NEW INEQUALITIES FOR RÈNYI ENTROPY ASSOCIATED TO TOMOGRAPHIC PROBABILITIES OF RADIATION BEAMS IN WAVEGUIDES

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Abstract

New entropic inequalities for Rènyi entropy are discussed. The inequalities are presented for functions of one continuous variable and functions of two continuous variables which are mapped onto tomographic-probability distributions using fractional Fourier transforms in one and two dimensions, respectively. The Shannon entropy and the entropic inequalities for the tomograms are studied. The minimum value of the Shannon entropy associated to saturation of entropic inequality is discussed. The discrete Fourier transform is used to obtain some inequalities for entropies associated to unitary matrices. The radiation beam modes in optical waveguides are described by tomographic-probability distributions containing complete information on the modes. The partial case of Hermite–Gauss modes is studied in detail.

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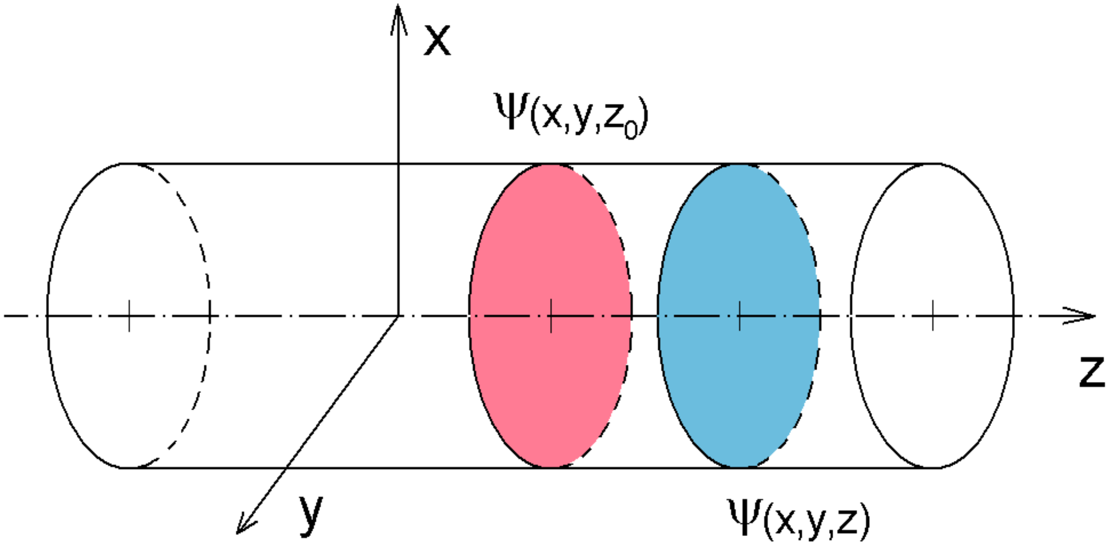

Figure 1: Wave function of light beam in optical fiber.

Introduction

The radiation beams in waveguides are described by the beam mode-profile function $\Psi(x, y, z)$ where x and y are transversal coordinates and z is the longitudinal coordinate. The complex function $\Psi(x, y, z)$ can be a solution to a linear or nonlinear equation. For, example, this function can be the soliton solution of nonlinear Schrödinger equation. In the case of linear propagation, the beam mode-profile function can be chosen as the solution to a linear Schrödinger-like equation.

There exist integral transforms which map the functions $\Psi(x, y, z)$ on the probability distribution densities. The transforms can be given as the so-called tomographic maps like symplectic tomogram or Fresnel tomogram. In these cases, the probability characteristics like entropy can be associated to the beams propagating in the media. The idea of this association is that for any probability distribution there exists a well-known construction of Shannon entropy or Rènyi entropy. Also for these entropies some nontrivial inequalities were found in an abstract context. In view of this, one can employ these known inequalities and implement them in a new domain of radiation beams propagating in optical waveguides.

