Quasideterminant solutions of a noncommutative mKP equation

Craig M. Sooman (`cms@maths.gla.ac.uk`)

Department of Mathematics, University of Glasgow

Joint work with Jon Nimmo and Claire Gilson

Noncommutative (nc) integrable systems

- Remove assumption that dependent variables commute.
- Still have Lax representation.
- Construct nc hierarchy.
- Results are valid for range of cases: matrix versions, quaternion versions, Moyal star product.
- Integrability is preserved. Can often find exact solutions.
- Find solutions from Darboux transformations.
- Solutions expressed as quasideterminants which are the natural replacement for determinants entries in a matrix do not commute.

ncmKP hierarchy

– Gelfand and Dickii 1976, Oevel and Rogers 1993

- Define pseudo-differential operator

$$T = \partial_x + w + w_1\partial_x^{-1} + w_2\partial_x^{-2} + w_3\partial_x^{-3} + \cdots,$$

- w and $w_s (s = 1, 2, \ldots)$ don't necessarily commute and depend on x and $t_q (q = 1, 2, \ldots)$, and ∂_x^i denotes the nth partial derivative operator $\frac{\partial^i}{\partial x^i}$.
- As in standard, commutative Sato theory, we define the ncmKP hierarchy as

$$T_{t_q} = [P_{\geq 1}(T^q), T], \qquad q = 1, 2, \ldots$$

- $P_{\geq 1}\left(\sum_i w_i \partial_x^i\right) = \sum_{i\geq 1} w_i \partial_x^i$ denotes projections of powers of the operator T onto the differential part.

ncmKP equation

– Wang and Wadati 2004, Dimakis and Müller-Hoissen 2006

- We set $t_2 = y$ and $t_3 = -4t$. From the ncmKP hierarchy:

$$w_y = w_{xx} + 2w_{1x} + 2ww_x + 2[w, w_1],$$
$$w_{1y} = w_{1xx} + 2w_{2x} + 2w_1 w_x + 2ww_{1x} + 2[w, w_2],$$
$$w_t = w_{xxx} + 3w_{1xx} + 3w_{2x} + 6ww_{1x} + 3w_1 w_x + 3w_x w_1 + 3ww_{xx} + 3w_x^2 + 3w^2 w_x + 3[w^2, w_1] + 3[w, w_2].$$

- Eliminating w_2 to get:

$$-\frac{1}{2}w_t - 2w_{xxx} - 3w_{1xx} - 6ww_{1x} - 3w_{1y} - 6w_x w_1 - 6ww_{xx} - 6w_x^2 - 6w^2 w_x - 6[w^2, w_1] = 0.$$

- Simplify via $w_1 = -\frac{1}{2}(w_x + w^2 - W)$.

ncmKP equation

- This gives the ncmKP equation

$$
\begin{aligned}
0 &= -4w_t + w_{xxx} - 6ww_xw + 3W_y + 3[w_x, W]_+ \\
&\quad -3[w_{xx}, w] - 3[W, w^2], \qquad (1) \\
0 &= W_x - w_y + [w, W]. \qquad (2)
\end{aligned}
$$

- This could also be obtained from the Lax pair:

$$
\begin{aligned}
L &= \partial_x^2 + 2w\partial_x - \partial_y, \\
M &= 4\partial_x^3 + 12w\partial_x^2 + 6(w_x + w^2 + W)\partial_x + 4\partial_t.
\end{aligned}
$$

- (2) is satisfied identically by change of variables

$$
w = -f_x f^{-1} (\neq -(\log f)_x), \quad W = -f_y f^{-1} (\neq -(\log f)_y).
$$

- $f = f(x, y, t)$ is invertible. It is not assumed that f and its derivatives commute.

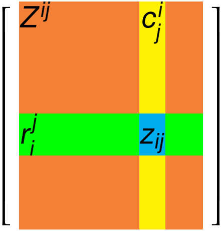
quasideterminants

– Gelfand and Retakh 1991

– Gelfand, Gelfand, Retakh and Wilson 2005

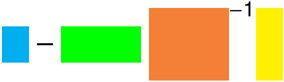
- An $n \times n$ matrix Z over a ring $\mathcal{R}$ (noncommutative, in general) has n^2 quasideterminants written as $|Z|_{ij}$.

