q-Calculus in Action.

From Vortex Images to Relativistic Integrable NLS

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Outlines

1. Vortex Images in Annular Domain
2. Integrable Relativistic Nonlinear Schrodinger Equations

1. O.K. Pashaev and O. Yilmaz, "Vortex images and q-elementary functions", J. Physics A: Mathematical and Theoretical, 41 (2008) 135207
2. R. Parwani and O.K. Pashaev, "Integrable hierarchies and information measures", J. Physics A: Mathematical and Theoretical, 41 (2008) 235207

Part I.
Vortex Images in Annular Domain

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Boundary Value Problem and Method of Images

Thomson 1845 - classical method of images $\Rightarrow$ powerful method solving BV problems in electrostatics and hydrodynamics (spheres, cylinders, half-spaces)
Greenhill 1877 - motion of one and two vortices inside and outside circular domain
Extension to multiply connected domains is not straightforward.
Doubly connected domain $\Rightarrow$ conformally mapped to annular region $r_1 < |z| < r_2$ unique up to linear map, $r_2/r_1 = r_2'/r_1'$

Milne-Thomson Circle Theorem

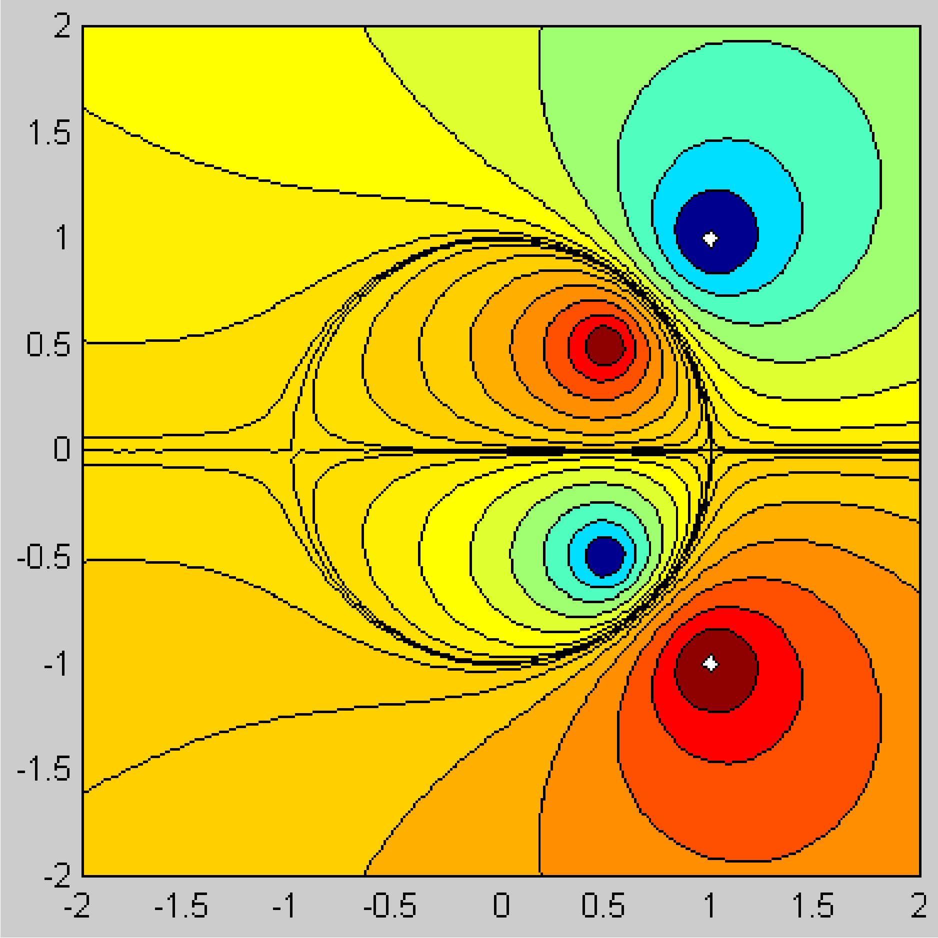
For complex velocity $\bar{V}(z) = u_1 - u_2 \Rightarrow$

$$\bar{V}(z) = \bar{v}(z) - \frac{r_1^2}{z^2}\, v\left(\frac{r_1^2}{z}\right)$$

$v(z)$ - complex velocity of flow in unbounded domain,
second term - correction of cylinder of radius r_1 placed at the origin
For vortex $\Gamma = -2\pi\kappa$ at z_0

$$\bar{V}(z) = \frac{i\kappa}{z - z_0} - \frac{i\kappa}{z - \frac{r_1^2}{\bar{z}_0}} + \frac{i\kappa}{z}$$

