Spectral analysis of elliptic sine-Gordon in the quarter plane

Beatrice Pelloni, University of Reading (UK)

www.personal.rdg.ac.uk/sms00bp

Gallipoli, June 2008

MOTIVATION:

Analysis of linear elliptic PDEs (Laplace, modified Helmholtz, Helmholtz) via spectral analysis of a suitable "Lax pair" : work of Fokas, Kapaev, Dassios ...

Solution obtained in quarter plane, semi-infinite strip, triangles and polygons in general - based on the analysis of a relation among boundary conditions, the global relation

This approach is also successful in the case of nonlinear integrable evolution equations

What I want to discuss are the first steps towards extending it to integrable elliptic case

Very little in the literature: Borisov and Kiseliev (1989), Gutshabash and Lipovskii (1990)

The problem: to characterise the function $q = q(x, y)$ that solves

$$q_{xx} + q_{yy} = \sin q, \qquad x \geq 0, \;\; y \geq 0.$$

and satisfies e.g. the boundary conditions

$$q(0, y) = d_1(y), \; y \geq 0, \quad q(x, 0) = d_2(x), \; x \geq 0.$$

The linearised problem: to characterise $q = q(x, y)$ solving

$$q_{xx} + q_{yy} = q, \qquad x \geq 0, \;\; y \geq 0.$$

and satisfying the boundary conditions

$$q(0, y) = d_1(y), \; y \geq 0, \quad q(x, 0) = d_2(x), \; x \geq 0.$$

To do this:

(1) write the linear equation in Lax pair form

(2) associate to the Lax pair a Riemann-Hilbert problem and solve it

(3) characterise the solution q in terms of the data of the data that determine the RH problems (the spectral functions)

(4) determine the Dirichlet to Neumann map: express the spectral functions only in terms of the given boundary conditions

In $z = x + iy$, $\bar{z} = x - iy$:

$$q_{z\bar{z}} = \frac{1}{4}q \Longleftrightarrow \begin{cases} \mu_z - \frac{i\lambda}{2}\mu = q_z + \frac{i\lambda}{2}q \\ \mu_{\bar{z}} + \frac{i}{2\lambda}\mu = -q_{\bar{z}} + \frac{i}{2\lambda}q \end{cases} \qquad \mu \sim -q \text{ as } \lambda \to \infty$$

(**Remark:** if we require $\mu \sim 1/\lambda$ at infinity, the rhs of the Lax pair is $2q_z$, $\frac{i}{\lambda}q$ respectively - no $q_{\bar{z}}$)

Equivalently:

$$W = \mathrm{e}^{-\frac{i}{2}(\lambda z - \frac{\bar{z}}{\lambda})}\left[(q_z + \frac{i\lambda}{2}q)dz - (q_{\bar{z}} - \frac{i}{2\lambda}q)d\bar{z}\right]$$

is a closed form. Hence $\Rightarrow$ **global relation**:

$$\int_{\mathcal{I}} W = 0, \quad \mathcal{I} = \{x \geq 0,\ y \geq 0\}.$$

Explicitly, the global relation is

$$\hat{q}_1(\lambda) + \hat{q}_2(\lambda) = 0, \quad \lambda \in (3) = \{Re(\lambda) \leq 0,\ Im(\lambda) \leq 0\}$$

$$\hat{q}_1(\lambda) = -\int_0^\infty \mathrm{e}^{\frac{1}{2}(\lambda+\frac{1}{\lambda})y}[iq_x(0,y) - \frac{1}{2}(\lambda - \frac{1}{\lambda})d_1(y)]dy, \quad Re(\lambda) \leq 0,$$
$$\hat{q}_2(\lambda) = \int_0^\infty \mathrm{e}^{-\frac{i}{2}(\lambda-\frac{1}{\lambda})x}[-iq_y(x,0) + \frac{i}{2}(\lambda + \frac{1}{\lambda})d_2(x)]dx, \quad Im(\lambda) \leq 0.$$

