Reciprocal transformations and reductions for a 2+1 integrable equation

P. G. Estévez.

Departamento de Fisica Fundamental.
Universidad de Salamanca SPAIN

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Summary

- An spectral problem in $2+1$ that includes equation previously studied

- Integrable cases

- Hodograph transformation I

- Reduction to $1+1$: The Degasperi-Procesi and Vakhnenko-Parkes equations

- Hodograph transformation II

- Reduction to $1+1$: The Vakhnenko-Parkes equation

Bibliography. Reciprocal Transformations

- **Rogers C and Nucci M C.** *"On reciprocal Bäcklund transformations ..."*. Physica Scripta 33 (1986)
- **Hone A.** *"Reciprocal transformations, Painlevé property and solutions of ..."*. Physics Letters A 249. (1998)
- **Euler N et al.** *"Auto-hodograph transformations for ....".* Jour. of Math.Anal. Aplic. 257 (2001)
- **Ferapontov E V and Pavlov M V.** *"Reciprocal transformations of Hamiltonian operators..."*. Jour. Math Phys. 44 (2003)
- **Rogers C and Schief W K.** *"Hidden symmetry in subsonic gasdynamics. Reciprocal..."*. Int.Jour. Eng. Sci. 42 (2004)
- **Abenda S and Grava T.** *"Reciprocal transformations and flat metrics on Hurwitz spaces"* . J. Phys.A: Math. Theor. 40 (2007)

Bibliography. Degasperis Procesi and Vakhnenko-Parkes equations

- **Vakhnenko V and Parkes J** *"The calculation of multisoliton....".* `Chaos, solitons and fractals 13 (2002)`
- **Vakhnenko V and Parkes J** *"A Bäcklund transformation....".* `Chaos, solitons and fractals 17 (2003)`
- **Degasperis A, Holm D D and Hone A.** *"A new integrable...".* `Theor. Math. Phys. 133 (2002)`
- **Hone A and Wang J P** *"Prolongation algebras and ..."* `Inv. Problems 19 (2003)`
- **Vakhnenko V and Parkes J** *"Periodic and solitary waves...".* `Chaos, solitons and fractals 20 (2004)`
- **Vakhnenko V and Parkes J** *"The connection of the Degasperis...".* `Proc. of Ins. of Math. Of NAS of Ukraine 50 (2004)`
- **Ragnisco O. and Popowicz Z** *"A two component ...".* `J. Phys. A: Math.Gen.39 (2006)`

CASE I: Estévez Prada. Journal of nonlinear Math. Phys. (2005)

$$\left(H_{x_2x_1x_1} + 3H_{x_2}H_{x_1} - \frac{3}{4}\frac{(H_{x_2x_1})^2}{H_{x_2}}\right)_{x_1} = H_{x_2x_3}$$

$$\Phi_{x_3} - \Phi_{x_1x_1x_1} - 3H_{x_1}\Phi_{x_1} - \frac{3}{2}H_{x_1x_1}\Phi = 0$$

$$\Phi_{x_1x_2} + H_{x_2}\Phi - \frac{1}{2}\frac{H_{x_1x_2}}{H_{x_2}}\Phi_{x_2} = 0$$

CASE II: Estévez Leble. Inverse Problems 11 (1995).

$$(H_{x_2x_1x_1} + 3H_{x_2}H_{x_1})_{x_1} = H_{x_2x_3}$$

$$\Phi_{x_3} - \Phi_{x_1x_1x_1} - H_{x_1}\Phi_{x_1} - 3H_{x_1x_1}\Phi = 0$$

$$\Phi_{x_1x_2} + H_{x_2}\Phi - \frac{H_{x_1x_2}}{H_{x_2}}\Phi_{x_2} = 0$$

There are particular cases of the equation

$$\Omega = \left(H_{x_2x_1x_1} + 3H_{x_2}H_{x_1} - \frac{(k-5)(k+1)}{12} \frac{(H_{x_2x_1})^2}{H_{x_2}} \right)$$

$$\Omega_{x_1} = H_{x_2x_3}$$

It has the Painlevé property for $k = 2$ (case I) or $k = -1$ (case II)

