Integrability of vector derivative NLS-type systems

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References

- arXiv: 0712.4373 [nlin.SI]

⋮

Introduction

In 1971–72, Zakharov–Shabat used the inverse scattering method and solved the nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}q_t + q_{xx} \pm |q|^2 q = 0. \tag{7}$$

In 1973, Manakov considered the natural vector generalization of the NLS equation (7),

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} + \langle \boldsymbol{q}, \boldsymbol{q}^* \rangle \boldsymbol{q} = \boldsymbol{0}, \quad \boldsymbol{q} = (q_1, q_2, \dots, q_m), \quad \langle \boldsymbol{q}, \boldsymbol{q}^* \rangle := \sum_{j=1}^{m} |q_j|^2,$$

and solved the two-component case $(m = 2)$ by the inverse scattering method. The extension to the general m-component case is straightforward ($e.g.$, Zakharov–Shabat: Funct. Anal. Appl. (1974)).

It is less well known that another vector generalization of the NLS equation (7) was studied by Kulish–Sklyanin in 1981 (Phy. Lett. A **84** 349):

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2\langle \boldsymbol{q}, \boldsymbol{q}^* \rangle \boldsymbol{q} - \langle \boldsymbol{q}, \boldsymbol{q} \rangle \boldsymbol{q}^* = \boldsymbol{0}.$$

This equation has been *rediscovered* again and again in the recent literature, up to a trivial linear transformation of the vector components.

It is now believed (*e.g.*, Sokolov–Wolf 2001) that these two equations exhaust the integrable vector generalizations of the NLS equation (7) (involving only scalar products between general m-component vectors).

In 1978, Kaup–Newell solved the derivative NLS equation,

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}(|q|^2 q)_x = 0, \tag{8}$$

by the inverse scattering method, followed by an important contribution of Kawata–Kobayashi–Inoue ('78,'79).

Note that the one-soliton solution of the derivative NLS equation (8) can be written as

$$q(x,t) = \frac{\partial}{\partial x}\left\{ \frac{\frac{1}{\mathrm{i}\lambda}\gamma^* \mathrm{e}^{\mathrm{i}\lambda x - \mathrm{i}\lambda^2 t}}{1 + \frac{\lambda}{2(\lambda-\lambda^*)^2}\gamma\gamma^* \mathrm{e}^{\mathrm{i}(\lambda-\lambda^*)x - \mathrm{i}(\lambda^2-\lambda^{*2})t}} \right\}, \quad \mathrm{Im}\,\lambda > 0.$$

The expression inside $\{\ \}$ is typical of the one-soliton form. This can be understood intuitively by introducing the potential variable $\hat{q}(x,t)$ as

$q =: \dfrac{\partial \hat{q}}{\partial x}$ and integrating (8) for x to get

$$\mathrm{i}\hat{q}_t + \hat{q}_{xx} + \mathrm{i}|\hat{q}_x|^2 \hat{q}_x = 0. \tag{9}$$

In fact, we can directly obtain the solutions of the potential derivative NLS (9) without performing the troublesome x-integration of the solutions of the derivative NLS (8).

The derivative NLS equation (8) is obtained as the "reality" reduction $r = -q^*$ of the system

$$\begin{cases} \mathrm{i}q_t + q_{xx} - \mathrm{i}(q^2 r)_x = 0, \\ \mathrm{i}r_t - r_{xx} - \mathrm{i}(r^2 q)_x = 0. \end{cases} \tag{10}$$

In the following, instead of the scalar equation (8) and its variants, we consider the non-reduced system (10) and its various extensions.

In addition to the Kaup–Newell system (10), there exist two important derivative NLS systems, that is,

Chen–Lee–Liu system:

$$\begin{cases} \mathrm{i}q_t + q_{xx} - \mathrm{i}qrq_x = 0, \\ \mathrm{i}r_t - r_{xx} - \mathrm{i}rqr_x = 0, \end{cases} \tag{11}$$

Gerdjikov–Ivanov (Ablowitz–Ramani–Segur) system:

$$\begin{cases} \mathrm{i}q_t + q_{xx} + \mathrm{i}qr_xq + \frac{1}{2}q^3r^2 = 0, \\ \mathrm{i}r_t - r_{xx} + \mathrm{i}rq_xr - \frac{1}{2}r^3q^2 = 0. \end{cases} \tag{12}$$

(11) and (12) also allow the reduction of complex conjugation $r = \pm q^*$. Note that the sign $\pm$ is not essential (cf. $x \mapsto -x$).