The Riemann-Hilbert problem:

$$z_1 = 0 + i\infty, \quad z_2 = 0, \quad z_3 = \infty + i0$$

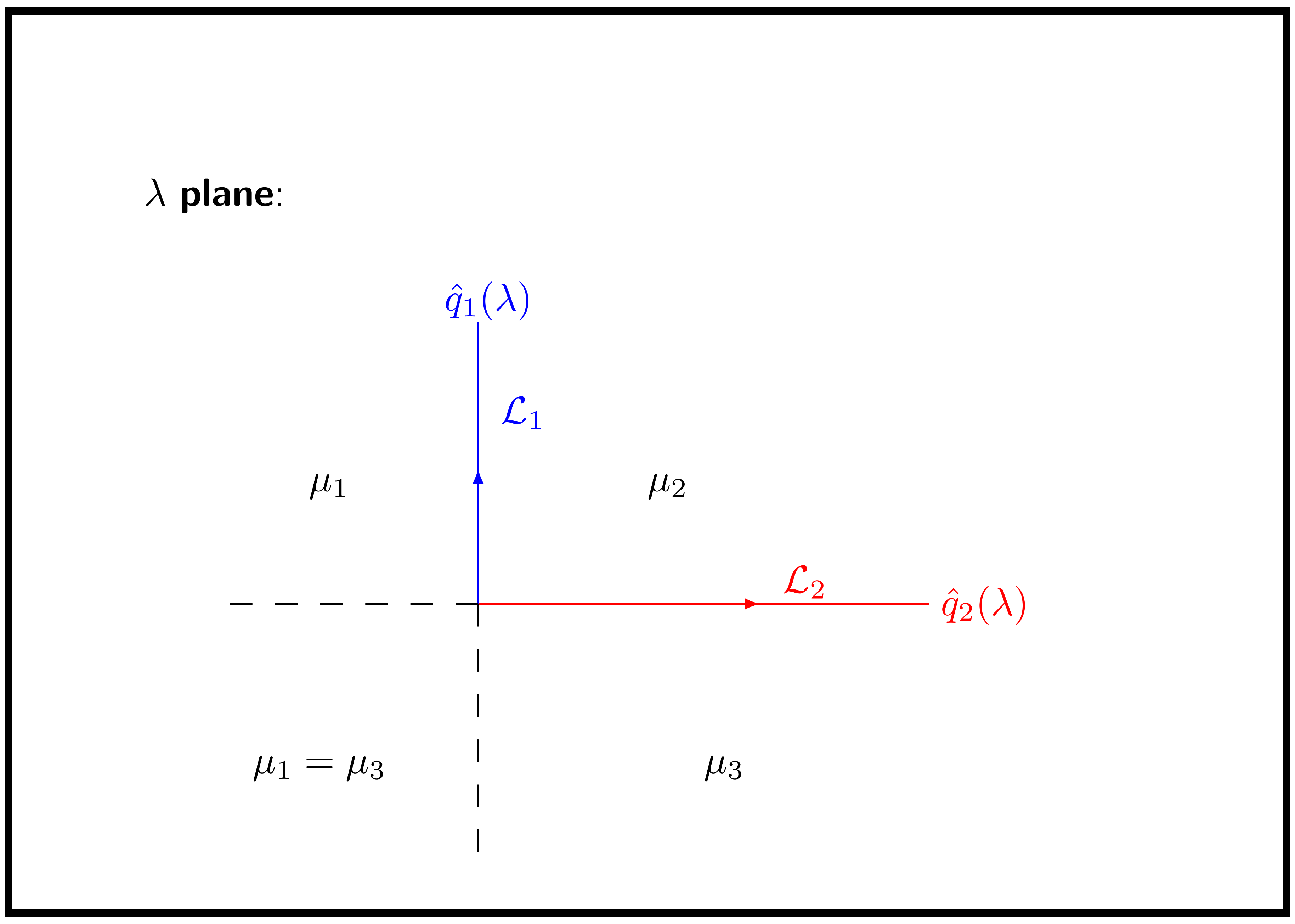
$$\mu_j = \int_{z_j}^{z} \mathrm{e}^{\frac{i}{2}(\lambda(z-\zeta) - \frac{(\bar{z}-\bar{\zeta})}{\lambda})} \left[(q_\zeta + \frac{i\lambda}{2} q) d\zeta - (q_{\bar{\zeta}} - \frac{i}{2\lambda} q) d\bar{\zeta} \right]$$