Two Vortices Outside Cylinder



Vortex in Annular Domain

$$\bar{V}(z) = \sum_{n=-\infty}^{\infty} \left[\frac{i\kappa}{z - z_0 q^n} - \frac{i\kappa}{z - \frac{r_1^2}{\bar{z}_0} q^n} \right]$$

N Vortices in Annular Domain

N point vortices - strengths $\kappa_1, ..., \kappa_N$ at positions $z_1, ..., z_N$ in annular domain $D : \{r_1 \leq |z| \leq r_2\}$ bounded by two concentric circles: $C_1 : z\bar{z} = r_1^2$ and $C_2 : z\bar{z} = r_2^2$. Complex velocity - Laurent series

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_{n+1}}{z^{n+1}}$$

boundary conditions

$$\left[z\,\bar{V}(z) + \bar{z}\,V(\bar{z})\right]|_{C_k} = 0,\; k = 1, 2$$

q-Elementary Functions

q-logarithm

$$Ln_q(1-x) \equiv -\sum_{n=1}^{\infty} \frac{x^n}{[n]}, \;\; |x| < q, \;\; q > 1$$

q-number

$$[n] \equiv 1 + q + q^2 + ... + q^{n-1} = \frac{q^n - 1}{q - 1}$$

q-exp functions (Jackson)(quantum dilogarithm- Kashaev)

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!}, \quad E_q^*(z) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{z^n}{[n]!}$$

Key Identity $q = r_2^2/r_1^2$

$$Ln_q(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{[n]} = (q-1)\sum_{n=1}^{\infty} \frac{z}{q^n + z}$$

Borwein- 1988, Sondow, Zudilin - 2006

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} +$$

$$\sum_{k=1}^{N} \frac{i\kappa_k}{z(q-1)} \left[Ln_q\left(1 - \frac{z}{z_k}\right) - Ln_q\left(1 - \frac{z\bar{z}_k}{r_1^2}\right)\right.$$

$$\left. + Ln_q\left(1 - \frac{r_2^2}{z\bar{z}_k}\right) - Ln_q\left(1 - \frac{z_k}{z}\right)\right]$$

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{k=1}^{N} \left[\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^n} + \sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^{-n}} \right]$$

$$- \sum_{k=1}^{N} \left[\sum_{n=0}^{\infty} \frac{i\kappa_k}{z - \frac{r_1^2}{\bar{z}_k} q^{-n}} + \sum_{n=0}^{\infty} \frac{i\kappa_k}{z - \frac{r_2^2}{\bar{z}_k} q^{n}} \right]$$

rearranging

$$\bar{V}(z) = \sum_{k=1}^{N} i\kappa_k \sum_{n=-\infty}^{\infty} \left[\frac{1}{z - z_k q^n} - \frac{1}{z - \frac{r_1^2}{\bar{z}_k} q^n} \right]$$

Vortex Images and q-Lattice

Pole singularities at the set of points

$$..., q^{-n} z_k , ..., q^{-2} z_k , q^{-1} z_k , z_k, q z_k , q^2 z_k , ..., q^n z_k , ...$$

- the q-chain. Set of vortex images for complex potential $V(z) = F'(z)$

$$F(z) = \sum_{k=1}^{N} i\kappa_k \sum_{n=-\infty}^{\infty} \left[\ln \frac{z - z_k q^n}{z - \frac{r_1^2}{\bar{z}_k} q^n} \right]$$

forms vortex q-lattice = two q-chains generated by vortex κ_k at z_k and its image $-\kappa_k$ at $r_1^2/\bar{z}_k$.

