Electroweak responses of few-body systems at low energies.

M. Viviani

INFN, Sezione di Pisa &
Department of Physics, University of Pisa
Pisa (Italy)

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Outline

Lectures 2 & 3: calculation of the $n-p$ radiative capture total cross section

Lecture 1

1. Notation
2. Electromagnetic transition
3. Generic direction of the photon
4. Lonw-wavelength approximation
5. Weak transition

Lecture 4: recent results

- $p-p$ & $p-{}^3\mathrm{He}$ fusion rates
- $n-d$ & $p-d$ radiative captures
- μ-capture

Notation (1)

Notation

- $\hbar = c = 1 \to$ "reduced quantities"
  - $m \equiv mc^2/(\hbar c)$ [fm^{-1}]; Ex. $M_p = 4.756$ [fm^{-1}]
  - $p \equiv pc/(\hbar c)$ [fm^{-1}]
  - etc.
- normalization of the momentum eigenfunctions
  - "box" of volume $\Omega = L^3$

$$\textit{Periodic boundary conditions} \quad \langle \boldsymbol{r} | \boldsymbol{p} \rangle = \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{r}}}{\sqrt{\Omega}}$$

  - $\boldsymbol{p} \equiv (2\pi/L)\{n_x, n_y, n_z\}$, $n_i = 0, \pm 1, \pm 2, \ldots$
  - $\langle \boldsymbol{p} | \boldsymbol{p}' \rangle = \delta_{\boldsymbol{p},\boldsymbol{p}'}$
  - $\sum_{\boldsymbol{p}} \to \Omega \int d^3p/(2\pi)^3$
- Clebsh-Gordan $[\psi_{j_1}\psi_{j_2}]_{JM} \equiv \sum_{m_1 m_2} (j_1 m_1 j_2 m_2 | JM) \psi_{j_1 m_1} \psi_{j_2 m_2}$

Notation (2)

Representations

- Time dependent perturbation theory
- Schroedinger representation $H = H_0 + V$
- Interaction representation $|\Psi_I(t)\rangle = \exp(iH_0 t)|\Psi_S(t)\rangle$
- $V_I(t) = \exp(iH_0 t) V \exp(-iH_0 t) \quad |\Psi_I(t)\rangle = U_I(t, t_0)|\Psi_I(t_0)\rangle$

$$i \frac{d}{dt} U_I(t, t_0) = V_I(t) U_I(t, t_0)$$

- $U_I(t, t_0) = 1 - i \int_{t_0}^{t} dt_1\ V_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1\ V_I(t_1) \int_{t_0}^{t_1} dt_2\ V_I(t_2) + \cdots$
- $P_{\alpha,\beta} = \lim_{T \to \infty} \frac{1}{T} |\langle\beta|U_I(+T/2, -T/2)|\alpha\rangle|^2 \to 2\pi\delta(E_\alpha - E_\beta)|\langle\beta|V|\alpha\rangle + \cdots|^2$
- First order $\to$ Fermi's Golden rule
- Matrix elements of V are now in Schroedinger representation

Electromagnetic transitions

The interaction Hamiltonian

- Reference text ([J.D. Walecka, *Theoretical Nuclear and Subnuclear Physics*])
- "Coulomb" gauge: $\boldsymbol{A}$ purely transverse, ρ static

$$V = -e \int d^3x \boldsymbol{J}(x) \cdot \boldsymbol{A}(x)$$

- EM field

$$\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{q}} \sum_{\lambda=\pm 1} \frac{1}{\sqrt{2\omega_{\mathbf{q}}\Omega}} \left[a_{\mathbf{q},\lambda} e^{i\mathbf{q}\cdot\mathbf{x}} \hat{\epsilon}_{\mathbf{q},\lambda} + a^\dagger_{\mathbf{q},\lambda} e^{-i\mathbf{q}\cdot\mathbf{x}} \hat{\epsilon}^*_{\mathbf{q},\lambda} \right]$$

