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The Cauchy problem of the Ward equation

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§1. Introduction

◇ 1.1 Background

The self-dual Yang-Mills equations are a fundamental tool in four-manifold topology. Taking a one dimension reduction, one obtains the three-dimensional monopole equation [Hitchin,1988], whilst a two dimension reduction yields harmonic maps into Lie groups [Uhlenbeck, 1989]. These equations have generated much interest and been extensively studied.

If we replace the background space by $R^{2,2}$, and take a two dimension reduction of the self-dual Yang-Mills equations on $R^{1,1}$, then the harmonic maps can be reduced to integrable systems - the sine-Gordon equation [Uhlenbeck, 1992] and the wave map equation [Terng-Uhlenbeck, 2004]. These hyperbolic equations can be solved by the inverse scattering method.

◇ 1.2 Ward Equation

The Ward equation [Ward,'88]

$$\left(J^{-1}J_t\right)_t - \left(J^{-1}J_x\right)_x - \left(J^{-1}J_y\right)_y - \left[J^{-1}J_t, J^{-1}J_y\right] = 0,$$
$$J : \mathrm{R}^3 \longrightarrow SU(n),$$

is obtained by a dimension reduction and a gauge fixing from the self-dual Yang-Mills equation on $R^{2,2}$. It differs only slightly from the wave map equation and is an integrable system with a Lax pair

$$\left[\lambda\partial_x - \partial_\xi - J^{-1}\partial_\xi J, \lambda\partial_\eta - \partial_x - J^{-1}\partial_x J\right] = 0,$$

$\forall\lambda \in \mathrm{C}$, with $\xi = \frac{t+y}{2}$, $\eta = \frac{t-y}{2}$. Setting $J^{-1}\partial_\xi J = -\partial_x Q$, $J^{-1}\partial_x J = -\partial_\eta Q$ and $(\eta, x, \xi) \to (x, y, t)$, then the **Ward equation** turns into [Fokas-Ioannidou,'01]

$$\partial_x\partial_t Q = \partial_y^2 Q + [\partial_y Q, \partial_x Q] \tag{2}$$

with a **Lax Pair**

$$\begin{aligned} (\partial_y - \lambda\partial_x)\,\Psi &= (\partial_x Q)\Psi, \\ (\partial_t - \lambda\partial_y)\,\Psi &= (\partial_y Q)\Psi. \end{aligned} \tag{3}$$

$\diamond$ 1.3 References

(1) Construction of **rational solutions**: [Ward, '88] ..., [Dai-Terng, '07].

- Ψ is mero. in $\lambda \in \mathrm{C}$ with finite many poles;
- *Only discrete scattering data appears.*

(2) Cauchy problem with **small** ini. data: [Villarroel, '90], [Fokas-Ioannidou, '01], [Dai-Terng-Uhlenbeck].

- Ψ has jump across R;
- *Only continuous scattering data appears.*

◇ 1.4 Goal

(1) Solve the Cauchy problem of the Ward eq. with

- **large** initial data;
- pure continuous scattering data

by **the Inverse Scattering Method**:

$$\begin{array}{ccc} \text{operator, } t=0 & \rightarrow & \text{scattering data, } t=0 \\ & & \downarrow \\ \text{operator, } t=T & \leftarrow & \text{scattering data, } t=T. \end{array}$$

(2) Solve the Cauchy problem of the Ward eq. with

- **mixed** (finite) scattering data

by **Backlund transformation**.

§2. Direct Problem

◇ 2.1 Theorem (Existence of the eigenfunction)

Let $Q \in \mathrm{P}_{\infty,\mathrm{k},0}$, $k \geq 2$. Then there is a bounded set $Z \subset \mathrm{C}$ such that $Z \cap (C\backslash \mathrm{R})$ is discrete in $C\backslash\mathrm{R}$, and

- For $\lambda \in \mathrm{C}\backslash(\mathrm{R} \cup \mathrm{Z})$, $\exists! \Psi$ s.t. $\Psi - 1 \in \mathrm{DH}^{\mathrm{k}}$,

$$(\partial_y - \lambda\partial_x)\,\Psi = (\partial_x Q)\Psi, \tag{3}$$

and $\Psi(x, y, \lambda) \to 1$, as $|x|$, $|y| \to \infty$.