The spectral problem can be summarized for both cases as:

$$\Phi_{x_1x_2} + H_{x_2}\Phi + \left(\frac{k-5}{6}\right) \frac{H_{x_1x_2}}{H_{x_2}} \Phi_{x_2} = 0$$

$$\Phi_{x_3} - \Phi_{x_1x_1x_1} - 3H_{x_1}\Phi_{x_1} + \left(\frac{k-5}{2}\right) H_{x_1x_1}\Phi = 0$$

Many different solutions as dromions, line solitons...etc. have been obtained for the two integrable cases in the above references

RECIPROCAL TRANSFORMATION I

$$dx_1 = \alpha(x,t,T)\,(dx - \beta(x,t,T)dt - \epsilon(x,t,T)dT)$$

$$x_2 = t, \qquad x_3 = T$$

$$\frac{\partial}{\partial x_1} = \frac{1}{\alpha}\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial x_3} = \frac{\partial}{\partial T} + \epsilon\frac{\partial}{\partial x}$$

$$\alpha_t + (\alpha\beta)_x = 0$$

$$\alpha_T + (\alpha\epsilon)_x = 0$$

$$\beta_T - \epsilon_t + \epsilon\beta_x - \beta\epsilon_x = 0$$

We select the transformation in the following form:

$$H_{x_2}(x_1, x_2, x_3) = \alpha(x,t,T)^k \qquad (k+1)(k-2) = 0$$

$$\Downarrow$$

$$H_{x_1}(x_1, x_2, x_3) = \frac{1}{3}\left(\frac{\Omega(x,t,T)}{\alpha^k} - k\frac{\alpha_{xx}}{\alpha} + (2k-1)\frac{\alpha_x^2}{\alpha^2}\right)$$

yielding the set of equations

$$\Omega_t = -\beta\Omega_x - k\Omega\beta_x + \alpha^{k-2}\left(-k\beta_{xxx} + (k-2)\beta_{xx}\frac{\alpha_x}{\alpha} + 3k\alpha^k\alpha_x\right)$$

$$\Omega_x = -k\alpha^{k+1}\epsilon_x$$

$$\alpha_t + (\alpha\beta)_x = 0, \qquad \alpha_T + (\alpha\epsilon)_x = 0$$

$$\beta_T - \epsilon_t + \epsilon\beta_x - \beta\epsilon_x = 0$$

LAX PAIR

$$\Phi(x_1, x_2, x_3) = \psi(x, t, T)\alpha^{\frac{2k-1}{3}}$$

$$\psi_{xt} = -\beta\psi_{xx} + \frac{2}{3}(k-2)\beta_x\,\psi_x + \left(\frac{2k-1}{3}\beta_{xx} - \alpha^{k+1}\right)\psi$$

$$\psi_T = \frac{\psi_{xxx}}{\alpha^3} + 2(k-2)\frac{\alpha_x}{\alpha^4}\,\psi_{xx} + \left(\frac{\Omega}{\alpha^{k+1}} - \epsilon + (k-2)\left(\frac{\alpha_{xx}}{\alpha^4} - 4\frac{\alpha_x^2}{\alpha^5}\right)\right)\psi_x$$

$$(k+1)(k-2) = 0$$

The condition of integrability can be written as:

$$A_1 A_2 = 0, \qquad A_1 + A_2 = 1$$

where

$$A_1 = \frac{k+1}{3}, \qquad A_2 = \frac{2-k}{3}$$

$A_1 = 1, \quad A_2 = 0$ **corresponds to the** $2+1$ **generalization of the Degasperis-Procesi equation**

$A_1 = 0, \quad A_2 = 1$ **corresponds to the** $2+1$ **generalization of the Vakhnenko-Parkes equation**