- circular polarization vectors

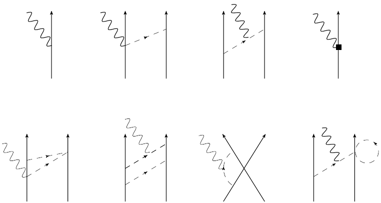
$$\hat{\epsilon}_{\mathbf{q},\pm 1} = \mp \frac{\hat{\epsilon}_{\mathbf{q},1} \pm i \hat{\epsilon}_{\mathbf{q},2}}{\sqrt{2}} \qquad \hat{\epsilon}_{\mathbf{q},\pm 0} = \hat{\boldsymbol{q}}$$

- $\hat{\boldsymbol{q}}$, $\hat{\epsilon}_{\mathbf{q},1}$, and $\hat{\epsilon}_{\mathbf{q},2}$ form an orthogonal basis
- Example: photon emission

$$\langle \boldsymbol{q}\lambda; J_f, M_f | V | 0; J_i, M_i \rangle = -\frac{e}{\sqrt{2\Omega\omega_{\mathbf{q}}}} \int d^3x \, \langle J_f, M_f | \boldsymbol{J}(\boldsymbol{x}) \cdot e^{-i\mathbf{q}\cdot\mathbf{x}} \hat{\epsilon}^*_{\mathbf{q},\lambda} | J_i, M_i \rangle$$

Nuclear current $\boldsymbol{J}(\boldsymbol{x})$

- $\boldsymbol{J}(x)$: constructed in terms of operators acting on the nucleon degrees of freedom
- From EFT [→ Epelbaum & Schiavilla talk's]
- [Kolling *et al.*, 2009], [Pastore *et al.*, 2009]



- $\boldsymbol{J}(\boldsymbol{x}) = \boldsymbol{J}^C(\boldsymbol{x}) + \boldsymbol{\nabla} \times \boldsymbol{\mu}(\boldsymbol{x})$
- LO: single-nucleon current, lowest order

$$\boldsymbol{J}^C(\boldsymbol{x}) = \sum_{j=1}^{A} \frac{1}{2M} e_j \left[\delta(\boldsymbol{x} - \boldsymbol{r}_j)\boldsymbol{p}_j + \boldsymbol{p}_j \delta(\boldsymbol{x} - \boldsymbol{r}_j)\right]$$

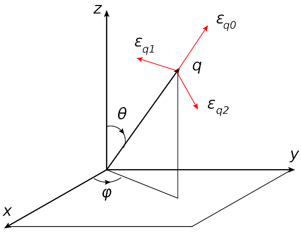
$$\boldsymbol{\mu}(\boldsymbol{x}) = \sum_{j=1}^{A} \frac{1}{2M} \mu_j \boldsymbol{\sigma}_j \delta(\boldsymbol{x} - \boldsymbol{r}_i)$$

- → it derives from $V_M = -\int d^3x\, \boldsymbol{B} \cdot \boldsymbol{\mu}$
- NLO: one-pion exchange contribution
- NNLO: $1/M^2$ relativistic correction to the LO
- N3LO: one-loop, tadpoles, contact minimal & non-minimal, . . .

Multipole analysis

Frames

- Lab frame ("L")
- $\boldsymbol{q}$ frame



$\boldsymbol{q}$ frame

- "Solid" spherical harmonics

$$\boldsymbol{Y}^{M}_{J\ell 1}(\hat{\boldsymbol{x}}) = \sum_{m=-\ell,\ell} \sum_{\lambda=0,\pm 1} (\ell, m, 1, \lambda | J, M) Y_{\ell m}(\hat{x}) \hat{\epsilon}_{\mathbf{q},\lambda} \qquad \hat{\epsilon}_{\mathbf{q},\lambda=0} = \hat{\boldsymbol{q}}$$

- $e^{i\boldsymbol{q}\cdot\boldsymbol{x}} = \sum_\ell i^\ell \sqrt{4\pi(2\ell+1)} Y_{\ell 0}(\hat{\boldsymbol{x}}) j_\ell(qx)$

$$\hat{\epsilon}_{\mathbf{q},\lambda} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} = \sum_{\ell,J} i^\ell \sqrt{4\pi(2\ell+1)} j_\ell(qx) (\ell, 0, 1, \lambda | J, \lambda) \boldsymbol{Y}^{\lambda}_{J\ell 1}(\hat{\boldsymbol{x}})$$

