac{2\pi i \int_{z_2}^{z_3} \frac{dx}{\sqrt{P_3}}}{\int_{z_1}^{z_2} \frac{dx}{\sqrt{P_3}}}, \tag{87}$$

and integrals dependence on z_i are expressed via the Weierstrass function of w. It is known, that

$$\theta(z + B) = \exp[-\frac{B}{2} - z]\theta(z), \tag{88}$$

so the ratio of the theta-functions has the period B.

Integrate (85) by the period of the Nahm equation solution X.

$$\gamma_N(t) = \int_0^X G(t, x, x) dx, \tag{89}$$

Next - use the definition of the generalized ζ-function

$$\zeta_N(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \Big(\gamma_{D_0}(t) - \gamma_N(t)\Big)\, dt. \tag{90}$$

SG kinks

The results for ϕ^4 model kinks are well-known, see, e.g. recent Leble,Zaitsev, where a table for the dimensions d=1,2,3,4 is evaluated by other technique (DT) and listed. These results fit the case B, which hence was strictly verified.

Kinks of the SG model. The substitution of all expressions from (51,66,56,58) yields a divergence of the integral for (68) $\widehat{\gamma}(p)$. To regularize the Green function let us divide its Laplace transform into two parts as

$$G(p,x) = G_c(p) + G_k(p,x), \tag{91}$$

where the part $G_c(p)$ is the Green function diagonal (the solution of (50) for a constant potential:

$$G_c = \frac{1}{2\sqrt{p+b^2}}. \tag{92}$$

The kink part is easily constructed via (51,66,56,58):

$$G_k = \frac{b^2 sech^2(bx)}{2p\sqrt{p+b^2}}. \tag{93}$$

The representation (91) results in two contributions of the quantum corrections to the kink (antikink) mass.

The first one coincides with (94) for $\nu = m^2$.

$$\gamma_{D_0}(t) = (4\pi)^{\frac{1-d}{2}} \frac{\Gamma(s+\frac{1-d}{2})}{\Gamma(s)} m^{d-1-2s}, \tag{94}$$

and the second one is obtained by the general scheme, namely

$$\gamma_k(p) = \int_{-\infty}^{\infty} G_k(p,x)dx. \tag{95}$$

Plugging (93) into (95) yields

$$\gamma_k(p) = \frac{b}{p\sqrt{p+k^2}}. \tag{96}$$

The function may be found directly from a table but we explain the result as an example for further development of the renormalization procedure. Note that the cut for the radical $\sqrt{p+b^2}$ is made along the Re p-axis from $-\infty$ to $-b^2$, the branch with $\sqrt{p+b^2} = i\sqrt{|p+b^2|}$ on the upper and $\sqrt{p+b^2} = -i\sqrt{|p+b^2|}$ on the lower bounds of the cut is chosen. After the deformation of the integral contour one has for $t > 0$

$$\gamma_L^R = 1 - \frac{b}{\pi}\int_0^\infty \frac{\exp(\xi+b^2)t}{(\xi+b^2)\sqrt{\xi}}d\xi. \tag{97}$$

The formula (97) gives an expansion of the resolvent of the operator $\partial_t + L$ by the operator L spectrum. The direct substitution of (97) leads to a divergent integrals. However in this case a renormalization is not necessary, the integral in the r.h.s. of (97) is expressed in terms of the error function

$$Erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2} d\tau.$$

namely, after some change of variables

$$\gamma_k(t) = Erf(b\sqrt{t}) = \frac{2b\sqrt{t}}{\sqrt{\pi}} \int_0^1 \exp[-b^2 t \tau^2] d\tau. \tag{98}$$

The Mellin transform of this gives the following expression of zeta function of the operator L.

$$\zeta_L^R(s) = \frac{2b^{-2s}}{\sqrt{\pi}}\frac{\Gamma(s+1/2)}{\Gamma(s)}\int_0^1 \tau^{-2s-1}d\tau = -\frac{b^{-2s}}{\sqrt{\pi}}\frac{\Gamma(s+1/2)}{\Gamma(s+1)}. \tag{99}$$

The integral converged only at $Res < 0$ but gives the analytical continuation for all $s \in C$ excluding the poles of $\Gamma(s+1)$.

