Phase transitions of random motion of hard-core particles

Based on two joint papers:

J. Harnad and A.Yu. Orlov, “Fermionic construction of tau functions and random processes”, Physica D 235 (2007) pp. 168-206

and

Random turn walk on a half line with creation of particles at the origin

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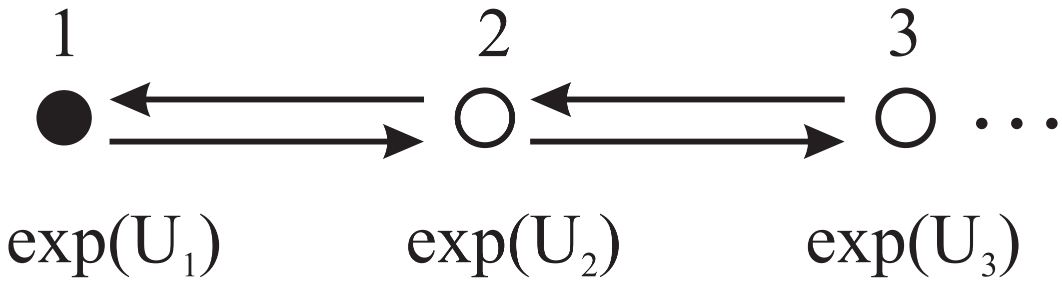
M.E. Fisher, “Walks, walls, wetting and melting”, *J. Stat. Phys.* **34** (1984) 667-729

M.E.Fisher introduced models of one-dimensional random walk of hard core particles on the lattice. We shall consider the models that Fisher called random turn walk models. They describe a motion of particles where at each tick of the clock a randomly chosen walker takes a random step.

In our case we consider a version of this model where particles move along a semi-line, and where also a particle may be created at the origin.

The model.



The probability is proportional:

- For the hop of a particle along an arrow $i \to j$: $e^{-U_j+U_i}$;
- For the birth process: $\frac{1}{\sqrt{2}}e^{-U_1}$;
- For the elimination process: $\frac{1}{2}e^{U_1}$.

The probability to come from a configuration $\lambda^{(0)}$ to a given configuration $\lambda^{(\mathsf{T})}$ in T steps is the ratio

$$p_{\lambda^{(0)} \to \lambda^{(\mathsf{T})}} = \frac{W_{\lambda^{(0)} \to \lambda^{(\mathsf{T})}}(\mathsf{T})}{Z_{\lambda^{(0)}}(\mathsf{T})} \tag{1}$$

where the denominator is a normalization function

$$Z_{\lambda^{(0)}}(\mathsf{T}) = \sum_{\mu} W_{\lambda^{(0)} \to \mu}(\mathsf{T}) \tag{2}$$

In what follows we shall omit superscripts for configurations.

Our goals are:

• To evaluate the asymptotic configuration of the particles in case the *hopping rate* $r(n) := e^{-U_n+U_{n-1}}$ does not depend on n.

• To demonstrate a sort of phase transition phenomenon.

• To link transition weights and the normalization function with tau functions of BKP hierarchy.

Neutral fermions.

E. Date, M. Jimbo, M. Kashiwara and T. Miwa, "Transformation groups for soliton equations, IV A new hierarchy of soliton equations of KP-type", *Physica* **4D** (1982) 343-365.

V. Kac and J. van de Leur, "The Geometry of Spinors and the Multicomponent BKP and DKP Hierarchies", CRM Proceedings and Lecture Notes **14** (1998) 159-202

$$[\phi_n, \phi_m]_+ = (-1)^n \delta_{n,-m},$$

where $[\ ,\ ]_+$ denotes the anticommutator. In particular, $(\phi_0)^2 = \frac{1}{2}$. The action on vacuum states:

$$\phi_n|0\rangle = 0, \quad \langle 0|\phi_{-n} = 0, \quad n < 0,$$

$$\phi_0|0\rangle = \frac{1}{\sqrt{2}}|0\rangle, \quad \langle 0|\phi_0 = \frac{1}{\sqrt{2}}\langle 0|,$$

The bases of Fock spaces are formed by vectors

$$|\lambda\rangle := \phi_{\lambda_1} \cdots \phi_{\lambda_N} |0\rangle$$

$$\langle\lambda| := (-1)^{|\lambda|} \langle 0| \phi_{-\lambda_N} \cdots \phi_{-\lambda_1} ,$$

where

$$\lambda_1 > \cdots > \lambda_N > 0$$

and $|\lambda| = \lambda_1 + \cdots + \lambda_N$.

