On classification of Camassa–Holm type equations

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Introduction

- Camassa–Holm equation:

$$m_t = 2mu_x + um_x, \quad m = u - u_{xx}$$

or

$$(1 - D_x^2)u_t = 3uu_x - 2u_xu_{xx} - uu_{xxx}$$

- Peakon solutions

$$u(x,t) = \sum_{j=1}^{N} p_j(t) \exp(-|x - q_j(t)|),$$

where

$$\begin{aligned} \dot{q}_j &= \sum_{k=1}^{N} \exp(-|q_j - q_k|), \\ \dot{p}_j &= p_j \sum_{k=1}^{N} p_k \,\mathrm{sign}(q_j - q_k) \exp(-|q_j - q_k|) \end{aligned}$$

- All the attributes of an integrable equation:
  - Lax representation
  - bi-Hamiltonian structure
  - infinite hierarchies of (local) symmetries and conservation laws

- The first local higher symmetry is

$$m_\tau = D_x(1 - D_x^2)m^{-\frac{1}{2}}$$

- The Camassa-Holm equation can be reduced to the first negative flow of the KdV hierarchy via a reciprocal transformation.

- Degasperis–Procesi equation:

$$m_t = 3mu_x + um_x, \quad m = u - u_{xx}$$

- The first local higher symmetry

$$m_\tau = D_x(1 - D_x^2)(4 - D_x^2)m^{-\frac{2}{3}}$$

- The Degasperis–Procesi equation can be related via a reciprocal transformation to the negative flow of the Kaup–Kupershmidt hierarchy.

- Other integrable equations of the form

$$(1 - D_x^2)u_t = F(u, u_x, \ldots)$$

or

$$m_t = F(u, m, u_x, m_x, \ldots), \quad m = u - u_{xx} \quad ?$$

- Is it possible to classify integrable equations of the Camassa–Holm type?

Theorem 1. *(Mikhailov–VN) If equation*

$$m_t = bmu_x + um_x, \quad m = u - u_{xx}$$

possesses an infinite hierarchy of (quasi-) local higher symmetries then $b = 2, 3$.

Generalisations of the Camassa–Holm type equation

We consider the equation of the form

$$
\begin{aligned}
(1-\epsilon^2 D_x^2)u_t &= c_1 u u_x + \epsilon\left[c_2 u u_{xx} + c_3 u_x^2\right] \\
&+ \epsilon^2\left[c_4 u u_{xxx} + c_5 u_x u_{xx}\right] \\
&+ \epsilon^3\left[c_6 u u_{xxxx} + c_7 u_x u_{xxx} + c_8 u_{xx}^2\right] \\
&+ \epsilon^4\left[c_9 u u_{xxxxx} + c_{10} u_x u_{xxxx} + c_{11} u_{xx} u_{xxx}\right]
\end{aligned}
\tag{1}
$$

- $c_1, \ldots, c_{11} \in \mathbb{C}$, $\epsilon \in \mathbb{C} \setminus \{0\}$,

- The right hand side of the equation is a homogeneous differential polynomial, if we assume weights

$$[D_x^i(u)] = i, \quad [\epsilon] = -1,$$

- The right hand side is quadratic in u and its x-derivatives.

Theorem 2. *Consider the equation (1) and suppose that either:*

$$c_2 \neq 0 \quad or \quad c_6 \neq 0 \quad or \quad c_9 \neq 0 \quad or \quad c_1 + c_4 \neq 0.$$

If the equation (1) possesses an infinite hierarchy of quasi-local higher symmetries then up to re-scaling $x \to \alpha x$, $t \to \beta t, u \to \gamma u$, $\alpha, \beta, \gamma = const$ *it is one of the list:*

$$(1-\epsilon^2 D_x^2)u_t = 3uu_x - 2\epsilon^2 u_x u_{xx} - \epsilon^2 u u_{xxx}, \quad (2)$$