Substituting the result (99) into one arising from (21) with account of (94) yields

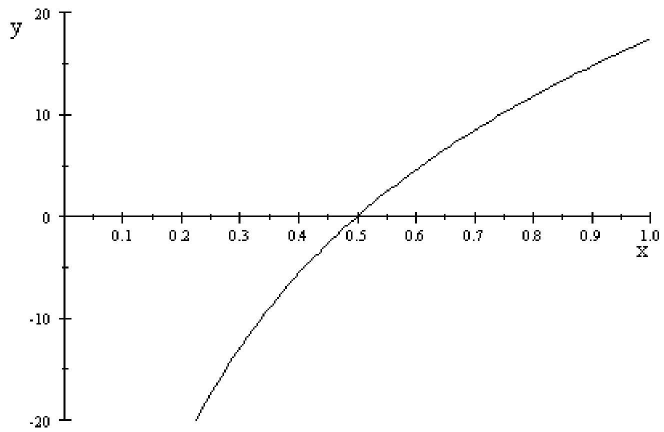
$$\zeta_L^D(s) = -4(4\pi)^{\frac{d}{2}} m^{d-1-2s} \frac{\Gamma(s+1-d/2)}{(2s+1-d)\Gamma(s)}. \tag{100}$$

The condition $b = m$ is taken into account.

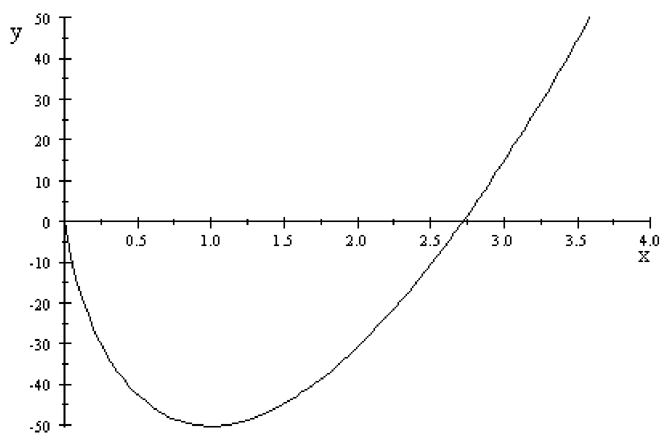
Differentiation of (100) by s at $s = 0$ give the desired correction

$$-\frac{1}{2}\frac{d\left(\zeta_L^D(s)\right)}{ds} = -2(4\pi)^{\frac{d}{2}} m^{d-2s-1} \frac{\Gamma\left(s-\frac{1}{2}d+1\right)}{\Gamma(s)(2s-d+1)^2} \left((d-2s-1)\left(\mathrm{Psi}\left(-\frac{1}{2}d+s+1\right) - 2\ln m - \mathrm{Psi}(s)\right) + 2\right) \tag{101}$$

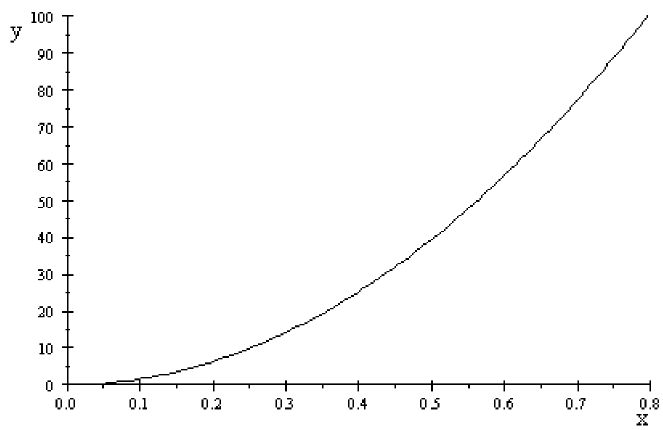
Plot the dependence of the mass correction $\frac{\hbar}{2}\frac{d\left(\zeta_L^D(0)\right)}{ds}$ on m of SG kink.



d=1, dependence of the correction $\frac{2}{\hbar}\delta m$ to mass on the parameter m



d=2, dependence of the correction $\frac{2}{\hbar}\delta m$ to mass on the parameter m



d=3, dependence of the correction $\frac{2}{\hbar}\delta m$ to mass on the parameter m

We would remind about the choice of the constant g=2 in the last sections.

Conclusions/Problems/Perspective

1. This approach allows to calculate one-loop corrections to the N-level reflectionless potential and solitons of SG and other models.

2. Main nearest perspective - applications of SG model to kink/soliton dynamics.

3. Embedding of Nahm model quantization to 3+1 YM theory.

4. Analyze SUSY partners (DT),

5. New possibilities; the quantum corrections to matrix models, Q-balls (Cervero-Estevez 1986), etc.