C. E. Shannon, *B*ell. Tech. J., **27**, 379 (1948).

A. Rènyi, *Probability Theory*, North-Holland, Amsterdam, 1970.

We review here the properties of tomographic invertible map of beam mode-profile functions onto the probability-distribution densities and, in view of the known inequalities of Shannon and Rènyi entropies, apply these inequalities for obtaining new characteristics and properties of the beams propagating in the waveguides. In addition, we discuss the new entropic inequalities for the functions of discrete variables which are analogs of spinors. Such functions can also be mapped onto probability distributions (called spin tomograms). For such functions, there exist discrete analogs of entropic inequalities and we apply these inequalities to associate them with properties of discretized "beam profiles." It can be considered as a model in which the beam-profile function of continuous coordinates is replaced by a finite set of functions depending on discrete variable. The issues of the talk are

- Schrödinger-like equation;
- Two-dimensional Hermite–Gauss modes in optical waveguides;
- Symplectic tomogram for two-dimensional mode-profile function;
- Fresnel tomograms;
- Entropic inequalities for light beams;
- Entropic inequalities for discrete variables.

Fock–Leontovich approximation for paraxial beams of electromagnetic radiation

In reality, it is not unusual that purely classical systems can be described by a quantumlike equation. For example, Fock and Leontovich used this ansatz in their study of the electromagnetic-wave propagation along the Earth's surface [Fock V A and Leontovich M A 1946 *Zh. Éksp. Teor. Fiz.* **16** 557]. They have shown that Maxwell's equations in nondispersive media for paraxial beams of the electromagnetic field can be reduced to a Schrödinger-like equation (the so-called paraxial approximation).

In fact, since it is the standard procedure for studying the electromagnetic waves, starting from Maxwell's equations, one obtains the Helmholz equation for the component of the electric field. The Helmholz equation is obtained for the wave with given frequency, neglecting the media dispersion and influence of polarization,

$$\frac{\partial^2 E}{\partial^2 x} + \frac{\partial^2 E}{\partial z^2} + k^2 n^2(x, z)\, E = 0 \tag{1}$$

(we consider a slab or planar waveguide configuration). In Eq. (1), $\lambda = 2\pi/k$ is the wavelength in vacuum, z is the longitudinal coordinate, and $n(x, z)$ is the refractive index.

For paraxial beams, we introduce the complex function $\Psi(x,z)$, which is a slowly varying amplitude of the electric field, using the following formula:

$$E(x,z) = n_0^{-1/2}(z)\,\Psi(x,z)\,\exp\left[ik\int_0^z n_0(\xi)\,d\xi\right]. \tag{2}$$

The ansatz (2) reduces the Helmholz equation (1) to the Schrödinger-like equation

$$i\lambda\frac{\partial\Psi(x,z)}{\partial z} = -\frac{1}{2n_0(z)}\frac{\partial^2\Psi(x,z)}{\partial x^2} + U(x,z)\,\Psi(x,z), \tag{3}$$

with $U(x,z)$ being an effective potential related to the refractive index of the medium $n(x,z)$ as

$$U(x,z) = \frac{1}{2\,n_0(z)}\left[n_0^2(z) - n^2(x,z)\right],$$

where $n_0(z) = n(0,z)$ is the refractive index of the medium at the beam axis. While deriving this equation, one neglects the second-order derivatives of ψ with respect to the coordinate z and the derivatives of the function $n_0(z)$, that can be done in the case of a slow variation of the refractive index along the beam axis.

Analogous Schrödinger-like equation can be obtained for the light propagating in optical fibers. This case corresponds to a quanutmlike system with two degrees of freedom.

Two-dimensional Hermite–Gauss modes in optical fibers

In the case of parabolic profile of refractive index in optical fibers (selfoc) where the potential is given in dimensionless variables as

$$U(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2},$$

one has the solutions of Schrödinger-like equations in the form of Hermite–Gauss functions (Hermite–Gauss modes) labeled by two integer numbers $m, n = 0, 1, 2 \ldots$ For given distance z_0, for example $z_0 = 0$, one has in dimensionless units

$$\Psi_{mn}(x, y) = f_m(x) f_n(y),$$

where

$$f_m(x) = \frac{e^{-x^2/2}}{\pi^{1/4}} \frac{1}{\sqrt{2^m m!}} H_m(x),$$

with $H_m(x)$ being the Hermite polynomial. The same ansatz is valid for $f_n(y)$.