If we apply the dependent variable transformation,

$$Q = q \exp\left(-2\mathrm{i}\delta \int^{x} qr \, \mathrm{d}x'\right),$$
$$R = r \exp\left(2\mathrm{i}\delta \int^{x} qr \, \mathrm{d}x'\right),$$

to the Gerdjikov–Ivanov system (12), we obtain a one-parameter family of derivative NLS systems:

$$\begin{cases} \mathrm{i}Q_t + Q_{xx} + \mathrm{i}(4\delta+1)Q^2R_x + 4\mathrm{i}\delta QRQ_x + (\delta+\frac{1}{2})(4\delta+1)Q^3R^2 = 0, \\ \mathrm{i}R_t - R_{xx} + \mathrm{i}(4\delta+1)R^2Q_x + 4\mathrm{i}\delta RQR_x - (\delta+\frac{1}{2})(4\delta+1)R^3Q^2 = 0. \end{cases}$$

This coincides with

the Kaup–Newell system (10) if $\delta = -\frac{1}{2}$,

the Chen–Lee–Liu system (11) if $\delta = -\frac{1}{4}$,

and thus (10)–(12) are mutually related by a change of variables.

The derivative NLS systems are homogeneous with respect to the following weighting scheme:

$$w(\partial_x) = 1, \quad w(\partial_t) = 2, \quad w(q) = w(r) = \frac{1}{2}.$$

Under this weighting scheme, the general ansatz for a homogeneous polynomial vector derivative NLS-type system takes the form

$$\begin{cases} \mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} + a_1\langle \boldsymbol{q}_x, \boldsymbol{r}\rangle \boldsymbol{q} + a_2\langle \boldsymbol{q}, \boldsymbol{r}_x\rangle \boldsymbol{q} + \cdots = \boldsymbol{0}, \\ \mathrm{i}\boldsymbol{r}_t - \boldsymbol{r}_{xx} + b_1\langle \boldsymbol{r}_x, \boldsymbol{q}\rangle \boldsymbol{r} + b_2\langle \boldsymbol{r}, \boldsymbol{q}_x\rangle \boldsymbol{r} + \cdots = \boldsymbol{0}. \end{cases}$$

As an integrability test, Sokolov–Wolf [J. Phys. A: Math. Gen. **34** (2001) 11139] assumed the existence of a third-order symmetry of the form

$$\begin{cases} \boldsymbol{q}_s + \boldsymbol{q}_{xxx} + c_1\langle \boldsymbol{q}_{xx}, \boldsymbol{r}\rangle \boldsymbol{q} + \cdots = \boldsymbol{0}, \\ \boldsymbol{r}_s + \boldsymbol{r}_{xxx} + d_1\langle \boldsymbol{r}_{xx}, \boldsymbol{q}\rangle \boldsymbol{r} + \cdots = \boldsymbol{0}. \end{cases}$$

They formulated the symmetry conditions $\boldsymbol{q}_{ts} = \boldsymbol{q}_{st}$ and $\boldsymbol{r}_{ts} = \boldsymbol{r}_{st}$ as a bilinear algebraic system, solved it, and obtained a complete list of vector derivative NLS-type systems possessing a third-order symmetry.

The list consists of the six systems (1)–(6) in the abstract booklet. The aim of this talk is to demonstrate that

- the two systems (1) and (3) are C-integrable (Calogero's terminology), i.e., linearizable by a change of variables;
- the other four systems (2), (4), (5) and (6) are S-integrable, i.e., they allow a Lax representation and are amenable to the inverse scattering method.

Due to the freedom of the phase transformation mentioned above, the listed systems contain free parameters. In particular, if we fix the parameters in the S-integrable systems at the Kaup–Newell case, they read as follows:

$$\begin{cases} \mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} - \mathrm{i}(\langle \boldsymbol{q}, \boldsymbol{r} \rangle \boldsymbol{q})_x = \boldsymbol{0}, \\ \mathrm{i}\boldsymbol{r}_t - \boldsymbol{r}_{xx} - \mathrm{i}(\langle \boldsymbol{r}, \boldsymbol{q} \rangle \boldsymbol{r})_x = \boldsymbol{0}, \end{cases}$$