We have *one to one correspondence* between configurations of hard core particles on the lattice $1,\ 2,\ 3, \ldots$ and the basis Fock vectors.

Consider the following operator:

$$B = B_1(U) + B_{-1}(U)$$

$$B_1(U) = \sum_{i>0}^{\infty} (-1)^{i+1} \phi_i \phi_{1-i} e^{-U_i + U_{i-1}} =$$

$$= \phi_1 \phi_0 e^{-U_1} - \phi_2 \phi_{-1} e^{-U_2 + U_1} + \phi_3 \phi_{-2} e^{-U_3 + U_2} - \dots$$

$$B_{-1}(U) = \sum_{i \geq 0}^{\infty} (-1)^{i+1} \phi_i \phi_{-1-i} e^{-U_i + U_{i+1}} =$$

$$= -\phi_0 \phi_{-1} e^{U_1} + \phi_1 \phi_{-2} e^{-U_1 + U_2} - \phi_2 \phi_{-3} e^{-U_2 + U_3} - \dots$$

Random procces as a sequence of Fock vectors

$$|\lambda'\rangle \to \left(B_1(U) + B_{-1}(U)\right)|\lambda'\rangle \to \cdots$$

$$\to \left(B_1(U) + B_{-1}(U)\right)^{\mathsf{T}}|\lambda'\rangle \to \cdots$$

describes an evolution of the initial (basis) Fock vector $|\lambda'\rangle$ - where the variable $\mathsf{T} = 0, 1, 2, \ldots$ plays a role of discrete time - to linear combinations of different basis Fock vectors.

Transition weight of the T-step random process

$$W_{\lambda' \to \lambda}(U; \mathsf{T}) := \langle \lambda | \left(B_1(U) + B_{-1}(U) \right)^{\mathsf{T}} | \lambda' \rangle$$

from an initial configuration described by coordinates $\lambda'_1, \ldots, \lambda'_{N'}$ to a target configuration $\lambda_1, \ldots, \lambda_N$.

N is not necessarily equal to N'.

Fermionic calculations yields:

Transition weights:

$$W_{0\to\lambda}(U;\mathsf{T}) =$$

$$= \begin{cases} 0 \\ \text{iff } \ \mathsf{T}-|\lambda| \quad \text{odd}, \\ \\ \dfrac{\mathsf{T}!}{\left(\frac{\mathsf{T}-|\lambda|}{2}\right)!} 2^{|\lambda|-\mathsf{T}-\frac{N}{2}} e^{-\sum_{i=1}^{N} U_{\lambda_i}} \prod_{i=1}^{N} \dfrac{1}{\lambda_i!} \prod_{i<j}^{N} \dfrac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j} \\ \text{iff } \ \mathsf{T}-|\lambda| \quad \text{even} \end{cases}$$

EXAMPLES

(1) $|\lambda| = 0$ corresponds to the returning to the initial position (time T is even),

$$W_{0\to 0}(\mathsf{T}) = \frac{\mathsf{T}!}{\left(\frac{\mathsf{T}}{2}\right)!}2^{-\mathsf{T}} = 2^{-\frac{\mathsf{T}}{2}}(\mathsf{T}-1)!!$$

(2) The case $\mathsf{T} = |\lambda|$ corresponds to the non-stop creation + forward motion processes. Then

$$W_{0\to\lambda}(\mathsf{T}) = 2^{-\frac{N}{2}} e^{-\sum_{i=1}^{N} U_{\lambda_i}} \prod_{i=1}^{N} \frac{1}{\lambda_i!} \prod_{i<j}^{N} \left| \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right|$$