μ_1 and μ_3 are both bounded analytic in (3), the third quadrant of the λ plane $\hat{q}_1 = \mu_1(0,0,\lambda)$, $\hat{q}_2 = -\mu_3(0,0,\lambda) \Rightarrow \boxed{\mu_1 = \mu_3,\ \lambda \in (3)}$

$$\mu_2 - \mu_1 = -\mathrm{e}^{\frac{i}{2}(\lambda z - \frac{1}{\lambda}\bar{z})} \hat{q}_1(\lambda)$$

$$\mu_2 - \mu_3 = \mathrm{e}^{\frac{i}{2}(\lambda z - \frac{1}{\lambda}\bar{z})} \hat{q}_2(\lambda)$$

$$\Rightarrow \mu + q = \frac{1}{2\pi i} \left\{ \int_{\mathcal{L}_1} \mathrm{e}^{\frac{i}{2}(\lambda' z - \frac{1}{\lambda'}\bar{z})} \frac{\hat{q}_1(\lambda')}{\lambda' - \lambda} d\lambda' + \int_{\mathcal{L}_2} \mathrm{e}^{\frac{i}{2}(\lambda' z - \frac{1}{\lambda'}\bar{z})} \frac{\hat{q}_2(\lambda')}{\lambda' - \lambda} d\lambda' \right\}$$



λ **plane**:
$\hat{q}_1(\lambda)$
$\mathcal{L}_1$
μ_1
μ_2
$\mathcal{L}_2$
$\hat{q}_2(\lambda)$
$\mu_1 = \mu_3$
μ_3

Representation of the solution:

$$q(z,\bar{z}) = \frac{1}{4\pi i}\left\{\int_{\mathcal{L}_1} \mathrm{e}^{\frac{i}{2}(\lambda z - \frac{1}{\lambda}\bar{z})}\frac{\hat{q}_1(\lambda)}{\lambda}d\lambda + \int_{\mathcal{L}_2} \mathrm{e}^{\frac{i}{2}(\lambda z - \frac{1}{\lambda}\bar{z})}\frac{\hat{q}_2(\lambda)}{\lambda}d\lambda\right\}$$

$$\begin{aligned}
\hat{q}_1(\lambda) &= -\int_0^\infty \mathrm{e}^{\frac{1}{2}(\lambda+\frac{1}{\lambda})y}[iq_x(0,y) - \frac{1}{2}(\lambda - \frac{1}{\lambda})d_1(y)]dy \\
&= -iU_1(\lambda) + \frac{1}{2}(\lambda - \frac{1}{\lambda})D_1(\lambda), \quad Re(\lambda) \leq 0, \\
\hat{q}_2(\lambda) &= \int_0^\infty \mathrm{e}^{-\frac{i}{2}(\lambda-\frac{1}{\lambda})x}[-iq_y(x,0) + \frac{i}{2}(\lambda + \frac{1}{\lambda})d_2(x)]dx \\
&= -iU_2(-i\lambda) + \frac{i}{2}(\lambda + \frac{1}{\lambda})D_2(-i\lambda), \quad Im(\lambda) \leq 0.
\end{aligned}$$

where $\quad F(\lambda) = \int_0^\infty \mathrm{e}^{\frac{1}{2}(\lambda+\frac{1}{\lambda})s} f(s)ds$

Dirichlet to Neumann map - manipulating the global relation

$$\boxed{\hat{q}_1(\lambda) + \hat{q}_2(\lambda) = 0, \quad \lambda \in (3)}$$

The two functions of λ appearing in all exponentials are

$\lambda \to \frac{1}{\lambda}$ $\qquad w_1(\lambda) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)$ $\qquad \lambda \to -\lambda$ $\qquad w_2(\lambda) = \frac{1}{2i}\left(\lambda - \frac{1}{\lambda}\right)$ $\qquad \lambda \to \frac{1}{\lambda}$

$\swarrow \searrow$ $\qquad\qquad \swarrow \searrow$

$w_1(\lambda) \quad -w_1(\lambda)$ $\qquad\qquad -w_2(\lambda) \quad -w_2(\lambda)$

Hence also

$$F(\frac{1}{\lambda}) = F(\lambda), \quad F(-i\frac{1}{\lambda}) = F(\frac{1}{i\lambda}) = F(i\lambda)$$