For example,

$$f_0(x) = \frac{e^{-x^2/2}}{\pi^{1/4}}, \qquad f_1(x) = f_0(x)\sqrt{2}x, \qquad \text{etc.}$$

Symplectic Tomography of Two Beams

We review tomographic approach to describe modes propagating in optical waveguides, e.g., optical fibers. The longitudinal coordinate z is considered for a moment to be fixed, e.g., $z = 0$. The profile of the mode field at this coordinate is given as a function $\Psi(x, y)$ of the fibers' transversal coordinates; it is considered to be normalized, i.e.,

$$\int |\Psi(x,y)|^2 \, dx \, dy = 1. \tag{4}$$

The tomographic-probability distribution (called also tomogram) is defined by Radon transform [J. Radon, *Berichte Sachsische Akademie der Wissenschaften, Leipzig, Mathematische-Physikalische Klasse*, **69** (1917), S. 262] generalized for two dimensions as symplectic tomogram [G.M. D'Ariano, S. Mancini, V.I. Man'ko, and P. Tombesi, *Quantum Semiclass. Opt.*, **8**, 1017 (1996)] and given as Fresnel integral squared in [Man'ko M A, Man'ko V I and Mendes R V 2001 *J. Phys. A: Math. Gen.* **34** 8321] as follows:

$$w(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = \frac{1}{4\pi^2|\nu_1\nu_2|} \left| \int \Psi(x,y) \exp\left[\frac{i}{2}\left(\frac{\mu_1}{\nu_1}x^2 + \frac{\mu_2}{\nu_2}y^2 - \frac{2X_1}{\nu_1}x - \frac{2X_2}{\nu_2}y\right)\right] dx \, dy \right|^2. \tag{5}$$

The symplectic tomogram (5) $w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)=$
$$\frac{1}{4\pi^2|\nu_1\nu_2|}\left|\int\Psi(x,y)\exp\left[\frac{i}{2}\left(\frac{\mu_1}{\nu_1}x^2+\frac{\mu_2}{\nu_2}y^2-\frac{2X_1}{\nu_1}x-\frac{2X_2}{\nu_2}y\right)\right]dx\,dy\right|^2$$
is nonnegative function of six real variables X_1, μ_1, ν_1, X_2, μ_2, and ν_2. It satisfies the normalization condition
$$\int w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)\,dX_1\,dX_2=1 \tag{6}$$
and can be interpreted as the joint probability density of two random variables
$$X_1=\mu_1x+\nu_1p_x,\qquad X_2=\mu_2y+\nu_2p_y, \tag{7}$$
where x and y are coordinates of an intersection point in transversal plane of the light ray and p_x and p_y are small angles determining the unit direction vector parallel to the light ray propagating along the fiber axis.

The tomographic-probability density determines the modulus and phase factor of the mode-profile function $\Psi(x,y)$ due to the inverse relation
$$\begin{aligned}\Psi(x,y)\Psi^*(x',y')&=\frac{1}{4\pi^2}\int dX_1\,dX_2\,d\mu_1\,d\mu_2\,d\nu_1\,d\nu_2\\&\times w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)\delta(\nu_1-x+x')\delta(\nu_2-y+y')\\&\times\exp\left\{\frac{i}{2}[2X_1-\mu_1(x+x')+2X_2-\mu_2(y+y')]\right\}.\end{aligned} \tag{8}$$

The symplectic tomogram (5) has the homogeneity property

$$w(\lambda_1 X_1, \lambda_1\mu_1, \lambda_1\nu_1, \lambda_2 X_2, \lambda_2\mu_2, \lambda_2\nu_2) = \frac{1}{|\lambda_1\lambda_2|} w(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2). \tag{9}$$

In view of this, it can be expressed in terms of optical tomogram depending on four real variables

$$\begin{aligned} w_{\mathrm{opt}}(X_1, \theta_1, X_2, \theta_2) \ &= \ \frac{1}{4\pi^2 |\sin\theta_1 \sin\theta_2|} \bigg| \Psi(x,y) \exp\bigg[\frac{i}{2}\Big(\cot\theta_1(x^2 + X_1^2) \\ &\quad + \cot\theta_2(y^2 + X_2^2) - \frac{2ixX_1}{\sin\theta_1} - \frac{2iyX_2}{\sin\theta_2}\Big)\bigg]\, dx\, dy \bigg|^2 . \end{aligned} \tag{10}$$

In fact, due to definitions of symplectic (5) and optical (10) tomograms, one has

$$w_{\mathrm{opt}}(X_1, \theta_1, X_2, \theta_2) = w(X_1, \cos\theta_1, \sin\theta_1, X_2, \cos\theta_2, \sin\theta_2) \tag{11}$$

and, due to the homogeneity property (9),

$$\begin{aligned} w(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) &= \frac{1}{\sqrt{(\mu_1^2 + \nu_1^2)\,(\mu_2^2 + \nu_2^2)}} \\ &\times w_{\mathrm{opt}} \left(\frac{X_1}{\sqrt{(\mu_1^2 + \nu_1^2)}}, \arctan\frac{\nu_1}{\mu_1}, \frac{X_2}{\sqrt{(\mu_2^2 + \nu_2^2)}}, \arctan\frac{\nu_2}{\mu_2} \right). \end{aligned} \tag{12}$$