In case the potential is a rapidly decreasing functions $U_{i-1} >> U_i$ (and, therefore, left hopping rates are much larger than right hopping rates), then the configurations where $\mathsf{T} = |\lambda|$ are dominant. Let us note that up without the factor $2^{-\frac{N}{2}} e^{-\sum_{i=1}^{N} U_{\lambda_i}}$ the number $W_{0\to\lambda}(\mathsf{T})$ is equal to the number of shifted standard tableau of shape λ, that is the number of ways the Young diagram of the strict partition λ may be created by adding box by box to the empty partition in a way that on each step we have the diagram of a strict partition.

(3) For given λ in large T limit (this means that $\mathsf{T} \gg |\lambda|$) by the Stirling's approximation we have

$$W_{0\to\lambda}(\mathsf{T}) = W_{0\to 0}(\mathsf{T})\mathsf{T}^{\frac{|\lambda|}{2}} 2^{-\frac{N}{2}} e^{-\frac{|\lambda|}{2}} 2^{|\lambda|} e^{o(\mathsf{T}^0)} e^{-E_\lambda},$$

where

$$E_\lambda = -\log \prod_{i=1}^{N} \frac{e^{-U_{\lambda_i}}}{\lambda_i!} \prod_{i<j}^{N} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$$

means an electrostatic energy of Coulomb particles (placed in an external field) which are attracted by their image. We see that in the large time limit the weight of a configuration increases with $|\lambda|$, and for given τ and $|\lambda|$ depends only on the E_λ.

The normalization function (partition function), counting weights for all possible target configurations, which may be achieved in the time duration T, is

$$Z(U;\mathsf{T}) := \sum_{N=0}^{\infty} \sum_{\lambda_1 > \cdots > \lambda_N > 0} \langle \lambda | (B_{-1} + B_1)^{\mathsf{T}} | 0 \rangle =$$

$$\sum_{N=0}^{\infty} 2^{-\frac{N}{2}} \sum_{\substack{\lambda_1 > \cdots > \lambda_N > 0 \\ \mathsf{T} - |\lambda| \ even}} \frac{2^{|\lambda| - \mathsf{T}}}{\Gamma(\frac{\mathsf{T} - |\lambda|}{2} + 1)} \prod_{i=1}^{N} \frac{e^{-U\lambda_i}}{\lambda_i!} \prod_{i,j}^{N} \left| \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right|$$

Due to the Gamma function this sum is finite.

Discrete version of a random matrix theory !

The probability to come to a configuration λ in T steps starting from the vacuum one is given by

$$p_{0\to\lambda}(U;\mathsf{T}) = \frac{W_{0\to\lambda}(U;\mathsf{T})}{Z(U;\mathsf{T})}$$

Asymptotic configuration of the particles in T $\to \infty$ limit

Saddle point method. The density of particles $\sigma(u)$, where $u = \frac{\lambda}{R}$ where R is the size of configuration. The density interpolates between full package state ($\sigma(u) = 1$), and empty state ($\sigma(u) = 0$):

$$0 \leq \sigma(u) \leq 1$$

The model: creation rate is $r(1) := \frac{e^{-U_1}}{\sqrt{2}} = r$ and the external potential is

$$U_n = -n \log r + (\beta - 1) \log n! \,, \quad n = 2, 3, \ldots$$

which means that creation and (the right) hopping rates are as follows

$$r(n) = r n^{1-\beta} \,, \quad n = 1, 2, 3, \ldots \qquad (3)$$

$\beta > 1$ describes a locking potential while $\beta < 1$ - driving particles to the right from the origin.

The case $\beta = 1$ may be considered as a discrete time version of the so-called asymmetric simple exclusion process (ASEP) on the half-line, now, the parameter r being an asymmetry parameter.