It allows us to write the Lax pair as

$$\psi_{xt} = A_1 \left[-\beta \psi_{xx} + \left(\beta_{xx} - \frac{1}{M} \right) \psi \right] + A_2 \left[-\beta \psi_{xx} - 2\beta_x \psi_x - (1 + \beta_{xx}) \psi \right]$$

$$\psi_T = A_1 \left[M\, \psi_{xxx} + (M\, \Omega - \epsilon)\, \psi_x \right] + A_2 \left[M\, \psi_{xxx} + 2M_x\, \psi_{xx} + (M_{xx} + \Omega - \epsilon)\, \psi_x \right]$$

where

$$M = \frac{1}{\alpha^3}$$

and the equations are:

$$A_1 \quad \left[M\Omega_t + M\beta\Omega_x + 2M\Omega\beta_x + 2M\beta_{xxx} + 2\frac{M_x}{M} \right] +$$
$$A_2 \quad [\Omega_t + \beta\Omega_x - \Omega\beta_x - M\beta_{xxx} - M_x\beta_{xx} - M_x] = 0$$

$$A_1 [M\Omega_x + \epsilon_x] + A_2 [\Omega_x - \epsilon_x] = 0$$

$$M_t = 3M\beta_x - \beta M_x$$

$$M_T = 3M\epsilon_x - \epsilon M_x$$

$$\beta_T - \epsilon_t + \epsilon\beta_x - \beta\epsilon_x = 0$$

Reduction independent on T: $\epsilon = 0$, $\Omega = a_o$, $\psi_T = \lambda\psi$

$$2MA_1\left[\beta_{xx} + a_0\beta - \frac{1}{M}\right]_x - A_2\left[M\beta_{xx} + a_o\beta + M\right]_x = 0$$

$$M_t = 3M\beta_x - \beta M_x$$

$A_2 = 0, a_o = -1$: DEGASPERIS PROCESI: $\beta_{xx} - \beta = \frac{1}{M} + q_0$

$$(\beta_{xx} - \beta)_t + \beta\beta_{xxx} + 3\beta_x\beta_{xx} - 4\beta\beta_x + 3q_0\beta_x = 0$$

$A_1 = 0, a_o = 0$: VAKHNENKO-PARKES: $\beta_{xx} + 1 = \frac{q_0}{M}$

$$[(\beta_t + \beta\beta_x)_x + 3\beta]_x = 0$$

Spatial part of the Lax pair

DEGASPERIS PROCESI for $a_0 = -1 \;\; A_2 = 0.$

$$A_1 \quad \left[\psi_{xxx} + a_o\psi_x - \frac{\lambda}{M}\psi\right] + \qquad\qquad \Uparrow$$

$$A_2 \quad \left[\psi_{xxx} + 2\frac{M_x}{M}\psi_{xx} + \frac{1}{M}(M_{xx} + a_o)\psi_x - \frac{\lambda}{M}\psi\right] = 0$$

$$\Downarrow$$

VAKHNENKO-PARKES for $A_1 = 0. \;\; a_0 = 0$

Temporal part of the Lax pair

$$4A_1 \quad [\lambda\psi_t + \psi_{xx} + \lambda\beta\psi_x + (a_0 - \lambda\beta_x)\psi] +$$

$$A_2 \quad [\lambda\psi_t + M\psi_{xx} + (\lambda\beta + M_x)\psi_x + (a_0 + \lambda\beta_x)\psi] = 0$$

There are two more interesting reductions that are the transformed of those studied in Estévez, Gandarias and Prada. Phys Lett A 343 (2005)

RECIPROCAL TRANSFORMATION II

$$x_1 = y, \qquad x_3 = T$$

$$dx_2 = \eta(y,z,T)\,(dx - u(y,z,T)dt - \omega(y,z,T)dT)$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y} + u\,\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial x_2} = \frac{1}{\eta}\,\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial x_3} = \frac{\partial}{\partial T} + \omega\,\frac{\partial}{\partial z}$$

$$\eta_y + (u\eta)_z = 0$$

$$\eta_T + (\eta\omega)_z = 0$$

$$u_T - \omega_y - u\omega_z + \omega u_z = 0$$

We select the transformation in the following form:

$$z = H(x_1, x_2, x_3) \Rightarrow dz = H_{x_1} dx_1 + H_{x_2} dx_2 + H_{x_3} dx_3$$

$$\Downarrow$$

$$H_{x_2}(x_1, x_2, x_3) = \frac{1}{\eta(y = x_1, z = H, T = x_3)}$$

$$H_{x_1}(x_1, x_2, x_3) = \frac{u(y = x_1, z = H, T = x_3)}{\eta(y = x_1, z = H, T = x_3)}$$

$$H_{x_3}(x_1, x_2, x_3) = \frac{\omega(y = x_1, z = H, T = x_3)}{\eta(y = x_1, z = H, T = x_3)}$$