Another tomogram called Fresnel tomogram of the light mode in the optical fiber is given by the integral transform

$$w_{\mathrm{F}}(X_1,\nu_1,X_2,\nu_2)=\frac{1}{4\pi^2}\frac{1}{|\nu_1\nu_2|}\left|\int\Psi(x,y)\exp\left[\frac{i(X_1-x)^2}{2\nu_1}+\frac{i(X_2-y)^2}{2\nu_2}\right]dx\,dy\right|^2. \tag{13}$$

Fresnel tomogram w_{F} is related to symplectic tomogram

$$w_{\mathrm{F}}(X_1,\nu_1,X_2,\nu_2)=w(X_1,1,\nu_1,X_2,1,\nu_2). \tag{14}$$

One can obtain symplectic tomogram $w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)$ in terms of Fresnel tomogram w_{F}

$$w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)=\frac{1}{|\mu_1\mu_2|}w_{\mathrm{F}}\left(\frac{X_1}{\mu_1},\frac{\nu_1}{\mu_1},\frac{X_2}{\mu_2},\frac{\nu_2}{\mu_2}\right). \tag{15}$$

In view of mutual tomogram relations one can find the mode-profile function $\Psi(x,y)$ either in terms of optical tomogram or in terms of Fresnel tomogram.

These tomograms and their mutual relations were considered, for example, in: [S. De Nicola, R. Fedele, M.A. Man'ko, and V.I. Man'ko, 'New uncertainty relation for tomographic entropy: Application to squeezed states and solitons," Eur. Phys. J. B, Vol. 52, pp. 191-198 (2006) [ArXiv quant-ph/0607200v1]; "New inequalities for tomograms in the probability representation of quantum states," Theor. Math. Phys., Vol. 152, pp. 1081–1086 (2007) [ArXiv quant-ph/0611114 v1]; "Symplectic entropy" (Feynman Festival, University of Maryland, August 2006), J. Phys. Conf. Ser., Vol. 70, 012007 (2007)].

For Hermite–Gauss modes of light propagating in optical waveguides, one has the tomograms:

(1) Optical tomogram

$$w_{\mathrm{opt}\,mn}(X_1,\theta_1,X_2,\theta_2)=\frac{1}{4\pi^2|\sin\theta_1\sin\theta_2|}\left|\int\frac{e^{-(x^2/2)-(y^2/2)}}{\sqrt{\pi}}\,H_m(x)H_n(y)\frac{1}{\sqrt{2^{m+n}m!n!}}\right.$$
$$\left.\times\exp\left[\frac{i}{2}\left(\cot\theta_1(x^2+X_1^2)+\cot\theta_2(y^2+X_2^2)-\frac{2ixX_1}{\sin\theta_1}-\frac{2iyX_2}{\sin\theta_2}\right)\right]dx\,dy\right|^2. \tag{16}$$

(2) Fresnel tomogram

$$w_{\mathrm{F}\,mn}(X_1,\nu_1,X_2,\nu_2)=\frac{1}{4\pi^2}\frac{1}{|\nu_1\nu_2|}\left|\int\frac{e^{-(x^2/2)-(y^2/2)}}{\sqrt{\pi}}\,H_m(x)H_n(y)\frac{1}{\sqrt{2^{m+n}m!n!}}\right.$$
$$\left.\times\exp\left[\frac{i(X_1-x)^2}{2\nu_1}+\frac{i(X_2-y)^2}{2\nu_2}\right]dx\,dy\right|^2. \tag{17}$$

Thus, we constructed in the explicit form tomographic-probability distributions for both optical and Fresnel tomogram of Hermite–Gauss modes in optical fibers.

