An analytical model for synthetic- and bio-polymers based on continuous elastic rods

Circular helix solutions

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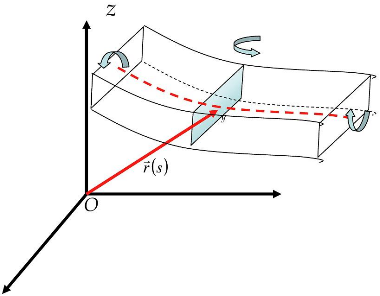
Nonlinear Physics, *Theory and Experiment. V*
Gallipoli, Lecce, June 12–21, 2008



For an unshearable elastic rod, in the *absence* of external efforts and momenta, at equilibrium we get

the static Kirchhoff's equations

$$\begin{cases} \dfrac{\mathrm{d}\vec{F}}{\mathrm{d}s} = \vec{0} \\[2ex] \dfrac{\mathrm{d}\vec{M}}{\mathrm{d}s} + \vec{d}_3 \times \vec{F} = \vec{0} \end{cases}$$

where

$$\left\{\vec{d}_1, \vec{d}_2, \vec{d}_3\right\}$$

are the three components of the generalized orthonormal Frenet frame: $\vec{d}_1$ and $\vec{d}_2$ define the plane of the transverse section of the rod while $\vec{d}_3$ defines the tangent direction to the central line.

At the equilibrium:

$$\vec{F}(s) = F_1(s)\,\vec{d}_1(s) + F_2(s)\,\vec{d}_2(s) + F_3(s)\,\vec{d}_3(s)$$

$$\vec{M}(s) = a_1(s)\;k_1(s)\;\vec{d}_1(s) + a_2(s)\;k_2(s)\;\vec{d}_2(s) + b(s)\;k_3(s)\;\vec{d}_3(s)$$

$a_1(s)$ and $a_2(s)$ are called *bending stiffnesses*

$b(s)$ is called *twist stiffness*.

They depend on

- the *elastic properties* of the constituting material of the rod
- the *shape* of the cross section of the rod
- the *mass density* along the cross section

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If we suppose

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to be constant (w.r.t. the arc-length), then the *bending stiffnesses* a_1 and a_2 and the *twist stiffness* b will be constant functions of the arc–length.

Static Kirchhoff's Equations (*arc–length independent elasticity case*)

$$\begin{cases} F_1' + k_2\,F_3 - k_3\,F_2 = 0 \\ F_2' + k_3\,F_1 - k_1\,F_3 = 0 \\ F_3' + k_1\,F_2 - k_2\,F_1 = 0 \\ a_1\,k_1' + (b - a_2)\,k_2\,k_3 = F_2 \\ a_2\,k_2' - (b - a_1)\,k_1\,k_3 = -F_1 \\ b\,k_3' - (a_1 - a_2)\,k_1\,k_2 = 0 \end{cases}$$

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The results so far obtained in the case of arc–length independent elasticity can be summarized as follows:

- Explicit solutions with constant κ, τ and φ only for rods with circular cross section
- For a generic cross section, explicit solutions with constant κ and τ only for $\varphi = n\,\frac{\pi}{2}$, $n \in \mathbb{Z}$ (Frenet helices)
- Equivalence between the static Kirchhoff's equations for particular choices of the constant parameters (a_1, a_2, b) and some particular integrable heavy tops (Lagrange–Poisson, Kovalevskaya, Goriachev, Chaplygin, Corlis–Field)

We need to suppose that the *bending stiffnesses* a_1 and a_2 and the *twist stiffness* b are arc–length dependent functions in order to:

- model anisotropic elastic rods where *the elastic properties*, *the shape of the cross section* and *the density* can vary along the arc–length, as it *actually* happens for most of the polymers in nature;
- characterize completely all the possible elastic rods in the shape of a circular helix (constant κ and τ) at equilibrium.

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to be arc–length dependent, then the *bending stiffnesses* a_1 and a_2 and the *twist stiffness* b will be functions of the arc–length.

Static Kirchhoff's Equations (*arc–length dependent elasticity case*)

$$\begin{cases} F_1' + k_2\, F_3 - k_3\, F_2 = 0 \\ F_2' + k_3\, F_1 - k_1\, F_3 = 0 \\ F_3' + k_1\, F_2 - k_2\, F_1 = 0 \\ a_1\, k_1' + a_1'\, k_1 + (b - a_2)\, k_2\, k_3 = F_2 \\ a_2\, k_2' + a_2'\, k_2 - (b - a_1)\, k_1\, k_3 = -F_1 \\ b\, k_3' + b'\, k_3 - (a_1 - a_2)\, k_1\, k_2 = 0 \end{cases}$$

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b\, k_3' + b'\, k_3 - (a_1 - a_2)\, k_1\, k_2 = 0
\end{cases}
$$

Again

$$
\begin{aligned}
k_1(s) &= \kappa(s) \sin[\varphi(s)] \\
k_2(s) &= \kappa(s) \cos[\varphi(s)] \\
k_3(s) &= \tau(s) + \varphi'(s)
\end{aligned}
$$

In order to have a cylindrical helix, we must solve the system for a_1, a_2, b, F_1, F_2, F_3 and φ with $\kappa(s)$ and $\tau(s)$ constant functions

$$\kappa(s) \equiv \kappa \quad \text{and} \quad \tau(s) \equiv \tau \ .$$

The system is obviously underdetermined, neverthless it turns out that under these assumptions it is possible to characterize completely its solutions.

$$
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$$\begin{aligned} k_1(s) &= \kappa(s)\,\sin[\varphi(s)] \\ k_2(s) &= \kappa(s)\,\cos[\varphi(s)] \\ k_3(s) &= \tau(s) + \varphi'(s) \end{aligned}$$

In order to have a cylindrical helix, we must solve the system for a_1, a_2, b, F_1, F_2, F_3 and φ with $\kappa(s)$ and $\tau(s)$ constant functions

$$\kappa(s) \equiv \kappa \text{ and } \tau(s) \equiv \tau \;.$$

The system is obviously underdetermined, neverthless it turns out that under these assumptions it is possible to characterize completely its solutions.

When $\tau \neq 0$ and $\kappa \neq 0$ we consider the following three cases:

I. $\varphi(s) \equiv n\frac{\pi}{2}$, $n \in \mathbb{Z}$;

II. $\varphi(s) = \varphi_0 - \tau s$;

III. $\varphi(s)$ not belonging to any of the previous cases.

When $\tau = 0$ and $\kappa \neq 0$ we consider the following three cases:

IV. $\varphi(s) \equiv n\frac{\pi}{2}$, $n \in \mathbb{Z}$;

V. $\varphi(s) \equiv \varphi_0$ with $\varphi_0 \neq n\frac{\pi}{2}$, $n \in \mathbb{Z}$;

VI. $\varphi(s)$ is not a constant function.

When $\tau = 0$ and $\kappa = 0$ we consider the following two cases:

VII. $\varphi(s) \equiv \varphi_0$;

VIII. $\varphi(s)$ is not a constant function.

These eight cases exhaust all the possible solutions that can be obtained assuming that $\kappa(s)$ and $\tau(s)$ are independent on the arc-length.

$\tau \neq 0$ and $\kappa \neq 0$, $\varphi(s) \equiv n\frac{\pi}{2}$, with n odd

$$\begin{cases} a_1(s) = (-1)^n \frac{F_1(0)}{\kappa\tau} + b_0 \\ a_2(s) \in \mathcal{C}^1(\mathbb{R}) \\ b(s) = b_0 \end{cases}$$

$$\begin{cases} k_1(s) = (-1)^{n+1}\kappa \\ k_2(s) = 0 \\ k_3(s) = \tau \end{cases}$$

$$\begin{cases} F_1(s) = F_1(0) \\ F_2(s) = 0 \\ F_3(s) = (-1)^{n+1} \frac{\tau F_1(0)}{\kappa} \end{cases}$$

We recall that $a_2(s) > 0\ \forall s$, with $a_2(0)$ fixed, and $b_0 = b(0)$ is a real positive constant. Furthermore, b_0 and $F_1(0)$ must be chosen so that $a_1(s) > 0\ \forall s$.

$\tau \neq 0$ and $\kappa \neq 0$, $\varphi(s) \equiv n\frac{\pi}{2}$, with n even

$$\begin{cases} a_1(s) \in \mathcal{C}^1(\mathbb{R}) \\ a_2(s) = (-1)^{n+1} \frac{F_2(0)}{\kappa\tau} + b_0 \\ b(s) = b_0 \end{cases}$$

$$\begin{cases} k_1(s) = 0 \\ k_2(s) = (-1)^n\kappa \\ k_3(s) = \tau \end{cases}$$

$$\begin{cases} F_1(s) = 0 \\ F_2(s) = F_2(0) \\ F_3(s) = (-1)^n \frac{\tau F_2(0)}{\kappa} \end{cases}$$

We recall that $a_1(s) > 0\ \forall s$, with $a_1(0)$ fixed, and $b_0 = b(0)$ is an arbitrary real positive constant. Furthermore, b_0 and $F_2(0)$ must be chosen so that $a_2(s) > 0\ \forall s$.

$\tau \neq 0,\ \kappa \neq 0,\ \kappa \neq \tau,\ \varphi(s) = \varphi_0 - \tau s$

$$\begin{cases} a_1(s) = \frac{\alpha\,\kappa^2}{\kappa^2-\tau^2} \\ a_2(s) = \frac{\alpha\,\kappa^2}{\kappa^2-\tau^2} \\ b(s) \in \mathcal{C}^1(\mathbb{R}) \end{cases}$$

$$\begin{cases} k_1(s) = \kappa\,\sin(\varphi_0 - \tau\,s) \\ k_2(s) = \kappa\,\cos(\varphi_0 - \tau\,s) \\ k_3(s) = 0 \end{cases}$$

$$\begin{cases} F_1(s) = -\frac{\alpha\,\kappa^3\,\tau}{\kappa^2-\tau^2}\,\sin(\varphi_0 - \tau\,s) \\ F_2(s) = -\frac{\alpha\,\kappa^3\,\tau}{\kappa^2-\tau^2}\,\cos(\varphi_0 - \tau\,s) \\ F_3(s) = -\frac{\alpha\,\kappa^2\,\tau^2}{\kappa^2-\tau^2} \end{cases}$$

where α is a real constant such that $a_1(s)$ and $a_2(s)$ are positive real constants and $b(s) > 0\ \forall s$, with $b(0)$ fixed.

$\tau \neq 0,\ \kappa \neq 0,\ \kappa = \tau,\ \varphi(s) = \varphi_0 - \tau s$

$$\begin{cases} a_1(s) = -\frac{F_3(0)}{\kappa^2} \\ a_2(s) = -\frac{F_3(0)}{\kappa^2} \\ b(s) \in \mathcal{C}^1(\mathbb{R}) \end{cases}$$

$$\begin{cases} k_1(s) = \kappa\,\sin(\varphi_0 - \kappa\,s) \\ k_2(s) = \kappa\,\cos(\varphi_0 - \kappa\,s) \\ k_3(s) = 0 \end{cases}$$

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where $F_3(0)$ is a real constant such that $a_1(s)$ and $a_2(s)$ are positive real constants and $b(s) > 0\ \forall s$, with $b(0)$ fixed.

$\tau \neq 0$ and $\kappa \neq 0$, $\varphi(s) \neq n\frac{\pi}{2}$ and $\varphi'(s) \neq -\tau$

$$\begin{cases} a_1(s) = \frac{\beta_0}{\tau} + \frac{2\,\beta_2}{\kappa^2}\left[\cos(s\omega)\cot\varphi(s) - \frac{1}{\sqrt{1+\gamma^2}}\sin(s\omega)\right] \\ a_2(s) = \frac{\beta_0}{\tau} - \frac{2\,\beta_2}{\kappa^2}\left[\cos(s\omega)\tan\varphi(s) + \frac{1}{\sqrt{1+\gamma^2}}\sin(s\omega)\right] \\ b(s) = \frac{\beta_1\omega + 2\beta_2\sin(s\omega)}{\omega\,(\tau+\varphi'(s))} \end{cases}$$

$$\begin{cases} k_1(s) = \kappa\,\sin\varphi(s) \\ k_2(s) = \kappa\,\cos\varphi(s) \\ k_3(s) = \tau + \varphi'(s) \end{cases}$$

$$\begin{cases} F_1(s) = \kappa\,(\beta_1 - \beta_0)\,\sin\varphi(s) \\ F_2(s) = \kappa\,(\beta_1 - \beta_0)\,\cos\varphi(s) \\ F_3(s) = \tau\,(\beta_1 - \beta_0) \end{cases}$$

where $\gamma = \frac{\kappa}{\tau}$ and $\omega = \sqrt{\kappa^2 + \tau^2}$. Furthermore, β_0, β_1 and β_2 are related to the initial conditions on $a_1(s)$, $a_2(s)$, $b(s)$ and $\varphi(s)$ in the following way:

$$\begin{cases} \beta_0 = \frac{\tau}{2}\,\{a_1(0) + a_2(0) - [a_1(0) - a_2(0)]\cos(2\varphi(0))\} \\ \beta_1 = [\tau + \varphi'(0)]\,b(0) \\ \beta_2 = \frac{\kappa^2}{4}\,[a_1(0) - a_2(0)]\sin(2\varphi(0)) \end{cases}$$

$\tau = 0$ and $\kappa \neq 0$, $\varphi(s) \equiv n\frac{\pi}{2}$, with n odd

$$\begin{cases} a_1(s) = a_0 - \frac{(-1)^{n+1} F_2(0)\sin(s\kappa) + F_3(0)\cos(s\kappa)}{\kappa^2} \\ a_2(s) \in \mathcal{C}^1(\mathbb{R}) \\ b(s) \in \mathcal{C}^1(\mathbb{R}) \end{cases}$$

$$\begin{cases} k_1(s) = (-1)^n \kappa \\ k_2(s) = 0 \\ k_3(s) = 0 \end{cases}$$

$$\begin{cases} F_1(s) = 0 \\ F_2(s) = (-1)^n F_3(0)\sin(s\kappa) + F_2(0)\cos(s\kappa) \\ F_3(s) = (-1)^{n+1} F_2(0)\sin(s\kappa) + F_3(0)\cos(s\kappa) \end{cases}$$

where $a_2(s) > 0$ and $b(s) > 0$ $\forall s$, with $a_2(0)$ and $b(0)$ fixed, and $a_0 = a_1(0) + \frac{F_3(0)}{\kappa^2}$ is an arbitrary real constant such that $a_0 > \frac{\sqrt{F_2(0)^2 + F_3(0)^2}}{\kappa^2}$, with $F_2(0)$ and $F_3(0)$ real constants.

$\tau = 0$ and $\kappa \neq 0$, $\varphi(s) \equiv n\frac{\pi}{2}$, with n even

$$\begin{cases} a_1(s) \in \mathcal{C}^1(\mathbb{R}) \\ a_2(s) = a_0 - \frac{(-1)^n F_1(0)\sin(s\kappa) + F_3(0)\cos(s\kappa)}{\kappa^2} \\ b(s) \in \mathcal{C}^1(\mathbb{R}) \end{cases}$$

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where $a_1(s) > 0$ and $b(s) > 0$ $\forall s$, with $a_2(0)$ and $b(0)$ fixed, and $a_0 = a_2(0) + \frac{F_3(0)}{\kappa^2}$ is an arbitrary real constant such that $a_0 > \frac{\sqrt{F_1(0)^2 + F_3(0)^2}}{\kappa^2}$, with $F_1(0)$ and $F_3(0)$ real constants.