$$(1-\epsilon^2 D_x^2)u_t = D_x\left(4-\epsilon^2 D_x^2\right)u^2, \quad (3)$$

$$(1-\epsilon^2 D_x^2)u_t = D_x\left[(4-\epsilon^2 D_x^2)u\right]^2, \quad (4)$$

$$(1-\epsilon^2 D_x^2)u_t = D_x(2+\epsilon D_x)\left[(2-\epsilon D_x)u\right]^2, \quad (5)$$

$$(1-\epsilon^2 D_x^2)u_t = D_x(2-\epsilon D_x)(1+\epsilon D_x)u^2, \quad (6)$$

$$(1-\epsilon^2 D_x^2)u_t = D_x(2-\epsilon D_x)\left[(1+\epsilon D_x)u\right]^2, \quad (7)$$

$$(1-\epsilon^2 D_x^2)u_t = D_x(1+\epsilon D_x)\left[(2-\epsilon D_x)u\right]^2, \quad (8)$$

$$(1-\epsilon^2 D_x^2)u_t = D_x\left[(2-\epsilon D_x)(1+\epsilon D_x)u\right]^2, \quad (9)$$

$$(1-\epsilon^2 D_x^2)u_t = (1-\epsilon^2 D_x^2)(\epsilon u u_{xx} - \frac{1}{2}\epsilon u_x^2 + c u u_x), \quad (10)$$

$$(1-\epsilon^2 D_x^2)u_t = (1-\epsilon D_x)\left[\epsilon S(u)S(u_{xx}) - \frac{1}{2}\epsilon(S(u_x))^2 - \frac{1}{2}cS(u)S(u_x)\right], \quad S = 1+\epsilon D_x. \quad (11)$$

Camassa-Holm equation (2). The equation (2) is the Camassa-Holm equation. It can be rewritten as

$$m_t = 2mu_x + um_x, \quad m = u - \epsilon^2 u_{xx}.$$

The Camassa-Holm equation possesses an infinite hierarchy of *local* higher symmetries and the first non-trivial local symmetry is

$$u_\tau = D_x(1 - \epsilon^2 u_{xx})^{-\frac{1}{2}}.$$

Degasperi–Procesi equation (3). The equation (3) is the Degasperis-Procesi equation and it can be rewritten as

$$m_t = 6mu_x + 2um_x, \quad m = (1 - \epsilon^2 D_x^2)u.$$

The Degasperis-Procesi equation also possesses an infinite hierarchy of local higher symmetries and the first non-trivial such a symmetry is

$$u_\tau = (4 - \epsilon^2 D_x^2) D_x (1 - \epsilon^2 u_{xx})^{-\frac{2}{3}}.$$

Equation (4). The first non-trivial symmetry of equation (4) is

$$u_\tau = D_x \left[\left(4 - \epsilon^2 D_x^2\right) (1 - \epsilon^2 D_x^2) u \right]^{-\frac{2}{3}}.$$

Equation (4) can be rewritten as

$$m_t = D_x \left(m + 3u\right)^2, \quad m = u - \epsilon^2 u_{xx}.$$

It is easy to see that the Degasperis-Procesi equation transforms into the equation (4) under the transformation

$$u \to (4 - \epsilon^2 D_x^2) u.$$

Equation (5). The first non-trivial symmetry of equation (5) is

$$u_\tau = (2+\epsilon D_x)D_x\left[(2-\epsilon D_x)(u-\epsilon^2 u_{xx})\right]^{-\frac{2}{3}}.$$

The Degasperis–Procesi equation transforms into (5) under the change of variables

$$u \to (2-\epsilon D_x)u.$$

Note that the other transformation $u \to (2+\epsilon D_x)u$ of Degasperis-Procesi gives the equation $(1-\epsilon^2 D_x^2)u_t = D_x(2-\epsilon D_x)\left[(2+\epsilon D_x)u\right]^2$, which transforms into (5) under the change $x \to -x,\ t \to -t$.