Saddle point equation for sum $Z(\mathsf{T})$ which will define the density function σ in the large time limit, $\mathsf{T} \longrightarrow \infty$. For $\lambda \in (0, R)$ we get

$$\log \frac{r}{\lambda^{\beta}} + P\int_0^R \frac{\sigma(xR^{-1})dx}{\lambda - x} - P\int_0^R \frac{\sigma(xR^{-1})dx}{\lambda + x} +$$

$$\frac{1}{2}\log 2\left(\mathsf{T} - \int_0^R x\sigma(xR^{-1})dx\right) = 0$$

where $P\int$ stands for the principal value. F.D.Gakhov, "Boundary Problems", ed. 2, Moscow, Nauka 1977

Asymptotic configuration of the particles. Given distance from the origin, λ, we find the particle density for $0 < \beta < 2$ as

$$\sigma(\lambda R^{-1}) = \frac{\beta}{\pi} \arccos \frac{\lambda}{R}, \quad \lambda \in [0, R]$$

where the size R and duration T are related as

$$\mathsf{T} = \frac{\beta}{8} R^2 + \frac{2^{2-2\beta}}{8r^2} R^{2\beta}$$

Thus, for large T, the dependence of R on T is different in regions $0 < \beta < 1$, $\beta = 1$ and $1 < \beta < 2$.

In the large T limit

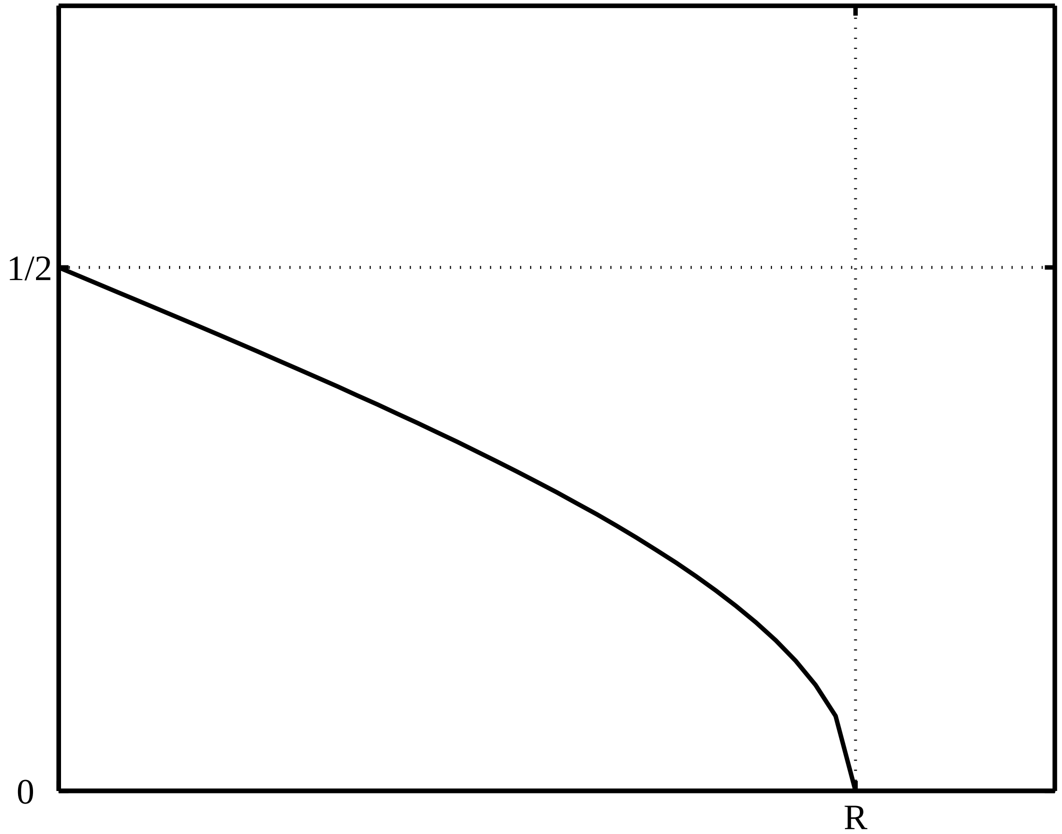
$$R = R(\beta, \mathsf{T}) = \begin{cases} \sqrt{\frac{8\mathsf{T}}{\beta}} & \text{if } \; 0 < \beta < 1 \\ \sqrt{\frac{8\mathsf{T}}{1+r^{-2}}} & \text{if } \; \beta = 1 \\ 2\left(2r^2\mathsf{T}\right)^{\frac{1}{2\beta}} & \text{if } 1 < \beta < 2 \end{cases}$$