$\tau = 0$ and $\kappa \neq 0$, $\varphi(s) \equiv \varphi_0$ with $\varphi_0 \neq n\frac{\pi}{2}$

$$\begin{cases} a_1(s) = \alpha_0 + \alpha_1 \cos(s\kappa) + \alpha_2 \sin(s\kappa) \\ a_2(s) = \alpha_0 + \alpha_1 \cos(s\kappa) + \alpha_2 \sin(s\kappa) \\ b(s) \in \mathcal{C}^1(\mathbb{R}) \end{cases}$$

$$\begin{cases} k_1(s) = \kappa \, \sin\varphi_0 \\ k_2(s) = \kappa \, \cos\varphi_0 \\ k_3(s) = 0 \end{cases}$$

$$\begin{cases} F_1(s) = \kappa^2 \left[\alpha_1 \sin(s\kappa) - \alpha_2 \cos(s\kappa)\right] \cos\varphi_0 \\ F_2(s) = -\kappa^2 \left[\alpha_1 \sin(s\kappa) - \alpha_2 \cos(s\kappa)\right] \sin\varphi_0 \\ F_3(s) = -\kappa^2 \left[\alpha_1 \cos(s\kappa) + \alpha_2 \sin(s\kappa)\right] \end{cases}$$

where $a_1(s) > 0$, $a_2(s) > 0$ and $b(s) > 0$ $\forall s$, with $a_1(0)$, $a_2(0)$ and $b(0)$ fixed, and α_0, α_1, α_2 are real constants such that $\alpha_0 > \sqrt{\alpha_1^2 + \alpha_2^2}$. Furthermore α_0, α_1, α_2 are defined in terms of the initial conditions $a_1(0)$, $a_2(0)$ and the vector $\vec{F}(0)$ as follows

$$\begin{cases} \alpha_0 = a_1(0) + \frac{F_3(0)}{\kappa^2} = a_2(0) + \frac{F_3(0)}{\kappa^2} \\ \alpha_1 = -\frac{F_3(0)}{\kappa^2} \\ \alpha_2 = \frac{F_2(0)}{\kappa^2 \, \sin\varphi_0} = -\frac{F_1(0)}{\kappa^2 \, \cos\varphi_0} \end{cases}$$

$\tau = 0$ and $\kappa \neq 0$, $\varphi(s) \equiv \varphi_0$

$$\begin{cases} a_1(s) = \gamma_0 + \gamma_1 \cos(s\kappa) + \gamma_2 \sin(s\kappa) + [\beta_0 \cos(s\kappa) + \beta_1 \sin(s\kappa)] \cot\varphi(s) \\ a_2(s) = \gamma_0 + \gamma_1 \cos(s\kappa) + \gamma_2 \sin(s\kappa) - [\beta_0 \cos(s\kappa) + \beta_1 \sin(s\kappa)] \tan\varphi(s) \\ b(s) = \frac{\kappa\,[\beta_0 \sin(s\kappa) - \beta_1 \cos(s\kappa) + \beta_2]}{\varphi'(s)} \end{cases}$$

$$\begin{cases} k_1(s) = \kappa \sin\varphi(s) \\ k_2(s) = \kappa \cos\varphi(s) \\ k_3(s) = \varphi'(s) \end{cases}$$

$$\begin{cases} F_1(s) = \beta_2\,\kappa^2 \sin\varphi(s) + \kappa^2\,[\gamma_1 \sin(s\kappa) - \gamma_2 \cos(s\kappa)] \cos\varphi(s) \\ F_2(s) = \beta_2\,\kappa^2 \cos\varphi(s) - \kappa^2\,[\gamma_1 \sin(s\kappa) - \gamma_2 \cos(s\kappa)] \sin\varphi(s) \\ F_3(s) = -\kappa^2\,[\gamma_1 \cos(s\kappa) + \gamma_2 \sin(s\kappa)] \end{cases}$$

where γ_0, γ_1, γ_2, β_0, β_1 and β_2 are real constants, expressed in terms of the initial conditions on $a_1(s)$, $a_2(s)$, $b(s)$, $\varphi(s)$ and on the vector $\vec{F}(s)$ as follows:

$$\begin{cases} \gamma_0 = a_2(0) + \frac{F_3(0)}{\kappa^2} + [a_1(0) - a_2(0)] \sin^2\varphi(0) \\ \gamma_1 = -\frac{F_3(0)}{\kappa^2} \\ \gamma_2 = -\frac{1}{\kappa^2} [F_1(0) \cos\varphi(0) - F_2(0) \sin\varphi(0)] \end{cases}$$

$$\begin{cases} \beta_0 = \frac{1}{2}[a_1(0) - a_2(0)] \sin(2\varphi(0)) \\ \beta_1 = \frac{1}{\kappa^2} [F_1(0) \sin\varphi(0) + F_2(0) \cos\varphi(0)] - \frac{b(0)}{\kappa}\,\varphi'(0) \\ \beta_2 = \frac{1}{\kappa^2} [F_1(0) \sin\varphi(0) + F_2(0) \cos\varphi(0)] \end{cases}$$

$\tau = 0$ and $\kappa = 0$, $\varphi(s) \equiv \varphi_0$

$$\begin{cases} a_1(s) \in \mathcal{C}^1(\mathbb{R}) \\ a_2(s) \in \mathcal{C}^1(\mathbb{R}) \\ b(s) \in \mathcal{C}^1(\mathbb{R}) \end{cases}$$

$$\begin{cases} k_1(s) = 0 \\ k_2(s) = 0 \\ k_3(s) = 0 \end{cases}$$

$$\begin{cases} F_1(s) = 0 \\ F_2(s) = 0 \\ F_3(s) = F_3(0) \end{cases}$$

We recall that $a_1(s) > 0$, $a_2(s) > 0$ and $b(s) > 0$ $\forall s$, with $a_1(0)$, $a_2(0)$ and $b(0)$ fixed.

$\tau = 0$ and $\kappa = 0$, $\varphi'(s) \neq 0$

$$\begin{cases} a_1(s) \in \mathcal{C}^1(\mathbb{R}) \\ a_2(s) \in \mathcal{C}^1(\mathbb{R}) \\ b(s) = \frac{\beta}{\varphi'(s)} \end{cases}$$

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We recall that $a_1(s) > 0$ and $a_2(s) > 0$ $\forall s$, with $a_1(0)$ and $a_2(0)$ fixed; moreover $\beta = b(0)\varphi'(0)$ must be chosen so that $b(s) > 0$ $\forall s$.

Since all the quantities involved in the previous solutions are physically constrained, we require that:

1. a_1, a_2 and b are real, positive and bounded;
2. the k_i's and F_i's, $i = 1, \ldots, 3$, are real and bounded.

The *second* condition will be automatically satisfied if φ is real and satisfying $|\varphi'(s)| \neq \infty$ $\forall s$.

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If $\beta_2 = 0$, for the *bending stiffnesses* a_1 and a_2 and the *twist stiffness* b we get:

$$\begin{cases} a_1(s) = \frac{\beta_0}{\tau} \\ a_2(s) = \frac{\beta_0}{\tau} \\ b(s) = \frac{\beta_1}{\tau + \varphi'(s)} \end{cases}$$

Real and positive values of $a_1(s)$ and $a_2(s)$ require $\beta_0 > 0$. At the same time real, positive and bounded $b(s)$ and bounded $k_3(s)$ require

$$\begin{cases} \varphi'(s) > -\tau\,, & \text{if } \beta_1 > 0 \\ \varphi'(s) < -\tau\,, & \text{if } \beta_1 < 0 \end{cases} .$$

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We recall that

$$\gamma = \frac{\kappa}{\tau} \text{ and } \omega = \sqrt{\kappa^2 + \tau^2} \, .$$

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$$\begin{cases} g_1\,(s, \varphi(s)) = \cos\,(s\omega)\cot\varphi(s) - \frac{\sin\,(s\omega)}{\sqrt{1+\gamma^2}} \\ g_2\,(s, \varphi(s)) = -\cos\,(s\omega)\tan\varphi(s) - \frac{\sin\,(s\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

we can rewrite the expressions for a_1, a_2 and b given in the 3^{rd} case as follows:

$$\begin{cases} a_1(s) = \frac{\beta_2}{2\kappa^2}\,\left[g_1(s, \varphi(s)) - \rho\right] \\ a_2(s) = \frac{\beta_2}{2\kappa^2}\,\left[g_2(s, \varphi(s)) - \rho\right] \\ b(s) = \frac{\beta_1\omega + 2\beta_2 \sin\,(s\omega)}{\omega(\tau + \varphi'(s))} \end{cases}$$

where

$$\rho = -2\tau\gamma^2\frac{\beta_0}{\beta_2} = 4\,\frac{a_1(0)\tan\varphi(0) + a_2(0)\cot\varphi(0)}{a_2(0) - a_1(0)} \, .$$

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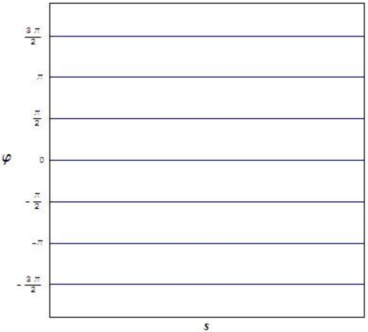

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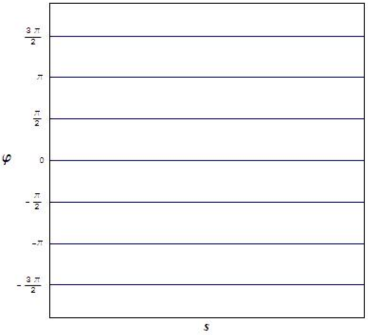
On the (s,φ)–plane we trace the lines at the points $\varphi = n\,\frac{\pi}{2}$ where the $\cot\varphi$ and $\tan\varphi$ functions in g_1 and g_2 diverge.

$\varphi(s)$ can assume the values $n\,\frac{\pi}{2}$ only at the points s such that $\cos(s\,\omega) = 0$, namely at the points where the coefficients of the $\cot\varphi$ and $\tan\varphi$ functions in g_1 and g_2 are null.

If the function $\varphi(s)$ intersects one of the horizontal lines on the (s,φ)–plane, a_1 and a_2 will remain bounded only if this intersection happens at one of the points $s^* = \frac{(2\,n+1)\,\pi}{2\,\omega}$ and, at that point, $\varphi'(s)|_{s=s^*} \neq 0$.



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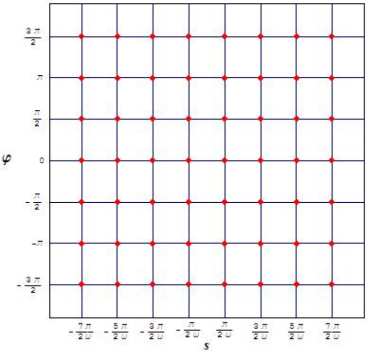

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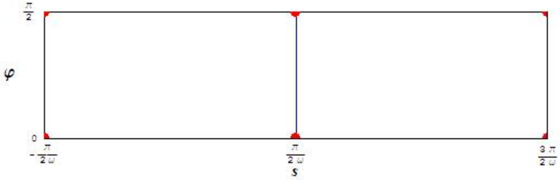
In this way we introduce a lattice on the (s, φ)–plane and reduced the study of the sign and of the boundedness of a_1 and a_2 on the minimal region composed by two adjacent tiles of the lattice.



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If

$$\rho \geq \rho_c$$
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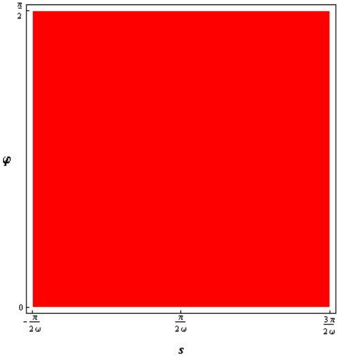
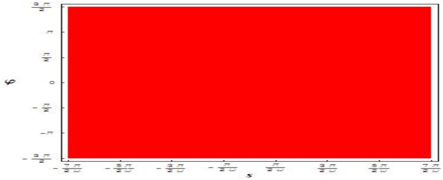
then, the functions a_1 and a_2 will not be positive for any point on the (s,φ)–plane.

In what follows, we will draw in red (respect. in white) the regions on the tiled (s,φ)–plane such that a_1 and a_2 are *negative* (respect. *positive*) real functions.





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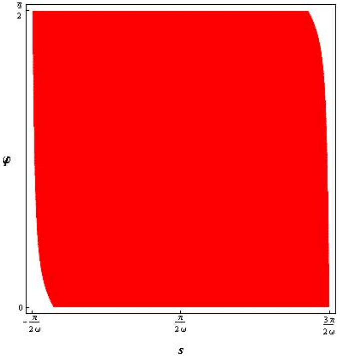

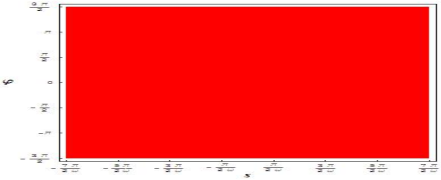

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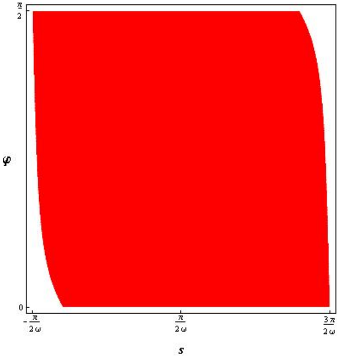

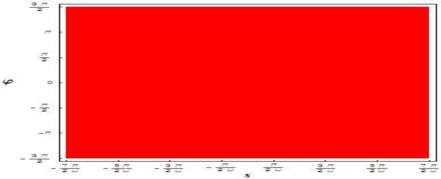

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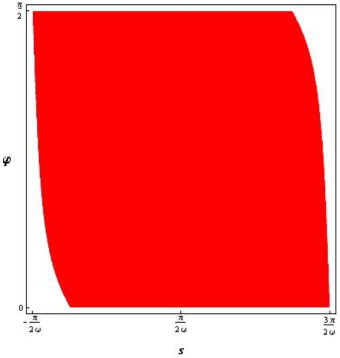

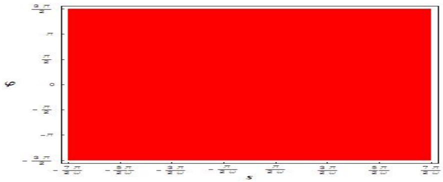

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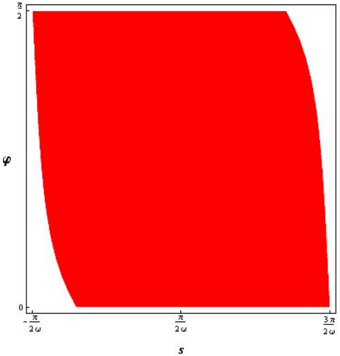

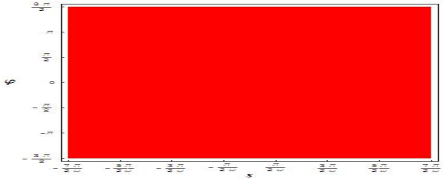

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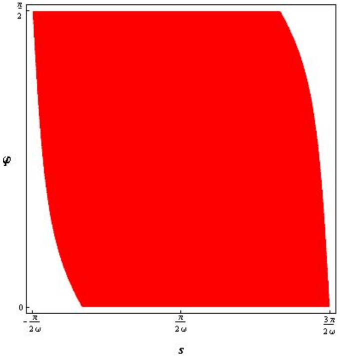

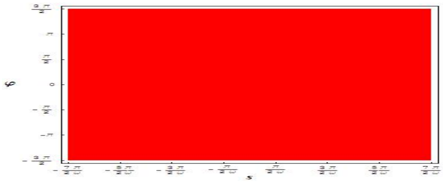

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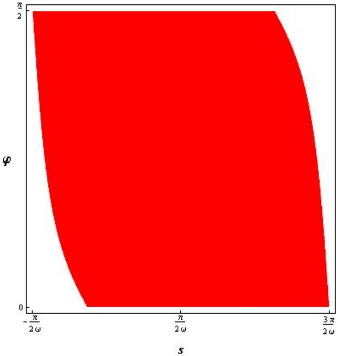

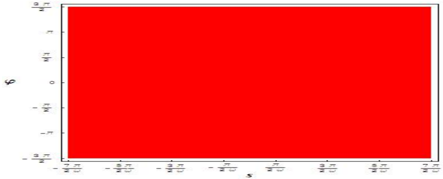

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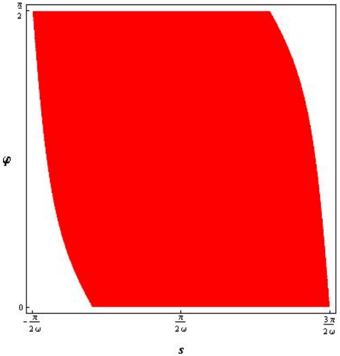

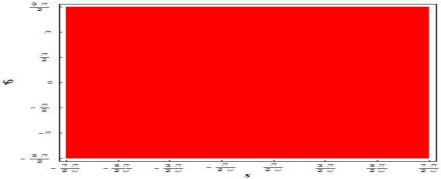

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$$\begin{cases} a_1(s) = \frac{\beta_2}{2\kappa^2}\,[g_1(s;\varphi) - \rho] \\ a_2(s) = \frac{\beta_2}{2\kappa^2}\,[g_2(s;\varphi) - \rho] \end{cases}$$

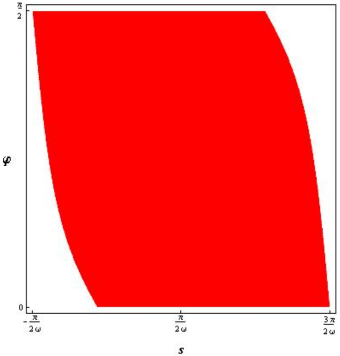
If

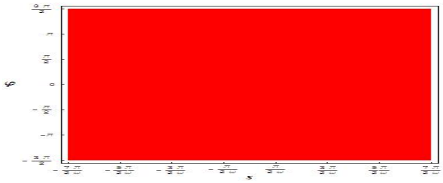
$$\rho \geq \rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

then, the functions a_1 and a_2 will not be positive for any point on the (s,φ)–plane.