Vortex Images and Zeroes of q- Exp Function

$$F(z) = \sum_{k=1}^{N} i\kappa_k \left[\ln(z - z_k) + \ln \frac{E_q\left(\frac{z}{(1-q)z_k}\right) E_q\left(\frac{z_k}{(1-q)z}\right)}{E_q\left(\frac{z\bar{z}_k}{(1-q)r_1^2}\right) E_q\left(\frac{r_2^2}{(1-q)z\bar{z}_k}\right)} \right]$$

all images are determined by red zeros of q-exponential functions.

$$F(z) = \sum_{k=1}^{N} i\kappa_k \ln \left[\frac{\Theta_1(i\frac{\tau-\tau_k}{2}, \tilde{q})}{\Theta_1(i\frac{\tau+\bar{\tau}_k}{2}, \tilde{q})} \right]$$

$\tau \equiv -\ln z$, $\tau_k \equiv -\ln z_k$, Θ_1- first Jacobi theta function,
$r_2 = 1$, $q = r_2^2/r_1^2 = 1/r_1^2 \equiv 1/\tilde{q}^2 \Rightarrow \tilde{q} < 1$

Point Vortex Motion

$$\dot{z}_0 = \frac{i\kappa}{\bar{z}_0(q-1)}\left[Ln_q\left(1-\frac{|z_0|^2}{r_1^2}\right) - Ln_q\left(1-\frac{r_2^2}{|z_0|^2}\right)\right]$$

Uniform rotation $z_0(t) = z_0(0)e^{i\omega t}$ with $\omega = \omega_1 + \omega_2$

$$\omega_1 = \sum_{n=1}^{\infty} \frac{\left(\frac{|z_0|}{r_1}\right)^{2n}}{[n]}, \quad \omega_2 = \sum_{n=1}^{\infty} \frac{\left(\frac{r_2}{|z_0|}\right)^{2n}}{[n]}$$

for $|z_0| = r_1 \Rightarrow \omega_1 = H(q)$, for $|z_0| = r_2 \Rightarrow \omega_2 = -H(q)$

$$\omega^{(N)} = H_N(q) = \sum_{n=1}^{N} \frac{1}{[n]_q}, \quad q - harmonic\ numbers$$

Frequency Irrationality

The problem of frequency rationality is related to the problem of q-logarithm rationality. Fix geometry by $q \geq 2$ and consider vortex at distance $|z_0|$, commensurable with one of radiuses r_1 or r_2 $\Rightarrow$ argument of logarithm $r = |z_0|^2/r_1^2$, or $r = r_2^2/|z_0|^2$ is non-zero rational.
If r- rational $\Rightarrow$ $Ln_q(1-r)$ irrational
for $q = 2$ - (Erdos), $q \geq 2$ - (Borwein)

$q \to \infty$ Limiting Cases

Two limiting geometries:

1. Single cylinder and vortex outside: $r_1 = const$, $r_2^2 = q r_1^2 \to \infty \Rightarrow$ the circle theorem

$$\bar{V}(z) = \frac{i\kappa}{z - z_0} - \frac{i\kappa}{z} \frac{r_1^2/\bar{z}_0}{z - r_1^2/\bar{z}_0}$$

2. Single cylinder and vortex inside: $r_2 = const$, $r_2^1 = r_2^2/q \to \infty$

$$\omega = -\frac{\kappa}{r_2^2 - |z_0|^2}$$

- Greenhill (1877)

N-vortex Dynamics

N - point vortices with circulations Γ_1,...,Γ_N, equations of motion: (k = 1,...,N)

$$\dot{\bar{z}}_k = \frac{1}{2\pi i}\sum_{j=1(j\neq k)}^{N}\frac{\Gamma_j}{z_k - z_j} + \frac{1}{2\pi i}\sum_{j=1}^{N}\sum_{n=\pm 1}^{\pm\infty}\frac{\Gamma_j}{z_k - z_j q^n}$$

$$-\frac{1}{2\pi i}\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}\frac{\Gamma_j}{z_k - \frac{r_1^2}{\bar{z}_k} q^n}$$

Hamiltonian Structure

$$\{f, g\} = \frac{2}{i} \sum_{j=1}^{N} \frac{1}{\Gamma_j} \left(\frac{\partial f}{\partial z_j} \frac{\partial g}{\partial \bar{z}_j} - \frac{\partial f}{\partial \bar{z}_j} \frac{\partial g}{\partial z_j} \right)$$

$$\dot{z}_k = \{z_k, H\}, \quad \dot{\bar{z}}_k = \{\bar{z}_k, H\}$$

Hamiltonian function

$$4\pi H = - \sum_{i,j=1 (i \neq j)}^{N} \Gamma_i \Gamma_j \ln |z_i - z_j| -$$

$$\sum_{i,j=1}^{N} \sum_{n=\pm 1}^{\pm\infty} \Gamma_i \Gamma_j \ln |z_i - z_j q^n| + \sum_{i,j=1}^{N} \sum_{n=-\infty}^{\infty} \Gamma_i \Gamma_j \ln |z_i \bar{z}_j - r_1^2 q^n|$$

q-Logarithmic form of Hamiltonian

$$H = -\frac{1}{4\pi}\sum_{i,j=1(i\neq j)}^{N} \Gamma_i\Gamma_j \ln|z_i - z_j| -$$

$$-\frac{1}{4\pi}\sum_{i,j=1}^{N}\Gamma_i\Gamma_j \ln\left|\frac{E_q\left(\frac{z_i}{(1-q)z_j}\right)E_q\left(\frac{z_j}{(1-q)z_i}\right)}{E_q\left(\frac{z_i\bar{z}_j}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z_i\bar{z}_j}\right)}\right|$$