NOTE: $\mathrm{e}^{\frac{i}{2}(\lambda z - \frac{1}{\lambda}\bar{z})}F(i\lambda)$ is bounded and analytic for $\lambda \in (1)$

$$iU_1(\lambda)+iU_2(-i\lambda)=\frac{1}{2}(\lambda-\frac{1}{\lambda})D_1(\lambda)+\frac{i}{2}(\lambda+\frac{1}{\lambda})D_2(-i\lambda),\quad \lambda\in\ (3)$$

$$\downarrow\ \lambda\to 1/\lambda$$

$$iU_1(\lambda)+iU_2(i\lambda)=-\frac{1}{2}(\lambda-\frac{1}{\lambda})D_1(\lambda)+\frac{i}{2}(\lambda+\frac{1}{\lambda})D_2(i\lambda),\quad \lambda\in\ (2)$$

$$\downarrow\ \lambda\to -\lambda\ \downarrow$$

$$iU_1(-\lambda)+iU_2(i\lambda)=-\frac{1}{2}(\lambda-\frac{1}{\lambda})D_1(-\lambda)-\frac{i}{2}(\lambda+\frac{1}{\lambda})D_2(i\lambda),\quad \lambda\in\ (1)$$

$$iU_1(-\lambda)+iU_2(-i\lambda)=\frac{1}{2}(\lambda-\frac{1}{\lambda})D_1(-\lambda)-\frac{i}{2}(\lambda+\frac{1}{\lambda})D_2(-i\lambda),\quad \lambda\in\ (4)$$

taking the difference:

$$iU_2(-i\lambda)-iU_2(i\lambda)=(\lambda-\frac{1}{\lambda})D_1(-\lambda)+\frac{i}{2}(\lambda+\frac{1}{\lambda})(D_2(i\lambda)-D_2(-i\lambda))$$

$$\lambda\in\ (1)\cap(4)=\mathcal{L}_2$$

The nonlinear problem

$$q_{z\bar{z}} = \frac{1}{4}\sin q$$

I consider a Lax pair that coincides with the linear one in the small q limit (different from the one in literature):

$$M_z - \frac{i\lambda}{4}[\sigma_3, M] = QM, \quad M_{\bar{z}} + \frac{i}{4\lambda}[\sigma_3, M] = \frac{i}{4\lambda}\tilde{Q}M$$

$$Q(z,\bar{z}) = \begin{pmatrix} 0 & \frac{iq_z}{2} \\ \frac{iq_z}{2} & 0 \end{pmatrix} = \frac{iq_z}{2}\sigma_1,$$
$$\tilde{Q}(z,\bar{z}) = (1-\cos q)\sigma_3 - (\sin q)\sigma_2.$$

This is normalised so that

$$M = I + \frac{m_1}{\lambda} + O\left(\frac{1}{\lambda}\right), \ |\lambda| \to \infty, \quad M = m_o + O(\lambda), \ |\lambda| \to 0$$

Equivalently,

$$W = \mathrm{e}^{-\frac{i}{4}(\lambda z - \frac{\bar{z}}{\lambda})\hat{\sigma}_3}[QMdz + \tilde{Q}Md\bar{z}], \quad \lambda \in ((1),(3))$$

is a closed form

Global relation:

$$\int_{\mathcal{I}} W = 0, \quad \lambda \in ((1),(3))$$

where $\mathcal{I} = \{x \geq 0,\ y \geq 0\}$

(Notation $\lambda \in ((1),(3))$ means: $\lambda \in (1)$ in the first column, $\lambda \in (3)$ in the second column)

In terms of the variables (x, y) this Lax pair is

$$M_x + \frac{w_2(\lambda)}{2}[\sigma_3, M] = Q_0(\lambda)M$$
$$M_y + \frac{w_1(\lambda)}{2}[\sigma_3, M] = iQ_0(-\lambda)M$$

with

$$Q_0(\lambda) = \frac{i(q_x - iq_y)}{4}\sigma_1 + \frac{i}{4\lambda}(1 - \cos q)\sigma_3 - \frac{i}{4\lambda}(\sin q)\sigma_2$$

NOTE:

$\det[w_2(\lambda)\sigma_3 - Q_0(\lambda)]$ is a function of $w_2(\lambda) = \frac{1}{2i}\left(\lambda - \frac{1}{\lambda}\right)$

$\det[w_1(\lambda)\sigma_3 - iQ_0(-\lambda)]$ is a function of $w_1(\lambda) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)$

The Riemann-Hilbert problem:

$$z_1 = 0 + i\infty, \quad z_2 = 0, \quad z_3 = \infty + i0$$

$$M_j = \int_{z_j}^{z} \mathrm{e}^{\frac{i}{4}(\lambda(z-\zeta)-\frac{(\bar{z}-\bar{\zeta})}{\lambda})\hat{\sigma}_3} \left[QMd\zeta + \tilde{Q}Md\bar{\zeta} \right]$$

with the jump relations

$$M_3(z,\bar{z},\lambda) = M_2(z,\bar{z},\lambda)\mathrm{e}^{\widehat{(\frac{i\lambda}{4}z-\frac{i}{4\lambda}\bar{z})\sigma_3}} M_3(0,0,\lambda), \quad \lambda \in (\mathbb{R}^-, \mathbb{R}^+)$$

$$M_1(z,\bar{z},\lambda) = M_2(z,\bar{z},\lambda)\mathrm{e}^{\widehat{(\frac{i\lambda}{4}z-\frac{i}{4\lambda}\bar{z})\sigma_3}} M_1(0,0,\lambda), \quad \lambda \in (i\mathbb{R}^-, i\mathbb{R}^+),$$

Global relation:

$$\int_{\mathcal{I}} W = 0, \quad W = \mathrm{e}^{-\frac{i}{4}(\lambda z - \frac{\bar{z}}{\lambda})\hat{\sigma}_3}[QMdz + \tilde{M}d\bar{z}], \quad \lambda \in ((1),(3))$$

where $\mathcal{I} = \{x \geq 0,\ y \geq 0\}$

Explicitly: $\boxed{S_1^{-1}S_0 = I,\ \ \lambda \in ((1),(3))}$

$$S_0(\lambda) = I - \int_0^\infty \mathrm{e}^{-\frac{i}{4}[\lambda - \frac{1}{\lambda}]x\hat{\sigma}_3} Q_0(\lambda) M_3(x,x,\lambda)dx = M_3(0,0,\lambda)$$

$$S_1(\lambda) = I - \int_0^\infty \mathrm{e}^{\frac{1}{4}[\lambda + \frac{1}{\lambda}]y\hat{\sigma}_3} iQ_0(-\lambda) M_1(iy,-iy,\lambda)dy = M_1(0,0,\lambda)$$

$$\Rightarrow M_1(z,\bar{z},\lambda) = M_3(z,\bar{z},\lambda), \quad \lambda \in ((1),(3)).$$

Symmetry of $Q_0 \Rightarrow$

$$S_0(\lambda) = \begin{pmatrix} a_0(\lambda) & -\overline{b_0(\bar{\lambda})} \\ b_0(\lambda) & \overline{a_0(\bar{\lambda})} \end{pmatrix}, \quad S_1(\lambda) = \begin{pmatrix} a_1(\lambda) & -\overline{b_1(\bar{\lambda})} \\ b_1(\lambda) & \overline{a_1(\bar{\lambda})} \end{pmatrix}.$$