Tomographic Entropies of Light Beams

In this section, we consider tomographic entropies associated to light beams in optical waveguides. There exists Shannon construction [C.E. Shannon, *B*ell. Tech. J., **27**, 379 (1948)] of entropy associated to a probability distribution function $P(n)$ of a discrete variable n

$$H = -\sum_n P(n)\ln P(n). \tag{18}$$

One can apply the Shannon construction to introduce tomographic entropy associated to tomographic probability distribution (5)

$$w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2) =$$
$$\frac{1}{4\pi^2|\nu_1\nu_2|}\left|\int\Psi(x,y)\exp\left[\frac{i}{2}\left(\frac{\mu_1}{\nu_1}x^2+\frac{\mu_2}{\nu_2}y^2-\frac{2X_1}{\nu_1}x-\frac{2X_2}{\nu_2}y\right)\right]dx\,dy\right|^2.$$

as follows:

$$H(\mu_1,\nu_1,\mu_2,\nu_2) = -\int w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)\ln w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)\,dX_1\,dX_2 \tag{19}$$

The above entropy is a new information characteristic of the light-beam profile in optical waveguides.

The generic symplectic entropy (19)

$$H(\mu_1,\nu_1,\mu_2,\nu_2) \equiv -\int w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)\ln w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)\,dX_1\,dX_2$$

yields also the optical tomographic entropy of the light beam

$$H_{\rm opt}(\theta_1,\theta_2) = -\int w_{\rm opt}(X_1,\theta_1,X_2,\theta_2)\ln w_{\rm opt}(X_1,\theta_1,X_2,\theta_2)\,dX_1\,dX_2 \qquad (20)$$

and the Fresnel tomographic entropy

$$H_{\rm F}(\nu_1,\nu_2) = -\int w_{\rm F}(X_1,\nu_1,X_2,\nu_2)\ln w_{\rm F}(X_1,\nu_1,X_2,\nu_2)\,dX_1\,dX_2. \qquad (21)$$

All three entropies (19), (20), and (21) are mutually related. An interesting property of tomographic entropies is connected with the uncertainty relation associated to the light-beam intensities determined by the mode-profile function $\Psi(x,y)$ and its Fourier transform

$$\tilde\Psi(p_x,p_y) = \frac{1}{2\pi}\int\Psi(x,y)\exp^{-i(p_xx+p_yy)}\,dx\,dy. \qquad (22)$$

This entropic uncertainty relation reads (see, e.g., I. Bialynicki-Birula, “Formulation of the uncertainty relations in terms of the Rènyi entropies,” *Phys. Rev. A*, **74**, 052101 (2006) [ArXiv quant-ph/0608116 v1])

$$-\int|\Psi(x,y)|^2\ln|\Psi(x,y)|^2\,dx\,dy - \int|\tilde\Psi(p_x,p_y)|^2\ln|\tilde\Psi(p_x,p_y)|^2\,dp_x\,dp_y \geq 2\ln(\pi e). \qquad (23)$$

The entropic uncertainty relation was generalized for the symplectic tomographic entropy as well as for the optical and Fresnel tomographic entropies. We consider below the example of optical tomographic entropy and introduce the function

$$R(\theta_1,\theta_2)=H_{\mathrm{opt}}(\theta_1,\theta_2)+H_{\mathrm{opt}}(\theta_1+\pi/2,\theta_2+\pi/2)-2\ln(\pi e), \qquad (24)$$

where $H_{\mathrm{opt}}(\theta_1,\theta_2)$ is given by (20)

$$H_{\mathrm{opt}}(\theta_1,\theta_2)\equiv-\int w_{\mathrm{opt}}(X_1,\theta_1,X_2,\theta_2)\ln w_{\mathrm{opt}}(X_1,\theta_1,X_2,\theta_2)\,dX_1\,dX_2.$$

The function (24) takes minimum value equal to zero. This value is realized for the light-beam profile in the fundamental Hermite–Gauss mode with $m=n=0$.

According to new entropic uncertainty relations, the function $R(\theta_1,\theta_2)$ must be nonnegative for all the values of angles θ_1 and θ_2, i.e.,

$$R(\theta_1,\theta_2)\geq 0. \qquad (25)$$

This means that, if one measures the modulus and phase of the mode-profile function $\Psi(x,y)$ by any method, the results of the measurement yield also the function (24) which must be nonnegative.

Nonnegativity of the function $R(\theta_1,\theta_2)$ for all the angles θ_1 and θ_2 can serve as an extra control of accuracy of the measurements.

The inequality constructed can be checked for the Hermite–Gaussian mode profiles in optical waveguides.