Useful formulas

Properties of the Clebsh-Gordan
& of the Bessel functions

$$
\begin{aligned}
(\ell,0,1,+1|\ell+1,+1) &= \sqrt{\frac{\ell+2}{2(2\ell+1)}} \\
(\ell,0,1,-1|\ell+1,-1) &= \sqrt{\frac{\ell+2}{2(2\ell+1)}}
\end{aligned}
$$

$$
\begin{aligned}
(\ell,0,1,+1|\ell,+1) &= -\sqrt{\frac{1}{2}} \\
(\ell,0,1,+1|\ell,-1) &= \sqrt{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
(1,+1,\ell,0|\ell-1,+1) &= \sqrt{\frac{\ell-1}{2(2\ell+1)}} \\
(1,-1,\ell,0|\ell-1,-1) &= \sqrt{\frac{\ell-1}{2(2\ell+1)}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{q}\boldsymbol{\nabla}\times j_J(qx)\boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}}) &= -i\Bigg(\sqrt{\frac{J+1}{2J+1}}j_{J-1}(qx)\boldsymbol{Y}^{\lambda}_{J,J-1,1}(\hat{\boldsymbol{x}}) \\
&\quad -\sqrt{\frac{J}{2J+1}}j_{J+1}(qx)\boldsymbol{Y}^{\lambda}_{J,J+1,1}(\hat{\boldsymbol{x}})\Bigg)
\end{aligned}
$$

$$\hat{\epsilon}_{\mathbf{q},\lambda} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} = \sum_J \Bigg[i^{J-1}\sqrt{4\pi} j_{J-1}(qx)\sqrt{\frac{J+1}{2}}\boldsymbol{Y}^{\lambda}_{J,J-1,1}(\hat{\boldsymbol{x}}) - i^J\sqrt{4\pi} j_J(qx)\lambda\sqrt{\frac{2J+1}{2}}\boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}}) + i^{J+1}\sqrt{4\pi} j_{J+1}(qx)\sqrt{\frac{J}{2}}\boldsymbol{Y}^{\lambda}_{J,J+1,1}(\hat{\boldsymbol{x}}) \Bigg]$$

$$= \sum_J i^J\sqrt{4\pi}\sqrt{\frac{2J+1}{2}} \Bigg[-\lambda j_J(qx)\boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}}) - i\Bigg(\sqrt{\frac{J+1}{2J+1}} j_{J-1}(qx)\boldsymbol{Y}^{\lambda}_{J,J-1,1}(\hat{\boldsymbol{x}}) - \sqrt{\frac{J}{2J+1}} j_{J+1}(qx)\boldsymbol{Y}^{\lambda}_{J,J+1,1}(\hat{\boldsymbol{x}}) \Bigg) \Bigg]$$

$$= \sum_J i^J\sqrt{2\pi(2J+1)} \Bigg[-\lambda j_J(qx)\boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}}) - \frac{1}{q}\boldsymbol{\nabla}\times j_J(qx)\boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}}) \Bigg]$$

- $\hat{\epsilon}^*_{\mathbf{q},\lambda} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} = [\hat{\epsilon}_{\mathbf{q},\lambda} e^{i\boldsymbol{q}\cdot\boldsymbol{x}}]^*$
- $\boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}})^\dagger = \boldsymbol{Y}^{-\lambda}_{J,J,1}(\hat{\boldsymbol{x}})$

Matrix element in the "$\boldsymbol{q}$" frame

- Final expression

$$\langle \boldsymbol{q}\lambda; J_f, M_f|V|0; J_i, M_i\rangle = \frac{e}{\sqrt{2\omega_q\Omega}}\sum_{J\geq 1}(-i)^J\sqrt{2\pi(2J+1)}\langle J_f, M_f|T^E_{J,-\lambda}+\lambda T^M_{J,-\lambda}|J_i, M_i\rangle_{\boldsymbol{q}}$$