- $\Psi(x, y, \cdot)$ is meromorphic in $\lambda \in \mathrm{C}\backslash\mathrm{R}$ with poles at $Z\backslash\mathrm{R}$, and $\Psi(x, y, \lambda) \to 1$ uniformly, as $|\lambda| \to \infty$.

- $$\det \Psi = 1, \quad \Psi(\lambda)\Psi^*(\bar{\lambda}) = 1.$$

$\diamond$ 2.2 Proof

• When $Q \in \mathrm{P}_{\infty,k,0}$ and $|\widehat{Q}_x(\xi,y)|_{L_1(d\xi dy)} < 1$. Then by [V], [F-I], [D-T-U], there exist Ψ and $Z = \phi$ which fullfill the theorem.

• Suppose $|\widehat{Q}_x(\xi,y)|_{L_1(d\xi dy)} < \left(\frac{3}{2}\right)^{N+1}$ and the theorem holds for N. Note

if $\Psi(x,y,\lambda)$ is an e.f. of $Q(x,y)$,

then $\Psi(x,y+y_0,\lambda)$ is an e.f. of $Q(x,y+y_0)$.

Hence wlog, we can assume

$$\begin{aligned} Q &= Q^- + Q^+, \\ Q^- &= 0, \text{ for } y \geq 1, \\ Q^+ &= 0, \text{ for } y \leq -1, \\ |\widehat{Q}_x^{\pm}(\xi,y)|_{L_1(d\xi dy)} &< \left(\frac{3}{2}\right)^N . \end{aligned}$$

So by induction, there exist $\Psi^{\pm}$ of $Q^{\pm}$ satisfying the theorem.

• Using the derivation property of $\partial_y - \lambda\partial_x$ and a change of variables $x + \lambda y = \tilde{x} + i\tilde{y} = z$, $\tilde{x}$, $\tilde{y} \in \mathrm{R}$. The existence of Ψ is eqivalent to solving the Riemann-Hilbert problem

$$
\begin{aligned}
&\partial_{\bar{z}} f(x, y, \lambda) = 0, \text{ for } z \in \mathrm{C}^{\pm}, \\
&f_{+}(x, 0, \lambda) = f_{-}(x, 0, \lambda) F(x, \lambda), \text{ for } z \in \mathrm{R} \\
&F(x, \lambda) = \Psi^{-}(x, 0, \lambda)^{-1} \Psi^{+}(x, 0, \lambda).
\end{aligned}
$$

◇ 2.3 Remarks

• The RH problem is a problem with non-small data in general. To solve it, we will first approximate the solution by a piecewise **rational** function. Then the correction is made by a solution of a Riemann-Hilbert problem with **small data** and a solution of a **finite linear system**.

• Since in the induction scheme,

(1) the eigenfunction derived in each step consists the data of *RH problems* in the next step;

(2) the nice asymptote is required to assure the solvability of the system of linear equations.

Main estimates

$$\partial_x^i\left(\Psi-1\right)\in L_2(dx)\ unif.,\ \lambda\in \mathrm{C}\backslash\left(\mathrm{R}\cup D_\epsilon(\lambda_j)\right), \tag{4}$$

$$\partial_x^i\left(\Psi-1\right)\to 0\ unif.\ in\ L_2(dx)\ as\ |\lambda|\to\infty. \tag{5}$$

• The coefficents of the linear system are **meromorephic** by induction. Hence the solvability of the linear system outside a bounded discrete set in C\R follows from the **asymptotic estimate (5)**.

$\diamond$ 2.4 Definition

$$
\begin{aligned}
&\mathrm{P}_{\infty,\mathrm{k}_1,\mathrm{k}_2}\\
=&\{q_x(x,y):\mathrm{R}\times\mathrm{R}\to\mathrm{su(n)}|\\
&\\
&|\xi^i y^s\widehat{q}|_{L_1(d\xi dy)},\ ||\xi^h\widehat{q}(\xi,y)|_{L_1(y)}|_{L_2(d\xi)},\\
&|\partial_x^j\partial_y^l q|_{L_\infty},\ \sup_y|\partial_x^j\partial_y^l q|_{L_1(dx)},\ |\partial_x^j\partial_y^l q|_{L_1(dxdy)}<\infty\\
&1\le i\le\max\{5,k_1\},\quad 0\le j,\, l\le\max\{5,k_1\},\\
&1\le h\le k_1,\, 0\le s\le k_2\,\}.
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{DH}^{\mathrm{k}}\\
=&\{f\,|\partial_x^i f(x,y)\ \textit{are uniformly bounded in}\ L_2(\mathrm{R},\mathrm{dx}),\\
&0\le i\le k.\}
\end{aligned}
$$