Kirchhoff-Routh Function

collecting to separating sum terms of vortex interaction with its own images

$$H = -\frac{1}{2\pi}\sum_{i<j}\Gamma_i\Gamma_j\left[\ln|z_i - z_j| + \ln\left|\frac{E_q\left(\frac{z_i}{(1-q)z_j}\right)E_q\left(\frac{z_j}{(1-q)z_i}\right)}{E_q\left(\frac{z_i\bar{z}_j}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z_i\bar{z}_j}\right)}\right|\right]$$

$$+\frac{1}{4\pi}\sum_{j=1}^{N}\Gamma_j \ln\left[E_q\left(\frac{|z_j|^2}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)|z_j|^2}\right)\right]$$

Green's Function

$G_I = G + G_H^{(k)}$ - hydrodynamic Green function

$$G_I(z, z_k) = -\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \ln \left| \frac{z - z_k q^n}{z - \frac{r_1^2}{\bar{z}_k} q^n} \right|$$

$n = 0$ term - single vortex Green's function,

$$G_H^{(k)} = -\frac{1}{2\pi} \sum_{n=\pm 1}^{\pm\infty} \ln |z - z_k q^n| + \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \ln \left| z - \frac{r_1^2}{\bar{z}_k} q^n \right|$$

-influence of boundaries

Green's Function in q-Exponential Form

$$G_I = -\frac{1}{2\pi}\ln|z - z_k| - \frac{1}{2\pi}\ln\left|\frac{E_q\left(\frac{z}{(1-q)z_k}\right)E_q\left(\frac{z_k}{(1-q)z}\right)}{r_2 E_q\left(\frac{z\bar{z}_k}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z\bar{z}_k}\right)}\right|$$

1. $G_I(z, z_k) = G_I(z_k, z)$ - symmetry
2. boundary values

$G_I(z, z_k)|_{C_2} = 0$ - at the outer circle

$G_I(z, z_k)|_{C_1} = \frac{1}{2\pi}\ln\left|\frac{r_2}{z_k}\right|$ - at the inner circle

Angular Momentum

N vortices in plane $\Rightarrow$ 4 integrals of motion: energy, 2 translations, 1 rotation
In annular domain $\Rightarrow$ 2 integrals of motion: energy, 1 rotation

$$\frac{d}{dt}\left(\sum_{k=1}^{N}\Gamma_k z_k \bar{z}_k\right) = 0$$

conservation of angular momentum

$$I = \sum_{k=1}^{N}\Gamma_k z_k \bar{z}_k$$

N-Polygon Solution

N vortices $\Gamma_k = \Gamma, k = 1, ..., N$, located at the same distance $r_1 < r < r_2$