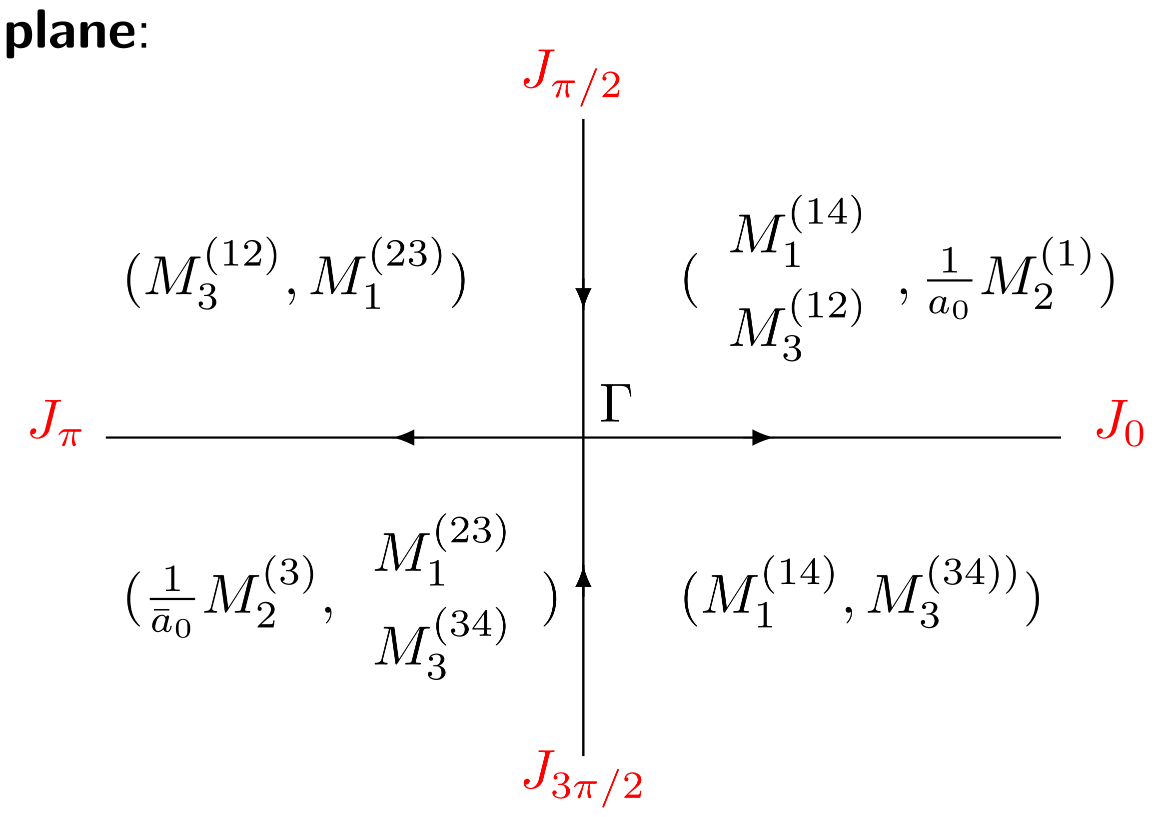
$$\text{also} \quad \overline{a(\bar{\lambda})} = a(-\lambda), \quad \overline{b(\bar{\lambda})} = b(-\lambda)$$

$$S_0 = (S_0^{(1\cup 2)}, S_0^{(3\cup 4)}), \quad S_1 = (S_1^{(1\cup 4)}, S_0^{(2\cup 3)})$$

Global relation $\Rightarrow$ $\boxed{a_1 = a_0, \qquad b_0 = b_1}$ $\lambda \in (1)$

Riemann-Hilbert problem - cont'd

λ **plane**:



$$M_- = M_+ J \qquad \Rightarrow M_+ - M_- = M_+(I - J), \;\; \lambda \in \Gamma$$

Global relation $\Rightarrow$ this problem is uniquely determined - J is triangular

$$J_0 = \begin{pmatrix} 1 & -\frac{\bar{b}_0}{a_0}\mathrm{e}^{i\theta(z,\bar{z})} \\ & \\ 0 & 1 \end{pmatrix}, \quad J_{\pi/2} = \begin{pmatrix} 1 & -\frac{\bar{b}_1}{a_0}\mathrm{e}^{i\theta(z,\bar{z})} \\ & \\ 0 & 1 \end{pmatrix}$$

$$J_\pi = \begin{pmatrix} 1 & 0 \\ & \\ \frac{b_0}{\bar{a}_1}\mathrm{e}^{-i\theta(z,\bar{z})} & 1 \end{pmatrix}, \quad J_{3\pi/2} = J_\pi J_{\pi/2}^{-1} J_0 = \begin{pmatrix} 1 & 0 \\ & \\ \frac{b_1}{\bar{a}_1}\mathrm{e}^{-i\theta(z,\bar{z})} & 1 \end{pmatrix}$$

$$\theta(z,\bar{z}) = \tfrac{1}{2}\left(\lambda z - \tfrac{1}{\lambda}\bar{z}\right)$$

$$M = I + \frac{1}{2\pi i}\int_\Gamma \frac{M_+(I-J)(\lambda')}{\lambda' - \lambda} d\lambda'$$