$\diamond$ 2.5 Remark

Our goal is to prove the inverse problem when $\sharp(Z)<\infty$ and $Z\subset\mathrm{C}\backslash\mathrm{R}$. In this case, by (4), for $\lambda\in\mathrm{R}$, $\Psi_\pm$ exists, and

$$\Psi_+(x,y,\lambda)=\Psi_-(x,y,\lambda)v(x+\lambda y,\lambda).$$

$\diamond$ 2.6 Definition

If $Q \in \mathrm{P}_{\infty,\mathrm{k},0}$, $k \geq 2$ and $\Psi_\pm$ exists, then $v(x+\lambda y, \lambda)$ is called the <u>continuous scattering data</u> of Q.

$\diamond$ 2.7 Theorem (Constraints on continuous SD)

For $Q \in \mathrm{P}_{\infty,k,1}$, $k \geq 7$, if $\Psi_\pm$ exist, then $\exists\,!\; v(x,y,\lambda)$ s.t.

$$\Psi_+(x,y,\lambda) = \Psi_-(x,y,\lambda)v(x+\lambda y,\lambda), \qquad \lambda \in R.$$

Moreover,

$$\det(v) \equiv 1, \quad v = v^* > 0,$$

and for $i + j \leq k - 4$,

$\partial_x^i \partial_y^j (v-1)$ *unif. bded in* $L_\infty \cap L_2(\mathrm{R}, \mathrm{d}\lambda) \cap \mathrm{L}_1(\mathrm{R}, \mathrm{d}\lambda)$
$\partial_x^i \partial_y^j (v-1) \to 0$ *unif. in* $L_\infty \cap L_2(\mathrm{R}, \mathrm{d}\lambda) \cap \mathrm{L}_1(\mathrm{R}, \mathrm{d}\lambda)$
$\partial_\lambda v \in L_2(\mathrm{R}, d\lambda)$ *and the norms depend conti. on* x, y.

§3. Inverse Problem for $\sharp(Z) = 0$

$\diamond$ 3.1 Theorem (Solvability of the RH problem)

If $v(x + \lambda y, \lambda)$ satis. constraints in Theorem 2.7, then

- $\exists\ !\ \Psi(x, y, \cdot)$ for the RH problem

$$\partial_{\bar{\lambda}} \Psi = 0, \ \ \lambda \in \mathrm{C} \backslash \mathrm{R}, \tag{6}$$

$$\Psi_+ = \Psi_- v, \ \ \lambda \in \mathrm{R}. \tag{7}$$

- $\Psi - 1$, $\partial_x \Psi$, $\partial_y \Psi$ *unif. bded in* $L_2(\mathrm{R}, \mathrm{d}\lambda)$,

and for $\lambda \notin \mathrm{R}$, and $i + j \leq k - 4$,

$$\partial_x^i \partial_y^j \Psi \in L_\infty(dxdy),$$
$$\partial_x^i \partial_y^j (\Psi - 1) \to 0 \ in \ L_\infty(dxdy)$$

-

$$\det \Psi(x, y, \lambda) \equiv 1,$$
$$\Psi(x, y, t, \lambda) \Psi(x, y, t, \bar{\lambda})^* \equiv 1.$$

$\diamond$ 3.2 Proof of Theorem 3.1

For $|x|, |y|$ large, the RH problem is a small data problem. Hence it can be solved. For $|x|, |y|$ not large, as before, we will first approximate the solution by a piecewise **rational** function. Then the correction is made by a solution of a Riemann-Hilbert problem with **small data** and a solution of a **finite linear system**. Note we obtain the global solvability by applying the Fredholm property and the reality condition $v = v^* > 0$.