$Z =$



- $1 \le i, j \le n$ and assume Z^{ij} is invertible
- Then $|Z|_{ij}$ exists and

$$|Z|_{ij} = z_{ij} - r^j_i (Z^{ij})^{-1} c^i_j =$$



quasideterminants

- For example, if $n = 2$, there are 4 quasideterminants.
- Let $Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$|Z|_{11} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}_{11} = a - bd^{-1}c$$

$$|Z|_{12} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}_{12} = b - ac^{-1}d$$

$$|Z|_{21} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}_{21} = c - db^{-1}a$$

$$|Z|_{22} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}_{22} = d - ca^{-1}b$$

quasideterminants

- It is more convenient to use another notation for quasideterminants by boxing the leading element z_{ij}.
- For example, if $n = 2$:

$$|Z|_{11} = \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = a - bd^{-1}c,$$

$$|Z|_{12} = \begin{vmatrix} a & \boxed{b} \\ c & d \end{vmatrix} = b - ac^{-1}d,$$

$$\ldots$$

- Note that the leading element can also be a matrix:

$$\begin{vmatrix} A_{n\times n} & B_{n\times l} \\ C_{m\times n} & \boxed{D_{m\times l}} \end{vmatrix} = D_{m\times l} - C_{m\times n}A^{-1}_{n\times n}B_{n\times l}.$$

Darboux transformations

- Let $\theta = \theta(x, y, t)$ be an eigenfunction for L and M, i.e. $L[\theta] = 0$ and $M[\theta] = 0$.
- It is not assumed that θ and its derivatives commute.
- Consider another pair of operators: $\widetilde{L} = G_\theta L G_\theta^{-1}$, $\widetilde{M} = G_\theta M G_\theta^{-1}$.
- G_θ is an invertible differential operator ($G_\theta[0] = 0$).
- $[\widetilde{L}, \widetilde{M}] = G_\theta [L, M] G_\theta^{-1} = 0$ if $[L, M] = 0$. So $\widetilde{L}, \widetilde{M}$ are compatible.
- For quasideterminant solutions, we take

$$G_\theta = ((\theta^{-1})_x)^{-1} \partial_x \theta^{-1} = 1 - \theta(\theta_x)^{-1} \partial_x.$$

Darboux transformations

- Let $\psi \neq \theta$ be another eigenfunction for L, M i.e. $L[\psi] = 0 = M[\psi]$.
- $\widetilde{L}[G_\theta[\psi]] = G_\theta L G_\theta^{-1}[G_\theta[\psi]] = G_\theta[L[\psi]] = G_\theta[0] = 0$. Similar for $\widetilde{M}$.
- So $\tilde{\psi} = G_\theta[\psi]$ is an eigenfunction for $\widetilde{L}, \widetilde{M}$.
- Quasideterminant structure is evident from

$$G_\theta[\psi] = \psi - \theta(\theta_x)^{-1}\psi_x = \begin{vmatrix} \theta & \boxed{\psi} \\ \theta_x & \psi_x \end{vmatrix}.$$

Darboux transformations

- Let θ_i, $i = 1, \dots, n$ be a particular set of eigenfunctions and introduce the notation $\Theta = (\theta_1, \theta_2, \dots, \theta_n)$.
- To iterate the Darboux transformation, let $\theta_{[1]} = \theta_1$ and $\psi_{[1]} = \psi$ be a general eigenfunction of $L_{[1]} = L$.
- Then $\psi_{[2]} := G_{\theta_{[1]}}[\psi_{[1]}]$ and $\theta_{[2]} = \psi_{[2]}|_{\psi \to \theta_2}$ are eigenfunctions for $L_{[2]} = G_{\theta_{[1]}} L_{[1]} G_{\theta_{[1]}}^{-1}$.
- In general, for $n \geq 1$ we define the nth Darboux transformation of ψ by

$$\psi_{[n+1]} = \psi_{[n]} - \theta_{[n]} (\theta_{[n]x})^{-1} \psi_{[n]x},$$

where $\theta_{[k]} = \psi_{[k]}|_{\psi \to \theta_k}$.