$$\begin{cases} \mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} - \mathrm{i}(2\langle \boldsymbol{q}, \boldsymbol{r} \rangle \boldsymbol{q} - \langle \boldsymbol{q}, \boldsymbol{q} \rangle \boldsymbol{r})_x = \boldsymbol{0}, \\ \mathrm{i}\boldsymbol{r}_t - \boldsymbol{r}_{xx} - \mathrm{i}(2\langle \boldsymbol{r}, \boldsymbol{q} \rangle \boldsymbol{r} - \langle \boldsymbol{r}, \boldsymbol{r} \rangle \boldsymbol{q})_x = \boldsymbol{0}, \end{cases}$$

$$\begin{cases} \mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} - 2\mathrm{i}\langle \boldsymbol{q}, \boldsymbol{r} \rangle \boldsymbol{q}_x - \mathrm{i}\langle \boldsymbol{q}, \boldsymbol{r}_{\boldsymbol{x}} \rangle \boldsymbol{q} = \boldsymbol{0}, \\ \mathrm{i}\boldsymbol{r}_t - \boldsymbol{r}_{xx} - 2\mathrm{i}\langle \boldsymbol{r}, \boldsymbol{q} \rangle \boldsymbol{r}_x - \mathrm{i}\langle \boldsymbol{r}, \boldsymbol{q}_{\boldsymbol{x}} \rangle \boldsymbol{r} = \boldsymbol{0}, \end{cases}$$

$$\begin{cases} \mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} - 2\mathrm{i}\langle \boldsymbol{q}, \boldsymbol{r} \rangle \boldsymbol{q}_x - \mathrm{i}\langle \boldsymbol{q}, \boldsymbol{q} \rangle \boldsymbol{r}_x = \boldsymbol{0}, \\ \mathrm{i}\boldsymbol{r}_t - \boldsymbol{r}_{xx} - 2\mathrm{i}\langle \boldsymbol{r}, \boldsymbol{q} \rangle \boldsymbol{r}_x - \mathrm{i}\langle \boldsymbol{r}, \boldsymbol{r} \rangle \boldsymbol{q}_x = \boldsymbol{0}. \end{cases}$$

The existence of x-differentiation in the nonlinear terms results in richer integrable vector generalizations of the derivative NLS than the NLS.

In the following, to make the comparison easier, we (try to) follow the original notation of Sokolov–Wolf (2001):

- the two vector unknowns in a system are expressed as U and V;
- some inessential parameters in the list can be scaled away, but they are often left as they were.

C-integrable systems

System (1):

$$\begin{cases} U_t = U_{xx} + 2\alpha\langle U, V\rangle U_x + 2\alpha\langle U, V_x\rangle U - \alpha\beta\langle U, V\rangle^2 U, \\ V_t = -V_{xx} + 2\beta\langle V, U\rangle V_x + 2\beta\langle V, U_x\rangle V + \alpha\beta\langle V, U\rangle^2 V. \end{cases}$$

We note that the system has the conservation law

$$\langle U, V\rangle_t = \left[\langle U_x, V\rangle - \langle U, V_x\rangle + (\alpha + \beta)\langle U, V\rangle^2\right]_x .$$

Then, by a change of variables,

$$\boldsymbol{u} := U \mathrm{e}^{\alpha \int^x \langle U, V\rangle \mathrm{d}x'}, \qquad \boldsymbol{v} := V \mathrm{e}^{-\beta \int^x \langle U, V\rangle \mathrm{d}x'},$$

(1) is converted to a pair of linear equations

$$\boldsymbol{u}_t = \boldsymbol{u}_{xx}, \qquad \boldsymbol{v}_t = -\boldsymbol{v}_{xx}.$$

System (3):

$$
\begin{cases} U_t = U_{xx} + 2\alpha\langle U, V\rangle U_x + 2\beta\langle U, V_x\rangle U + 2(\beta-\alpha)\langle U_x, V\rangle U - \alpha\beta\langle U, V\rangle^2 U, \\ V_t = -V_{xx} + 2\alpha\langle V, U\rangle V_x + 2\alpha\langle V, U_x\rangle V + \alpha\beta\langle V, U\rangle^2 V. \end{cases}
$$

Similarly, by a change of variables,

$$
\boldsymbol{u} := U\mathrm{e}^{\alpha\int^x \langle U,V\rangle \mathrm{d}x'}, \qquad \boldsymbol{v} := V\mathrm{e}^{-\alpha\int^x \langle U,V\rangle \mathrm{d}x'},
$$