$\diamond$ 3.3 Theorem (Solv. of the inverse problem)

If $v(x + \lambda y, \lambda)$ satis. the constraints in Theorem 2.7, then

$$(\partial_y - \lambda \partial_x)\, \Psi = (\partial_x Q)\Psi$$

with

$$Q(x, y) = \frac{1}{2\pi i} \int_{\mathrm{R}} \Psi_-(v - 1) d\xi$$

and $\partial_x Q(x, y) \in su(n)$, for $i + j \le k - 4$, $i > 0$,

$$\begin{aligned}
&\partial_x^i \partial_y^j Q,\ \partial_y Q,\ Q \in L_\infty, \\
&\partial_x^i \partial_y^j Q,\ \partial_y Q,\ Q \to 0 \ as\ x\ or\ y \to \infty.
\end{aligned}$$

$\diamond$ 3.4 Proof of Theorem 3.3

By Theorem 3.1,

$$\begin{aligned}\Psi =& 1+\frac{1}{2\pi i}\int_{\mathrm{R}}\frac{\Psi_{-}(x,y,\zeta)\left(v(x+\zeta y,\zeta)-1\right)}{\zeta-\lambda}d\zeta\\ =& 1+C\Psi(v-1)\\ =& 1+C_v\Psi.\end{aligned}$$

Hence $L_\lambda v=0$ yields

$$\begin{aligned}L_\lambda\Psi =& L_\lambda C\Psi(v-1)\\ =& \left[L_\lambda,C\right]\Psi(v-1)+CL_\lambda(\Psi(v-1))\\ =& \partial_x\left(\frac{1}{2\pi i}\int_{\mathrm{R}}\Psi_{-}(v-1)d\xi\right)+C\left(L_\lambda\Psi\right)(v-1)\\ =& \partial_x Q+C_v(L_\lambda\Psi),\end{aligned}$$

with

$$Q(x,y)=\frac{1}{2\pi i}\int_{\mathrm{R}}\Psi_{-}(v-1)d\xi.$$

Finally

$$\begin{aligned}L_\lambda\Psi =& (1-C_v)^{-1}(\partial_x Q)\\ =& (\partial_x Q)(1-C_v)^{-1}1\\ =& (\partial_x Q)\Psi\end{aligned}$$

⋄ 3.7 The spectral flow

From previous observation, we find the algebraic constraint $v(x, y, \lambda) = v(x + \lambda y, \lambda)$ is key to derive (3). Hence if we let

$$v(x, y, t, \lambda) = v(x + \lambda y + \lambda^2 t, \lambda).$$

By analogy, we will obtain

$$(\partial_t - \lambda \partial_y)\, \Psi = (\partial_y Q) \Psi.$$

Theorem (Solvability of the Cauchy problem)

If $Q_0 \in \mathrm{P}_{\infty,k,1}$, $k \geq 7$, and there are no poles of the eigenfunction Ψ_0 of Q_0, then the Cauchy problem of the Ward equation with initial condition $Q(x, y, 0) = Q_0(x, y)$ admits a global solution where for $i + j + h \leq k - 4$, $i^2 + j^2 > 0$,

$$\begin{aligned}
&\partial_x Q(x, y, t) \in su(n), \\
&\partial_x^i \partial_y^j \partial_t^h Q,\ \partial_t Q,\ Q \in L_\infty, \\
&\partial_x^i \partial_y^j \partial_t^h Q,\ \partial_t Q,\ Q \to 0,\ \mathit{as}\ x,\ y,\ t \to \infty.
\end{aligned}$$

$\diamond$ 3.6 Remarks on examples

- Schwartz potentials with $|\widehat{Q_x}|_{L_1(d\xi dy)} < 1$;
- Let $v(x, y, \lambda) = v(x + \lambda, \lambda)$ satisfy

$$\begin{aligned}
&\det(v) = 1, \\
&v = v^* > 0, \\
&v - 1 \in \text{Schwartz}, \\
&v \text{ satisfies } (4'), (5'), (8).
\end{aligned}$$

Then we can solve Ψ, and justify that

$$\partial_x^i \partial_y^j \partial_\lambda^h (\Psi - 1) \text{ is Schwartz.}$$

Moreover, by using the fomula

$$Q(x, y) = \frac{1}{2\pi i} \int_{\mathrm{R}} \Psi_-(v - 1) d\xi,$$

one obtains that Q is Schwartz.