- $|J_f, M_f\rangle$ and $|J_i, M_i\rangle$ "quantized" along $\boldsymbol{q}$

Electric and magnetic operators

$$T^E_{J,M} = \frac{1}{q}\int d^3x\, \boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{\nabla}\times j_J(qx)\boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}})$$

$$T^M_{J,M} = \int d^3x\, \boldsymbol{J}(\boldsymbol{x})\cdot j_J(qx)\boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}})$$

Wigner-Eckart $\rightarrow$ reduced matrix elements (RMEs)

$$\langle J_f, M_f|T^X_{J,M}|J_iM_i\rangle = \frac{(-)^{J_i-M_i}}{\sqrt{2J+1}}(J_f, M_f, J_i, -M_i|J, M)X^{J_f,J_i}_J$$

Matrix element in the "$\boldsymbol{q}$" frame

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Selection rules

Under Parity

- $\boldsymbol{x} \to -\boldsymbol{x}$, $\boldsymbol{J} \to -\boldsymbol{J}$, $\boldsymbol{Y} \to (-)^J \boldsymbol{Y}$
- $T^E_{J,M} \to (-)^J T^E_{J,M}$
- $T^M_{J,M} \to (-)^{J+1} T^M_{J,M}$

Selection rules

- "EJ" transitions: $\langle J_f, M_f | T^E_{J,-\lambda} | J_i M_i \rangle \neq 0$ if
  - $|J_f - J_i| \leq J \leq J_f + J_i$
  - $\Pi_f \Pi_i = (-)^J$
- "MJ" transitions: $\langle J_f, M_f | T^M_{J,-\lambda} | J_i M_i \rangle \neq 0$ if
  - $|J_f - J_i| \leq J \leq J_f + J_i$
  - $\Pi_f \Pi_i = (-)^{J+1}$
- Examples
  - $0^+ \to 0+$ "forbidden"
  - $1^+ \to 0^+$ M1 transition $1^- \to 0^+$ E1 transition
  - $2^+ \to 0^+$ E2 transition $2^- \to 0^+$ M2 transition

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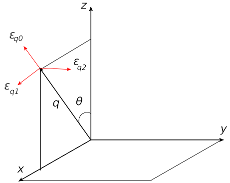
Generic direction of the photon

Frames

- Lab & $\boldsymbol{q}$ frames
- For simplicity: $\boldsymbol{q}$ in the xz plane



- Use the properties of $\boldsymbol{J}$ to express $|J,M\rangle_L$ in terms of $|J,M\rangle_{\boldsymbol{q}}$
- $[J_i, J_j] = i\epsilon_{ijk} J_k$
- $J_x e^{-i\theta J_y} = e^{-i\theta J_y}\left(\cos\theta J_x + \sin\theta J_z\right)$
- $J_z e^{-i\theta J_y} = e^{-i\theta J_y}\left(-\sin\theta J_x + \cos\theta J_z\right)$
- Exercise: $\boldsymbol{J}\cdot\hat{\boldsymbol{q}} e^{-i\theta J_y}|J,M\rangle_L = M e^{-i\theta J_y}|J,M\rangle_L$
- Therefore $|J,M\rangle_{\boldsymbol{q}} = e^{-i\theta J_y}|J,M\rangle_L$ or alternatively $|J,M\rangle_L = e^{+i\theta J_y}|J,M\rangle_{\boldsymbol{q}}$
- Rotation matrices $d^J_{M',M}(\theta) = \langle J,M'|e^{-i\theta J_y}|J,M\rangle$

$$|J,M\rangle_L = \sum_{M'} |J,M'\rangle_{\boldsymbol{q}}\ \langle J,M'|_{\boldsymbol{q}} e^{+i\theta J_y}|J,M\rangle_{\boldsymbol{q}} = \sum_{M'} |J,M'\rangle_{\boldsymbol{q}}\ d^J_{M',M}(-\theta)$$