(3) is reduced to a triangular system,

$$
\boldsymbol{u}_t = \boldsymbol{u}_{xx} + 2(\beta-\alpha)\langle \boldsymbol{u}, \boldsymbol{v}\rangle_x \boldsymbol{u}, \qquad \boldsymbol{v}_t = -\boldsymbol{v}_{xx}.
$$

If $\alpha = \beta$, this system is already linear. If $\alpha \neq \beta$, this system is a vector reduction of the matrix system proposed by Olver–Sokolov (1998) and solved by Tsuchida–Wadati (1999).

Indeed, its "conservation law" guarantees that the linear system for a square matrix A,

$$\begin{cases} A_x = (\beta - \alpha)\boldsymbol{v}^T \boldsymbol{u} A, \\ A_t = \left[(\beta - \alpha)(\boldsymbol{v}^T \boldsymbol{u}_x - \boldsymbol{v}_x^T \boldsymbol{u}) + (\beta - \alpha)^2 (\boldsymbol{v}^T \boldsymbol{u})^2 \right] A, \end{cases}$$

is compatible. Then, we can linearize the equation for $\boldsymbol{u}$ as

$$(\boldsymbol{u}A)_t = (\boldsymbol{u}A)_{xx}.$$

S-integrable systems

System (2):

$$\begin{cases} U_t = U_{xx} + 2\alpha\langle U, V\rangle U_x + 2\beta\langle U, V_x\rangle U + \beta(\alpha - 2\beta)\langle U, V\rangle^2 U, \\ V_t = -V_{xx} + 2\alpha\langle V, U\rangle V_x + 2\beta\langle V, U_x\rangle V - \beta(\alpha - 2\beta)\langle V, U\rangle^2 V. \end{cases}$$

System (4):

$$\begin{cases} U_t = U_{xx} + 2\alpha\langle U, V\rangle U_x + 2\alpha\langle U, V_x\rangle U + 2\beta\langle U_x, V\rangle U - \alpha(\alpha - \beta)\langle U, V\rangle^2 U, \\ V_t = -V_{xx} + 2\alpha\langle V, U\rangle V_x + 2\alpha\langle V, U_x\rangle V + 2\beta\langle V_x, U\rangle V + \alpha(\alpha - \beta)\langle V, U\rangle^2 V. \end{cases}$$

Both systems are already known, $e.g.$,

Kundu–Strampp–Oevel (1995 J. Math. Phys.),

Hisakado (1998 J. Phys. Soc. Jpn.),

Tsuchida–Wadati (1999 Phys. Lett. A).

We start with the simplest matrix generalization of the Chen–Lee–Liu system (11) (van der Linden–Capel–Nijhoff 1989 Physica A)

$$\begin{cases} Q_t = Q_{xx} - 2aQRQ_x, \\ R_t = -R_{xx} - 2aR_xQR. \end{cases}$$

Considering the (row, column) vector reduction

$$Q = (q_1, \dots, q_m), \quad R = (r_1, \dots, r_m)^T,$$

and the (column, row) vector reduction

$$Q = (q_1, \dots, q_m)^T, \quad R = (r_1, \dots, r_m),$$

we obtain two vector analogues of the Chen–Lee–Liu system:

$$\begin{cases} \boldsymbol{q}_t = \boldsymbol{q}_{xx} - 2a\langle \boldsymbol{q}, \boldsymbol{r} \rangle \boldsymbol{q}_x, \\ \boldsymbol{r}_t = -\boldsymbol{r}_{xx} - 2a\langle \boldsymbol{r}, \boldsymbol{q} \rangle \boldsymbol{r}_x, \end{cases} \tag{13}$$

$$\begin{cases} \boldsymbol{q}_t = \boldsymbol{q}_{xx} - 2a\langle \boldsymbol{q}_x, \boldsymbol{r} \rangle \boldsymbol{q}, \\ \boldsymbol{r}_t = -\boldsymbol{r}_{xx} - 2a\langle \boldsymbol{r}_x, \boldsymbol{q} \rangle \boldsymbol{r}. \end{cases} \tag{14}$$