In what follows, we will draw in red (respect. in white) the regions on the tiled (s,φ)–plane such that a_1 and a_2 are *negative* (respect. *positive*) real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

$$\begin{cases} a_1(s) = \frac{\beta_2}{2\kappa^2}\,[g_1(s;\varphi) - \rho] \\ a_2(s) = \frac{\beta_2}{2\kappa^2}\,[g_2(s;\varphi) - \rho] \end{cases}$$

If

$$\rho \geq \rho_c$$

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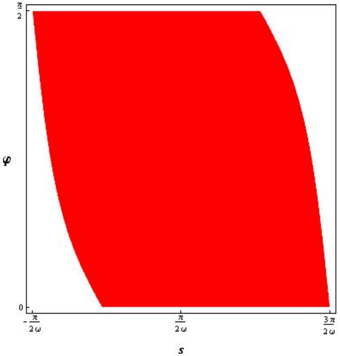
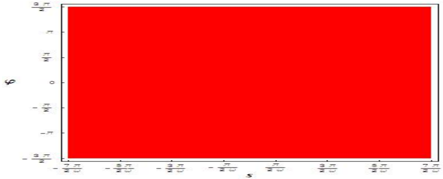
In what follows, we will draw in red (respect. in white) the regions on the tiled (s,φ)–plane such that a_1 and a_2 are *negative* (respect. *positive*) real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

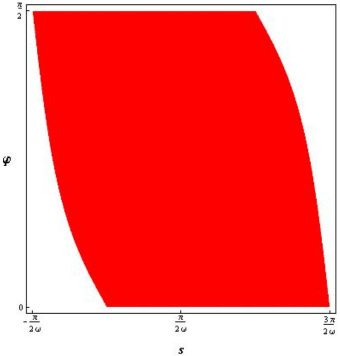
$$\begin{cases} a_1(s) = \frac{\beta_2}{2\kappa^2}\,[g_1(s;\varphi) - \rho] \\ a_2(s) = \frac{\beta_2}{2\kappa^2}\,[g_2(s;\varphi) - \rho] \end{cases}$$

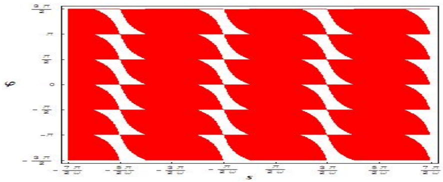
If

$$-\rho_c < \rho < \rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the *white* regions in the tiles on the (s,φ)–plane are not connected, so it is not possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \dfrac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \dfrac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

$$\begin{cases} a_1(s) = \dfrac{\beta_2}{2\kappa^2}\,[g_1(s;\varphi) - \rho] \\ a_2(s) = \dfrac{\beta_2}{2\kappa^2}\,[g_2(s;\varphi) - \rho] \end{cases}$$

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$$-\rho_c < \rho < \rho_c$$
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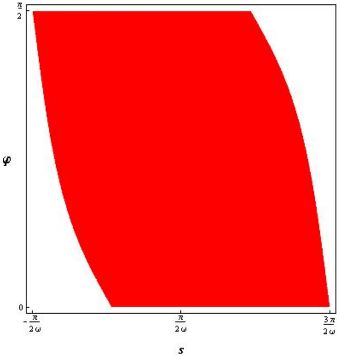
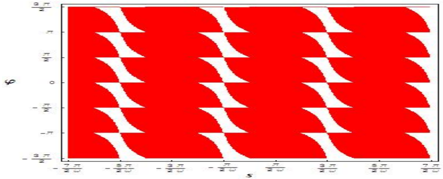
the *white* regions in the tiles on the (s,φ)–plane are not connected, so it is not possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

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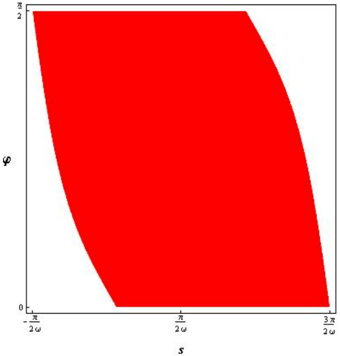
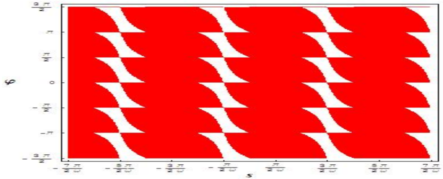
If

$$-\rho_c < \rho < \rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the *white* regions in the tiles on the (s,φ)–plane are not connected, so it is not possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

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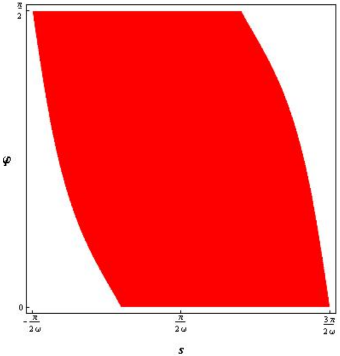
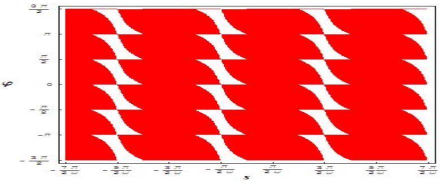
If

$$-\rho_c < \rho < \rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the *white* regions in the tiles on the (s, φ)–plane are not connected, so it is not possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

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If

$$-\rho_c < \rho < \rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

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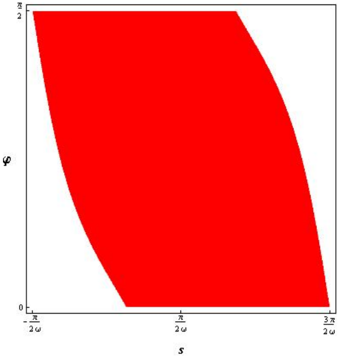
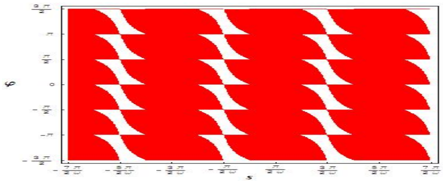




$$
\begin{cases}
g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\
g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}}
\end{cases}
$$

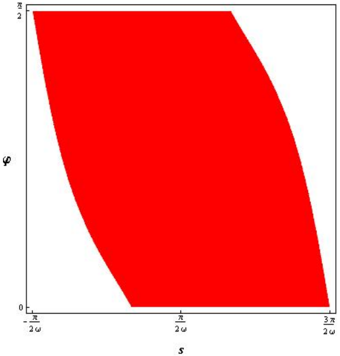
$$
\begin{cases}
a_1(s) = \frac{\beta_2}{2\kappa^2}\,[g_1(s;\varphi)-\rho] \\
a_2(s) = \frac{\beta_2}{2\kappa^2}\,[g_2(s;\varphi)-\rho]
\end{cases}
$$

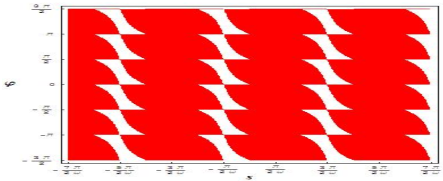
If

$$-\rho_c < \rho < \rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the *white* regions in the tiles on the (s,φ)–plane are not connected, so it is not possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

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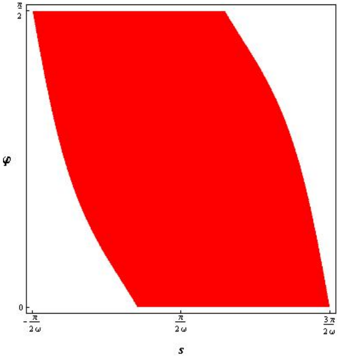

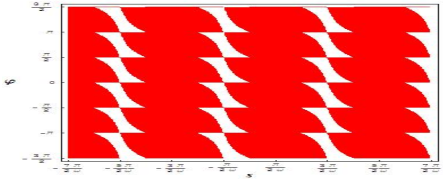

$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

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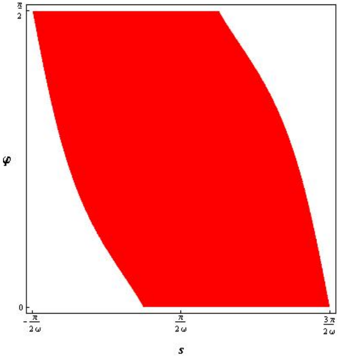
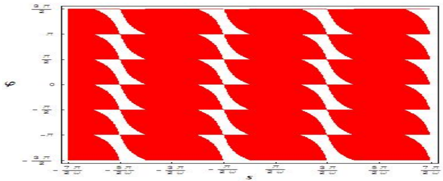
If

$$-\rho_c < \rho < \rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the *white* regions in the tiles on the (s, φ)–plane are not connected, so it is not possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

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If

$$-\rho_c < \rho < \rho_c$$

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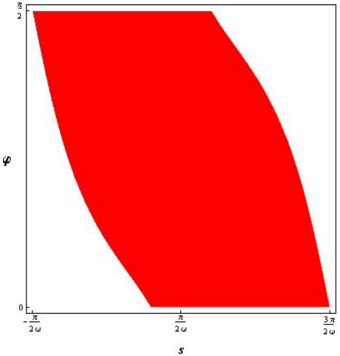
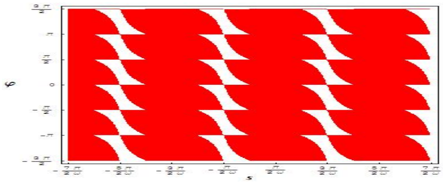
the *white* regions in the tiles on the (s,φ)–plane are not connected, so it is not possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

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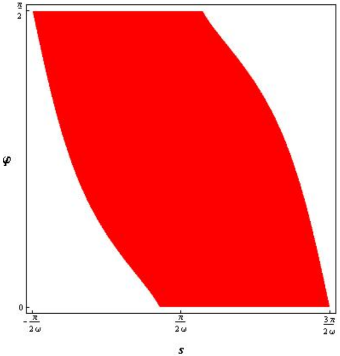
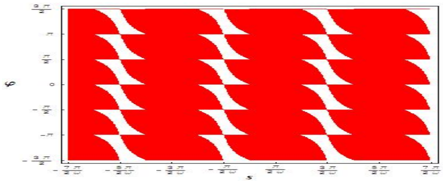
If

$$-\rho_c < \rho < \rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the *white* regions in the tiles on the (s,φ)–plane are not connected, so it is not possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





$$\begin{cases} g_1(s;\varphi) = \cos(s\,\omega)\cot\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \\ g_2(s;\varphi) = -\cos(s\,\omega)\tan\varphi - \frac{\sin(s\,\omega)}{\sqrt{1+\gamma^2}} \end{cases}$$

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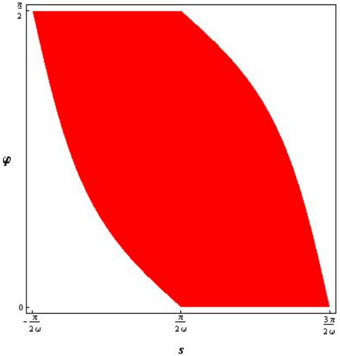
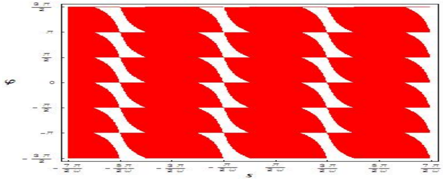
If

$$-\rho_c < \rho < \rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the *white* regions in the tiles on the (s,φ)–plane are not connected, so it is not possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





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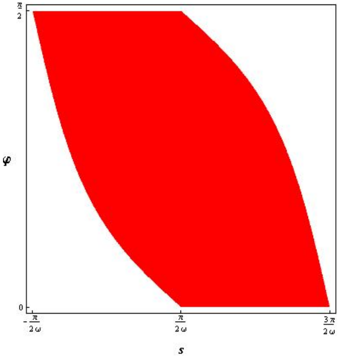
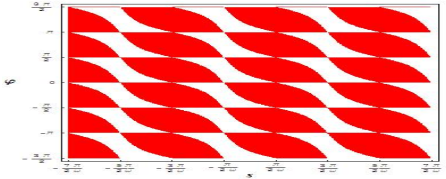
$$\begin{cases} a_1(s) = \frac{\beta_2}{2\kappa^2}\,[g_1(s;\varphi) - \rho] \\ a_2(s) = \frac{\beta_2}{2\kappa^2}\,[g_2(s;\varphi) - \rho] \end{cases}$$

If

$$\rho = -\rho_c$$
$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the white regions in the tiles on the (s,φ)–plane get connected in the points of the lattice.





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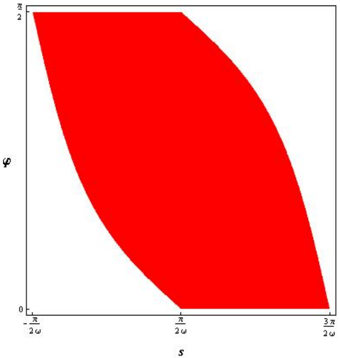
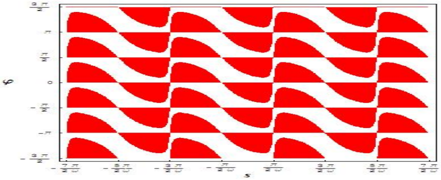
If

$$\rho < -\rho_c$$

$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the *white* regions in the tiles on the (s,φ)–plane start to get wider and connected, so it is possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





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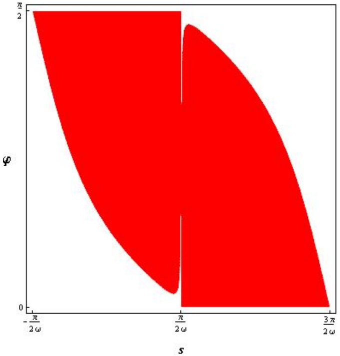
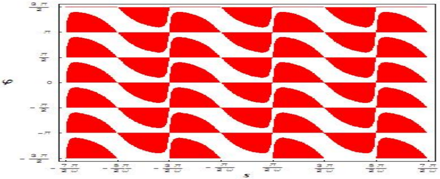
If

$$\rho < -\rho_c$$
$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

the *white* regions in the tiles on the (s,φ)–plane start to get wider and connected, so it is possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





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$$\begin{cases} a_1(s) = \frac{\beta_2}{2\kappa^2}\,[g_1(s;\varphi) - \rho] \\ a_2(s) = \frac{\beta_2}{2\kappa^2}\,[g_2(s;\varphi) - \rho] \end{cases}$$

If

$$\rho < -\rho_c$$

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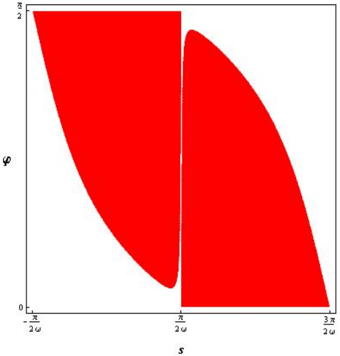
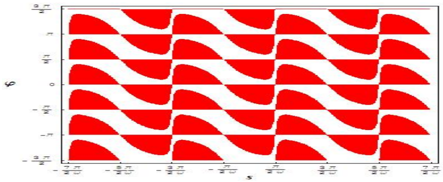
the *white* regions in the tiles on the (s,φ)–plane start to get wider and connected, so it is possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





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If

$$\rho < -\rho_c$$
$$\rho_c = \frac{1}{\sqrt{1+\gamma^2}}$$

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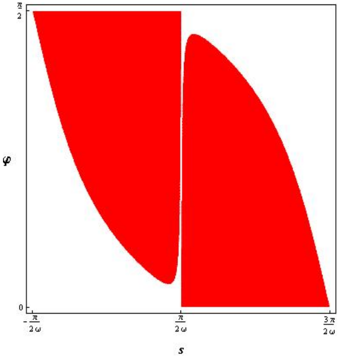

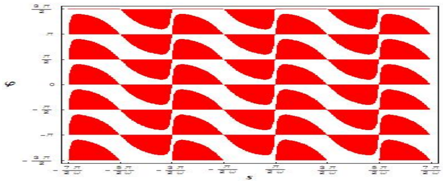

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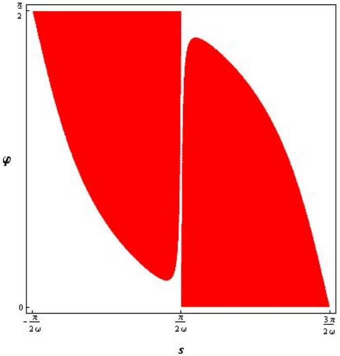

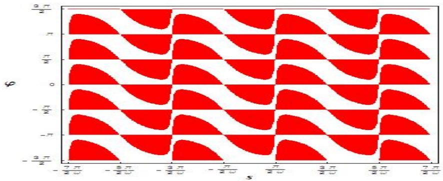

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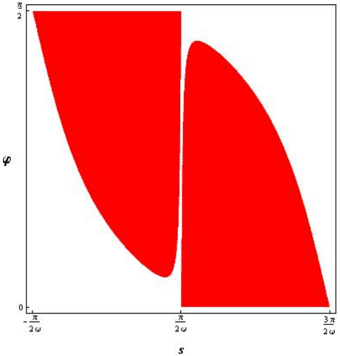