Equation (6). Equation (6) possesses a hierarchy of local higher symmetries and the first non-trivial one is

$$u_\tau = D_x\left[(1-\epsilon D_x)u\right]^{-1}.$$

Equation (7). The higher symmetries of this equation are quasi-local and the first non-trivial one is

$$(1+\epsilon D_x)u_\tau = D_x\left[(1-\epsilon^2 D_x^2)u\right]^{-1}.$$

However, the equation (7) can be rewritten as

$$m_t = D_x(2-\epsilon D_x)\left[(1+\epsilon D_x)u\right]^2, \quad m = u-\epsilon^2 u_{xx}$$

and the latter equation possesses an infinite hierarchy of local higher symmetries in dynamical variable m. One can easily check that the first such a symmetry is

$$m_\tau = D_x(1-\epsilon D_x)m^{-1}.$$

Equations (6) and (7) are related by the transformation $u \to (1+\epsilon D_x)u$. It is clear that this transformation does not preserve the locality of higher symmetries of equation (6).

Equation (8). The first higher symmetry of this equation is

$$u_\tau = D_x \left[(2 - \epsilon D_x)(1 - \epsilon D_x)u\right]^{-2}.$$

It possesses an infinite hierarchy of local higher symmetries.

Equation (9). The first non-trivial higher symmetry of this equation is quasi-local

$$(1 + \epsilon D_x)u_\tau = D_x \left[(2 - \epsilon D_x)(1 - \epsilon^2 D_x^2)u\right]^{-2}.$$

We can rewrite this equation as

$$m_t = D_x \left[(2 - \epsilon D_x)(1 + \epsilon D_x)u\right]^2, \quad m = u - \epsilon^2 u_{xx}$$

and the latter equation possesses an infinite hierarchy of local higher symmetries (in m variable). Equation (9) can be obtained from (8) via the transformation $u \to (1 + \epsilon D_x)u$.

Perturbative symmetry approach in the symbolic representation.

Consider an evolutionary equation

$$u_t = F[u] \in \mathcal{R} \qquad (12)$$

$\mathcal{R}$ denotes a differential ring of polynomials in u and its x-derivatives over $\mathbb{C}$. The ring has a natural gradation in degrees of nonlinearity

$$\mathcal{R} = \bigoplus_{n>0} \mathcal{R}_n,$$

where $\mathcal{R}_n$ denotes a linear space of differential polynomials of degree n.

Definition 1. *A differential polynomial $G \in \mathcal{R}$ is called a generator of a symmetry of the equation (12) if a differential equation*

$$u_\tau = G$$

is compatible with (12) $F_\tau - G_t = 0$.

Every differential polynomial $F \in \mathcal{R}$ can be expressed as

$$F = F_1[u] + F_2[u] + F_3[u] \cdots, \quad F_i[u] \in \mathcal{R}_i.$$

It is convenient to introduce a notion of "little oh" as

$$f = o(R_p) \Leftrightarrow F \in \bigoplus_{i>p} R_p.$$

Definition 2. *A differential polynomial $G \in \mathcal{R}$ is called a generator of an approximate symmetry of degree $p > 0$ if a differential equation $u_\tau = G$ is compatible with the equation (12) up to terms of degree p*

$$F_\tau - G_t = o(\mathcal{R}_p).$$

- An integrable equation possesses infinitely many approximate symmetries of any degree.

- Any equation with $u_t = F_1[u] + F_2[u] + \cdots, \ F_1[u] \neq 0$ possesses infinitely many approximate symmetries of degree 1 − these are the symmetries of a linear equation $u_t = F_1[u]$.

- A condition of existence of approximate symmetries of degree 2 imposes strong restrictions on the equation.

- An equation may possess infinitely many approximate symmetries of degree 2 but fail to possess approximate symmetries of degree 3.

- Under some technical conditions on the equation the condition of existence of infinitely many approximate symmetries of degree 3 and the existence of at least one exact symmetry is sufficient for integrability.

Symbolic representation

Let us introduce a notation $u_i := D_x^i(u), \ i = 0, 1, 2, 3, \ldots$. Also we shall often write u_0 as u.

Symbolic representation is nothing more than a simplified form of notations of a Fourier transform.