New inequalities for Rènyi operator symbol entropies

In this section, we continue the study of tomographic entropies along the line of our previous work: M.A. Man'ko, V.I. Man'ko, and R. Vilela Mendes, "A probabilistic operator symbol framework for quantum information," J. Russ. Laser Res., Vol. 27, pp. 507-532 (2006) [ArXiv quant-ph/0602189 v1] and derive new inequalities for spin tomographic entropies related to quantum Fourier transform. For continuous conjugate variables (position and momentum), the inequalities for Rènyi entropy associated with probability densities in the position and momentum were obtained in [I. Bialynicki-Birula, *Phys. Rev. A*, **74**, 052101 (2006) [ArXiv quant-ph/0608116 v1]. In this work, for N-dimensional Hilbert space an analog of the uncertainty relation for the Rènyi entropies was given in the form

$$\frac{1}{1-\alpha}\ln\left(\sum_{k=1}^{N}\tilde{p}_k^{\alpha}\right)+\frac{1}{1-\beta}\ln\left(\sum_{l=1}^{N}p_l^{\beta}\right)\geq\ln N, \tag{26}$$

where α and β are real numbers, $\tilde{p}_k=\mid\tilde{a}_k\mid^2$, $p_l=\mid a_l\mid^2$, $(1/\alpha)+(1/\beta)=2$, and the complex numbers $\tilde{a}_k$ and a_l are connected by quantum Fourier transform

$$\tilde{a}_k=\frac{1}{\sqrt{N}}\sum_{l=1}^{N}\exp\left(\frac{2\pi ikl}{N}\right)a_l=\sum_{l=1}^{N}F_{kl}a_l, \tag{27}$$

while F_{kl} are the matrix element of quantum Fourier transform matrix F.

The quantum Fourier transform matrix F has the form

$$F_{m'm} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a & a^2 & \cdots & a^{N-1} \\ 1 & a^2 & a^4 & \cdots & a^{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a^{N-1} & a^{N-2} & \cdots & a \end{pmatrix}, \qquad a = \exp\left(\frac{2\pi i}{N}\right). \tag{28}$$

Below we use the inequality of Bialynicki-Birula to obtain new inequalities for Shannon and Rènyi entropies associated with the so-called unitary spin tomograms.

The unitary spin tomogram of a j-spin state with the density $N\times N$-matrix ρ can be considered as a column probability N-vector $\vec{w}(u)$ $N = 2j+1$ depending on unitary $N\times N$-matrix u with the components

$$w_m(u) \equiv w(m,u) = \langle m \mid u^\dagger \rho u \mid m\rangle,$$

$$m = -j, -j+1, -j+2, \ldots, j-1, j, \qquad j = 0, 1/2, 1, 3/2, \ldots$$

Then we can introduce another N-vector with components $p_m(u) = \sqrt{w_m(u)}$.

Applying the B-B inequality (26)

$$\frac{1}{1-\alpha}\ \ln\left(\sum_{k=1}^{N}\tilde{p}_k^{\alpha}\right)+\frac{1}{1-\beta}\ \ln\left(\sum_{l=1}^{N}p_l^{\beta}\right)\geq \ln N$$

to these vectors and using the notation

$$\left|\sum_{m'=-j'}^{j}F_{mm'}\sqrt{w(m',u)}\right|=\sqrt{w_F(m,u)}, \tag{29}$$

where $F_{mm'}$ is matrix with matrix elements of quantum Fourier transform and $w_F(m,u)$ is the probability distribution given by (29), we obtain inequality

$$\frac{1}{1-\alpha}\ \ln\left(\sum_{m=-j}^{j}w(m,u)^{\alpha}\right)+\frac{1}{1-\beta}\ \ln\left(\sum_{m=-j}^{j}w_F(m,u)^{\beta}\right)\geq \ln N. \tag{30}$$

In this sum, the first term in this sum is called Rènyi entropy $R_\alpha(u)$ and second term is called Rènyi entropy $R_\beta(u)$.

For $\alpha\rightarrow 1$, Rènyi entropy $R_\alpha(u)$ has as the limit Shannon entropy $H(u)$.