As we see the discontinuity appears at $\beta = 1$ in the large T limit. The same behavior has the number of particles:

$$N(r, \beta, \mathsf{T}) = R\int_0^1 \left(\frac{\beta}{\pi} \arccos u\right) du = \frac{\beta R(\beta, \mathsf{T})}{\pi}$$

The weight of the asymptotic configuration for large enough T is

$$|\lambda(\mathsf{T})| = \frac{\beta R^2}{8} \approx \begin{cases} \mathsf{T} + O\left(\mathsf{T}^{\beta}\right) & \text{if } \ 0 < \beta < 1 \\ \frac{\mathsf{T}}{1+r^{-2}} & \text{if } \ \beta = 1 \\ \frac{\beta}{2}\left(2r^2\mathsf{T}\right)^{\frac{1}{\beta}} & \text{if } 1 < \beta < 2 \end{cases}$$



Asymptotic density of particles.

$$\frac{1}{2}(\mathsf{T} - |\lambda|) = \begin{cases} O\left(\mathsf{T}^{\beta}\right) & \text{if } \ 0 < \beta < 1 \\ \frac{\mathsf{T}}{2(1+r^2)} & \text{if } \ \beta = 1 \\ \frac{\mathsf{T}}{2} - \frac{\beta}{4}\left(2r^2\mathsf{T}\right)^{\frac{1}{\beta}} & \text{if } 1 < \beta < 2 \end{cases}$$

Then, as we see the normalization function $Z(\mathsf{T})$ which according to the saddle point method has the same leading term in the large T limit as $W_{0\to\lambda(\mathsf{T})}(r,\beta;\mathsf{T})$

has a discontuinity at $\beta = \mathtt{1}$ which may be interpreted as a sort of the first kind phase transition in our non-equilibrium system.

Now we can evaluate the type of asymptotic of probability to achieve a given configuration in $\mathsf{T} \longrightarrow \infty$ steps. We have $Z(\mathsf{T}) = W_{0\to\lambda(\mathsf{T})}(\mathsf{T})e^{O(\ln R)}$, where the last factor originates from the Gaussian integral around the saddle point configuration $\lambda(\mathsf{T})$. Then for $\mathsf{T} \gg |\lambda|$ we have

$$p_{0\to\lambda}(r,\beta,\mathsf{T}) \approx \frac{W_{0\to\lambda}(r,\beta,\mathsf{T})}{W_{0\to\lambda(\mathsf{T})}(r,\beta,\mathsf{T})} \approx$$

$$\approx \mathsf{T}^{\frac{|\lambda|}{2}} e^{-\frac{|\lambda|}{2}} 2^{|\lambda|} e^{-E_\lambda(r,\beta)} e^{\omega(\mathsf{T},r,\beta)}$$

where ω does not depend on λ.

The answer depends on the region of β

$$e^{\omega(\mathsf{T},r,\beta)} = \begin{cases} e^{\frac{\beta-1}{2}\mathsf{T}\ln\mathsf{T}-\mathsf{T}\left(\frac{\beta}{2}\ln\frac{8}{\beta}+\frac{\beta}{2}+\ln r)\right)+\frac{\mathsf{T}}{2}\ln(2e)+O(\sqrt{\mathsf{T}})} & \text{if } 0 \\ e^{-\mathsf{T}\left\{\frac{\ln 2(1+r^2)}{2}+\frac{b}{2\left(1+r^{-2}\right)}\right\}+O(\sqrt{\mathsf{T}})} & \text{if } \beta \\ e^{\frac{\beta}{4}\left(2r^2\mathsf{T}\right)^{\frac{1}{\beta}}(1-\beta)+O\left(\mathsf{T}^{\frac{1}{2\beta}}\right)} & \text{if } 1 \end{cases}$$

where $b = 2\ln\,2 - 1$. As we see, in each case, in the large T limit e^{ω} is vanishing.