1) Linear monomials u_n:

$$u_n \to \widehat{u}\xi_1^n,$$

2) Quadratic monomials $u_n u_m$:

$$u_n u_m \to \frac{\widehat{u}^2}{2}(\xi_1^n \xi_2^m + \xi_1^m \xi_2^n).$$

3) General monomial:

$$u_0^{n_0} u_1^{n_1} \cdots u_p^{n_p} \to$$

$$\to \widehat{u}^n \langle \xi_1^0 \xi_2^0 \cdots \xi_{n_0}^0 \xi_{n_0+1}^1 \cdots \xi_{n_0+n_1}^1 \cdots \xi_n^p \rangle_\xi$$

$$n_0 + n_1 + \cdots + n_p = n.$$

4) Multiplication: $f, g \in \mathcal{R}$

$$f \to u^n a(\xi_1, \ldots, \xi_n), \ g \to u^p b(\xi_1, \ldots, \xi_p)$$

$$fg \quad \to \quad u^{n+p} \langle a(\xi_1, \ldots, \xi_n)\, b(\xi_{n+1}, \ldots, \xi_{n+p}) \rangle_\xi,$$

5) Derivation: $f \in \mathcal{R}$

$$f \to \widehat{u}^n a(\xi_1, \ldots, \xi_n)$$

$$D_x^N(f) \to \widehat{u}^n (\xi_1 + \cdots + \xi_n)^N a(\xi_1, \ldots, \xi_n).$$

For example, if

$$f = uu_2 \Longrightarrow f \to \frac{\widehat{u}^2}{2}(\xi_1^2 + \xi_2^2), \;\; D_x^n(f) \to \frac{\widehat{u}^2}{2}(\xi_1 + \xi_2)^n(\xi_1^2 + \xi_2^2)$$

5) Pseudo-differential operators in the symbolic form:

$$D_x \to \eta$$

$$\eta^k(\widehat{u}^n a(\xi_1, \ldots, \xi_n)) = \widehat{u}^n a(\xi_1, \ldots, \xi_n)(\xi_1 + \cdots + \xi_n)^k, \quad k \in \mathbb{Z}$$

6) Formal series in the symbolic form:

Let we have two operators fD^q and gD^s such that f and g have symbols $u^i a(\xi_1, ..., \xi_i)$ and $u^j b(\xi_1, ..., \xi_j)$ respectively. Then

$$fD^q \to u^i a(\xi_1, ..., \xi_i)\eta^q, gD^s \to u^j b(\xi_1, ..., \xi_j)\eta^s$$

and

$$fD^q \circ gD^s \to$$

$$u^{i+j}\langle a(\xi_1, ..., \xi_i)(\eta + \sum_{m=i+1}^{i+j} \xi_m)^q b(\xi_{i+1}, ..., \xi_{i+j})\eta^s\rangle \, .$$

More general, in the symbolic representation instead of formal series in powers of D_x we consider formal series in powers on nonlinearity:

$$B = b(\eta) + ub_1(\xi_1, \eta) + u^2 b_2(\xi_1, \xi_2, \eta) + \cdots \, .$$

Here $b_j(\xi_1, \ldots, \xi_j, \eta)$ are some functions of their arguments, symmetric with respect to permutations of arguments ξ_i, but not necessarily argument η.

A function $b_n(\xi_1, \ldots, \xi_n, \eta)$ is called local if all the coefficients of its expansion

$$b_n(\xi_1, \ldots, \xi_n, \eta) = \sum_{j<s} b_{nj}(\xi_1, \ldots, \xi_n)\eta^j, \quad \eta \to \infty$$

are symmetric polynomials in $\xi_1, \ldots, \xi_n$.

7) The symbolic representation of the Frechét derivative of the element $f \to u^n a(\xi_1, ..., \xi_n)$ is

$$f_* \to nu^{n-1} a(\xi_1, ..., \xi_{n-1}, \eta) \, .$$

Symmetry Approach in symbolic representation

Let the right hand side of equations (12) be a differential polynomial. In the symbolic representation it can be written as

$$\widehat{u}_t = \widehat{u}\omega(\xi_1) + \frac{\widehat{u}^2}{2}a_1(\xi_1,\xi_2) + \frac{\widehat{u}^3}{3}a_2(\xi_1,\xi_2,\xi_3) + \cdots = F\,, \quad (13)$$

where $\omega(\xi_1), a_n(\xi_1, ..., \xi_{n+1})$ are symmetrical polynomials. We will also assume that $\deg \omega(\xi_1) \geq 2$.