Darboux transformations

$$\psi_{[2]} = \psi - \theta_1(\theta_{1x})^{-1}\psi_x = \begin{vmatrix} \theta_1 & \boxed{\psi} \\ \theta_1^{(1)} & \psi^{(1)} \end{vmatrix},$$

$$\psi_{[3]} = G_{\theta_{[2]}}[\psi_{[2]}] = \cdots = \begin{vmatrix} \theta_1 & \theta_2 & \boxed{\psi} \\ \theta_1^{(1)} & \theta_2^{(1)} & \psi^{(1)} \\ \theta_1^{(2)} & \theta_2^{(2)} & \psi^{(2)} \end{vmatrix},$$

- $^{(k)}$ denotes kth x-derivative. After n iterations we have

$$\psi_{[n+1]} = \begin{vmatrix} \Theta & \boxed{\psi} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \psi^{(n-1)} \\ \Theta^{(n)} & \psi^{(n)} \end{vmatrix}.$$ Call this a quasiwronskian

- Proof is by induction.

Darboux transformations

- The transformed operator

$$\widetilde{L} = \underbrace{((\theta^{-1})_x)^{-1}\partial_x\theta^{-1}}_{G_\theta} L \underbrace{\theta\partial_x^{-1}(\theta^{-1})_x}_{G_\theta^{-1}}$$

  preserves the structure of w.

- The coefficient

$$\tilde{w} = -\tilde{f}_x\tilde{f}^{-1} = -(-\theta(\theta_x)^{-1}f)_x(-\theta(\theta_x)^{-1}f)^{-1}$$

  satisfies the ncmKP equation.

- $\tilde{f}$ can be expressed as the quasideterminant

$$\tilde{f} = -\theta(\theta_x)^{-1}f = \begin{vmatrix} \theta & \boxed{0} \\ \theta_x & 1 \end{vmatrix} f.$$

Darboux transformations

- Let $f = f_{[1]}$. For the nth Darboux transformation of f we have

$$f_{[n+1]} = \begin{vmatrix} \Theta & \boxed{0} \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix} f.$$

- An analogous transformation can be made on f^{-1}.
- Let $g = f^{-1}$. This gives $w = -(g^{-1})_x g = g^{-1} g_x$.
- From $\widetilde{L}$ we get

$$\tilde{w} = \tilde{g}^{-1}\tilde{g}_x = (g\theta_x\theta^{-1})^{-1}(g\theta_x\theta^{-1})_x,$$

Darboux transformations

- $\tilde{g}$ can be expressed as the quasideterminant

$$\tilde{g} = g\theta_x\theta^{-1} = -g\begin{vmatrix} \theta & 1 \\ \theta_x & \boxed{0} \end{vmatrix}.$$

- Let $g = g_{[1]}$. For the nth Darboux transformation of g we have

$$g_{[n+1]} = -g\begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}.$$

binary Darboux transformations

– Matveev and Salle 1991, Oevel and Schief 1993

- Consider the formal adjoint operator $G_\theta^\dagger$.
- Typically working with matrices. Formal adjoint is the linear operation $(a\partial_x^i)^\dagger = (-1)\partial_x^i a^\dagger$.
- For operators, $(G_1 G_2)^\dagger = G_2^\dagger G_1^\dagger$.
- $\widetilde{L} = G_\theta L G_\theta^{-1}$ so $\widetilde{L}^\dagger = {G_\theta^\dagger}^{-1} L^\dagger G_\theta^\dagger$, i.e. $L^\dagger = G_\theta^\dagger \widetilde{L}^\dagger {G_\theta^\dagger}^{-1}$.
- So G_θ induces and adjoint Darboux transformation from $\widetilde{L}^\dagger, \widetilde{M}^\dagger$ to $L^\dagger, M^\dagger$.

$$L, M \xrightarrow{G_\theta} \widetilde{L}, \widetilde{M}$$

$$L^\dagger, M^\dagger \xleftarrow{G_\theta^\dagger} \widetilde{L}^\dagger, \widetilde{M}^\dagger$$

binary Darboux transformations

Construction

$$\begin{array}{ccccc} L, M & \xrightarrow{G_\theta} & \widetilde{L}, \widetilde{M} & \xleftarrow{G_{\hat\theta}} & \widehat{L}, \widehat{M} \\ \theta & & & & \hat\theta \end{array}$$

$$\begin{array}{ccccc} L^\dagger, M^\dagger & \xleftarrow{G_\theta^\dagger} & \widetilde{L}^\dagger, \widetilde{M}^\dagger & \xrightarrow{G_{\hat\theta}^\dagger} & \widehat{L}^\dagger, \widehat{M}^\dagger \\ \phi & & i(\theta), i(\hat\theta) & & \end{array}$$