§4. The Ward Solitons

◇ **4.1 Definition** (Loop Groups)

$\mathrm{D} = \{f : \mathrm{C} \to M_n(\mathrm{C})$ *satisfies* :

- $f(\bar{\lambda})^* f(\lambda) = 1, \ \lim_{\lambda \to \infty} f = I.$
- f *is mero. in* $\mathrm{C}\backslash\mathrm{R}$ *w. finitely many poles.*
- $f_\pm$ *exist.*$\}$;

$\mathrm{D}_c = \{f \in \mathrm{D}$ *s.t.* f *is holo. in* $\mathrm{C}\backslash\mathrm{R}$ *and* $f_\pm$ *exist.*$\}$;

$\mathrm{D}_r = \{f \in \mathrm{D}$ *s.t.* f *is rational.* $\}$.

◇ **4.2 Theorem** (Uhlenbeck '89)

For $\psi \in \mathrm{D_r}$ and $Z(f) \subset \mathrm{C}^+$, we can factorize

$$\psi = f_1 \cdots f_p.$$

Where

$$f_i = g_{z_i, \pi_{i_1}} \cdots g_{z_i, \pi_{i_m}}$$

and $g_{z_i, \pi_{i_k}}$ is a simplest element of the form

$$g_{z_i, \pi_{i_k}} = 1 + \frac{z_i - \bar{z}_i}{\lambda - z_i} \pi_{i_k}^{\perp},$$

$$\pi_{i_k}^2 = \pi_{i_k}, \ \pi_{i_k}^* = \pi_{i_k},$$

$$z_i \in \mathrm{C}^+ \ \textit{and} \ z_i \neq z_j \ \textit{if} \ i \neq j.$$

$\diamond$ **4.3 Theorem** (Dai-Terng '07)

Suppose $\exists A_0(x,y)$, $\Psi_0(x,y,\lambda)$ such that

$$(\partial_y - \lambda\partial_x)\Psi_0 = A_0\Psi_0,$$

with

$$\begin{gathered}\Psi_0(x,y,\cdot) \in \mathrm{D_r} \textit{ with only pole } z \textit{ of multi.} \, k,\\ \partial_x^i\partial_y^j\Psi_0(x,y,\lambda) \in L_\infty,\\ \lim_{|x|,\,|y|\to\infty} \partial_x^i\partial_y^j(\Psi_0(x,y,\lambda) - C_1) = 0,\end{gathered}$$

for $i+j \leq N$, $\lambda \in \mathrm{C}\backslash\{\mathrm{z}\}$. Then there exist $\Psi(x,y,t,\lambda)$, $A(x,y,t)$, $B(x,y,t)$ s.t.

$$\begin{aligned}(\partial_y - \lambda\partial_x)\Psi &= A\Psi, \quad (9)\\ (\partial_t - \lambda\partial_y)\Psi &= B\Psi, \quad (10)\end{aligned}$$

$$\Psi(x,y,0,\lambda) = \Psi_0,\ A(x,y,0) = A_0,$$

and

$$\begin{gathered}\Psi(x,y,t,\cdot) \in \mathrm{D_r} \textit{ with only pole } z \textit{ of multi. } k,\\ \partial_x^i\partial_y^j\partial_t^h\Psi(x,y,t,\lambda) \in L_\infty,\\ \lim_{|x|,\,|y|,\,|t|\to\infty} \partial_x^i\partial_y^j\partial_t^h(\Psi(x,y,t,\lambda) - C_1) = 0,\end{gathered}$$

for $i+j+h \leq N$, $\lambda \in \mathrm{C}\backslash\{\mathrm{z}\}$. Where C_1 is independent of x, y, t.