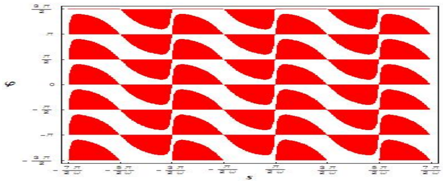

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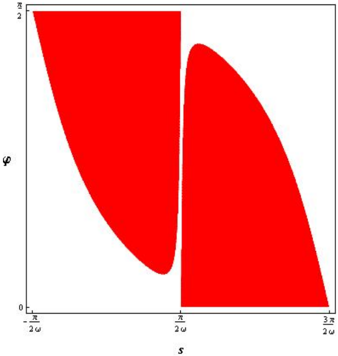

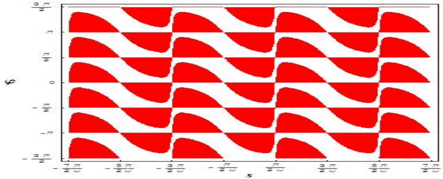

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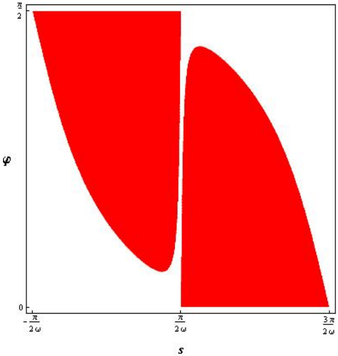

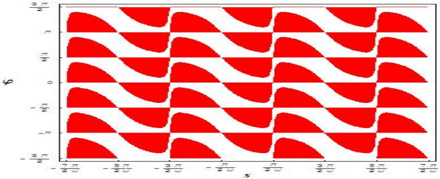

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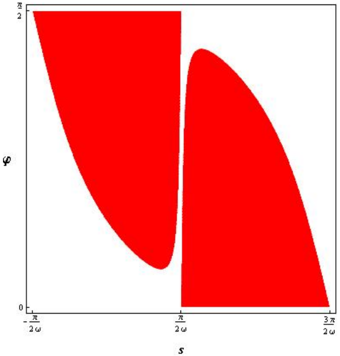

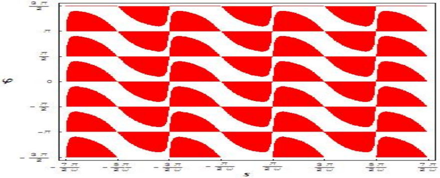

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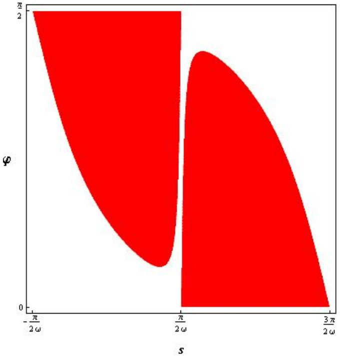

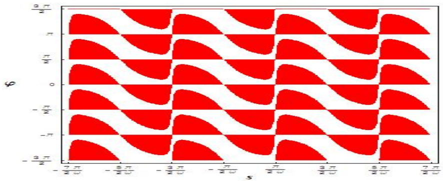

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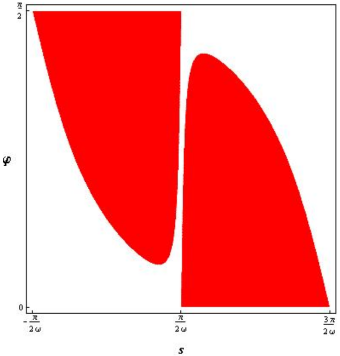

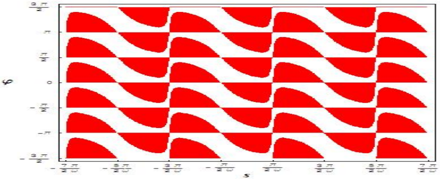

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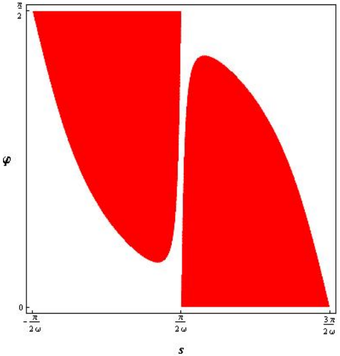

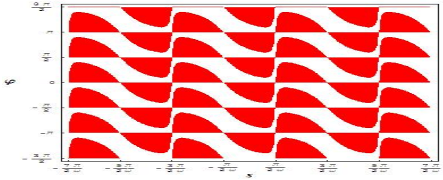

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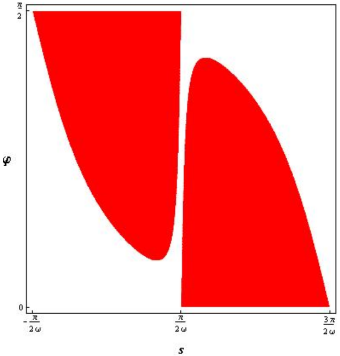

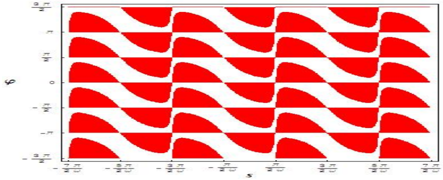

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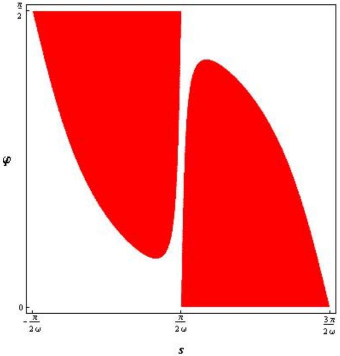

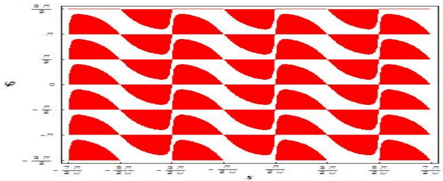

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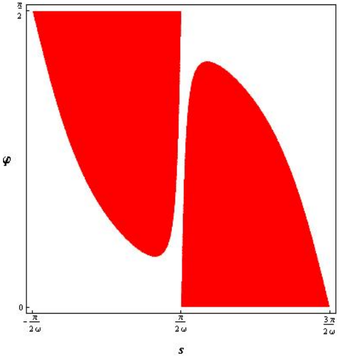

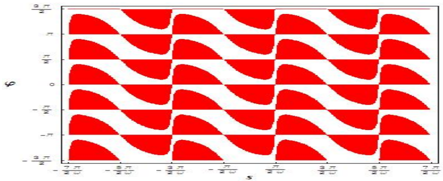

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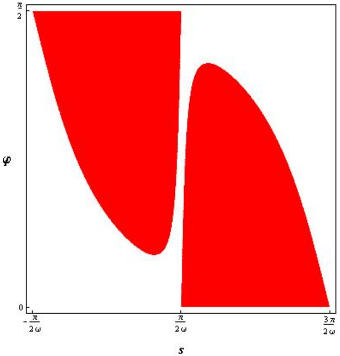

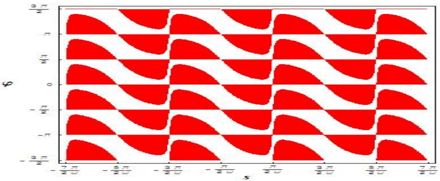

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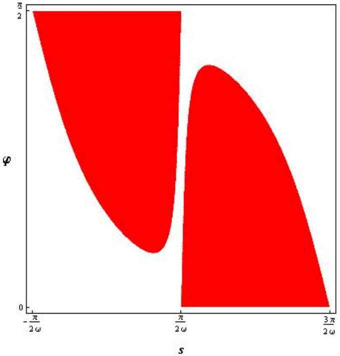

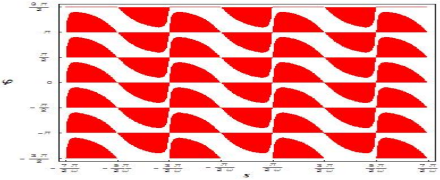

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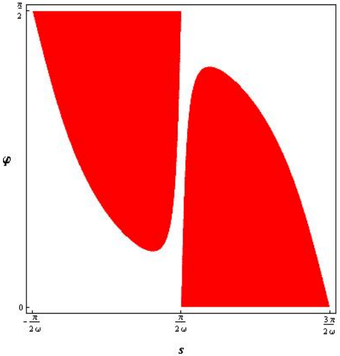
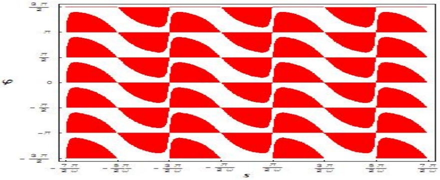
the *white* regions in the tiles on the (s,φ)–plane start to get wider and connected, so it is possible to trace a continuous function $\varphi(s)$ on this plane such that a_1 and a_2 are positive real functions.





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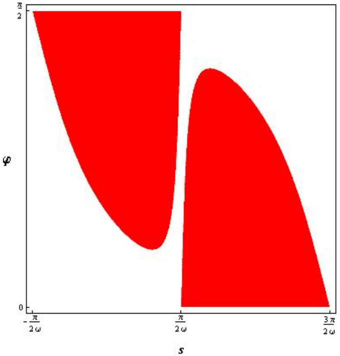

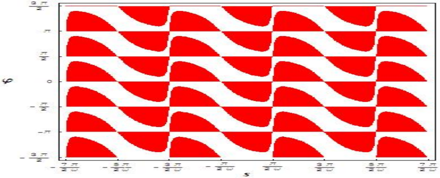

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If

$$\rho < -\rho_c$$

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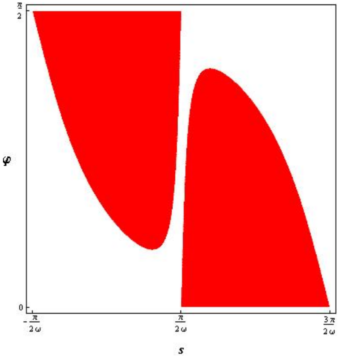
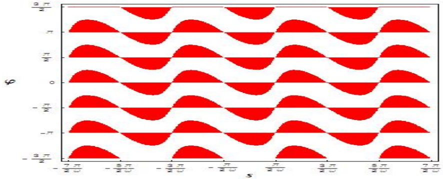
the *red* regions in the single minimal tile on the (s,φ)–plane are separated and hump-shaped, the height of the humps getting lower as the value of ρ decreases.





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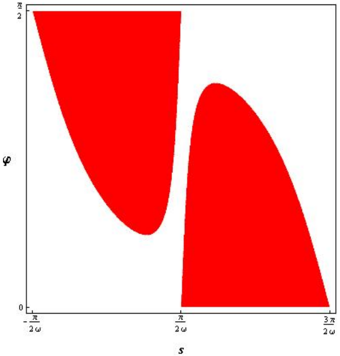

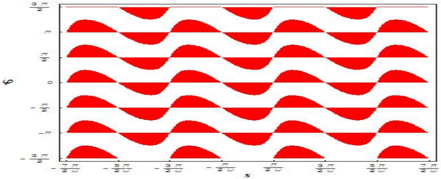

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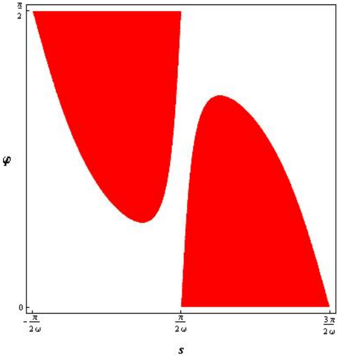

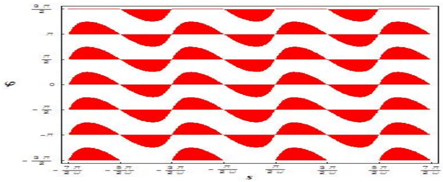

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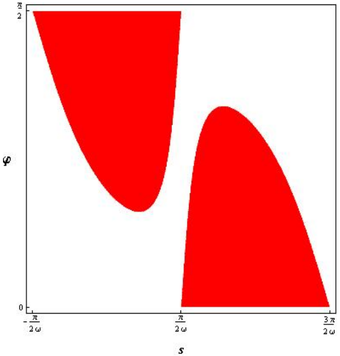

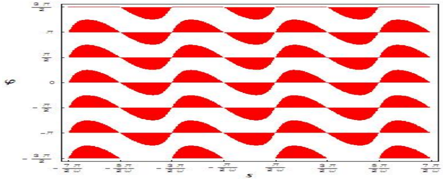

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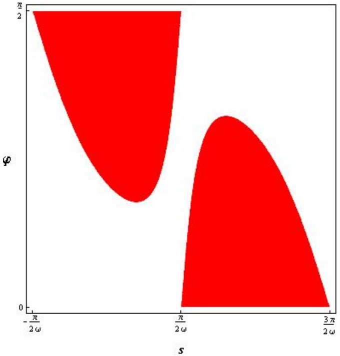

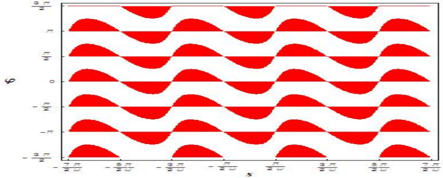

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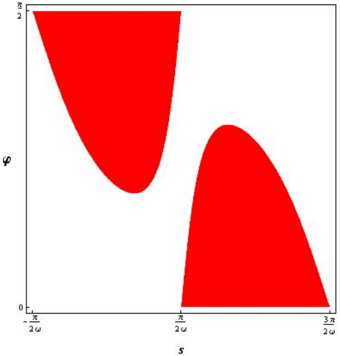

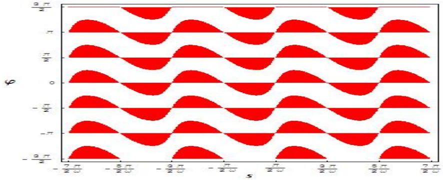

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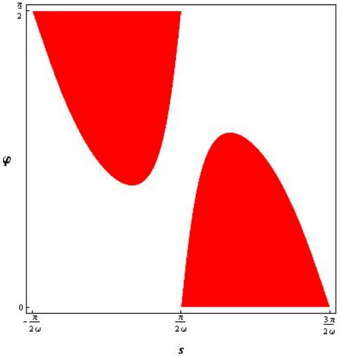

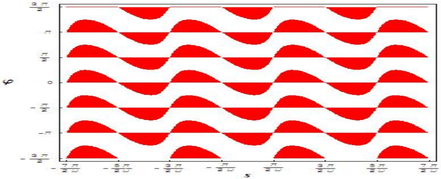

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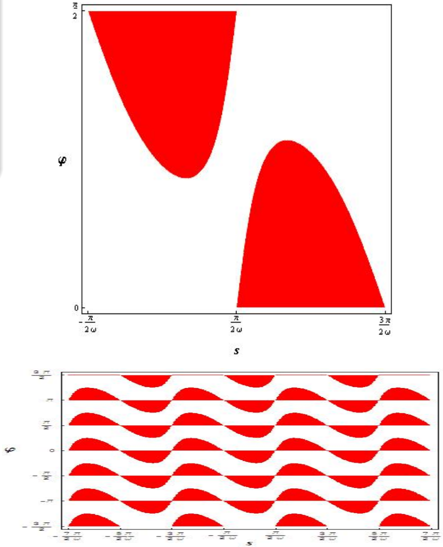

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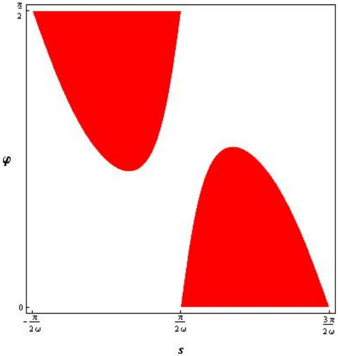

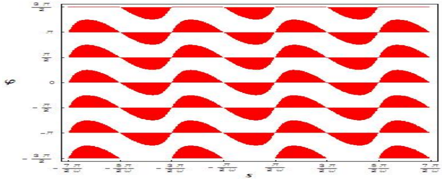

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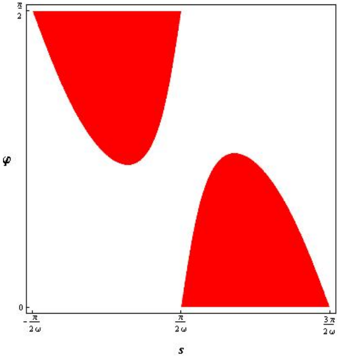

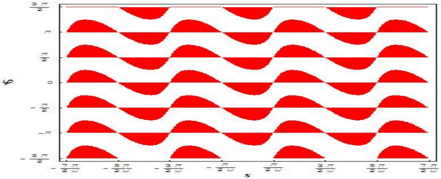

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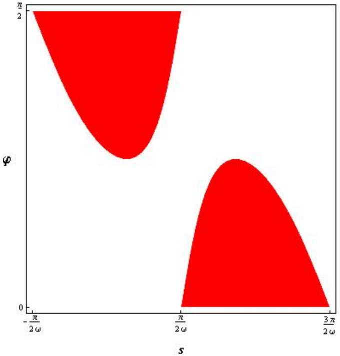

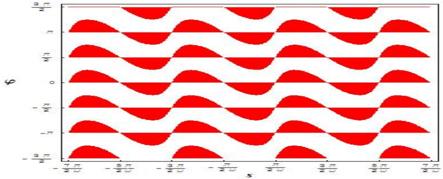

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If

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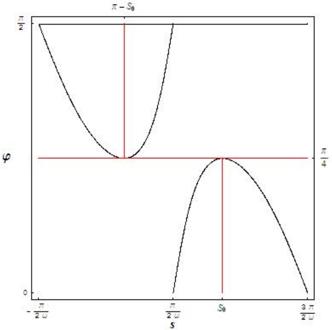
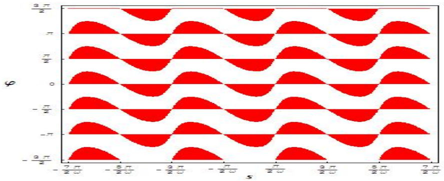
the two *red* humped regions in the single minimal tile on the (s,φ)–plane have the same height $\frac{\pi}{4}$.