Phase transitions

(i) As we see in case of decreasing potential (or, the same, increasing rightward hopping rate), $0 < \beta < 1$, the weight of the asymptotic configuration is equal to $\mathtt{T}$ which means that asymptotic configuration is created by only creating events at the origin and rightward hops, there were no elimination events and backward hops in the history of this configuration.

For $\beta < 0$ solution does not exists. In $\beta \to +0$ limit the number of particles vanishes, while the weight is equal to T. Indeed, the external potential U_n is decreasing so rapidly that the largest weight has the one particle configuration where the particle moves in the ballistic way: is is located at the distance T to the origin. $\beta = 1$ first kind transition.

(ii) When $\beta > 2$ we have a locking potential which forces particles to form a sort of a condensate of particles (the region of full package) which fills a region near the origin The size of the condensed phase is defined by β. $\beta = 2$ third kind transition.

This problem is treated by a method similar to M.Douglas and V.Kazakov, “Large N phase transitions in continuum QCD_2 ”, arXiv: hep-th/9305047

Along this way we can show that the solution is given in terms of elliptic integrals of the first and third kind.

(ii) The other model where the injection rate is a free parameter may be considered and an analog of the phase transition

Relation to the BKP tau function

The link of the described stochastic system to integrable equations is two-fold.

I. A BKP tau function as generating function for transition weights $W_{\lambda'\to\lambda}(\mathsf{T})$.

$$\text{BKP } \tau - \text{function:} \quad \tau(s,\bar{s};U,z) =$$

$$\sum_{\mathsf{T},N,N'} 2^{-\frac{N}{2}-\frac{N'}{2}} \sum_{\substack{\lambda_1>\cdots>\lambda_N>0\\ \lambda'_1>\cdots>\lambda'_{N'}>0}} \frac{z^{\mathsf{T}}}{\mathsf{T}!} Q_\lambda\left(\frac{s}{2}\right) Q_{\lambda'}\left(\frac{\bar{s}}{2}\right) W_{\lambda'\to\lambda}(U;\mathsf{T}),$$

where $Q_\lambda\left(\frac{s}{2}\right)$, $\lambda = (\lambda_1,\lambda_2,\dots)$, are projective Schur functions,

II. Generating function for "partition functions":

$$Z(U;z) = e^{\frac{z^2}{4}} \langle 0| e^{H(t)} e^{C(z)} \left(1 + \sqrt{2} \sum_{n\geq 0} \frac{z^n}{n!} \phi(n)\phi_0 \right) |0\rangle$$

$$C(z) = \frac{1}{2} \sum_{n,m\geq 0} \frac{z^{n+m}}{n!m!} \phi(n)\phi(m) sign(m-n),$$

$$U_n = - \sum_{m=1,3,5,\dots}^{\infty} n^m t_m, \quad \phi(x) := \sum_{k=-\infty}^{+\infty} x^k \phi_k$$

BKP τ − function: $Z(U;z) = \sum_{\mathsf{T}=0}^{\infty} \frac{z^{\mathsf{T}}}{\mathsf{T}!} Z(U;\mathsf{T})$

This is another example of the BKP tau function which may be related to the so-called resonant multi-soliton solution where the number of solitons is infinite and momentum of solitons are nonnegative numbers. Thus, the tau function of this (dual) BKP hierarchy is a generating function for normalization functions. The higher times $t = (t_1, t_3, t_5, \dots)$ of this dual BKP hierarchy parametrize hopping rates of the particles of our stochastic model.