$\diamond$ 4.4 Ideas of the proof

- If $k = 1$, one can set

$$\Psi_0 = 1 + \frac{z - \bar{z}}{\lambda - z} \pi_0^{\perp}(x, y), \quad \pi_0^* = \pi_0, \, \pi_0^2 = \pi_0.$$

Using the initial condi. $(\partial_y - \lambda \partial_x)\Psi_0 = A_0 \Psi_0$, one can justify that

$$A_0(x, y) = (z - \bar{z})\partial_x \pi_0,$$
$$\mathrm{Im}\pi_0 \textit{ is spanned by columns of } V(x + zy),$$
$$V : \mathrm{C} \to M_{n\times r} \textit{ is holo. in } \mathrm{C}/\{p_1, \cdots, p_s\},$$

and $M_{n\times r}$ denotes the space of rank r complex $n \times r$ matrices. Therefore, define

$$\Psi = 1 + \frac{z - \bar{z}}{\lambda - z} \pi^{\perp}, \quad \pi^* = \pi, \; \pi^2 = \pi,$$
$$\mathrm{Im}\pi \textit{ is spanned by columns of } V(x + zy + z^2 t).$$

One can justify Ψ solve the Cauchy problem with

$$A(x, y, t) = (z - \bar{z})\partial_x \pi, \qquad B(x, y, t) = (z - \bar{z})\partial_y \pi.$$

- $k > 1$, the loop group factorization theorem implies

$$\Psi_0(x,y,\lambda) = g_{z,\pi_{k+1,0}}\Psi_{k,0}(x,y,\lambda),$$
$$g_{z,k+1,0}(x,y,\lambda) = 1 + \frac{z-\bar{z}}{\lambda - z}\pi^{\perp}_{k+1,0}(x,y),$$
$$(\partial_y - \lambda\partial_x)\Psi_{k,0} = A_{k,0}(x,y)\Psi_{k,0},$$
$$\Psi_{k,0} \in \mathrm{D_r},\ Z(\Psi_{k,0}) = \{z\} \textit{ with multiplicity } k.$$

By induction, $\exists \Psi_k(x,y,t,\lambda)$ such that

$$(\partial_y - \lambda\partial_x)\Psi_k(x,y,t,\lambda) = A_k(x,y,t)\Psi(x,y,t,\lambda),$$
$$(\partial_t - \lambda\partial_y)\Psi_k(x,y,t,\lambda) = B_k(x,y,t)\Psi(x,y,t,\lambda),$$
$$\Psi_k(x,y,0,\lambda) = \Psi_{k,0}(x,y,\lambda),\ \ A_k(x,y,0) = A_{k,0}(x,y),$$

and

$$\Psi(x,y,t,\cdot) \in \mathrm{D_r} \textit{ with only pole } z \textit{ of multiplicity } k,$$
$$\cdots$$

On the other hand, by the Backlund transformation, one has

$$\tau_{1,0}(x,y),\cdots,\tau_{n_{k+1},0}(x,y) \textit{ span } \mathrm{Im}\,\pi_{k+1,0},$$
$$(\partial_y - z\partial_x)\tau_{j,0} = A_{k,0}(x,y)\tau_{j,0}.$$

Hence $\tau_{j,0}(x,y) = \eta_{k,0}(x,y)H_j(x+zy)$. Where $\eta_{k,0}$ is a fundmental solution of $(\partial_y - z\partial_x)f = A_{k,0}(x,y)f$.

Dai and Terng show that $\exists \eta_k(x,y,t)$ such that

$$
\begin{aligned}
&(\partial_y - z\partial_x)\eta_k = A_k(x,y,t)\eta_k,\\
&(\partial_t - z\partial_y)\eta_k = B_k(x,y,t)\eta_k,\\
&\eta_k(x,y,0) = \eta_{k,0}(x,y),\ A_k(x,y,0) = A_{k,0}(x,y).
\end{aligned}
$$

So $\tau_{j,0}$ can be extended by

$$\tau_j(x,y,t) = \eta_k(x,y,t)H(x+zy+z^2t),$$

then $g_{z,\pi_{k+1},0} = g_{z,\pi_{k+1,0}}\Psi_{k,0}$ can extended, too. Finally, let

$$\Psi(x,y,t,\lambda) = g_{z,\pi_{k+1}}\Psi_k(x,y,t,\lambda).$$

Applying the theory of Backlund transformation, one will show that Ψ satisfies the Cauchy problem of the Theorem.