Generic direction of the photon (2)

Matrix element in the Lab frame

$$
\begin{aligned}
&\langle \boldsymbol{q}\lambda; J_f, M_f | V | 0; J_i, M_i\rangle_L = \\
&= -\frac{e}{\sqrt{2\omega_q \Omega}} \langle J_f, M_f|_L \int d^3x \boldsymbol{J}(\boldsymbol{x}) \cdot \hat{\epsilon}^*_{\mathbf{q},\lambda} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} |J_i, M_i\rangle_L \\
&= -\frac{e}{\sqrt{2\omega_q \Omega}} \langle J_f, M_f|_{\boldsymbol{q}} e^{-i\theta J_y} \int d^3x \boldsymbol{J}(\boldsymbol{x}) \cdot \hat{\epsilon}^*_{\mathbf{q},\lambda} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} e^{i\theta J_y} |J_i, M_i\rangle_{\boldsymbol{q}} \\
&= -\frac{e}{\sqrt{2\omega_q \Omega}} \sum_{M_f', M_i'} d^{J_f}_{M_f', M_f}(-\theta)^* d^{J_i}_{M_i', M_i}(-\theta) \langle J_f, M_f'|_{\boldsymbol{q}} \int d^3x \boldsymbol{J}(\boldsymbol{x}) \cdot \hat{\epsilon}^*_{\mathbf{q},\lambda} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} |J_i, M_i'\rangle_{\boldsymbol{q}} \\
&= \frac{e}{\sqrt{2\omega_q \Omega}} \sum_{M_f', M_i'} d^{J_f}_{M_f', M_f}(-\theta)^* d^{J_i}_{M_i', M_i}(-\theta) \langle J_f, M_f'|_{\boldsymbol{q}} \sum_J (-i)^J \sqrt{2\pi(2J+1)} \\
&\qquad \times \left[T^E_{J,-\lambda} + \lambda T^M_{J,-\lambda} \right] |J_i, M_i'\rangle_{\boldsymbol{q}} \\
&= \frac{e}{\sqrt{2\omega_q \Omega}} \langle J_f, M_f|_{\boldsymbol{q}} e^{-i\theta J_y} \sum_J (-i)^J \sqrt{2\pi(2J+1)} \left[T^E_{J,-\lambda} + \lambda T^M_{J,-\lambda} \right] e^{i\theta J_y} |J_i, M_i\rangle_{\boldsymbol{q}}
\end{aligned}
$$

Properties of an operator of rank ℓm

$$e^{-i\theta J_y} T_{\ell,m} e^{i\theta J_y} = \sum_{m'} d^{\ell}_{m',m}(-\theta) T_{\ell,m'}$$

Example

Exercise: Take $\boldsymbol{r} = (x, y, z)$ and $\boldsymbol{L} = -i\boldsymbol{r} \times \boldsymbol{\nabla}$: Show that

$$e^{-i\theta L_y} r_+ e^{i\theta L_y} = \sum_{m'} d^{1}_{m',+1}(-\theta) r_{m'}$$

where

$$r_+ = -\frac{x+iy}{\sqrt{2}} \qquad r_0 = z \qquad r_- = +\frac{x-iy}{\sqrt{2}}$$

is a tensor of rank 1. Note that

$$d^1_{m',m}(\theta) = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}$$

Finally

$$\langle \boldsymbol{q}\lambda; J_f, M_f | V | 0; J_i, M_i \rangle_L =$$

$$= \frac{e}{\sqrt{2\omega_q \Omega}} \langle J_f, M_f |_{\boldsymbol{q}} e^{-i\theta J_y} \sum_J (-i)^J \sqrt{2\pi(2J+1)} \left[T^E_{J,-\lambda} + \lambda T^M_{J,-\lambda} \right] e^{i\theta J_y} | J_i, M_i \rangle_{\boldsymbol{q}}$$

$$= \frac{e}{\sqrt{2\omega_q \Omega}} \langle J_f, M_f |_{\boldsymbol{q}} \sum_{J,M} (-i)^J \sqrt{2\pi(2J+1)} d^J_{M,-\lambda}(-\theta) \left[T^E_{J,M} + \lambda T^M_{J,M} \right] | J_i, M_i \rangle_{\boldsymbol{q}}$$