Also using for pure state N-vector $\mid \psi\rangle$ the definition of spin tomogram $w(m,u) = |\langle m \mid u \mid \psi\rangle|^2$, we obtain another similar inequality

$$\frac{1}{1-\alpha}\ln\left(\sum_{m=-j}^{j} w(m,u)^{\alpha}\right) + \frac{1}{1-\beta}\ln\left(\sum_{m=-j}^{j} w(m,Fu)^{\beta}\right) \geq \ln N, \tag{31}$$

where F is quantum Fourier transform matrix with matrix elements

$$F_{m'm} = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a & a^2 & \cdots & a^{N-1} \\ 1 & a^2 & a^4 & \cdots & a^{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a^{N-1} & a^{N-2} & \cdots & a \end{pmatrix}, \qquad a = \exp\left(\frac{2\pi i}{N}\right).$$

Thus one has for Rènyi entropy the inequality for each unitary matrix

$$R_{\alpha}(u) + R_{\beta}(Fu) \geq \ln N. \tag{32}$$

Thus the unitary spin tomogram of the particle with spin j for the state with $N\times N$ density matrix ρ, where $N = 2j+1$, must satisfy inequality (31).
In the limit $\alpha \to 1$, $\beta \to 1$, one gets inequalities for Shannon entropy of spin state

$$H(u) + H(Fu) \geq \ln N, \tag{33}$$

with the first term on the left-hand side being Shannon entropy and the second one, its quantum Fourier transform.

Another inequality reads

$$H(u) + H_F(u) \geq \ln N, \tag{34}$$

where $H_F(u)$ is Shannon entropy associated with probability distribution $w_F(m,u)$.

For minimum value of the Shannon entropy realized for unitary matrix u_0, one has the von Neuman entropy defined by the density matrix ρ as $S_{\rm vN} = -\mathrm{Tr}\,\rho\ln\rho$, i.e., the equality

$$H(u_0) = S_{\rm vN}. \tag{35}$$

This means that the von Neuman entropy $S_{\rm vN}$ of a quantum state can be found as the minimum value of Shannon entropy $H(u)$ where the minimum takes place for a specific unitary matrix u_0.

Inequality (33) $H(u) + H(Fu) \geq \ln N$ written for unitary matrix u_0

$$H(u_0) + H(Fu_0) \geq \ln N \tag{36}$$

provides a new inequality for von Neuman entropy

$$S_{\rm vN} + H(Fu_0) \geq \ln N, \tag{37}$$

where $H(Fu_0)$ is a new entropy.

The inequality has the following physical interpretation.

If the density operator $\hat{\rho}$ of the quantum state of spin is given in the form of spectral decomposition

$$\hat{\rho} = \sum_{q=-j}^{j} \lambda_q \mid q\rangle\langle q \mid, \tag{38}$$

one can choose the eigenstate $\mid q\rangle$ of the density operator $\hat{\rho}$ and identify it with a discrete "position" state. Then the states

$$\mid p\rangle = \hat{F} \mid q\rangle, \tag{39}$$

where $\hat{F}$ is the Fourier transform operator, are interpreted as "momentum" eigenstates. The matrix elements

$$\langle p \mid \hat{F} \mid q\rangle = F_{pq} \tag{40}$$

provide the matrix F which coincides with the quantum Fourier transform matrix.

Now position and momentum became discrete numbers but the relation between the position and momentum vectors is analogous to the relation between these vectors in the case of continuous position and momentum which is known to be given by the standard Fourier transform.

Thus we have the interpretation of the new inequality in the same manner as it was done in the case of continuous variables.

The new entropy $H(Fu_0)$ in (37)

$$S_{\rm vN} + H(Fu_0) \geq \ln N,$$

is the Shannon entropy for "momentum" distribution, if we identify the standard von Neuman entropy with Shannon entropy for "position" distribution.

Let us illustrate the inequalities on the example of pure qubit state, i.e., the spin state $s = 1/2$ and $s_z = 1/2$ with density matrix $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

"Position" operator $\hat{q}$ is identified with Pauli σ_z matrix, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and "momentum" operator $\hat{p}$ is identified with Pauli σ_x matrix, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Two position eigenvectors $\mid q\rangle$, where $q = \pm 1/2$, are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and two momentum eigenvectors $\mid p\rangle$, where $p = \pm 1/2$, are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The quantum Fourier transform matrix F reads: $F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

For our case, matrix $u_0 = 1$.