The transformed equations are

$$A_1\left[u_{zy}+uu_{zz}+\frac{1}{4}u_z^2+3u-G\right]+A_2\left[u_{zy}+uu_{zz}+u_z^2+3u-G\right]$$

$$G_y=(\omega-uG)_z$$

$$u_T=\omega_y-\omega u_z+u\omega_z$$

$$\eta_y+(u\eta)_z, \quad \eta_T+(\omega\eta)_z$$

$$A_1A_2=0, \quad A_1+A_2=1$$

and the Lax pair is:

$$A_1 \quad \left[\Phi_{zy} + u\Phi_{zz} + \Phi + \frac{1}{2}u_z\Phi_z\right] +$$
$$A_2 \quad [\Phi_{zy} + u\Phi_{zz} + \Phi] = 0$$

$$A_1 \quad \left[\Phi_T - \Phi_{yyy} - u^3\Phi_{zzz} + 3u\left(u_y - \frac{1}{2}uu_z\right)\Phi_{zz} + \frac{3}{2}u_y\Phi\right] +$$
$$A_1 \quad \left[\omega - \frac{uG}{2} - u_{yy} - \frac{u^2u_{zz}}{2} + \frac{u_yu_z}{2} - \frac{uu_z^2}{8} - \frac{3u^2}{2}\right]\Phi_z$$
$$+$$
$$A_2 \quad \left[\Phi_T - \Phi_{yyy} - u^3\Phi_{zzz} + 3uu_y\Phi_{zz}\right] +$$
$$A_2 \quad \left[\omega - 2uG - u_{yy} + u^2u_{zz} - u_yu_z + uu_z^2 + 3u^2\right]\Phi_z = 0$$

Reduction independent on T: $\omega = 0$, $\psi_T = \lambda\psi$

$$G_y + (uG)_z = 0, \qquad N = u_{zz} - 2$$

$$A_1\left[u_{zy} + uu_{zz} + \frac{1}{4}u_z^2 + 3u - G\right] + A_2\left[u_{zy} + uu_{zz} + u_z^2 + 3u - G\right] = 0$$

VAKHNENKO-PARKES $\Uparrow$ $A_1 = 0\,, G = 0$

Lax pair

$$A_1 \quad \left[\lambda\Phi_{zzz} + \left(G - \lambda\frac{N_z}{N}\right)\Phi_{zz} + \left(\frac{3}{2}G_z - \frac{N_z}{N}G\right)\Phi_z\right] +$$
$$A_1 \quad \left[\left(G_{zz} + \frac{N}{2}u_{zz} + N - \frac{N_z}{N}G_z\right)\frac{\Phi}{2}\right] +$$
$$A_2 \quad [\lambda\Phi_{zzz} + G\Phi_{zz} + G_z\Phi_{zz} + (u_{zz} + 1)\,\Phi] = 0$$

$$A_1 \quad \left[\Phi_y - \frac{2\lambda}{N}\Phi_{zz} + \left(u - \frac{2G}{N}\right)\Phi_z - \frac{1}{2}\left(u_z + \frac{G_z}{N}\right)\Phi\right] +$$
$$A_2 \quad [\Phi_y - \lambda\Phi_{zz} + (u - G)\Phi_z - u_z\Phi] = 0$$

Conclusions

- Different reciprocal transformations for an spectral problem in 2+1 dimensions are investigated.

- Reductions of the transformed spectral problems are presented.

- The Vakhnenko-Parkes and Degasperis-Procesi equation appear as two of the different possible reductions.