In terms of the RMEs

$$\langle J_f, M_f | T^X_{J,M} | J_i M_i \rangle = \frac{(-)^{J_i - M_i}}{\sqrt{2J+1}} (J_f, M_f, J_i, -M_i | J, M) X_J^{J_f, J_i}$$

Summary

$\boldsymbol{q}$ frame

$$\langle \boldsymbol{q}\lambda; J_f, M_f | V | 0; J_i, M_i \rangle_{\boldsymbol{q}} = $$
$$= \frac{e}{\sqrt{2\omega_q \Omega}} \sum_{J \geq 1} (-i)^J \sqrt{2\pi} (-)^{J_i - M_i} (J_f, M_f, J_i, -M_i | J, -\lambda) \left[E_J^{(J_f, J_i)} + \lambda M_J^{(J_f, J_i)} \right]$$

Lab frame

$$\langle \boldsymbol{q}\lambda; J_f, M_f | V | 0; J_i, M_i \rangle_L = $$
$$= \frac{e}{\sqrt{2\omega_q \Omega}} \sum_{J \geq 1, M} (-i)^J \sqrt{2\pi} d^J_{M,-\lambda}(-\theta) (-)^{J_i - M_i} (J_f, M_f, J_i, -M_i | J, M) \left[E_J^{(J_f, J_i)} + \lambda M_J^{(J_f, J_i)} \right]$$

The RMEs are the same!

Gamma decay rate

- Probability of decay for unit time

$$dw = 2\pi\delta(E_i - E_f - \omega_q)|\langle \boldsymbol{q}\lambda; J_f, M_f|V|0; J_i, M_i\rangle|^2$$

- Total decay probability

$$w = \frac{1}{2J_i + 1}\sum_{\boldsymbol{q},\lambda} 2\pi\delta(E_i - E_f - \omega_q)|\langle \boldsymbol{q}\lambda; J_f, M_f|V|0; J_i, M_i\rangle|^2$$

Example

Exercise: Show that

$$w = \frac{1}{2J_i + 1} 8\pi\alpha\omega \sum_{J\geq 1}\left[|E_J^{(J_f, J_i)}|^2 + |M_J^{(J_f, J_i)}|^2\right]$$

$\alpha =$ fine structure constant $= e^2/(4\pi)$

Lonf-wavelenght approximation

- Since usually $\omega_q \equiv |\boldsymbol{q}| = E_i - E_f \approx 1$ MeV $\to 1/200$ fm^{-1}
- $qx \leq 1/20$ so we can expand $j_J(qx) \approx (qx)^J/(2J+1)!!$
- Assuming $\boldsymbol{J}(\boldsymbol{x}) = \boldsymbol{J}_C(\boldsymbol{x}) + \boldsymbol{\nabla} \times \boldsymbol{\mu}(\boldsymbol{x})$
- After some lengthy passage

$$
\begin{aligned}
T^E_{J,M} &= -i\frac{q^{J-1}}{(2J+1)!!}\sqrt{\frac{J+1}{J}}\int d^3x \left[\boldsymbol{\nabla}\cdot\boldsymbol{J}_C(\boldsymbol{x}) + \frac{q^2}{J+1}\boldsymbol{\nabla}\cdot\Big(\boldsymbol{x}\times\boldsymbol{\mu}(\boldsymbol{x})\Big)\right]x^J Y_{JM}(\hat{\boldsymbol{x}})\\
T^M_{J,M} &= i\frac{q^{J}}{(2J+1)!!}\sqrt{\frac{J+1}{J}}\int d^3x \left[\boldsymbol{\mu}(\boldsymbol{x}) + \frac{1}{J+1}\Big(\boldsymbol{x}\times\boldsymbol{J}_C(\boldsymbol{x})\Big)\right]\cdot\boldsymbol{\nabla}\Big(x^J Y_{JM}(\hat{\boldsymbol{x}})\Big)
\end{aligned}
$$