Symmetries of equation (13), if they exist, can be found recursively:

Proposition 1. *Expression*

$$\widehat{u}_\tau = \widehat{u}\Omega(\xi_1) + \sum_{j\geq 1} \frac{\widehat{u}^{j+1}}{j+1} A_j(\xi_1, ..., \xi_{j+1}) = G \qquad (14)$$

is a symmetry of (13) if and only if functions $A_j(\xi_1, ..., \xi_{j+1})$ *determined as follows are polynomials in* $\xi_1, \ldots, \xi_{j+1}$

$$A_1(\xi_1,\xi_2) = \frac{\Omega(\xi_1+\xi_2) - \Omega(\xi_1) - \Omega(\xi_2)}{\omega(\xi_1+\xi_2) - \omega(\xi_1) - \omega(\xi_2)} a_1(\xi_1,\xi_2),$$

$$A_2(\xi_1,\xi_2,\xi_3) =$$

$$= \frac{\Omega(\xi_1+\xi_2+\xi_3) - \Omega(\xi_1) - \Omega(\xi_2) - \Omega(\xi_3)}{\omega(\xi_1+\xi_2+\xi_3) - \omega(\xi_1) - \omega(\xi_2) - \omega(\xi_3)} a_2(\xi_1,\xi_2,\xi_3) +$$

$$+ \frac{3}{2} \frac{\langle A_1(\xi_1,\xi_2+\xi_3) a_1(\xi_2,\xi_3) - a_1(\xi_1,\xi_2+\xi_3) A_1(\xi_2,\xi_3)\rangle}{\omega(\xi_1+\xi_2+\xi_3) - \omega(\xi_1) - \omega(\xi_2) - \omega(\xi_3)}$$

$$A_m(\xi_1, ..., \xi_{m+1}) = \frac{G^{\Omega}(\xi_1, ..., \xi_{m+1})}{G^{\omega}(\xi_1, ..., \xi_{m+1})} a_m(\xi_1, ..., \xi_{m+1}) +$$

$$G^{\omega}(\xi_1, ..., \xi_{m+1})^{-1} \cdot \Big[$$

$$\langle \sum_{j=1}^{m-1} \frac{m+1}{m-j+1} A_j(\xi_1, ..., \xi_j, \sum_{k=j+1}^{m+1} \xi_k) a_{m-j}(\xi_{j+1}, ..., \xi_{m+1}) -$$

$$- \sum_{j=1}^{m-1} \frac{m+1}{j+1} a_{m-j}(\xi_1, ..., \xi_{m-j}, \sum_{k=m-j+1}^{m+1} \xi_k) \cdot A_j(\xi_{m-j+1}, ..., \xi_{m+1}) \rangle \Big]$$

where

$$G^{\omega}(\xi_1, ..., \xi_m) = \omega(\sum_{n=1}^{m} \xi_n) - \sum_{n=1}^{m} \omega(\xi_n),$$

$$G^{\Omega}(\xi_1, ..., \xi_m) = \Omega(\sum_{n=1}^{m} \xi_n) - \sum_{n=1}^{m} \Omega(\xi_n)\,.$$

Definition 3. *A formal series*

$$\Lambda=\phi(\eta)+\widehat{u}\phi_1(\xi_1,\eta)+\widehat{u}^2\phi_2(\xi_1,\xi_2,\eta)+\widehat{u}^3\phi_3(\xi_1,\xi_2,\xi_3,\eta)+\cdots,$$

where $\phi(\eta)$ is a non-constant polynomial in η is called a formal recursion operator for the equation (13) if it satisfies the equation

$$\Lambda_t=F_*\circ\Lambda-\Lambda\circ F_*$$

and all its coefficients are local functions.