We know

$$M = I + \frac{m_1}{\lambda} + O\left(\frac{1}{\lambda}\right),\ |\lambda| \to \infty,$$

hence

$$m_1 = I - \frac{1}{2\pi i}\int_\Gamma M_+(I-J)(\lambda)d\lambda$$

$$\Rightarrow q_z(z,\bar{z}) = (m_1)_{21}, \quad \cos q(z,\bar{z}) = 1 + 4i\frac{\partial (m_1)_{11}}{\partial x} + 2[(m_1)_{21}]^2$$

E.g.

$$q_z(z,\bar{z}) = \frac{1}{2\pi i}\int_{\Gamma_-}(M_+)_{22}\frac{b(\lambda)}{a_0(-\lambda)}\mathrm{e}^{-\frac{i}{2}(\lambda z - \frac{\bar{z}}{\lambda})}d\lambda$$

($b = b_0$ on $\mathbb{R}^-$, $b = b_1$ on $i\mathbb{R}^-$, $a_0(-\lambda) = a_1(-\lambda)$ for $\lambda \in (3)$)

Theorem Given smooth functions $d_1(y)$ $u_1(y)$, $d_2(x)$, $u_2(x)$ defined on $[0,\infty)$, such that $d_1(0)=d_2(0)$, $d_1'(0)=u_2(0)$, $d_2'(0)=u_1(0)$, define the column vectors $(a_0(\lambda),b_0(\lambda)^\tau$, $(a_1(\lambda),b_1(\lambda)^\tau$ by

$$\begin{pmatrix} a_0(\lambda) \\ b_0(\lambda) \end{pmatrix} = \psi_3(0,\lambda),\ \lambda\in(12), \qquad \begin{pmatrix} a_1(\lambda) \\ b_1(\lambda) \end{pmatrix} = \psi_1(0,\lambda),\ \lambda\in(14)$$

where ψ_1, ψ_3 are the unique solutions of

$$\psi_3(x,\lambda)_x + w_2(\lambda)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\psi_3(x,\lambda) = Q_0(x,0,\lambda)\psi_3(x,\lambda),$$

$$\psi_1(y,\lambda)_y + w_1(\lambda)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\psi_1(y,\lambda) = iQ_0(0,y,-\lambda)\psi_1(y,\lambda),$$

$$\lim_{x\to\infty}\psi_3(x,\lambda)\to(1,0)^\tau, \qquad \lim_{y\to\infty}\psi_1(y,\lambda)\to(1,0)^\tau,$$

Suppose that the prescribed functions are such that the elements of these column vectors coincide in (1):

$$a_0 = a_1, \qquad b_0 = b_1, \quad \lambda \in (1) \quad (a_0 \neq 0 \text{ in } (1))$$

Define $M(z, \bar{z}, \lambda)$ as the solution of the Riemann-Hilbert problem above, and $m_1 = \lim_{\lambda \to \infty}(\lambda(M - I))$.

Then the functions $q(x, y)$ defined by

$$\cos q(x, y) = 1 + 4i \frac{\partial (m_1)_{11}}{\partial x} + 2[(m_1)_{21}]^2$$

satisfies $q_{xx} + q_{yy} = \sin q$, as well as

$$q(0, y) = d_1(y) \quad q_x(0, y) = u_1(y) \qquad q(x, 0) = d_2(x), \quad q_y(x, 0) = u_2(x).$$

Dirichlet to Neumann map

Evolution problems (e.g. NLS) for some special BC (called **linearisable**) it is possible to express the spectral functions in terms of the spectral functions associated with the initial conditions ONLY

Elliptic problems *conjecture: no linearisable conditions exist* (at least through this Lax representation). The necessary condition for similarity transformation restricts the possibilities to constant $q(x,0)$ and $q(0,y)$ but then the global relation cannot be decoupled

However, the global relation can be used to derive a nonlinear **Volterra integral equation** for unknown derivatives at the boundary.

For more complicated boundary conditions (Robin type): need to determine an additional RH problem for unknown boundary data(?)

work in progress...