$$z_k(t) = r e^{i\omega t + i\frac{2\pi}{N}k}$$

rotation frequency

$$\omega = \frac{\Gamma}{2\pi r^2}\frac{N-1}{2} +$$

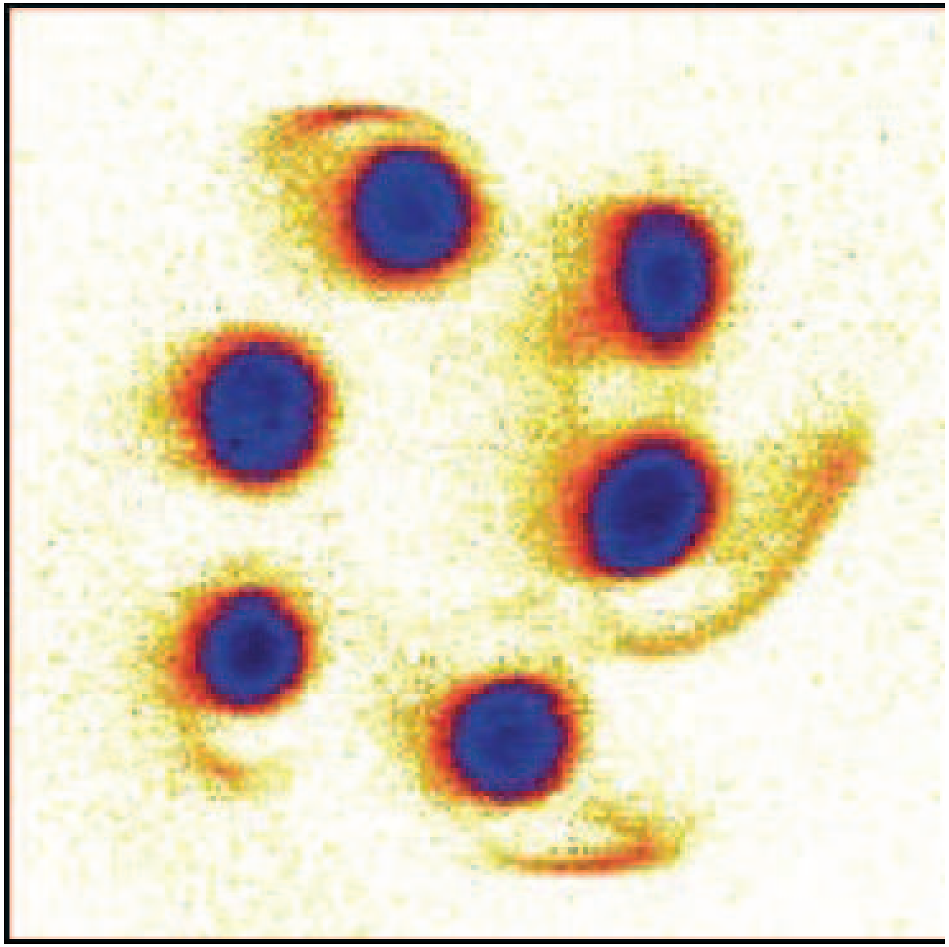
$$\frac{\Gamma}{2\pi r^2(q-1)}\sum_{j=1}^{N}\left[Ln_q\left(1-\frac{r_2^2}{r^2}\,e^{i\frac{2\pi}{N}j}\right) - Ln_q\left(1-\frac{r^2}{r_1^2}\,e^{-i\frac{2\pi}{N}j}\right)\right]$$

2D Vortex Paterns and Magnetized Electron Columns

Flow vorticity $\Rightarrow$ Electron density
Electrons mimic ideal two-dimensional fluid

Electron Column

Fajans,...



Instability in Rotating System

Influence of the radius ratio on the flow pattern

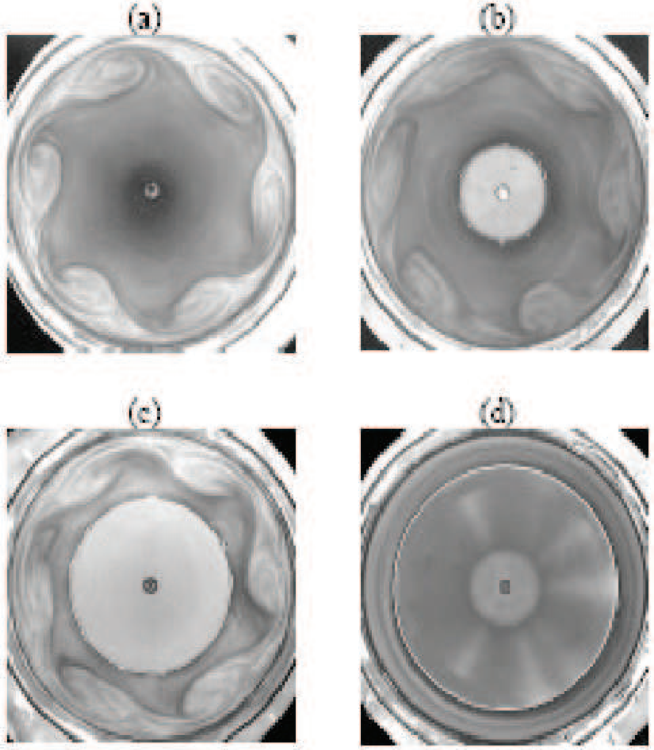


Fig. 16 Influence of the radius ratio s on the flow pattern for $G = 0.0429$ and $Re = 36945$ (spin-up): (a) $s = 0$, (b) $s = 0.286$, (c) $s = 0.536$, (d) $s = 0.75$.