We introduce a new pair of vector variables by

$$U := \boldsymbol{q} \mathrm{e}^{-b \int^x \langle \boldsymbol{q}, \boldsymbol{r} \rangle \mathrm{d}x'}, \qquad V := \boldsymbol{r} \mathrm{e}^{b \int^x \langle \boldsymbol{q}, \boldsymbol{r} \rangle \mathrm{d}x'}.$$

Then, (13) is transformed to the system

$$\begin{cases} U_t = U_{xx} + 2b\langle U, V_x\rangle U + 2(b-a)\langle U, V\rangle U_x - b(a+b)\langle U, V\rangle^2 U, \\ V_t = -V_{xx} + 2b\langle V, U_x\rangle V + 2(b-a)\langle V, U\rangle V_x + b(a+b)\langle V, U\rangle^2 V. \end{cases}$$

With a new parametrization: $a = -\alpha + \beta,\ b = \beta,$ this system coincides with system (2).

In the same way, (14) is transformed to the system

$$\begin{cases} U_t = U_{xx} - 2a\langle U_x, V\rangle U + 2b\langle U, V_x\rangle U + 2b\langle U, V\rangle U_x - b(a+b)\langle U, V\rangle^2 U, \\ V_t = -V_{xx} - 2a\langle V_x, U\rangle V + 2b\langle V, U_x\rangle V + 2b\langle V, U\rangle V_x + b(a+b)\langle V, U\rangle^2 V. \end{cases}$$

With a new parametrization: $a = -\beta,\ b = \alpha,$ this system coincides with system (4).

System (5):

$$
\begin{cases}
U_t = U_{xx} + 4\alpha\langle U,V\rangle U_x + 2(\alpha-\beta)\langle U,U\rangle V_x + 4\beta\langle U,V_x\rangle U \\
\qquad + 4\beta(\alpha-2\beta)\langle U,V\rangle^2 U - 2\beta(\alpha-\beta)\langle U,U\rangle\langle V,V\rangle U \\
\qquad - 4\beta(\alpha-\beta)\langle U,U\rangle\langle U,V\rangle V, \\
V_t = -V_{xx} + 4\alpha\langle V,U\rangle V_x + 2(\alpha-\beta)\langle V,V\rangle U_x + 4\beta\langle V,U_x\rangle V \\
\qquad - 4\beta(\alpha-2\beta)\langle V,U\rangle^2 V + 2\beta(\alpha-\beta)\langle V,V\rangle\langle U,U\rangle V \\
\qquad + 4\beta(\alpha-\beta)\langle V,V\rangle\langle V,U\rangle U.
\end{cases}
$$

We start with a matrix generalization of the Gerdjikov–Ivanov (Ablowitz–Ramani–Segur) system (van der Linden–Capel–Nijhoff 1989 Physica A):

$$
\begin{cases}
Q_t = Q_{xx} + 2aQR_xQ - 2a^2 QRQRQ, \\
R_t = -R_{xx} + 2aRQ_xR + 2a^2 RQRQR.
\end{cases}
\tag{15}
$$

We introduce a set of $2^{M-1} \times 2^{M-1}$ matrices $\{e_1, \dots, e_{2M-1}\}$ (generators of the Clifford algebra) satisfying the anti-commutation relation

$$\{e_i, e_j\} := e_i e_j + e_j e_i = -2\delta_{ij} I.$$

Then, the matrices Q and R written as

$$Q = q_1 I + \sum_{j=1}^{2M-1} q_{j+1} e_j, \quad R = r_1 I - \sum_{j=1}^{2M-1} r_{j+1} e_j, \tag{16}$$

satisfy the following relations:

$$\begin{aligned} QRQ &= \frac{1}{2}\{Q, \{Q, R\}\} - \frac{1}{2}\{Q^2, R\} \\ &= 2\langle \boldsymbol{q}, \boldsymbol{r}\rangle Q - \langle \boldsymbol{q}, \boldsymbol{q}\rangle \bar{R}, \end{aligned} \tag{17a}$$

$$RQR = 2\langle \boldsymbol{r}, \boldsymbol{q}\rangle R - \langle \boldsymbol{r}, \boldsymbol{r}\rangle \bar{Q}. \tag{17b}$$

Here, the vectors $\boldsymbol{q}$, $\boldsymbol{r}$ and the bar ("Clifford conjugation") are given by

$$\boldsymbol{q} = (q_1, \dots, q_{2M}), \quad \boldsymbol{r} = (r_1, \dots, r_{2M}),$$

and

$$\bar{Q} = q_1 I - \sum_{j=1}^{2M-1} q_{j+1} e_j, \quad \bar{R} = r_1 I + \sum_{j=1}^{2M-1} r_{j+1} e_j.$$