Binary Darboux transformation

$$L, M \xrightarrow{G_{\theta,\phi} = G_{\hat\theta}^{-1} G_\theta} \widehat{L}, \widehat{M}$$

- Must determine $\hat\theta$.
- From $\ker G_\theta^\dagger$ we get $i(\theta)$.

binary Darboux transformations

- Introduce $\Omega = \partial_x^{-1}(\phi^\dagger \theta_x)$.
- $G_\theta^\dagger[i(\theta)] = 0$ is satisfied by $i(\theta) = ({\theta^\dagger}^{-1})_x$.
- Then $i(\hat{\theta}) = ({\hat{\theta}^\dagger}^{-1})_x = {G_\theta^\dagger}^{-1}[\phi_x]$. So $\hat{\theta} = \theta\Omega^{-1}$.
- Now $G_{\theta,\phi_x} = G_{\hat{\theta}}^{-1} G_\theta = 1 - \theta\Omega^{-1}\partial_x^{-1}\phi^\dagger\partial_x$.
- Then we define the binary Darboux transformation by

$$\psi_{[n+1]} = \psi_{[n]} - \theta_{[n]}\Omega(\rho_{[n]}, \theta_{[n]})^{-1}\Omega(\rho_{[n]}, \psi_{[n]}),$$
$$\phi_{[n+1]} = \phi_{[n]} - \rho_{[n]}\Omega(\rho_{[n]}, \theta_{[n]})^{\dagger^{-1}}\Omega(\phi_{[n]}, \theta_{[n]})^\dagger$$

- $\theta_{[n]} = \psi_{[n]}|_{\psi\to\theta_n}$, $\rho_{[n]} = \phi_{[n]}|_{\phi\to\rho_n}$.

binary Darboux transformations

- Let $\mathrm{P} = (\rho_1, \ldots, \rho_n)$. For $n \geq 1$

$$\psi_{[n+1]} = \begin{vmatrix} \Omega(\mathrm{P}, \Theta) & \Omega(\mathrm{P}, \psi) \\ \Theta & \boxed{\psi} \end{vmatrix}$$

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\mathrm{P}, \Theta)^\dagger & \Omega(\phi, \Theta)^\dagger \\ \mathrm{P} & \boxed{\phi} \end{vmatrix}.$$

- Proof is by induction. We call these quasigrammians.
- The effect of $\widehat{L} = G_{\theta,\phi_x} L G_{\theta,\phi_x}^{-1}$ is that

$$\hat{f} = \begin{vmatrix} \Omega & \rho^\dagger \\ \theta & \boxed{1} \end{vmatrix} f.$$

- After n Darboux transformations

$$f_{[n+1]} = \begin{vmatrix} \Omega(\mathrm{P}, \Theta) & \mathrm{P}^T \\ \Theta & \boxed{\mathrm{I}} \end{vmatrix} f.$$

matrix mKP equation

– Goncharenko and Veselov 2001

- For the trivial vacuum $f = 1$ (giving $w = W = 0$), we get

$$F = \begin{vmatrix} \Omega(\mathrm{P}, \Theta) & \mathrm{P}^T \\ \Theta & \boxed{\mathrm{I}} \end{vmatrix}.$$

- The equations $L[\theta] = M[\theta] = 0$ and $L^\dagger[\rho] = M^\dagger[\rho] = 0$ have nontrivial solutions

$$\theta = A\, e^{k(x+ky-4k^2t)},$$
$$\rho = B\, e^{-q(x+qy-4q^2t)}.$$

- A and B are $d \times m$ matrices. $k, q \in \mathbb{R}$.

matrix mKP equation

- $n = d = 2$ gives a two soliton solution. Change of polarization and phase.
- Here $A_j = r_j P_j$, where r_j is a scalar, and P_j is a projection matrix, i.e. $P_j^2 = P_j$. Take $B = \mathtt{I}$.
- We assume that the P_j are the rank-1 projection matrices

$$P_j = \frac{u_j \otimes v_j}{(u_j, v_j)} = \frac{u_j v_j^T}{u_j^T v_j} \quad \text{where} \quad (u_j, v_j) \neq 0.$$