- Using Current Conservation

$$
\boldsymbol{\nabla}\cdot\boldsymbol{J}(x) + i\Big[H_0,\rho(\boldsymbol{x})\Big] = 0 \qquad \langle J_f, M_f|\Big[H_0,\rho(\boldsymbol{x})\Big]|J_i, M_i\rangle = (E_f - E_i)\langle J_f, M_f|\rho(\boldsymbol{x})|J_i, M_i\rangle
$$

- ... but $(E_f - E_i) = q$, therefore the electric RMEs can be approximated by

$$
T^E_{J,M} \approx \frac{q^J}{(2J+1)!!}\sqrt{\frac{J+1}{J}}\int d^3x\ \rho(\boldsymbol{x})x^J Y_{JM}(\hat{\boldsymbol{x}})
$$

- We can assume with good approximation $\rho(\boldsymbol{x}) = \sum_{i=1}^{A} e_i\delta(\boldsymbol{x} - \boldsymbol{r}_i)$

Weak transitions

Hamiltonian

- Current-current interaction

$$V_W = \frac{G_V}{\sqrt{2}} \int d^3x \; \mathcal{J}_\mu(x)\mathcal{J}^\mu(x)$$

- $\mathcal{J}_\mu(x) = \mathcal{J}_\mu^{(l)}(x) + \mathcal{J}_\mu^{(h)}(x)$
- $\mathcal{J}_\mu^{(l)}(x)$ can be taken directly from the standard model
- $\mathcal{J}_\mu^{(h)}(x) = \mathcal{V}_\mu(x) - \mathcal{A}_\mu(x)$ vector and axial currents

Leptonic matrix element $= l_\mu e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}$

- Beta decay ${}^A_Z X \to {}^A_{Z+1} Y + e^- + \bar\nu_e$

$$\langle e^-;\bar\nu_e|\mathcal{J}_\mu(x)^{(l)}|0\rangle = \frac{\bar u_e}{\sqrt{2E_e\Omega}}\gamma_\mu(1-\gamma^5)\frac{v_{\nu_e}}{\sqrt{2E_{\nu_e}\Omega}}e^{-i(\boldsymbol{p}_e+\boldsymbol{p}_{\nu_e})\cdot\boldsymbol{x}}$$

- Muon capture $\mu^- + {}^A_Z X \to {}^A_{Z-1} Y + \nu_\mu$

$$\langle \nu_\mu|[\mathcal{J}_\mu(x)^{(l)}]^\dagger|\mu^-\rangle = \frac{\bar u_{\nu_\mu}}{\sqrt{2E_{\nu_\mu}\Omega}}\gamma_\mu(1-\gamma^5)\frac{u_\mu}{\sqrt{2E_\mu\Omega}}e^{i(\boldsymbol{p}_\mu-\boldsymbol{p}_{\nu_\mu})\cdot\boldsymbol{x}}$$

- We can deconpose

$$\boldsymbol{l} = \sum_{\lambda=0,\pm1} l_\lambda \hat{\epsilon}^*_{\mathbf{q},\lambda}$$

- It is easy ti see that

$$l_{\lambda=\pm1} = \mp\frac{1}{\sqrt{2}}(l_1 \pm i l_2) \qquad l_{\lambda=0} = l_3$$

- In general, thus, the matrix element of V_W will be of the type

$$\langle J_f, M_f|V_W|J_i, M_i\rangle = \frac{G_V}{\sqrt{2}}\langle J_f, M_f|\int d^3x \left[l_0\rho^{(h)}(\boldsymbol{x}) - \sum_\lambda l_\lambda \hat{\epsilon}^*_{\mathbf{q},\lambda}\cdot \boldsymbol{J}^{(h)}(\boldsymbol{x})\right]e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}|J_i, M_i\rangle$$

- For the part $\hat{\epsilon}^*_{\mathbf{q},\lambda}e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}$ we can use the same formulas as derived for the EM current