The points where we get the maxima are

$$s_n = \frac{(2\,n+1)\,\pi}{\omega} + \frac{1}{\omega}\arcsin\left(\frac{1}{\rho\sqrt{1+\gamma^2}}\right)$$





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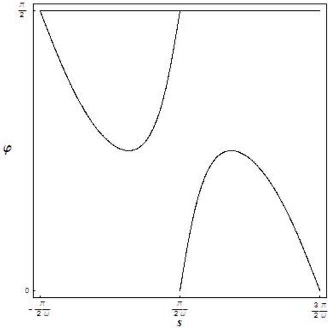

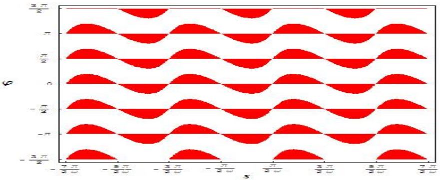

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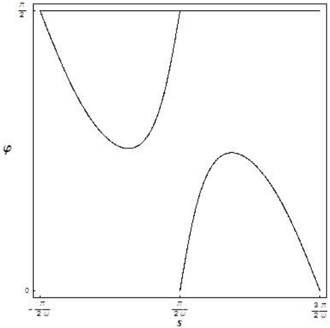

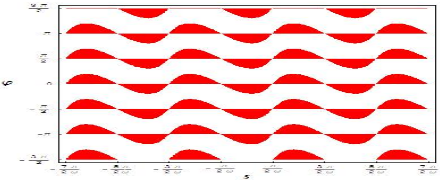

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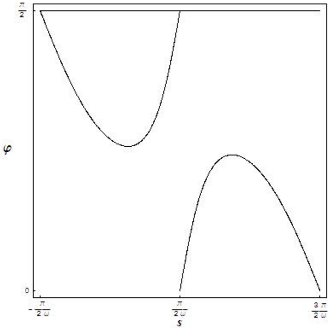

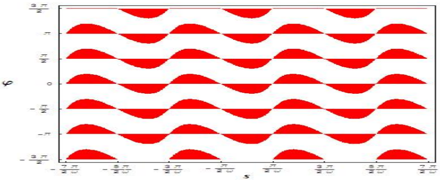

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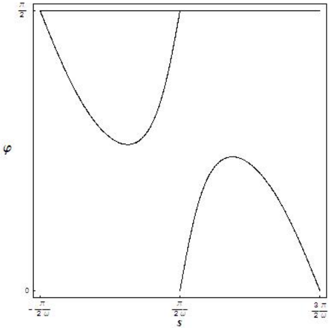

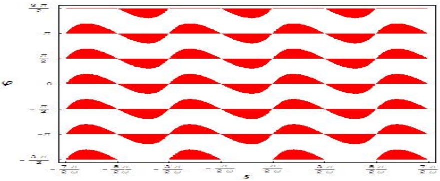

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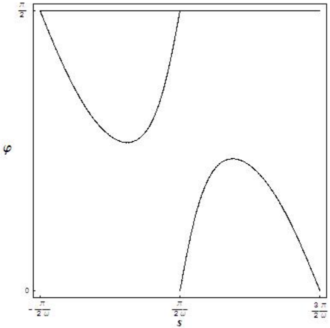

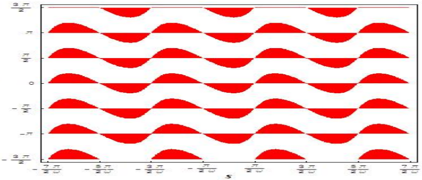

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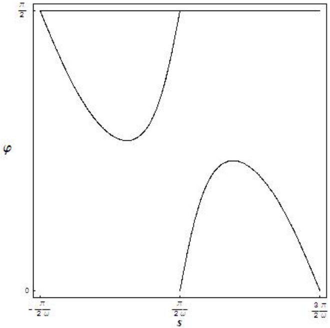

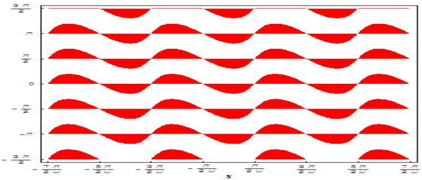

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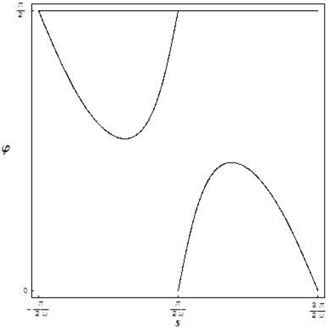
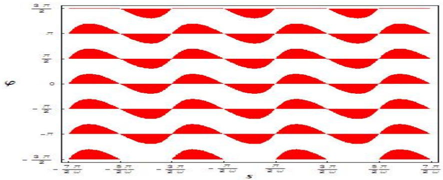
$$\rho = -\sqrt{1 + \frac{1}{1+\gamma^2}}$$

the two *red* humped regions in the single minimal tile on the (s,φ)–plane have the same height $\frac{\pi}{4}$.

The points where we get the maxima are

$$s_n = \frac{(2\,n+1)\,\pi}{\omega} + \frac{1}{\omega}\arcsin\left(\frac{1}{\rho\sqrt{1+\gamma^2}}\right)$$





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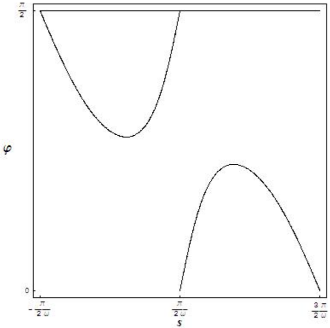

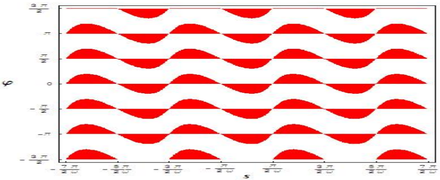

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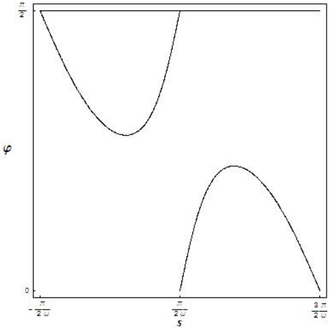

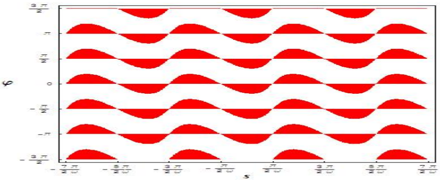

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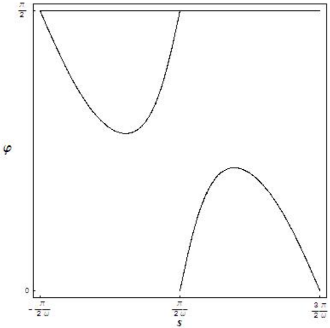

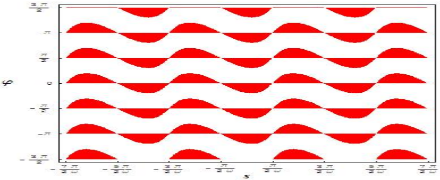

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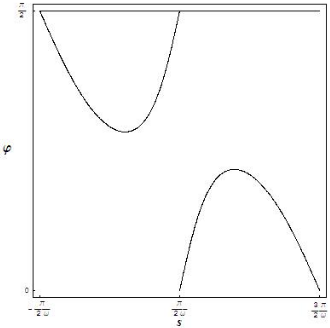

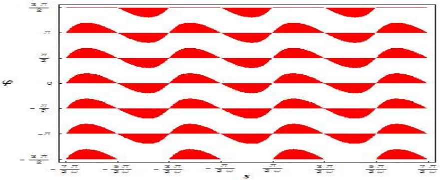

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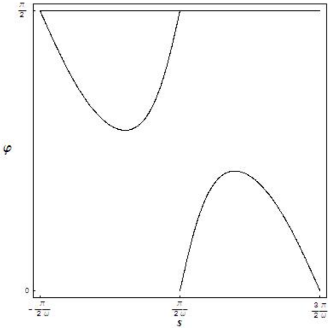

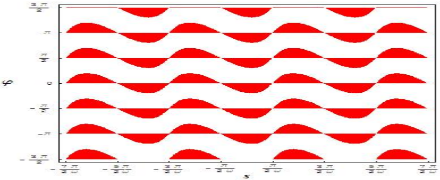

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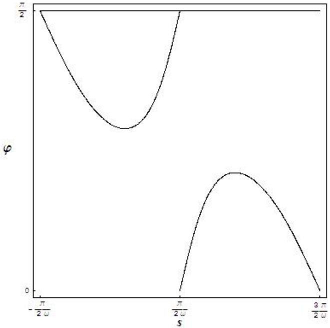

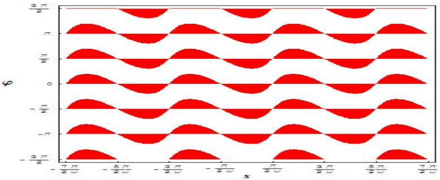

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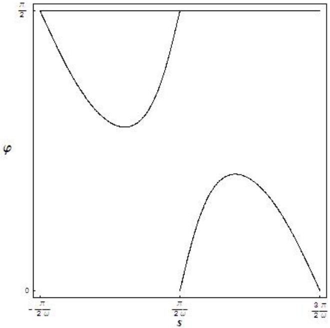

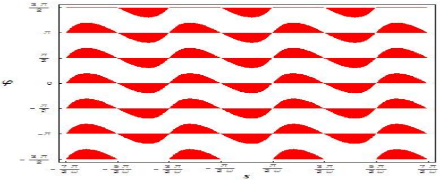

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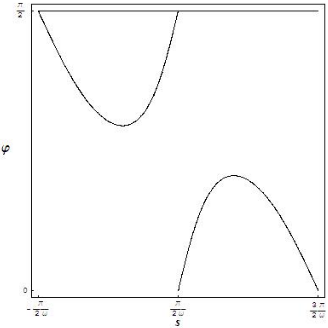

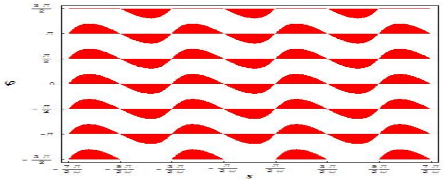

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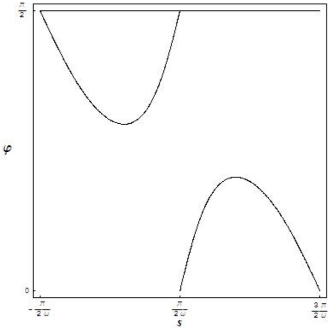
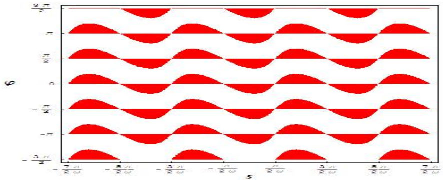

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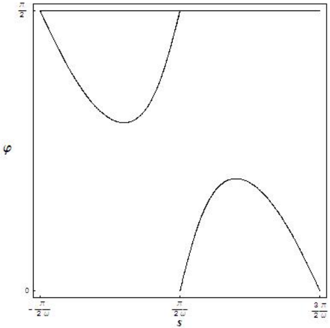

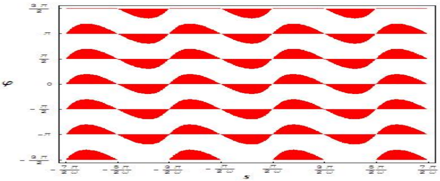

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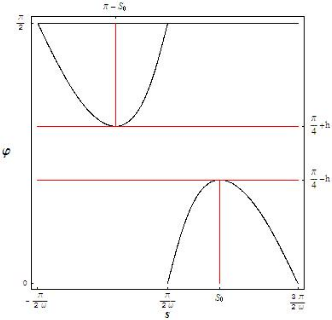
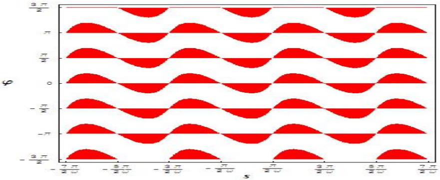
$$\rho < -\sqrt{1 + \frac{1}{1+\gamma^2}}$$

the two *red* humped regions in the single minimal tile on the (s,φ)–plane have height $\frac{\pi}{4} - h$ where

$$h = \frac{\pi}{4} + \text{arcot}\left(\rho\sqrt{1 - \frac{1}{(1+\gamma^2)\,\rho^2}}\right)$$





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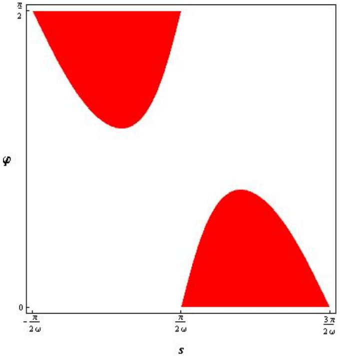

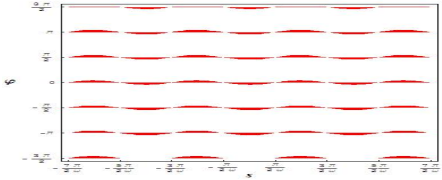

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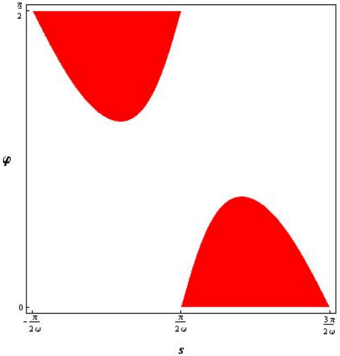

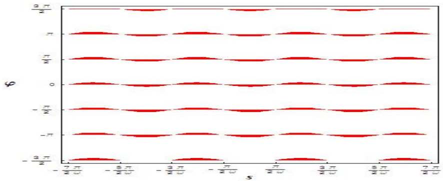

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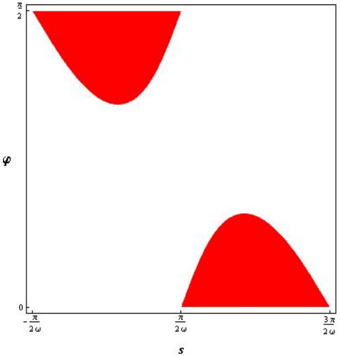

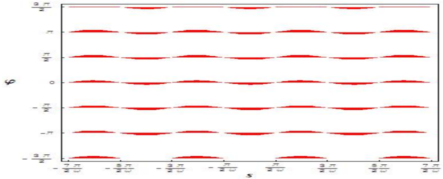

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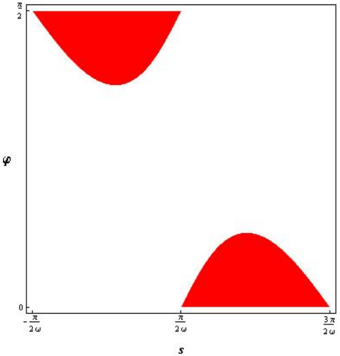