Repeated use of (17a) and (17b) provides the relations

$$QRQRQ = (4\langle \boldsymbol{q}, \boldsymbol{r} \rangle^2 - \langle \boldsymbol{q}, \boldsymbol{q} \rangle \langle \boldsymbol{r}, \boldsymbol{r} \rangle) Q - 2 \langle \boldsymbol{q}, \boldsymbol{q} \rangle \langle \boldsymbol{q}, \boldsymbol{r} \rangle \bar{R}, \tag{18a}$$

$$RQRQR = (4\langle \boldsymbol{r}, \boldsymbol{q} \rangle^2 - \langle \boldsymbol{r}, \boldsymbol{r} \rangle \langle \boldsymbol{q}, \boldsymbol{q} \rangle) R - 2 \langle \boldsymbol{r}, \boldsymbol{r} \rangle \langle \boldsymbol{r}, \boldsymbol{q} \rangle \bar{Q}. \tag{18b}$$

Thus, considering the reduction (16) of (15) and using the formulas (17) and (18), we obtain a vector analogue of the Gerdjikov–Ivanov system (V. E. Adler: nlin/0011039):

$$\begin{cases} \boldsymbol{q}_t = \boldsymbol{q}_{xx} + 4a\langle \boldsymbol{q}, \boldsymbol{r_x}\rangle \boldsymbol{q} - 2a\langle \boldsymbol{q}, \boldsymbol{q}\rangle \boldsymbol{r}_x - 8a^2\langle \boldsymbol{q}, \boldsymbol{r}\rangle^2 \boldsymbol{q} + 2a^2\langle \boldsymbol{q}, \boldsymbol{q}\rangle\langle \boldsymbol{r}, \boldsymbol{r}\rangle \boldsymbol{q} \\ \qquad + 4a^2\langle \boldsymbol{q}, \boldsymbol{q}\rangle\langle \boldsymbol{q}, \boldsymbol{r}\rangle \boldsymbol{r}, \\ \boldsymbol{r}_t = -\boldsymbol{r}_{xx} + 4a\langle \boldsymbol{r}, \boldsymbol{q_x}\rangle \boldsymbol{r} - 2a\langle \boldsymbol{r}, \boldsymbol{r}\rangle \boldsymbol{q}_x + 8a^2\langle \boldsymbol{r}, \boldsymbol{q}\rangle^2 \boldsymbol{r} - 2a^2\langle \boldsymbol{r}, \boldsymbol{r}\rangle\langle \boldsymbol{q}, \boldsymbol{q}\rangle \boldsymbol{r} \\ \qquad - 4a^2\langle \boldsymbol{r}, \boldsymbol{r}\rangle\langle \boldsymbol{r}, \boldsymbol{q}\rangle \boldsymbol{q}. \end{cases} \tag{19}$$

System (19) has the conservation law

$$\langle \boldsymbol{q}, \boldsymbol{r}\rangle_t = \left[\langle \boldsymbol{q_x}, \boldsymbol{r}\rangle - \langle \boldsymbol{q}, \boldsymbol{r_x}\rangle + 2a\langle \boldsymbol{q}, \boldsymbol{r}\rangle^2 - a\langle \boldsymbol{q}, \boldsymbol{q}\rangle\langle \boldsymbol{r}, \boldsymbol{r}\rangle\right]_x.$$

We introduce a new pair of vector variables by

$$U := \boldsymbol{q}\mathrm{e}^{-2b\int^x \langle \boldsymbol{q}, \boldsymbol{r}\rangle \mathrm{d}x'}, \qquad V := \boldsymbol{r}\mathrm{e}^{2b\int^x \langle \boldsymbol{q}, \boldsymbol{r}\rangle \mathrm{d}x'}.$$

Then, using the above conservation law, (19) is transformed to the system

$$\begin{cases} U_t = U_{xx} + 4b\langle U, V\rangle U_x - 2a\langle U, U\rangle V_x + 4(a+b)\langle U, V_x\rangle U \\ \qquad - 4(a+b)(2a+b)\langle U, V\rangle^2 U + 2a(a+b)\langle U, U\rangle\langle V, V\rangle U \\ \qquad + 4a(a+b)\langle U, U\rangle\langle U, V\rangle V, \\ V_t = -V_{xx} + 4b\langle V, U\rangle V_x - 2a\langle V, V\rangle U_x + 4(a+b)\langle V, U_x\rangle V \\ \qquad + 4(a+b)(2a+b)\langle V, U\rangle^2 V - 2a(a+b)\langle V, V\rangle\langle U, U\rangle V \\ \qquad - 4a(a+b)\langle V, V\rangle\langle V, U\rangle U. \end{cases}$$

With a new parametrization: $a = -\alpha + \beta,\ b = \alpha$, this system coincides with system (5).