Part II.
Integrable Relativistic Nonlinear Schrodinger Equations

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Compressible Fluid in One Dimension

Isentropic Fluid

$$v_t + v v_x + (\mathcal{E}(\rho))_x = 0, \quad \rho_t + (\rho v)_x = 0$$

for velocity potential $v = S_x$ and enthalpy potential $\mathcal{E}(\rho) = dV(\rho)/d\rho$

$$S_t + \frac{(S_x)^2}{2} + \frac{dV(\rho)}{d\rho} = 0, \quad \rho_t + (\rho S_x)_x = 0$$

Action

$$A = \int \left(\rho S_t + \frac{\rho (S_x)^2}{2} + V(\rho) \right) dx\, dt$$

Information Theory and Fluid Dynamics

$$\int (\rho(x,t) - \rho_0) dx = 1 \;\; normalization\, of\, probability$$

Fisher information measure

$$I_F = \int \frac{(\rho_x)^2}{\rho} dx = 4 \int (\sqrt{\rho})_x (\sqrt{\rho})_x \, dx$$

variational functional

$$A + \frac{\lambda^2}{8} I_F$$

λ - the Lagrange multiplier

Nonlinear Schrödinger Equation

Madelung representation for the wave function

$$\psi = \sqrt{\rho}\, e^{\frac{i}{\hbar} S}$$

Weak nonlinearity $\Rightarrow$ NLS

$$i\psi_t + \psi_{xx} + 2\kappa^2 |\psi|^2 \psi = 0$$

Zakharov-Shabat problem

$$\frac{\partial}{\partial x} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} p & -\kappa^2 \bar{\psi} \\ \psi & \frac{i}{2} p \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

NLS Hierarchy

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_{t_N} = \mathcal{R}^N \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad N = 1, 2, ...$$

AKNS Recursion operator

$$\mathcal{R} = i\sigma_3 \begin{pmatrix} \partial_x + 2\kappa^2 \psi \int^x \bar{\psi} & -2\kappa^2 \psi \int^x \psi \\ \\ -2\kappa^2 \bar{\psi} \int^x \bar{\psi} & \partial_x + 2\kappa^2 \bar{\psi} \int^x \psi \end{pmatrix}$$

AKNS Linear Problem

$$\frac{\partial}{\partial t_N}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -iA_N & -\kappa^2 \bar{C}_N \\ C_N & -iA_N \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} C_N \\ \bar{C}_N \end{pmatrix} = (p^{N-1} + p^{N-2}\mathcal{R} + ... + \mathcal{R}^{N-1})\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

q-number operator

$$1 + q + q^2 + ... + q^{N-1} \equiv [N]_q$$

$q = \mathcal{R}/p$ - operator

$$1 + \frac{\mathcal{R}}{p} + \left(\frac{\mathcal{R}}{p}\right)^2 + ... + \left(\frac{\mathcal{R}}{p}\right)^{N-1} \equiv [N]_{\mathcal{R}/p}$$

$$\begin{pmatrix} C_N \\ \bar{C}_N \end{pmatrix} = p^{N-1}[N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \frac{\mathcal{R}^N - p^N}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

$$A_N = -\frac{p^N}{2} - i\kappa^2\, p^{N-1}\Big(\int^x \bar{\psi}, -\int^x \psi\Big)\,[N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

Hierarchy of Information Measures

$$I_2 = \int (\sqrt{\rho})_x (\sqrt{\rho})_x \, dx, \quad \color{red}{Fisher\ measure}$$

$$I_4 = \int (\sqrt{\rho})_{xx} (\sqrt{\rho})_{xx} \, dx$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$I_{2n} = \int (\sqrt{\rho})_{x..x} (\sqrt{\rho})_{x...x} \, dx$$

Integrable Nonlinearization

Classical particle energy-momentum relation

$$E = E(p) = E_0 + E_1 p + E_2 p^2 + \dots$$

Canonical quantization $p \rightarrow -i\hbar\frac{\partial}{\partial x}$
$\Rightarrow$ time-dependent Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = H\left(-i\hbar\frac{\partial}{\partial x}\right)\psi$$

In spinor form

$$i\hbar\sigma_3\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix} = H\left(-i\hbar\sigma_3\frac{\partial}{\partial x}\right)\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}$$