$\diamond$ 4.5 **Theorem** (Dai-Terng '07)

Suppose

- $f(x, y, t, \lambda)$ and $g(x, y, t, \lambda)$ satisfy (9), (10) with different A, B.
- $Z(g) = \{z\}$ with multiplicity of k, and f is holomorphic and non-degenerate at z, $\bar{z}$.

Then there exist unique $\tilde{f}$ and $\tilde{g}$ s.t.

- $\Psi = \tilde{f} g = \tilde{g} f$ satisfies (9), (10) w. new A, B.
- $Z(\tilde{g}) = \{z\}$ with mult. of k, $\tilde{f}$ is holo. and non-degenerate at z, $\bar{z}$.

Moreover, if for $i + j + h \leq N$, $\lambda \in \mathrm{C} \backslash \{z\}$,

$$\partial_x^i \partial_y^j \partial_t^h f,\ \partial_x^i \partial_y^j \partial_t^h g \in L_\infty$$
$$\lim_{|x|,\,|y|,\,|t| \to \infty} \partial_x^i \partial_y^j \partial_t^h (f - C_1) = 0,$$
$$\lim_{|x|,\,|y|,\,|t| \to \infty} \partial_x^i \partial_y^j \partial_t^h (g - C_2) = 0,$$

then ....

$\diamond$ 4.6 Ideas of the proof

- If $k = 1$, then by assumption, $\exists \pi$ such that $g = 1 + \frac{z-\bar{z}}{\lambda - z}\pi^{\perp}(x,y)$. To find Ψ, we first need to find $\tilde{g}$,

$$\tilde{g} = 1 + \frac{z-\bar{z}}{\lambda - z}\tilde{\pi}^{\perp}(x,y)$$

and

$$\tilde{f} = (\tilde{g}f)\,g^{-1} \textit{ is idependent of } \lambda. \qquad (11)$$

Computing the residue of $(\tilde{g}f)\,g^{-1}$, we find that (11) is equiv. to that $\tilde{\pi}$ is the hermitian projection of C^n onto $f(x,y,t,z)\,(\mathrm{Im}\,(\pi(x,y)))$.

Secondly, we need to justify

$$((\partial_y - \lambda\partial_x)\Psi)\,\Psi^{-1}, \quad ((\partial_t - \lambda\partial_y)\Psi)\,\Psi^{-1}$$

is independent of λ. Since $\Psi = \tilde{g}f$,

$$\begin{aligned}&[(\partial_y - \lambda\partial_x)\Psi]\Psi^{-1}\\ =&[(\partial_y - \lambda\partial_x)\tilde{g}]\tilde{g}^{-1} + \tilde{g}[((\partial_y - \lambda\partial_x)f)f^{-1}]\tilde{g}^{-1}\end{aligned}$$

So $[(\partial_y - \lambda\partial_x)\Psi]\Psi^{-1}$ is holo. on $\mathrm{C}\backslash\{z,\bar{z}\}$ and is bdd at $\lambda = \infty$. On the other hand, using $\Psi = \tilde{f}g$, one has

$$\begin{aligned}&[(\partial_y - \lambda\partial_x)\Psi]\Psi^{-1}\\ =&[(\partial_y - \lambda\partial_x)\tilde{f}]\tilde{f}^{-1} + \tilde{f}[((\partial_y - \lambda\partial_x)g)g^{-1}]\tilde{f}^{-1}\end{aligned}$$

and can show that $[(\partial_y - \lambda\partial_x)\Psi]\Psi^{-1}$ is holo. at $\{z,\bar{z}\}$

- If $k > 1$, ...

§5. The Cauchy problem w. mixed scattering data

⋄ 5.1 Theorem

If $Q_0 \in \mathrm{P}_{\infty,k,1}$, $k \geq 7$, $Z(\Psi_0) \subset \mathrm{C}\backslash\mathrm{R}$, $\sharp(Z(\Psi_0)) < \infty$, then the Cauchy problem of the Ward equation with initial condition $Q(x,y,0) = Q_0(x,y)$ admits a global solution where for $i+j+h \leq k-4$, $i^2+j^2 > 0$,

$$\begin{aligned}
&\partial_x Q(x,y) \in su(n),\\
&\partial_x^i \partial_y^j \partial_t^h Q,\ \partial_t Q,\ Q \in L_\infty,\\
&\partial_x^i \partial_y^j \partial_t^h Q,\ \partial_t Q,\ Q \to 0,\ \ as\ x,\ y,\ t \to \infty.
\end{aligned}$$