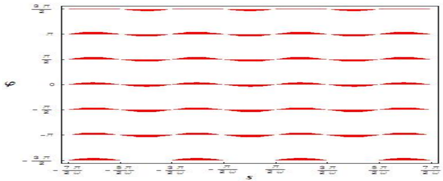

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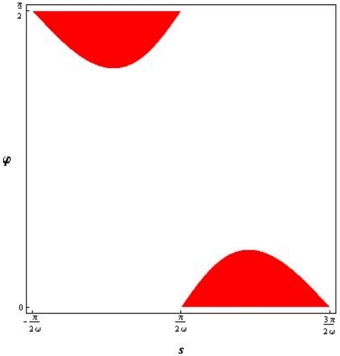

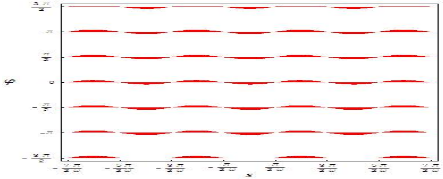

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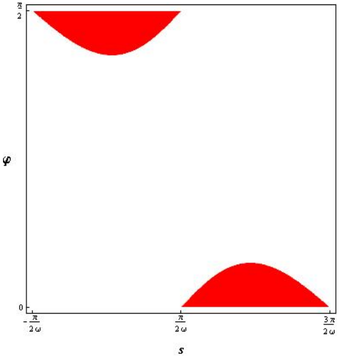

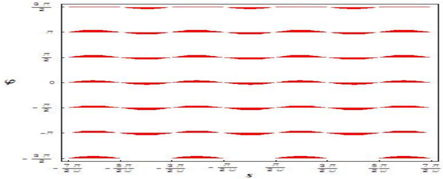

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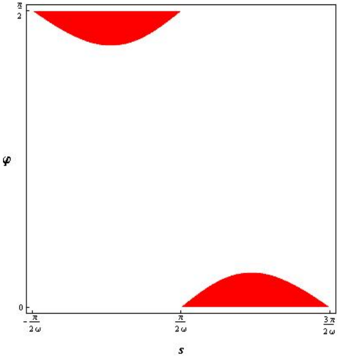

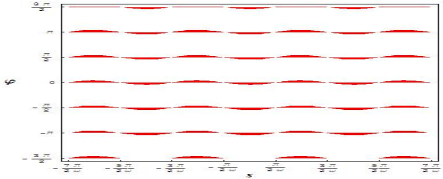

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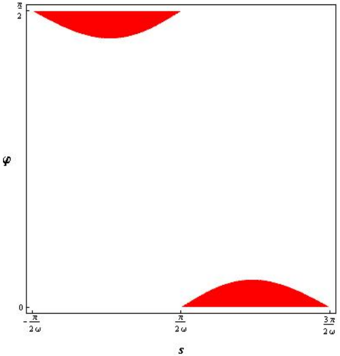

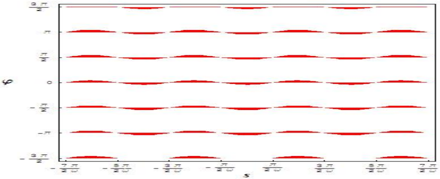

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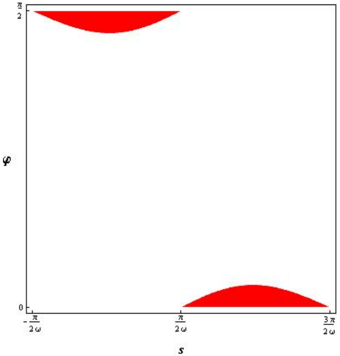

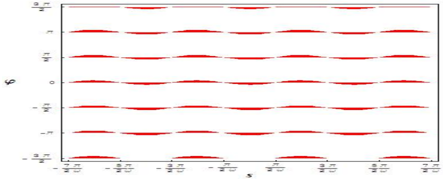

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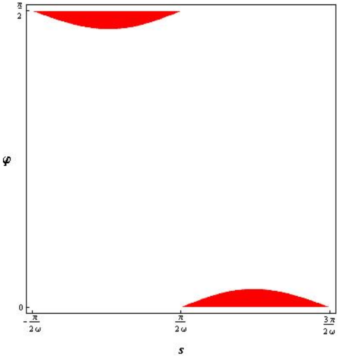

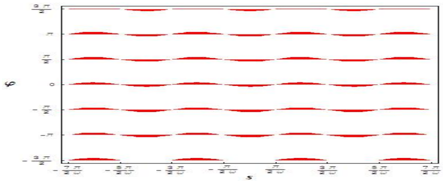

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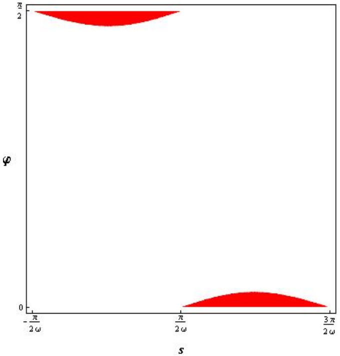

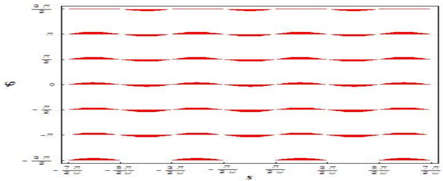

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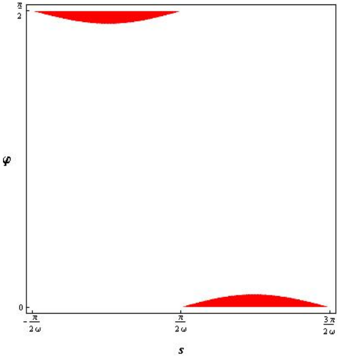

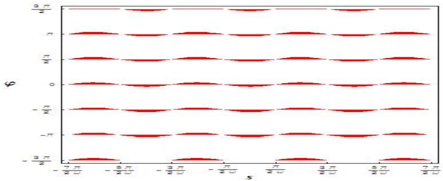

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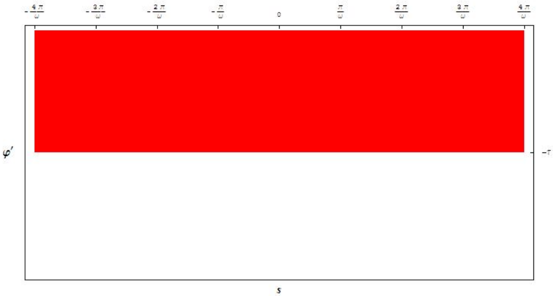

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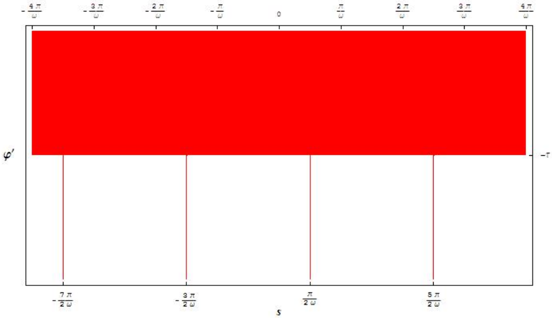

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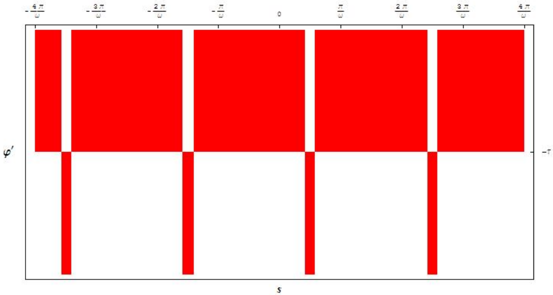

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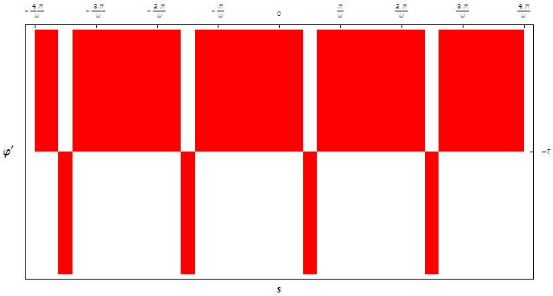

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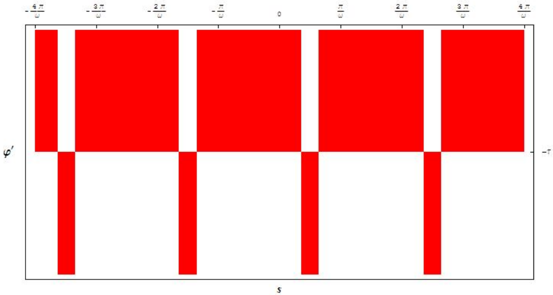

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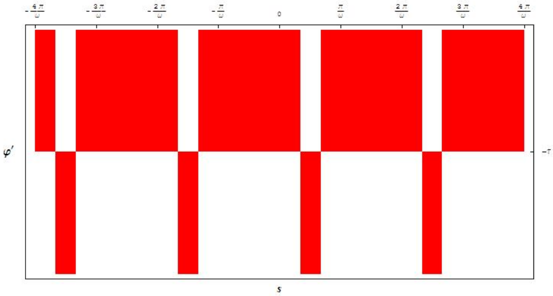

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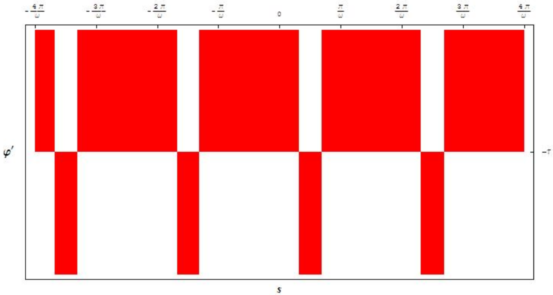

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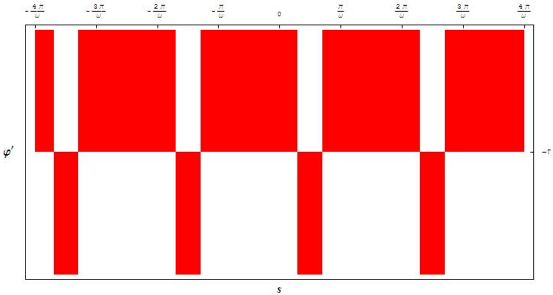

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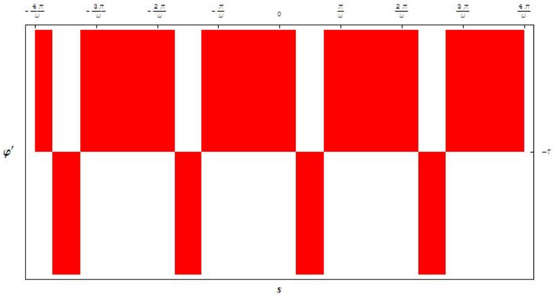

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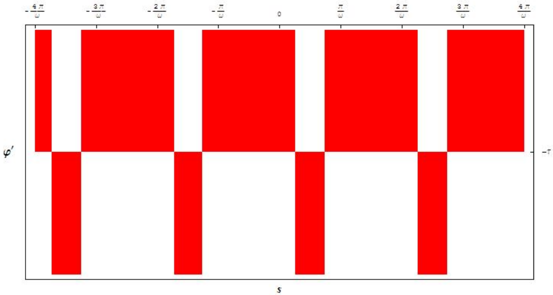

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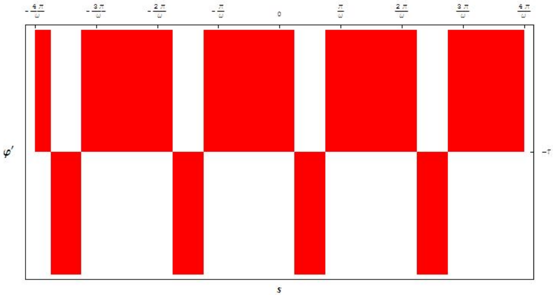

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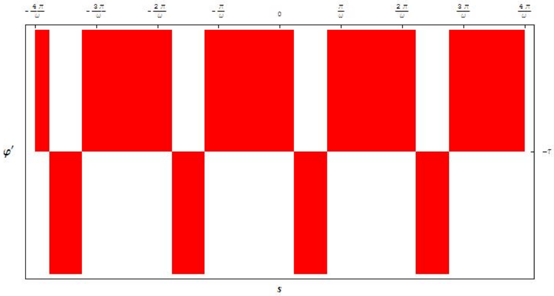

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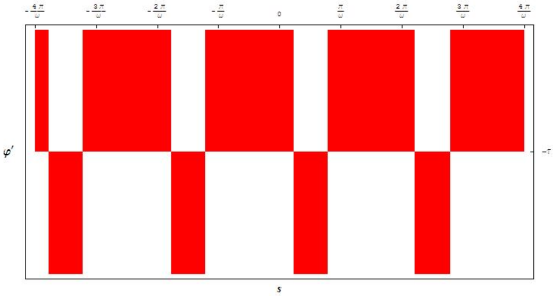

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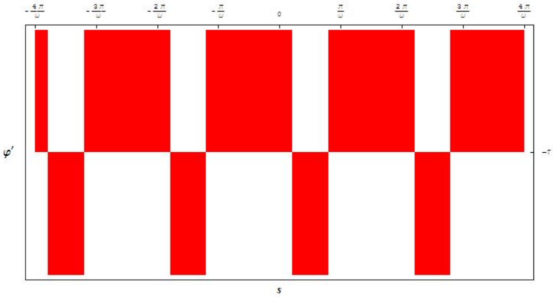

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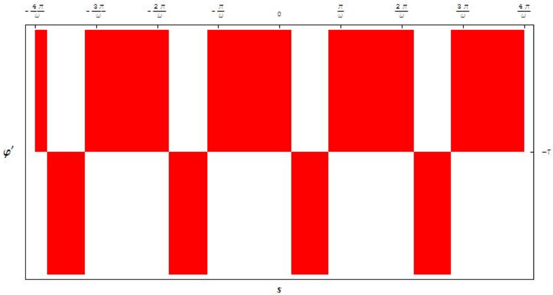

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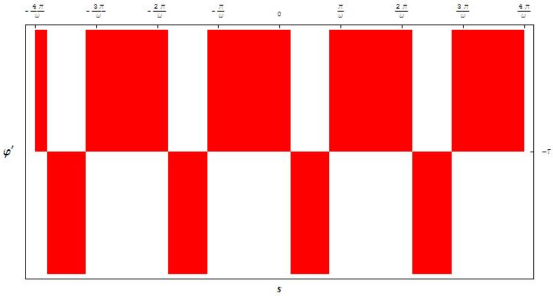

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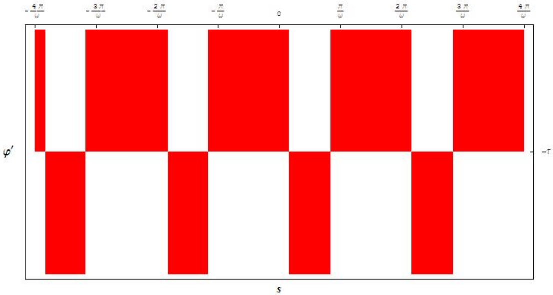

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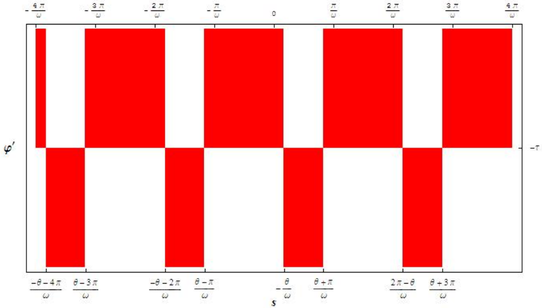

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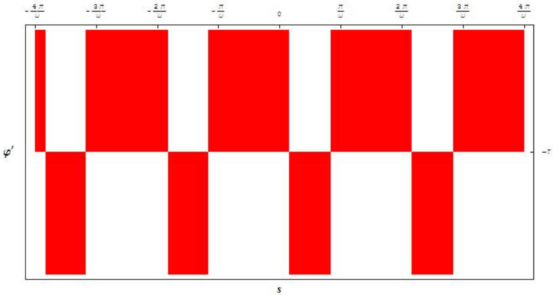

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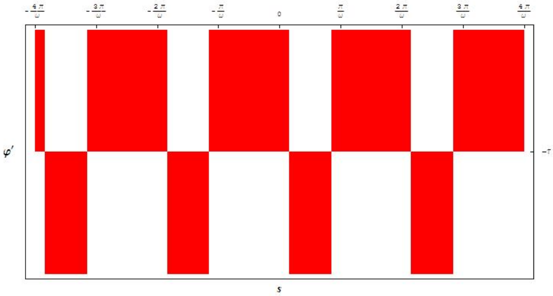

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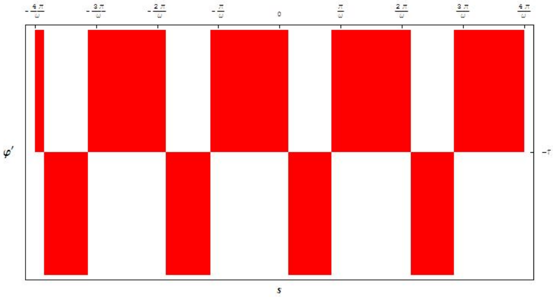