Linear and Nonlinear Quantization

Momentum operator $\hat{p} = \mathcal{R}_0 = i\sigma_3 \frac{\partial}{\partial x}$ the recursion operator in linear approximation $\Rightarrow$ Linear Schrodinger equation

$$i\hbar\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = H(-\hbar\mathcal{R}_0) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

Momentum operator $\hat{p} = \mathcal{R}$ the recursion operator $\Rightarrow$ Nonlinear integrable Schrodinger equation

$$i\hbar\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = H(-\hbar\mathcal{R}) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

Lax Representation

By q-derivative and q-number

$$D_q^{(\zeta)} f(\zeta) = \frac{f(q\zeta) - f(\zeta)}{(q-1)\zeta}, \quad [N]_q = 1 + q + q^2 + \ldots + q^{N-1}$$

for the operator valued $q = \mathcal{R}/p$

$$D_{\mathcal{R}/p}^{(p)} \zeta^N = [N]_{\mathcal{R}/p}\, p^{N-1} \quad (1)$$

$$\begin{pmatrix} C \\ \bar{C} \end{pmatrix} = \sum_{N=1}^{\infty} E_N\, p^{N-1} [N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \sum_{N=1}^{\infty} E_N D_{\mathcal{R}/p}^{(p)} p^N \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

Linearity and Dispersion

$$\begin{pmatrix} C \\ \bar{C} \end{pmatrix} = D^{(p)}_{\mathcal{R}/p} \sum_{N=0}^{\infty} E_N \, p^N \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = D^{(p)}_{\mathcal{R}/p} \, E(p) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

$$\begin{pmatrix} C \\ \bar{C} \end{pmatrix} = \frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

$$\frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} = E_1 + E_2(\mathcal{R} + p) + E_3(\mathcal{R}^2 + \mathcal{R}p + p^2) + \ldots$$

$$A = -\frac{1}{2} E(p) - i\kappa^2 \left(\int^x \bar{\psi}, - \int^x \psi \right) \frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

Semi-relativistic NLS

Expanding the relativistic dispersion relation $E = \sqrt{m^2c^4 + p^2c^2}$ for low momenta

$$E = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \ldots$$

$\Rightarrow$ “semi-relativistic" Schrödinger equation as a formal power series

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi = mc^2\left(1 - \frac{\hbar^2}{2m^2c^2}\frac{\partial^2}{\partial x^2} - \frac{\hbar^4}{8m^4c^4}\frac{\partial^4}{\partial x^4} + \ldots\right)\psi \qquad (2)$$

Semi-relativistic Hartree-Fock Equation

$$i\psi_t = mc^2\sqrt{1 - \frac{1}{m^2c^2}\Delta}\ \psi + (V_\gamma * |\psi|^2)\psi$$

$*$ - convolution in R^n, $V_\gamma = \lambda|x|^{-\gamma}$, $0 < \gamma < n$.
Boson Stars with Coulomb potential - Lieb, Yau - (high velocities of bosons - incorporating special relativistic effects) $\Rightarrow$ explore the collapse and structure formation of bosonic matter
Existence of traveling solitary waves in R^3 - Fröhlich,...
Relativistic Quarks in Nuclei - Nickisch, Durand., nuclear many-body problem as relativistic system of baryons and mesons - (structure of exotic nuclei with extreme isospin values)
None of of those models is known to be integrable

Integrable Relativistic Nonlinear Schrodinger Equation

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = mc^2 \sqrt{1 + \frac{1}{m^2c^2}\left(i\sigma_3 \frac{\partial}{\partial x}\right)^2} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \qquad (3)$$

replacing $\mathcal{R}_0 = i\sigma_3 \frac{\partial}{\partial x} \Rightarrow \mathcal{R}$

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = mc^2 \sqrt{1 + \frac{1}{m^2c^2}\mathcal{R}^2} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \qquad (4)$$

by formal power series expansion

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = mc^2 \left(1 + \frac{\mathcal{R}^2}{2m^2c^2} - \frac{\mathcal{R}^4}{8m^4c^4} + \frac{\mathcal{R}^6}{16m^6c^6} \pm \ldots\right) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

The Linear Problem

$$\frac{\partial}{\partial x}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2}p & -\kappa^2\bar{\psi} \\ \psi & \frac{i}{2}p \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad (5)$$