Inequality (37) for the von Neuman entropy $S_{\mathrm{vN}}+H(Fu_0)\geq \ln N$ is saturated since

$$S_{\mathrm{vN}}=0,\qquad H(F)=\ln 2 \tag{41}$$

and

$$S_{\mathrm{vN}}+H(F)=\ln 2\geq \ln 2. \tag{42}$$

Also inequality (32) for Rènyi entropy $R_\alpha(u)+R_\beta(Fu)\geq \ln N$ is saturated

$$R_\alpha(u_0)+R_\beta(Fu_0)=\ln 2\geq \ln 2. \tag{43}$$

To illustrate in more detail the inequalities obtained, let us now discuss the example of mixed state of spin $s=1/2$ state (qubit) with diagonal density matrix with real nonnegative matrix elements

$$\rho=\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix},\qquad a+b=1. \tag{44}$$

Then inequality (33) $H(u)+H(Fu)\geq \ln N$ can be visualized as follows.
Von Neuman entropy of this state reads: $S_{\mathrm{vN}}=-a\ln a-b\ln b$.
The density matrix subjected by quantum Fourier transform

$$F=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1\\ 1 & -1\end{pmatrix}\qquad \text{reads}\qquad F^\dagger\rho F=\begin{pmatrix} 1/2 & (a-b)/2\\ (a-b)/2 & 1/2\end{pmatrix}. \tag{45}$$

Its tomographic entropy

$$H(Fu_o) = \ln 2, \qquad u_0 = 1. \tag{46}$$

Thus inequality (33) $\quad H(u) + H(Fu) \geq \ln N \quad$ looks as follows:

$$-a \ln a - b \ln b + \ln 2 \geq \ln 2, \tag{47}$$

which only means that von Neuman entropy is nonnegative.

But inequality (34) $\quad H(u) + H_F(u) \geq \ln N \quad$ gives better estimation, as we see below, since the number $\ln 2$ is replaced by a smaller number.

In fact, the tomographic-probability vector of the qubit state

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{48}$$

is associated to the probability-amplitude vector with positive components

$$\vec{W} = \begin{pmatrix} \sqrt{a} \\ \sqrt{b} \end{pmatrix}. \tag{49}$$

Then after making the quantum Fourier transform of this vector, we get the column vector

$$\vec{W}_F = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{a} + \sqrt{b} \\ \sqrt{a} - \sqrt{b} \end{pmatrix}. \tag{50}$$

The probability-distribution vector associated to the above probability-amplitude vector reads

$$\vec{w}_F = \begin{pmatrix} (1/2) + \sqrt{ab} \\ (1/2) - \sqrt{ab} \end{pmatrix}. \tag{51}$$

Thus we apply inequality relating Shannon entropies to two vectors (49)

$$\vec{W} = \begin{pmatrix} \sqrt{a} \\ \sqrt{b} \end{pmatrix}$$

and (51) and obtain

$$-a\ln a - b\ln b - \left(\frac{1}{2} + \sqrt{ab}\right)\ln\left(\frac{1}{2} + \sqrt{ab}\right) - \left(\frac{1}{2} - \sqrt{ab}\right)\ln\left(\frac{1}{2} - \sqrt{ab}\right) \geq \ln 2, \tag{52}$$

or

$$S_{\rm vN} - \left(\frac{1}{2} + \sqrt{ab}\right)\ln\left(\frac{1}{2} + \sqrt{ab}\right) - \left(\frac{1}{2} - \sqrt{ab}\right)\ln\left(\frac{1}{2} - \sqrt{ab}\right) \geq \ln 2. \tag{53}$$

This inequality looks more complicated though we know that $S_{\rm vN} \geq 0$.

Some inequalities for unitary matrix can be obtained.

Let us consider unitary $N\times N$-matrix u_{jk}. In view of the formalism developed, one has the inequality

$$-\sum_{j=1}^{N}\left(|u_{jk}|^2\ln|u_{jk}|^2+|(Fu)_{jk}|^2\ln|(Fu)_{jk}|^2\right)\geq \ln N \tag{54}$$

or

$$-\sum_{j=1}^{N}\sum_{k=1}^{N}\left(|u_{jk}|^2\ln|u_{jk}|^2+|(Fu)_{jk}|^2\ln|(Fu)_{jk}|^2\right)\geq N\ln N, \tag{55}$$

where F_{jk} is the Fourier transform matrix.

We demonstrated on the example of qubit ($s=1/2$) that, for the tomograms of spin states connected by quantum Fourier transform, one has constraints in the form of inequalities for Shannon tomographic entropies.

One can demonstrate analogous constraints for Rènyi tomographic entropies too.

Conclusions

The new entropic inequalities are discussed for tomograms of the light-beam profiles in optical fibers. The discrete analogs of the entropic inequalities are obtained. These inequalities can be interpreted as inequalities for tomographic probabilities associated to spin states (qudits).