Proposition 2. *The coefficients of the formal recursion operator can be determined recursively*

$$\phi_1(\xi_1,\eta)=G^{\omega}(\xi_1,\eta)^{-1}a_1(\xi_1,\eta)(\phi(\eta+\xi_1)-\phi(\eta))$$

$$\phi_m(\xi_1,...,\xi_m,\eta)=$$

$$G^{\omega}(\xi_1,...,\xi_m,\eta)^{-1}\{(\phi(\eta+\sum_{k=1}^{m}\xi_k)-\phi(\eta))a_m(\xi_1,...,\xi_m,\eta)+$$

$$\sum_{n=1}^{m-1}\langle\frac{n}{m-n+1}\phi_n(\xi_1,...,\xi_{n-1},\sum_{k=n}^{m}\xi_k,\eta)a_{m-n}(\xi_n,...,\xi_m)+$$

$$\phi_n(\xi_1,...,\xi_n,\eta+\sum_{k=n+1}^{m}\xi_k)a_{m-n}(\xi_{n+1},...,\xi_m,\eta)-$$

$$a_{m-n}(\xi_{n+1},...,\xi_m,\eta+\sum_{k=1}^{n}\xi_k)\phi_n(\xi_1,...,\xi_n,\eta)\rangle\}.$$

Theorem 3. *(Mikhailov-VN) Suppose equation (13) has an infinite hierarchy of symmetries*

$$\hat{u}_{t_i} = \hat{u}\Omega_i(\xi_1) + \sum_{j\geq 1} \frac{\hat{u}^{j+1}}{j+1} A_{ij}(\xi_1, \ldots, \xi_{j+1}) = G_i\,, \quad i = 1, 2, \ldots$$

where $\Omega_i(\xi_1)$ *is a polynomial of degree* $m_i = \deg(\Omega_i(\xi_1))$ *and* $m_1 < m_2 < \cdots < m_i < \cdots$. *Then the coefficients* $\phi_m(\xi_1, ..., \xi_m, \eta)$ *of the formal recursion operator*

$$\Lambda = \eta + \hat{u}\phi_1(\xi_1, \eta) + \hat{u}^2\phi_2(\xi_1, \xi_2, \eta) + \cdots$$

are local.

Integrability test:

- Find a first few coefficients $\phi_n(\xi_1, ..., \xi_n, \eta)$ (first three nontrivial coefficients ϕ_n were sufficient to analyse in all known to us cases).

- Expand these coefficients in series of $1/\eta$

$$\phi_n(\xi_1, ..., \xi_n, \eta) = \sum_{s=s_n} \Phi_{ns}(\xi_1, ..., \xi_n)\eta^{-s} \qquad (15)$$

  and find the corresponding functions $\Phi_{ns}(\xi_1, ..., \xi_n)$.

- Check that functions $\Phi_{ns}(\xi_1, ..., \xi_n)$ are polynomials (not rational functions).

Nonlocal extension to the Camassa-Holm type equations.

$$\begin{aligned} u_t &= \Delta\Big(c_1 u u_x + \epsilon\left[c_2 u u_{xx} + c_3 u_x^2\right] \\ &+ \epsilon^2\left[c_4 u u_{xxx} + c_5 u_x u_{xx}\right] \\ &+ \epsilon^3\left[c_6 u u_{xxxx} + c_7 u_x u_{xxx} + c_8 u_{xx}^2\right] \\ &+ \epsilon^4\left[c_9 u u_{xxxxx} + c_{10} u_x u_{xxxx} + c_{11} u_{xx} u_{xxx}\right]\Big) = F, \end{aligned} \quad (16)$$

where

$$\Delta = (1 - \epsilon^2 D_x^2)^{-1}.$$

We extend the differential ring $\mathcal{R}$

$$\mathcal{R}^0_\Delta = \mathcal{R}\,, \quad \mathcal{R}^1_\Delta = \overline{\mathcal{R}^0_\Delta \bigcup \Delta(\mathcal{R}^0_\Delta)}\,, \quad \mathcal{R}^{n+1}_\Delta = \overline{\mathcal{R}^n_\Delta \bigcup \Delta(\mathcal{R}^n_\Delta)}\,,$$