⋄ 5.2 Remarks on the proof

- $\Psi_{0,\pm}$ exists and $\exists v$ such that $\Psi_{0,+} = \Psi_{0,-}v$.
- Solve the RH problem with data v, $\exists f$ s. t.

$$\begin{aligned}
&\partial_{\bar{\lambda}} f = 0,\ \ \lambda \in \mathrm{C}\backslash\mathrm{R},\\
&f_+ = f_- v.
\end{aligned}$$

Hence

$$\Psi_0 = \tilde{g} f, \quad \tilde{g} \in \mathrm{D_r}.$$

- By analogy,

$$\exists \tilde{f} \in \mathrm{D_c},\ \tilde{f}_+ = \tilde{v}\tilde{f}_-,$$

and

$$\Psi_0 = \tilde{f} g, \quad g \in \mathrm{D_r}.$$

- Since

$$L_\lambda \Psi_0 = A_0 \Psi_0, \quad \Psi_0 = \tilde{g} f = \tilde{f} g.$$

We conclude

$$L_\lambda f = A_{0,c} f, \quad f \in \mathrm{D_c} \tag{12}$$

$$L_\lambda g = A_{0,r} g, \quad g \in \mathrm{D_r} \tag{13}$$

- (Induction) If $Z(g) = \{z\}$, then:
  Using the solvability of the Cauchy problem with pure. conti. data, $\exists F$,

$$\begin{aligned}
&L_\lambda F = A_c F, \\
&M_\lambda F = B_c F, \\
&F(x, y, 0, \lambda) = f(x, y, \lambda),\ A_c(x, y, 0) = A_{0,c}(x, y).
\end{aligned}$$

Using the solvability of the Cauchy problem with pure. discrete data, $\exists G$,

$$\begin{aligned}
&L_\lambda G = A_r G, \\
&M_\lambda G = B_r G, \\
&G(x, y, 0, \lambda) = g(x, y, \lambda),\ A_r(x, y, 0) = A_{0,r}(x, y).
\end{aligned}$$

- Using the Backlund transformation, $\exists\,!\,\tilde{F},\ \tilde{G}$ s.t.

$$\Psi = \tilde{F}G = \tilde{G}F$$

and

$$L_\lambda \Psi = A\Psi, \quad M_\lambda \Psi = B\Psi,$$

$$\begin{aligned} &\Psi(x,y,0,\lambda) \\ =&\tilde{F}(x,y,0,\lambda)g(x,y,\lambda) = \tilde{G}(x,y,0,\lambda)f(x,y,\lambda) \\ =&\tilde{f}(x,y,\lambda)g(x,y,\lambda) = \tilde{g}(x,y,\lambda)f(x,y,\lambda) \\ =&\Psi_0(x,y,\lambda). \end{aligned}$$

- (Induction) If $Z(g) = \{z_1, \cdots, z_k\}$, $k > 1$, then by the loop group factorization theorem, we have

$$\begin{aligned} &\Psi_0' = \tilde{f}_{k-1,0}\, g_0'' = \tilde{g}_0'' f_{k-1,0}, \\ &Z(g_0'') = \{\alpha\},\ \alpha \notin Z(\tilde{f}_{k-1,0}), \\ &Z(\tilde{g}_0'') = \{\beta\},\ \beta \notin Z(f_{k-1,0}). \end{aligned}$$

Where

$$\begin{aligned} &\tilde{f}_{k-1,0} = \tilde{f}_0 g_{0,1}' \cdots g_{0,k-1}', \\ &f_{k-1,0} = \tilde{g}_{0,2}' \cdots \tilde{g}_{0,k}' f_0. \end{aligned}$$

Using the induction hypo.(solv. of the Cauchy problem with mixed scattering data with distinct $k-1$ poles), and the solv. of the Cauchy problem with pure. discrete data, we obatin $G''(x,y,t,\lambda)$, $F_{k-1}(x,y,t,\lambda)$ ......

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