System (6):

$$
\begin{cases}
U_t = U_{xx} + 4\alpha\langle U, V\rangle U_x - 2\beta\langle U, U\rangle V_x + 4\alpha\langle U, V_x\rangle U \\
\qquad + 4\beta\langle U_x, V\rangle U - 4\beta\langle U, U_x\rangle V - 4\alpha(\alpha-\beta)\langle U, V\rangle^2 U \\
\qquad + 6\beta(\alpha-\beta)\langle U, U\rangle\langle V, V\rangle U - 4\beta(\alpha-\beta)\langle U, U\rangle\langle U, V\rangle V, \\
V_t = -V_{xx} + 4\alpha\langle V, U\rangle V_x - 2\beta\langle V, V\rangle U_x + 4\alpha\langle V, U_x\rangle V \\
\qquad + 4\beta\langle V_x, U\rangle V - 4\beta\langle V, V_x\rangle U + 4\alpha(\alpha-\beta)\langle V, U\rangle^2 V \\
\qquad - 6\beta(\alpha-\beta)\langle U, U\rangle\langle V, V\rangle V + 4\beta(\alpha-\beta)\langle V, V\rangle\langle V, U\rangle U.
\end{cases}
$$

We start with the simplest matrix generalization of the Kaup–Newell system ($e.g.$ Konopelchenko 1981):

$$
\begin{cases}
Q_t = Q_{xx} + 2c(QRQ)_x, \\
R_t = -R_{xx} + 2c(RQR)_x.
\end{cases}
\tag{20}
$$

Considering the reduction (16) of (20) and using the relations (17), we obtain a vector analogue of the Kaup–Newell system (Adler–Svinolupov–Yamilov 1999 PLA; cf. Kulish–Sklyanin 1981):

$$\begin{cases} \boldsymbol{q}_t = \boldsymbol{q}_{xx} + (4c\langle \boldsymbol{q}, \boldsymbol{r}\rangle \boldsymbol{q} - 2c\langle \boldsymbol{q}, \boldsymbol{q}\rangle \boldsymbol{r})_x, \\ \boldsymbol{r}_t = -\boldsymbol{r}_{xx} + (4c\langle \boldsymbol{r}, \boldsymbol{q}\rangle \boldsymbol{r} - 2c\langle \boldsymbol{r}, \boldsymbol{r}\rangle \boldsymbol{q})_x. \end{cases} \tag{21}$$

Besides the $2^M \times 2^M$ Lax representation based on the Clifford algebra, this system also possesses a $(2M+2) \times (2M+2)$ Lax representation.

System (21) has the conservation law

$$\langle \boldsymbol{q}, \boldsymbol{r}\rangle_t = \left[\langle \boldsymbol{q_x}, \boldsymbol{r}\rangle - \langle \boldsymbol{q}, \boldsymbol{r_x}\rangle + 6c\langle \boldsymbol{q}, \boldsymbol{r}\rangle^2 - 3c\langle \boldsymbol{q}, \boldsymbol{q}\rangle\langle \boldsymbol{r}, \boldsymbol{r}\rangle \right]_x.$$

We introduce a new pair of vector variables by

$$U := \boldsymbol{q}\mathrm{e}^{-2d\int^x \langle \boldsymbol{q},\boldsymbol{r}\rangle \mathrm{d}x'}, \qquad V := \boldsymbol{r}\mathrm{e}^{2d\int^x \langle \boldsymbol{q},\boldsymbol{r}\rangle \mathrm{d}x'}.$$