Symbolic representation of operator Δ is $\Delta \to \frac{1}{1-\epsilon^2\eta^2}$. The symbolic representation of elements of differential rings $\mathcal{R}^n_\Delta$ is obvious. For example if $A \in \mathcal{R}^0_\Delta$ and

$$A \to \widehat{u}^n a(\xi_1, ..., \xi_n)$$

then

$$\Delta(A) \to \widehat{u}^n \frac{a(\xi_1, \ldots, \xi_n)}{1 - \epsilon^2(\xi_1 + \cdots + \xi_n)^2}$$

.

Performing shift $u \to u + 1$ we bring equation (16) to the form

$$\begin{aligned} u_t &= \Delta(F_1[u] + F_2[u]), \quad (17) \\ F_1[u] &= c_1 u_x + \epsilon c_2 u_{xx} + \epsilon^2 c_4 u_{xxx} + \epsilon^3 c_6 u_{xxxx} + \epsilon^4 c_9 u_{xxxxx}, \\ F_2[u] &= F \end{aligned}$$

Theorem 4. *Consider the equation (17) and suppose that either:*

$$c_2 \neq 0 \quad \text{or} \quad c_6 \neq 0 \quad \text{or} \quad c_9 \neq 0 \quad \text{or} \quad c_1 + c_4 \neq 0.$$

If the equation (17) possesses a formal recursion operator with quasi-local coefficients then up to re-scaling $x \to \alpha x$, $t \to \beta t, u \to \gamma u$, $\alpha, \beta, \gamma = const$ it is one of the list:

$$(1-\epsilon^2 D_x^2)u_t = 3uu_x - 2\epsilon^2 u_x u_{xx} - \epsilon^2 u u_{xxx}, \tag{18}$$

$$(1-\epsilon^2 D_x^2)u_t = D_x \left(4 - \epsilon^2 D_x^2\right) u^2, \tag{19}$$

$$(1-\epsilon^2 D_x^2)u_t = D_x \left[(4 - \epsilon^2 D_x^2)u\right]^2, \tag{20}$$

$$(1-\epsilon^2 D_x^2)u_t = D_x(2+\epsilon D_x)\left[(2-\epsilon D_x)u\right]^2, \tag{21}$$

$$(1-\epsilon^2 D_x^2)u_t = D_x(2-\epsilon D_x)(1+\epsilon D_x)u^2, \tag{22}$$

$$(1-\epsilon^2 D_x^2)u_t = D_x(2-\epsilon D_x)\left[(1+\epsilon D_x)u\right]^2, \tag{23}$$

$$(1-\epsilon^2 D_x^2)u_t = D_x(1+\epsilon D_x)\left[(2-\epsilon D_x)u\right]^2, \tag{24}$$

$$(1-\epsilon^2 D_x^2)u_t = D_x\left[(2-\epsilon D_x)(1+\epsilon D_x)u\right]^2, \tag{25}$$

$$(1-\epsilon^2 D_x^2)u_t = (1-\epsilon^2 D_x^2)(\epsilon u u_{xx} - \frac{1}{2}\epsilon u_x^2 + c u u_x), \tag{26}$$

$$\begin{aligned}(1-\epsilon^2 D_x^2)u_t = {} & (1-\epsilon D_x)\Big[\epsilon S(u)S(u_{xx}) - \frac{1}{2}\epsilon(S(u_x))^2 \\ & -\frac{1}{2}c S(u)S(u_x)\Big], \quad S = 1+\epsilon D_x.\end{aligned} \tag{27}$$

Camassa-Holm type equations with cubic nonlinearity

- Zhijun Qiao's equation

$$m_t = D_x \left[m(u^2 - u_x^2) \right], \quad m = u - u_{xx}.$$

- Another equation of this form

$$m_t = u^2 m_x + 3uu_x m, \quad m = u - u_{xx}.$$

  The corresponding structures and solutions of this equation have been studied recently by A.N.W. Hone and Jing Ping Wang.