Vector analogue of Toda lattice.

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Laplace method:

$u_{xy}+a(x,y)u_x+b(x,y)u_y+c(x,y)u=0,\ \ u=u(x,y).$

Laplace invariants: $h=a_x+ab-c$ and $k=b_y+ab-c$. If $h,\ k$ are non-zeroes:

$$u=\frac{1}{h}\left(\frac{\partial}{\partial x}+b\right)u_1$$

launches the Laplace transformation chain. Discrete case: R.Hirota, R.S. Ward, A.V. Zhiber, V.V. Sokolov, I.M. Andersson, N. Kamran, V.E. Adler, S. Ya. Startsev:

$(Lu)_{ij}=0,\ L=T_iT_j+\xi_{ij}T_i+\nu_{ij}T_j+\eta_{ij},\ T_iu_i=u_{i+1}.$

Hierarchy of operators $L_k,\ k=0,1,2,...,\ L_0=L$:

1) $L_k=(T_i+a_{i,j+k})(T_j+b^k_{i-1,j})+H^k_{ij}=$

$$(T_j + b_{ij}^k)(T_i + a_{i,j+k-1}) + K_{ij}^k.$$

Iterative procedure:

$$2) L_{k+1}(T_j + b_{i-1,j}^k) = (T_j + b_{ij}^{k+1}) L_k$$

Laplace invariant (over transform $u_{ij} \to \lambda_{ij} u_{ij}$):

$$h_{ij}^k = \frac{H_{ij}^k}{a_{i,j+k} b_{i-1,j}^k}$$

Chain of Laplace invariants: h_{ij}^k, $k \in Z$. If $h_{ij}^0 = 0$, then the linear equation solves in quadratures.

Nonlinear problems:

$$u_{i+1,j+1} = f(u_{ij}, u_{i+1,j}, u_{i,j+1})$$

and its linearization:

$$L = T_i T_j - \frac{\partial f}{\partial u_{i+1,j}} T_i - \frac{\partial f}{\partial u_{i,j+1}} T_j - \frac{\partial f}{\partial u_{ij}}$$

The chain of Laplace invariants is finite (zeroes at both ends) $\Leftrightarrow$ there exist integrals in characteristic directions: $\Delta_i I = \Delta_j J = 0$. Problem: collect nontrivial examples of Liouville type difference equations. In the continuous case there is a relation among the invariants:

$$(ln h_j)_{xy} = h_{j+1} - 2h_j + h_{j-1}$$

(two-dimensional Toda chain). Various integrable boundary conditions yield integrable models:

$$1)\ ...h_{-2} = h_{-1} = 0 = h_1 = h_2 = ...$$

leads to the Liouville equation,

$$2)\ ...h_{-2} = h_{-1} = 0 = h_N = h_{N+1} = ...$$

gives the Toda lattice

$$h^i_{xy} = \sum_{j=0}^{N-1} a_{ij} exp(h^j),\ i = 0, ..., N-1,$$

a_{ij} is Cartan matrix of simple Lie algebra A_N. Similar boundary problems yield Toda chains associated with Lie algebras B,C. The same situation arises for discrete equations on four points of the grid: appropriate boundary conditions lead to integrable difference equations.

Five-point case:

$$u_{i+1,j+1} = f(u_{ij}, u_{i+1,j}, u_{i,j+1}, u_{i,j-1}),$$

linearization

$$L_0 = T_i T_j - \frac{\partial f}{\partial u_{i+1,j}} T_i - \frac{\partial f}{\partial u_{i,j+1}} T_j -$$

$$\frac{\partial f}{\partial u_{ij}} - \frac{\partial f}{\partial u_{i,j-1}} T_{-j},$$

1) $L_k = (T_i + a_{i,j+k} + c_{ij}^k T_{-j})(T_j + b_{i-1,j}^k) + H_{ij}^k =$

$$(T_j + b_{ij}^k)(T_i + a_{i,j+k-1} + c_{ij}^{k-1} T_{-j}) + K_{ij}^k,$$

2) $(T_j + b_{ij}^{k+1}) = L_{k+1}(T_j + b_{i-1,j}^k),$

$$b_{ij}^{k+1} = b_{i-1,j}^{k} \frac{H_{i,j+1}^{k}}{H_{ij}^{k}},$$

$$c_{ij}^{k+1} = c_{ij}^{k} \frac{b_{ij}^{k+1}}{b_{i-1,j-1}^{k+1}},$$

$$H_{ij}^{k+1} = H_{i,j+1}^{k} + a_{i,j+k} b_{ij}^{k+1} - a_{i,j+k+1} b_{i-1,j}^{k+1} +$$

$$c_{i,j+1}^{k} - c_{ij}^{k+1}.$$

Two chains of Laplace invariants:

$$P_{ij}^{k} = \frac{H_{ij}^{k}}{a_{i,j+k} b_{i-1,j}^{k}}, \quad q_{ij}^{k} = \frac{H_{ij}^{k}}{c_{ij}^{k}}, \ k \in Z,$$

and the relation on Laplace invariants does not reduce to a scalar equation and yields the following system:

$$\left(p_{i,j+1}^{k-1} + 1 + \frac{p_{i,j+1}^{k-1}}{q_{i,j+2}^{k-1}} \right) \left(p_{i+1,j}^{k+1} + 1 + \frac{p_{i+1,j}^{k+1}}{q_{i+1,j}^{k+1}} \right) =$$

$$p^k_{ij}p^k_{i+1,j+1}\left(1+\frac{1}{p^k_{i+1,j}}+\frac{1}{q^k_{i+1,j+1}}\right)\times$$

$$\left(1+\frac{1}{p^k_{i,j+1}}+\frac{1}{q^k_{i,j+1}}\right)$$

$$\frac{p^{k-1}_{ij}p^{k+1}_{i+1,j}}{q^{k-1}_{i,j+1}q^{k+1}_{i+1,j}}=\frac{p^k_{i,j-1}p^k_{i+1,j+1}}{q^k_{i+1,j}q^k_{i,j+1}}$$

Under $c^k_{ij}=0$ $\left(\frac{1}{q}=0\right)$ the system transforms to two-dimensional Toda lattice. What is integrability for 5-point systems of nonlinear discrete equations? Iterative procedure:

$$L_k=(T_i+A_{i,j+k}+C^k_{ij}T_{-j})(T_j+B^k_{i-1,j})+H^k_{ij}=$$

$$(T_j+B^k_{ij})(T_i+A_{i,j+k-1}+C^{k-1}_{ij}T_{-j})+K^k_{ij},$$

$$(T_j+B^{k+1}_{ij})L_k=L_{k+1}(T_j+B^k_{i-1,j})$$

$$B^{k+1}_{ij}X^k_{ij}=X^k_{i,j+1}B^k_{i-1,j},$$

where

$$X_{ij}^{k} = H_{i+k,j}^{k} H_{i+k-1,j}^{k-1} \dots H_{ij}^{0}$$

is the generalized invariant.

$$C_{ij}^{k+1} B_{i-1,j-1}^{k+1} = B_{ij}^{k+1} C_{ij}^{k},$$

$$H_{ij}^{k+1} = H_{i,j+1}^{k} + B_{ij}^{k+1} A_{i,j+k} - A_{i,j+k+1} B_{i-1,j}^{k+1} + C_{i,j+1}^{k} - C_{ij}^{k+1}$$

Example. Simple boundary condition for two-dimensional Toda lattice

$$p_{ij}^{k} = p, \quad q_{ij}^{k} = q, \quad k \neq 0,$$

p, q are nonzero constants. Then we have a system on two sequences $p_{ij} = p_{ij}^{0}$, $q_{ij} = q_{ij}^{0}$:

$$p_{ij}p_{i+1,j+1}\left(1+\frac{1}{p_{i+1,j}}+\frac{1}{q_{i+1,j+1}}\right)\times$$

$$\left(1+\frac{1}{p_{i,j+1}}+\frac{1}{q_{i,j+1}}\right)=$$

$$p^2\left(1+\frac{1}{p}+\frac{1}{q}\right)^2,$$

$$\frac{p_{i,j-1}p_{i+1,j+1}}{q_{i+1,j}q_{i,j+1}}=\left(\frac{p}{q}\right)^2,$$

which is of Liouville type (chain of Laplace invariants is finite).

1. Lemma on solvability.

Systems on matrices B^{k+1}, C^{k+1} are solvable $\Leftrightarrow$

$$B^0_{i-1,j}KerX^k_{ij}\subset KerX^k_{i,j+1},$$

$$C_{ij}^{k} Ker B_{i-1,j-1}^{k+1} Ker B_{ij}^{k+1}.$$

2. Lemma on uniqueness.

Let

1) $A_{i+l+1,j+l}^{T} Ker(X_{i+1,j}^{l})^{T} \subset Ker(X_{ij}^{l})^{T}$,

$$l = 0, 1, ...k;$$

2) $Ker(X_{ij}^{0})^{T} \subset Ker(X_{i-1,j-1}^{1})^{T} \subset ...$

$$\subset Ker(X_{i-k,j-k}^{k})^{T};$$

3) $for\ \forall B_{ij}^{l+1}\ Ker(B_{i+l,j-1}^{l+1})^{T} \subset Ker(X_{ij}^{l})^{T}$,

$$l = 0, 1, ..., k.$$

Then the generalized invariant X_{ij}^{k+1} does not depend on choice of matrices B_{ij}^{l}, C_{ij}^{l}.