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -iA & -\kappa^2\bar{C} \\ C & -iA \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad (6)$$

$$\begin{pmatrix} C \\ \bar{C} \end{pmatrix} = \frac{\sqrt{m^2c^4+\mathcal{R}^2c^2}-\sqrt{m^2c^4+p^2c^2}}{\mathcal{R}-p}\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \qquad (7)$$

spectral parameter p - classical momentum

$$A = -\frac{1}{2}\sqrt{m^2c^4 + p^2c^2}$$

$$-i\kappa^2 \left(\int^x \bar{\psi}, - \int^x \psi \right) \frac{\sqrt{m^2c^4 + \mathcal{R}^2c^2} - \sqrt{m^2c^4 + p^2c^2}}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

Integrability at any order of $1/c^2$

$$i\psi_t = mc^2\sqrt{1-\frac{1}{m^2c^2}\frac{\partial^2}{\partial x^2}}\,\psi + \frac{\kappa^2}{m}|\psi|^2\psi$$

$$-\frac{\kappa^2}{4m^3c^2}[(2|\psi_x|^2\psi+4|\psi|^2\psi_{xx}+\bar{\psi}_{xx}\psi^2+3\bar{\psi}\psi_x^2)+3\kappa^2|\psi|^4\psi]+O(\frac{1}{c^4})$$

- integrable relativistic corrections to the NLS equation at any order.
Fisher information has appear as non-relativistic approximation of relativistic information measure hierarchy

Relativistic Quantum Mechanics in Rapidity Variables

Relativistic dispersion in rapidity variables χ

$$E = mc^2 \cosh \chi, \quad p = mc \sinh \chi$$

Relativistic Hamiltonian

$$H = mc^2 \int \bar{\psi} \cosh \left(i\lambda \frac{\partial}{\partial x} \right) \psi \, dx$$

$\lambda = \hbar/(mc)$ - Compton wave-length of relativistic particle.

$$\cosh(i\lambda \frac{\partial}{\partial x}) = 1 + \frac{1}{2!} \left(i\lambda \frac{\partial}{\partial x} \right)^2 + \frac{1}{4!} \left(i\lambda \frac{\partial}{\partial x} \right)^4 + ...$$

Nonlinear Integrable Relativistic Schrodinger Equation

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_t = mc^2 \cosh\left(i\lambda\sigma_3 \frac{\partial}{\partial x} \right) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

integrable nonlinearization $\Rightarrow$

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_t = mc^2 \cosh\left(\lambda\mathcal{R}\right) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

Zakharov-Shabat problem with AKNS evolution and

$$\begin{pmatrix} C \\ \bar{C} \end{pmatrix} = mc^2 \frac{\cosh(\lambda\mathcal{R}) - \cosh(\lambda p)}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}$$

Wootters Measure

Free Hamiltonian - finite difference operator

$$H = \frac{mc^2}{2} \int \bar{\psi} \, (e^{L \frac{\partial}{\partial x}} + e^{-L \frac{\partial}{\partial x}}) \, \psi \, dx =$$

$$= \frac{mc^2}{2} \int (\bar{\psi}(x)\psi(x+L) + \bar{\psi}(x)\psi(x-L)) \, dx$$

$\Rightarrow$ Wootters type measure

$$I_W = \int (\sqrt{\rho(x)}\sqrt{\rho(x+L)} + \sqrt{\rho(x)}\sqrt{\rho(x-L)}) \, dx$$

Relativistic Burgers Equations I

$$i\psi_t = mc^2\sqrt{1 - \frac{1}{m^2c^2}\frac{\partial^2}{\partial x^2}}\psi$$

Complex Cole-Hopf transformation = Madelung Representation

$$v = (\ln\psi)_x \quad \textit{or} \quad \psi = e^{\int^x v}$$

Semi-relativistic Burgers Equation

$$iv_t = mc^2\left(\sqrt{1 - \frac{1}{m^2c^2}\left(\frac{\partial}{\partial x} + v\right)^2} \cdot 1\right)_x$$

Relativistic Burgers Equations II

$$i\psi_t = mc^2 \cosh\left(i\lambda \frac{\partial}{\partial x}\right)\psi$$

or

$$i\psi(x,t)_t = \frac{mc^2}{2}[\psi(x+i\lambda) + \psi(x-i\lambda)]$$

$$v = (\ln \psi)_x$$

Relativistic Burgers Equation in rapidity variables

$$iv_t = mc^2 \left(\cosh i\lambda \left(\frac{\partial}{\partial x} + v\right) \cdot 1\right)_x$$

Non-relativistic Limit

For $c \to \infty \Rightarrow$ Burgers equation

$$i v_t + v_{xx} + 2 v v_x$$