Decomposition of the longitudinal part ($\boldsymbol{q}$ frame)

- Longitudinal part

$$\hat{\epsilon}_{\mathbf{q},0}e^{i\boldsymbol{q}\cdot\boldsymbol{x}} = -\frac{i}{q}\sum_{J\geq0}\sqrt{4\pi(2J+1)}i^J\boldsymbol{\nabla}\left[j_J(qx)Y_{J,0}(\hat{\boldsymbol{x}})\right]$$

Decomposition of the charge part ($\boldsymbol{q}$ frame)

- Charge part

$$e^{i\boldsymbol{q}\cdot\boldsymbol{x}} = \sum_{J\geq 0} i^J \sqrt{4\pi(2J+1)} j_J(qx) Y_{J,0}(\hat{\boldsymbol{x}})$$

Summary

- charge, longitudinal, electric and magmetic operators

$$\begin{aligned}
T^C_{JM} &= \int d^3x\ \rho^{(h)}(\boldsymbol{x}) \left[j_J(qx) Y_{J,0}(\hat{\boldsymbol{x}}) \right] \\
T^L_{JM} &= \frac{i}{q} \int d^3x\ \boldsymbol{J}^{(h)}(\boldsymbol{x}) \cdot \boldsymbol{\nabla} \left[j_J(qx) Y_{J,0}(\hat{\boldsymbol{x}}) \right] \\
T^E_{J,M} &= \frac{1}{q} \int d^3x\ \boldsymbol{J}^{(h)}(\boldsymbol{x}) \cdot \boldsymbol{\nabla} \times j_J(qx) \boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}}) \\
T^M_{J,M} &= \frac{1}{q} \int d^3x\ \boldsymbol{J}^{(h)}(\boldsymbol{x}) \cdot j_J(qx) \boldsymbol{Y}^{\lambda}_{J,J,1}(\hat{\boldsymbol{x}})
\end{aligned}$$

$$\begin{aligned}
\langle J_f, M_f | V_W | J_i, M_i \rangle_{\boldsymbol{q}} &= \frac{G_V}{\sqrt{2}} \langle J_f, M_f |_{\boldsymbol{q}} \Big\{ - \sum_{J\geq 1} \sqrt{2\pi(2J+1)} (-i)^J \sum_{\lambda=\pm 1} l_\lambda \left[T^E_{J,-\lambda} + \lambda T^M_{J,-\lambda} \right] \\
&\quad + \sum_{J\geq 0} \sqrt{4\pi(2J+1)} (-i)^J \left[l_3 T^L_{J,-\lambda} - l_0 T^C_{J,-\lambda} \right] \Big\} | J_i, M_i \rangle_{\boldsymbol{q}}
\end{aligned}$$

Vector and axial RMEs

RMEs

- $\mathcal{J}_\mu^{(h)} = \mathcal{V}_\mu^{(h)} - \mathcal{A}_\mu^{(h)}$
  - $\mathcal{V}_\mu^{(h)} \equiv \{\rho^V, \boldsymbol{J}^V\}$, ρ^V scalar, $\boldsymbol{J}^V$ vector
  - $\mathcal{A}_\mu^{(h)} \equiv \{\rho^A, \boldsymbol{J}^A\}$, ρ^A pseudoscalar, $\boldsymbol{J}^A$ axial-vector
- RMEs ($X = C, L, E, M$)

$$\langle J_f, M_f | T^X_{J,M} | J_i M_i \rangle = \frac{(-)^{J_i - M_i}}{\sqrt{2J+1}} (J_f, M_f, J_i, -M_i | J, M) \left[X_J^{J_f, J_i}(V) - X_J^{J_f, J_i}(A) \right]$$

Selection rules

- $0^+ \to 0^+$: $C_0(V)$ & $L_0(V)$
- $0^- \to 0^+$: $C_0(A)$ & $L_0(A)$
- Etc.

- Generic direction of $\boldsymbol{q}$ ...
- Cross section & polarization observables

End of Lecture 1

Thank for your attention