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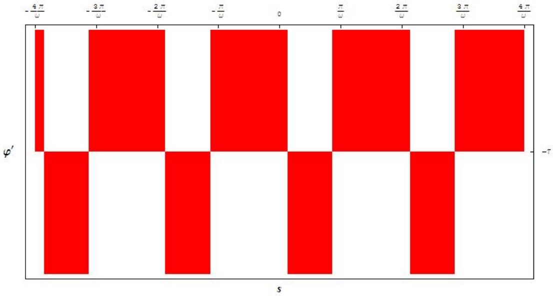

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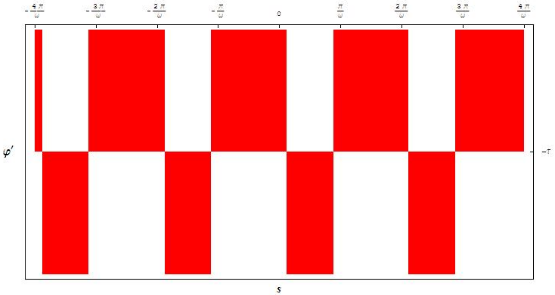

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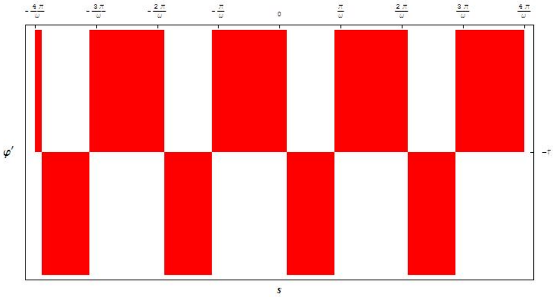

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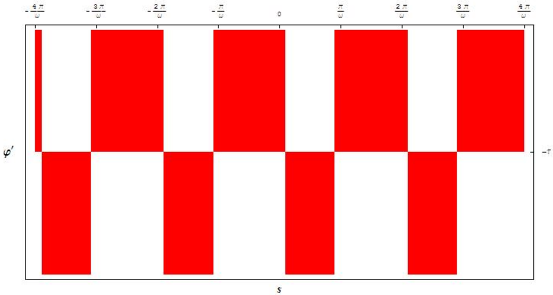

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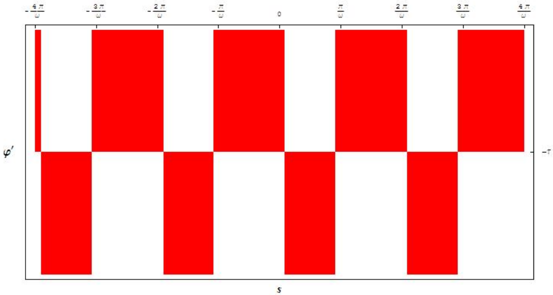

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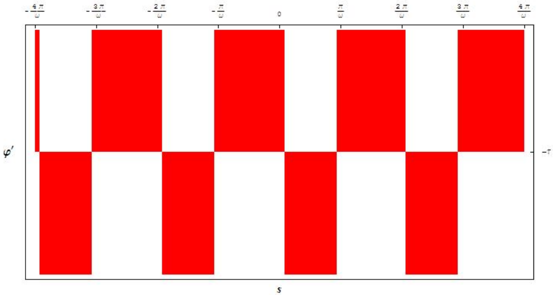

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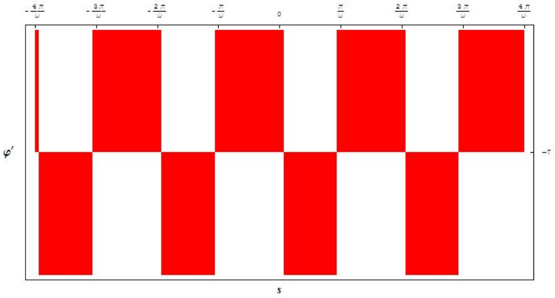

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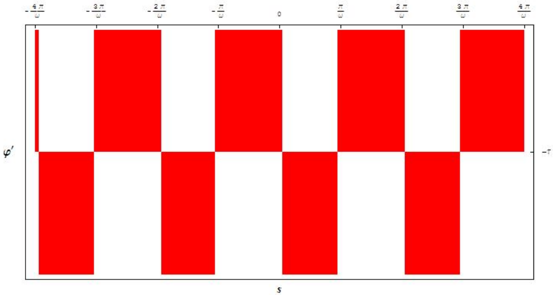

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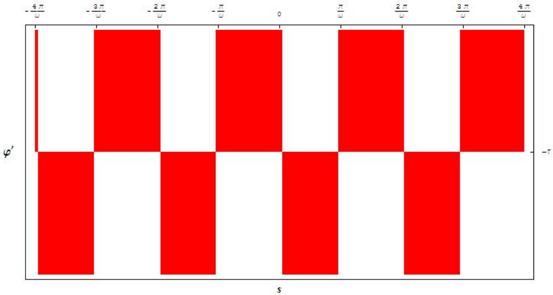

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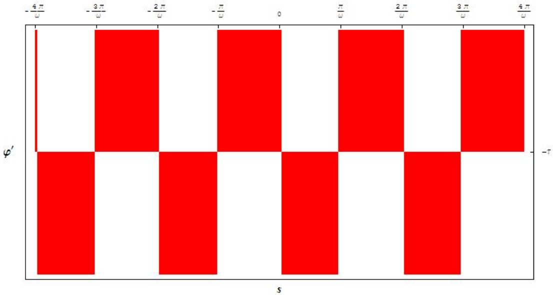

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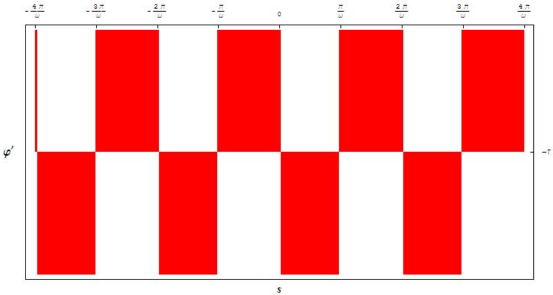

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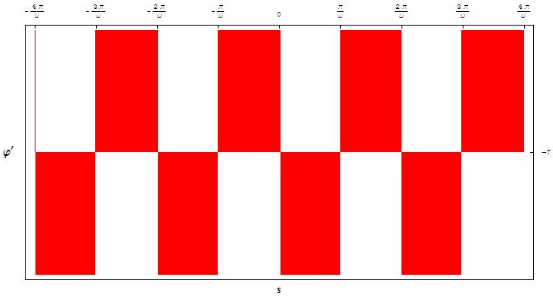

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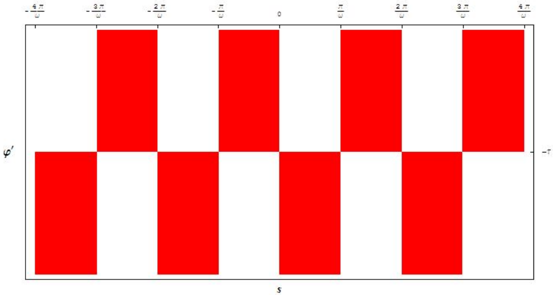

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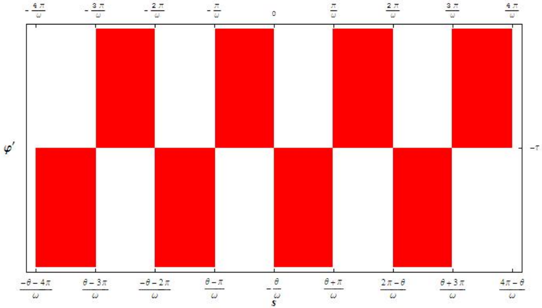

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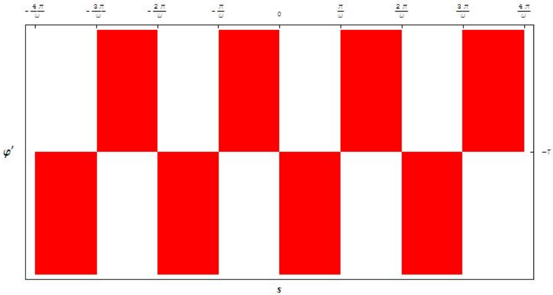

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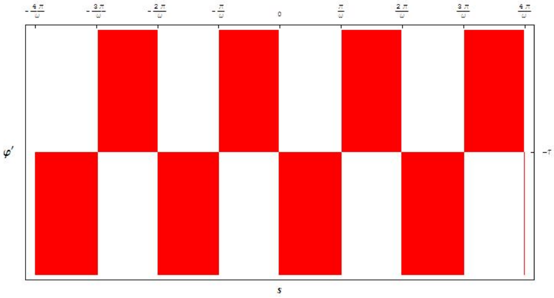

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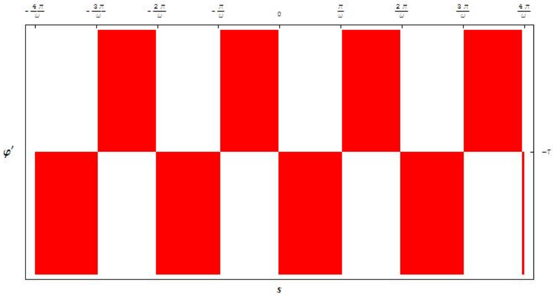

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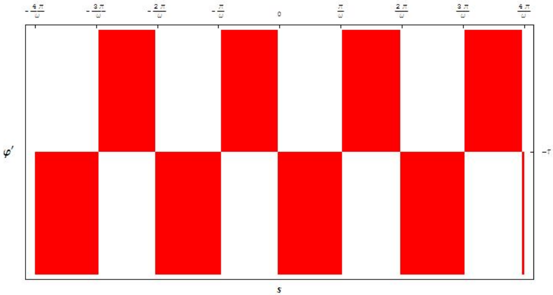

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Twisting stiffness

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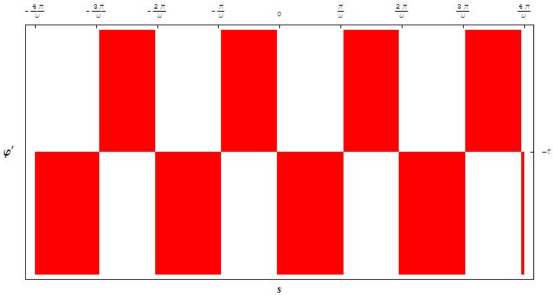

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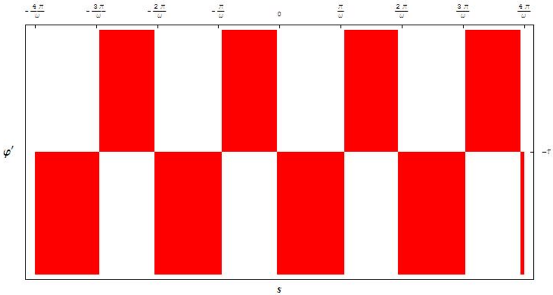

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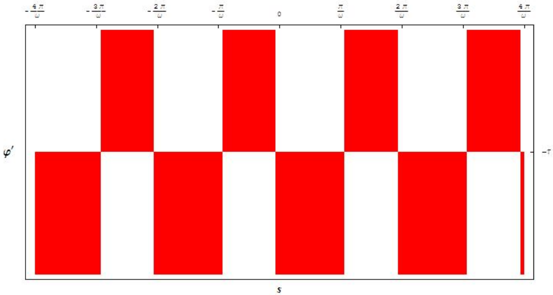

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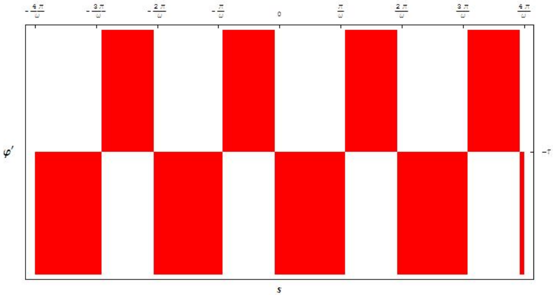

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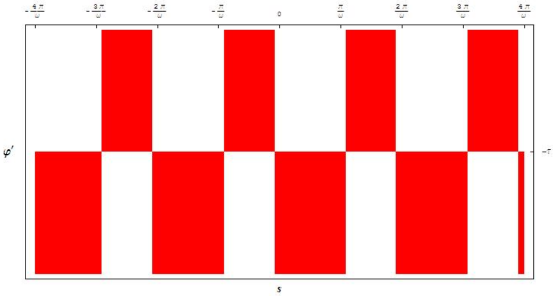

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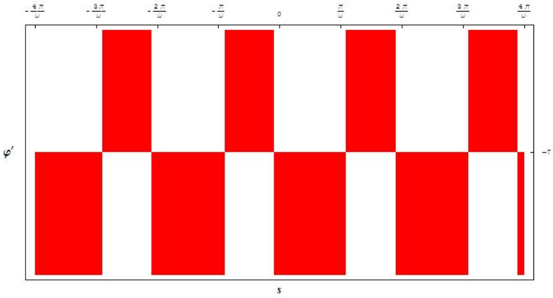

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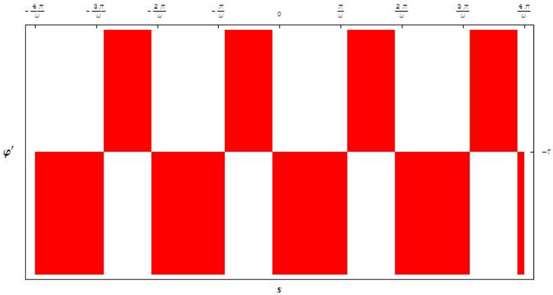

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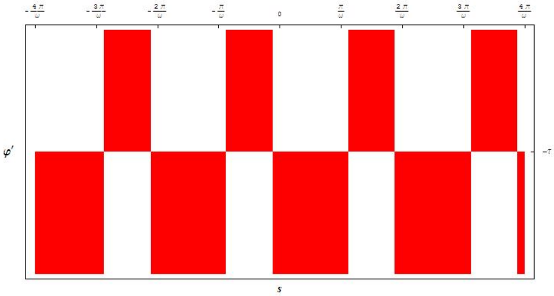

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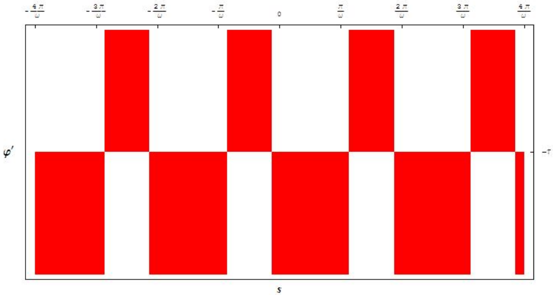

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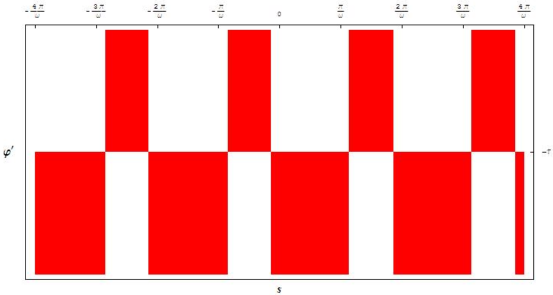

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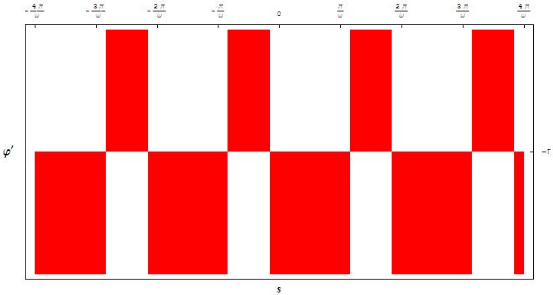

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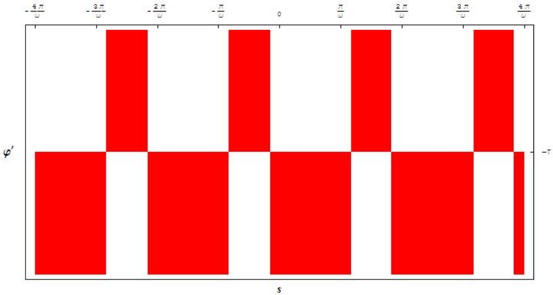

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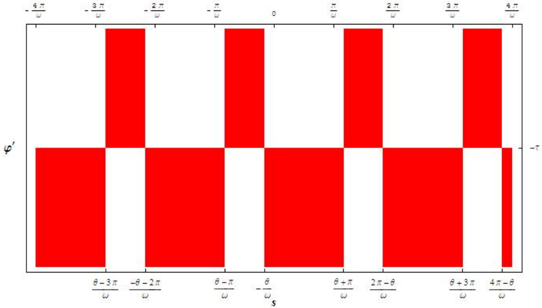