Then, using the above conservation law, (21) is transformed to the system

$$\begin{cases} U_t = U_{xx} + 4(c+d)\langle U, V\rangle U_x - 2c\langle U, U\rangle V_x + 4(c+d)\langle U, V_x\rangle U \\ \qquad + 4c\langle U_x, V\rangle U - 4c\langle U, U_x\rangle V - 4d(c+d)\langle U, V\rangle^2 U \\ \qquad + 6cd\langle U, U\rangle\langle V, V\rangle U - 4cd\langle U, U\rangle\langle U, V\rangle V, \\ V_t = -V_{xx} + 4(c+d)\langle V, U\rangle V_x - 2c\langle V, V\rangle U_x + 4(c+d)\langle V, U_x\rangle V \\ \qquad + 4c\langle V_x, U\rangle V - 4c\langle V, V_x\rangle U + 4d(c+d)\langle V, U\rangle^2 V \\ \qquad - 6cd\langle U, U\rangle\langle V, V\rangle V + 4cd\langle V, V\rangle\langle V, U\rangle U. \end{cases}$$

With a new parametrization: $c = \beta,\ d = \alpha - \beta$, this system coincides with system (6).

Summary

- In this talk, we considered the six vector derivative NLS-type systems in the Sokolov–Wolf list.

- Two of the six systems turned out to be linearizable by a change of variables.

- Two of the remaining four systems are obtained by applying a phase transformation to a column/row vector reduction of the matrix generalization of the Chen–Lee–Liu system.

- Finally, the remaining two systems are obtained as a reduction of the matrix generalization of the Gerdjikov–Ivanov (ARS) system and the Kaup–Newell system, respectively. The matrix variables are expanded in terms of the generators of the Clifford algebra (cf. Eichenherr–Pohlmeyer (1979, PLB) & Kulish–Sklyanin (1981, PLA)).

- The matrix Chen–Lee–Liu, Gerdjikov–Ivanov and (potential) Kaup–Newell systems are all solvable by the inverse scattering method associated with the matrix NLS system. The quantity $\exp\left(\int^x \langle \boldsymbol{q}, \boldsymbol{r} \rangle \mathrm{d}x'\right)$ for the phase transformation of the vector derivative NLS systems can be computed explicitly within the inverse scattering formalism.

For example, the solutions to the matrix Kaup–Newell equation

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}(qq^\dagger q)_x = O, \tag{22}$$

tending to zero as $x \to +\infty$, can be constructed through the compact formula

$$q(x,t) = \frac{\partial \mathcal{K}(x,x;t)}{\partial x},$$

$$\mathcal{K}(x,y) = \bar{G}(y) - \frac{\mathrm{i}}{2}\int_x^\infty \mathrm{d}s_1 \int_x^\infty \mathrm{d}s_2 \frac{\partial \mathcal{K}(x,s_1)}{\partial s_1}\frac{\partial \bar{G}(s_1+s_2-x)^\dagger}{\partial s_2}\frac{\partial \bar{G}(s_2+y-x)}{\partial y},$$

$$y \geq x,$$

where

$$\mathrm{i}\bar{G}_t + \bar{G}_{xx} = O.$$

The N-soliton solution of the matrix Kaup–Newell equation (22) is written as

$$q(x,t) = \frac{\partial}{\partial x}\left\{ (\underbrace{I\, I\, \ldots\, I}_{N}) \begin{pmatrix} S_{11} & \ldots & S_{1N} \\ \vdots & \ddots & \vdots \\ S_{N1} & \ldots & S_{NN} \end{pmatrix}^{-1} \begin{pmatrix} C_1 \mathrm{e}^{\mathrm{i}\lambda_1 x - \mathrm{i}\lambda_1^2 t} \\ \vdots \\ C_N \mathrm{e}^{\mathrm{i}\lambda_N x - \mathrm{i}\lambda_N^2 t} \end{pmatrix} \right\}, \tag{23}$$

where the block matrix elements S_{jk} are given by

$$S_{jk} := \delta_{jk} I + \sum_{l=1}^{N} \frac{\lambda_j \lambda_k \lambda_l^*}{2(\lambda_j - \lambda_l^*)(\lambda_k - \lambda_l^*)} C_j C_l^\dagger \mathrm{e}^{\mathrm{i}(\lambda_j - \lambda_l^*)x - \mathrm{i}(\lambda_j^2 - \lambda_l^{*2})t}, \quad 1 \le j,k \le N.$$

A different (somewhat implicit) formula for the solutions was already presented by Nijhoff, Capel, Quispel and van der Linden ('83 & '89), though they did not construct explicit solutions.