- u_j, v_j are d-vectors. Further,

$$\Omega = A_j \frac{k_i e^{K_i + Q_j}}{k_i - q_j} + \delta_{i,j} \mathtt{I}.$$

- $K_i = k_i(x + k_i y - 4k_i^2 t)$, $Q_i = -q_i(x + q_i y - 4q_i^2 t)$.

matrix mKP equation

- Expanding F we get

$$\begin{aligned} F &= \mathtt{I} - \begin{bmatrix} A_1 e^{K_1} & A_2 e^{K_2} \end{bmatrix} \begin{bmatrix} A_j \frac{k_i e^{K_i+Q_j}}{k_i - q_j} + \delta_{i,j}\mathtt{I} \end{bmatrix}^{-1}_{2\times 2} \begin{bmatrix} \mathtt{I}e^{Q_1} \\ \mathtt{I}e^{Q_2} \end{bmatrix} \\ &= \mathtt{I} - \begin{bmatrix} L_1 e^{-Q_1} & L_2 e^{-Q_2} \end{bmatrix} \begin{bmatrix} \mathtt{I}e^{Q_1} \\ \mathtt{I}e^{Q_2} \end{bmatrix} \\ &= \mathtt{I} - L_1 - L_2. \end{aligned}$$

- Therefore

$$\begin{aligned} L_1 \left(\mathtt{I} + \frac{k_1 r_1 e^{K_1+Q_1}}{k_1 - q_1} P_1 \right) &= e^{K_1+Q_1} A_1 - \frac{k_2 e^{K_1+Q_1}}{k_1 - q_2} L_2 A_1, \\ L_2 \left(\mathtt{I} + \frac{k_2 r_2 e^{K_2+Q_2}}{k_2 - q_2} P_2 \right) &= e^{K_2+Q_2} A_2 - \frac{k_1 e^{K_2+Q_2}}{k_2 - q_1} L_1 A_2. \end{aligned}$$

matrix mKP equation

- Using

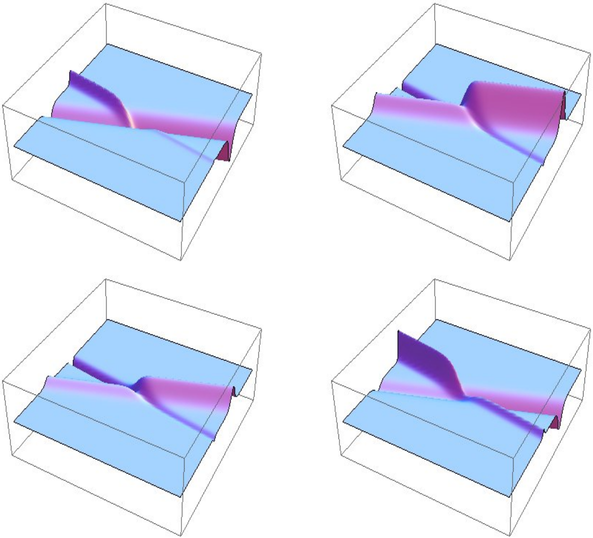
$$(\mathtt{I} - aP)^{-1} = \mathtt{I} + aP(1-a)^{-1}$$

for a scalar $a \neq 1$ we have

$$L_1 = \frac{(k_2 - q_1)}{h}((k_1 - q_2)h_2\mathtt{I} - p_2 A_2)A_1,$$
$$L_2 = \frac{(k_1 - q_2)}{h}((k_2 - q_1)h_1\mathtt{I} - p_1 A_1)A_2.$$

- $h_i = e^{-(K_i+Q_i)} + \frac{k_i r_i}{(k_i - q_i)}$.
- $h = h_1 h_2 (k_1 - q_2)(k_2 - q_1) - \alpha k_1 k_2 r_1 r_2$.
- $\alpha = \frac{(u_j, v_i)(u_i, v_j)}{(u_i, v_i)(u_j, v_j)}$ and $i = 1, 2$.

Plot of $w = -F_x F^{-1}$ at $t = 0$



Conclusions/Further work

- Quasideterminants are the natural structure to use to describe solutions of iterated Darboux transformations.
- The quasideterminant solutions hold for any case:
  - reduction to scalar case under commutative limit
  - allows for explicit nc examples.
- Solutions can be directly verified using properties of quasideterminants. This is sometimes easier than in the commutative case.
- What about other nc integrable systems? Harry Dym, reciprocal links expressed as quasideterminants? nc Painleve analysis?