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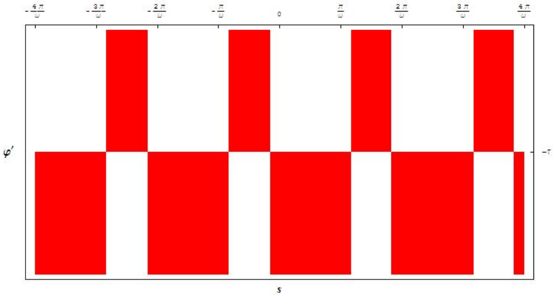

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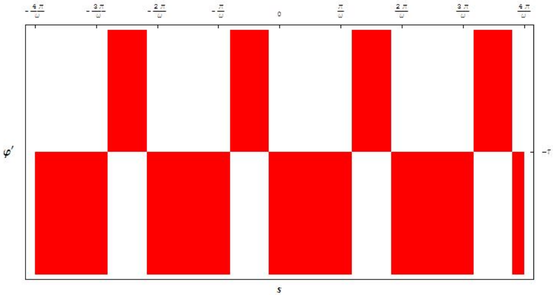

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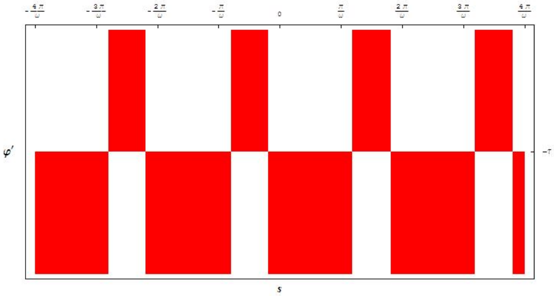

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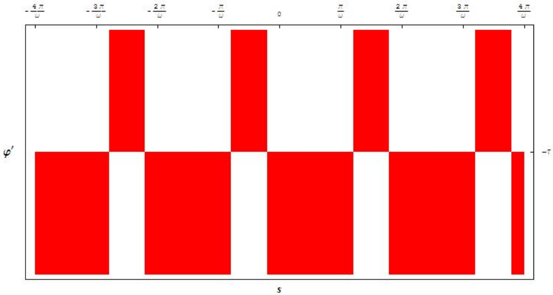

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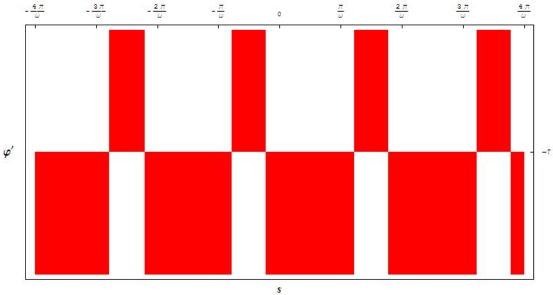

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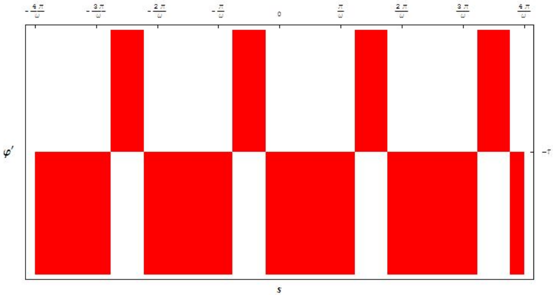

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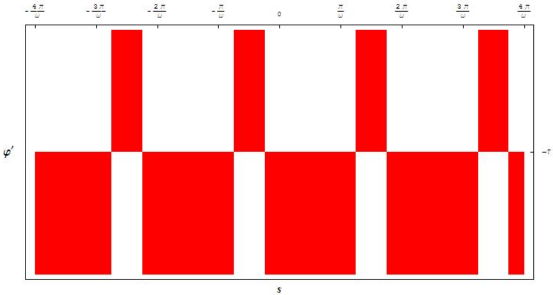

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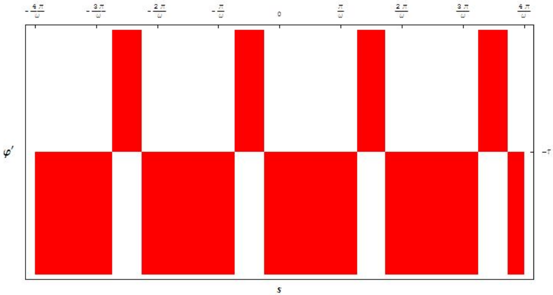

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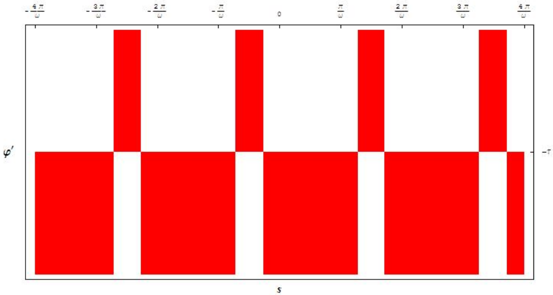

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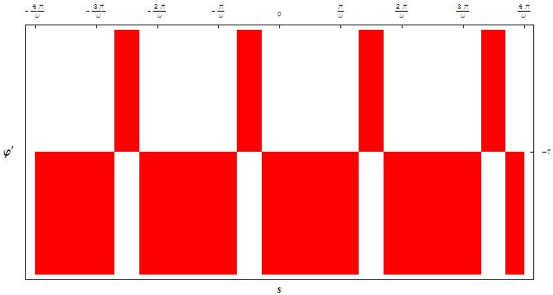

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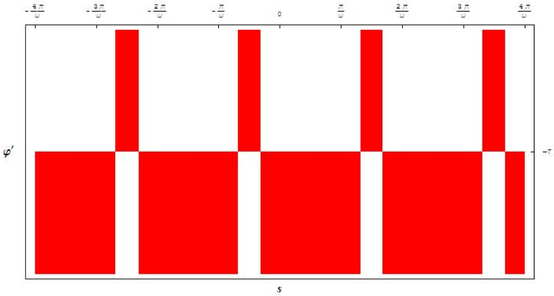

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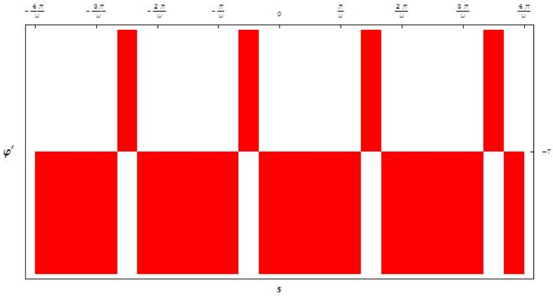

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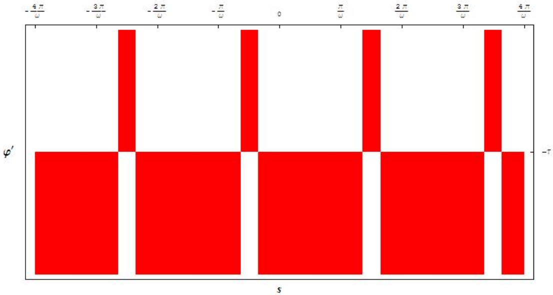

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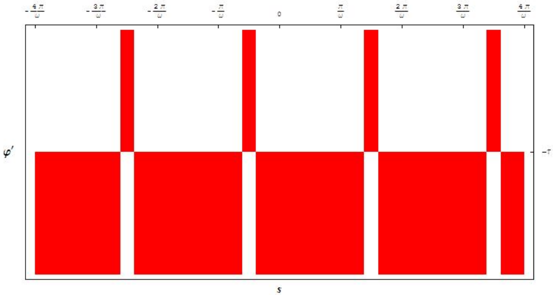

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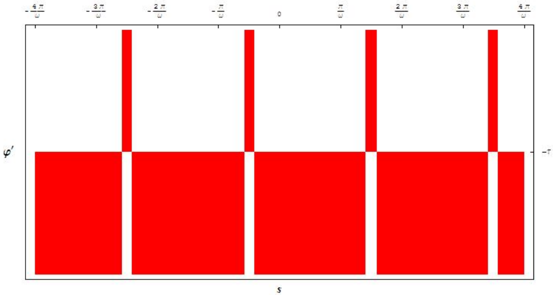
$$b(s) = \frac{\beta_1 \omega + 2\beta_2 \sin(s\omega)}{\omega(\tau + \varphi'(s))}$$

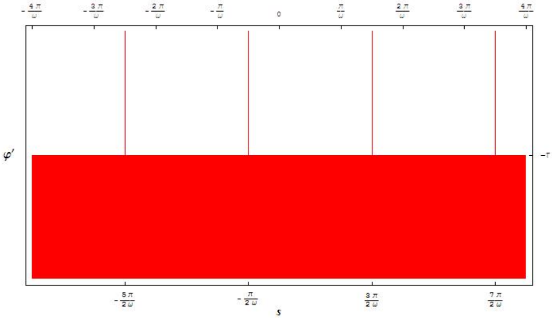
$$-\frac{\omega\beta_2}{2} < \beta_1 < \frac{\omega\beta_2}{2}$$



$$\theta = \sin^{-1}\left(\frac{\omega\beta_1}{2\beta_2}\right)$$

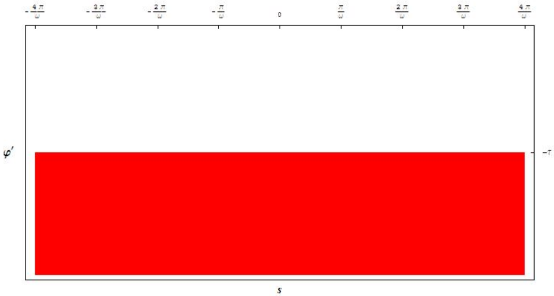
$$b(s) = \frac{\beta_1 \omega + 2\beta_2 \sin(s\omega)}{\omega(\tau + \varphi'(s))}$$

$$\beta_1 = \frac{\omega \beta_2}{2}$$



$$b(s) = \frac{\beta_1 \omega + 2\beta_2 \sin(s\omega)}{\omega(\tau + \varphi'(s))}$$

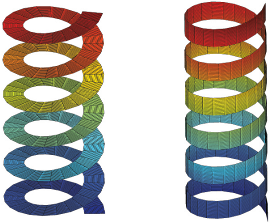
$$\beta_1 > \frac{\omega \beta_2}{2}$$



If we are interested on a constant φ ($\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = 0$) we have that Kirchhoff's equations admit solutions only in the following cases:

$$\varphi(s) \equiv n\frac{\pi}{2},\ n \in \mathbb{Z}, \text{ and } \kappa \neq 0$$

This case describes a ring-like rod if $\tau = 0$, whereas, if $\tau \neq 0$, it describes two different kinds of Frenet helices, whether n is even or odd.
We recall that a helix featuring zero intrinsic twist with $\tau \neq 0$ is usually referred to as a *Frenet helix*.
The case $\varphi = \pi$ is trivially mapped into the case $\varphi = 0$ by a π-rotation of the cross section; in the same way $\varphi = \frac{3\pi}{2}$ is mapped into $\varphi = \frac{\pi}{2}$.



If we are interested on a constant φ ($\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = 0$) we have that Kirchhoff's equations admit solutions only in the following cases:

$$\boldsymbol{\varphi(s) \equiv \varphi_0,\ \varphi_0 \neq n\frac{\pi}{2},\ n \in \mathbb{Z},\ \kappa \neq 0 \text{ and } \tau \neq 0}$$

In this case one of the following three sets of conditions must hold:

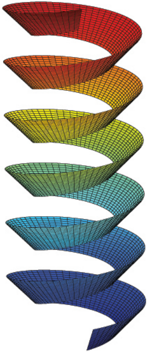
$$\begin{cases} \beta_2 > 0 \\ \rho \leq -\sqrt{1 + \frac{1}{1+\gamma^2}} \\ \left|\varphi - (2n+1)\frac{\pi}{4}\right| < h\,, \quad n \in \mathbb{Z} \\ \beta_1 > \frac{\omega\beta_2}{2} \end{cases}$$

$$\begin{cases} \beta_2 = 0 \\ \beta_1 > 0 \end{cases}$$

$$\begin{cases} \beta_2 < 0 \\ \rho \geq \sqrt{1 + \frac{1}{1+\gamma^2}} \\ \left|\varphi - (2n+1)\frac{\pi}{4}\right| < h\,, \quad n \in \mathbb{Z} \\ \beta_1 < \frac{\omega\beta_2}{2} \end{cases}$$

where

$$h = \frac{\pi}{4} + \operatorname{arcot}\left(\rho\sqrt{1 - \frac{1}{(1+\gamma^2)\,\rho^2}}\right).$$



If we are interested on a constant φ ($\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = 0$) we have that Kirchhoff's equations admit solutions only in the following cases:

$$\varphi(\mathbf{s}) \equiv \varphi_{\mathbf{0}},\ \varphi_{\mathbf{0}} \neq \mathbf{n}\frac{\pi}{\mathbf{2}},\ \mathbf{n} \in \mathbb{Z},\ \kappa \neq \mathbf{0} \text{ and } \tau = \mathbf{0}$$

This case describes a ring-like rod featuring a generic angle $\varphi \neq n\frac{\pi}{2}$. In the present context this type of configuration can be regarded as a degeneration of the circular helix.

If we are interested on a constant φ ($\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = 0$) we have that Kirchhoff's equations admit solutions only in the following cases:

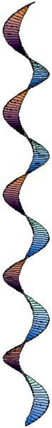
$$\varphi(\mathbf{s}) \equiv \varphi_{\mathbf{0}} \text{ and } \kappa = \tau = \mathbf{0}$$

This case describes a straight rod featuring a generic angle $\varphi(s) \equiv \varphi_0$. In the present context also this type of configuration can be regarded as a degeneration of the circular helix.

The results herein presented are part of a big work in progress. Among the future main tasks we need to

1. Rephrase all these results in terms of an energy minimization principle
2. Implement all the obtained calculations in order to have numerical simulations of different helices
3. Compare these results with the experimental data and molecular dynamics simulations
4. Study the time dependent problem

$3/10$–Helix ($\varphi = 0$)



3/10–Helix ($\varphi = \frac{\pi}{2}$)

1) P.M. Chaikin and T.C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, U.K., 2000)
2) J. Hafner, C. Wolverton, G. Ceder, MRS Bullettin **31**, 659 (2006)
3) S. Hammes-Shiffer, S.J. Benkovic, Ann. Rev. Biochem. **75**, 519 (2006)
4) D. Eisenberg, Proc. Natl. Acad. Sci. USA **100**, 11207 (2003)
5) R. Improta, K.N. Kudin, G.E. Scuseria, V. Barone, J. Am. Chem. Soc. **123**, 3311 (2001)
6) M. Doi, S.F. Edwards, *The Theory of Polymer Dynamics* (Clarendon, New York, 1991)
7) J.R. Banavar, T.X. Hoang, A. Maritan, F. Seno, A. Trovato, Phys. Rev. E **70**, 041905 (2004)
8) G. Kirchhoff, *Vorlesungen über Mathematische Physik: Mechanik*, 3rd edition (Teubner Verlag, Leipzig, 1883)
9) A. Clebsh, *Theorie der Elastizität Fester Körper* (Teubner Verlag, Leipzig, 1862)
10) S.S. Antman, *Nonlinear Problems of Elasticity* (Springer Verlag, New York, 1995)
11) A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover Publications, New York, 1926)

12) L.D. Landau and E.M. Lifshitz, *Theory of Elasticity* (Course of Theoretical Physics, Vol 7) (Pergamon Press Oxford, 1986)

13) S. Zhao, S. Zhang, Z. Yao, L. Zhang, Phys. Rev. E **74**, 032801(4) (2006)

14) S. Kehrbaum and J.H. Maddocks, Phil. Trans.: Math. Phys. Eng. Sci. **355**, 2117 (1997)

15) A. Goriely, M. Nizette and M. Tabor, J. Nonl. Sci. **11**, 3 (2001)

16) N. Chouaieb and J. Maddocks, J. Elasticity **77**, 221 (2004)

17) A. Goriely and M. Nizette, J. Math. Phys. **40**, 2830 (1999)

18) A. Goriely and M. Nizette, R. C. Dynamics **5**, 1 (2000)

29) A. Goriely and P. Shipman, Phys. Rev. E **61**, 2830 (2000)

20) D.J. Struik, *Lectures on Classical Differential Geometry* (Dover Publications, New York, 1988)

21) G. Odian, *Principles of Polymerization*, 4th edition (Wiley-Interscience, Hoboken, NJ, 2004)

22) S. Zhang, X. Zuo, M. Xia, S. Zhao, E. Zhang, Phys. Rev. E **70**, 051902(6) (2004)