Short star-products for filtered quantizations (with E. Rains and D. Stryker)

A - comm. algebra, $A=\bigoplus_{i\geq 0} A_i$, $A_0=\mathbb{C}$, $\dim A_i<\infty$, $\{,\}$ Poisson bracket of degree -2.

$\mathbb{Z}/2$ acts on A by $(-1)^d$

Def. A ($\mathbb{Z}/2$-equiv.) star-product on A is $a*b=ab+\hbar C_1(a,b)+\hbar^2 C_2(a,b)+\dots$ $\quad \hbar=1$

$C_i: A\otimes A\to A$ of degree $-2i$

associative, $C_1(a,b)-C_1(b,a)=\{a,b\}$

Def. $*$ is even if $C_i(a,b)=(-1)^i C_i(b,a)$.

$\Rightarrow C_1(a,b)=\frac{1}{2}\{a,b\}$.

Ex. Moyal $*$-product $A=\mathbb{C}[x,p]$ $\quad m: A\otimes A\to A$

$$a*b = m\left(e^{\frac{1}{2}(\partial_p\otimes\partial_x-\partial_x\otimes\partial_p)}(a\otimes b)\right)=$$

$$=\sum \frac{1}{2^n n!}(-1)^k\binom{n}{k}\partial_p^k\partial_x^{n-k}a\cdot\partial_x^k\partial_p^{n-k}b$$

This is even.

More generally, if V is a f.d. symplectic vector space and $\pi\in\Lambda^2 V$ the Poisson bivector then for $A=\mathbb{C}[V]$ we have

$$a*b=m\left(e^{\frac{1}{2}\pi}(a\otimes b)\right) \qquad \text{(even)}$$

Def. (Beem, Peelaers, Rastelli, 2016)

$*$ is short if $\forall a \in A_m, b \in A_n$

$C_i(a,b) = 0 \quad \forall i > \min(m,n)$.

(Automatically : $\forall i > \frac{m+n}{2}$)

Def. A filtered ($\mathbb{Z}/2$ equiv.) quantization of $(A, \{,\})$ is a $\mathbb{Z}_{\geq 0}$-filtered associative algebra $\mathcal{A} = \bigcup_{i \geq 0} F_i\mathcal{A}$ s.t. $\mathrm{gr}\,\mathcal{A} = A$ and an autom $s: \mathcal{A} \to \mathcal{A}$ (pres. filtr.) s.t. $\mathrm{gr}(s) = (-1)^d$ $s^2 = 1$.

Ex. If $*$ is a star-product on A then it gives rise to a filtered quantiz: $\mathcal{A} = A$, operation is $*$.

Def. A quantization map is a linear filtr. pres. map $\phi: A \to \mathcal{A}$ commuting with $\mathbb{Z}/2$-action and such that $\mathrm{gr}(\phi) = \mathrm{id}$. $(\phi(a) = \hat{a})$

If $\mathcal{A}$ is equipped with a quant. map then can construct $*$ on $A = \mathrm{gr}\,\mathcal{A}$ by $a * b = \phi^{-1}(\phi(a)\phi(b))$.

Lemma This is a bijection.

Lemma. $*$ even $\Longleftrightarrow$ $\exists\ \sigma: \mathcal{A} \to \mathcal{A}$ antiautom. pres. filtr., $\sigma^2 = s$, $gr(\sigma) = i^d$, $\phi \circ i^d = \sigma \circ \phi$.

Ex. Moyal product is short & even.

Ex. $\mathfrak{g} = sl_2$, $\mathcal{A} = U_\chi = U(\mathfrak{g})/(C = \chi)$.

$gr\,\mathcal{A} = \mathbb{C}[\mathcal{N}]$ $\mathcal{N}$ = cone $xy = z^2$ in $\mathbb{C}^3$,

$s = 1$, $A = V_0 \oplus V_2 \oplus V_4 \oplus V_6 \oplus \cdots$ $A_i = V_i$,

$\exists!$ sl_2-invariant $\phi: A \to \mathcal{A}$.

$V_m \otimes V_n = V_{|m-n|} \oplus$ higher $\Rightarrow$ if $a \in A_m$, $b \in A_n$ then $a * b \in A_{|m-n|} \oplus$ higher.

$\Rightarrow$ it is <u>short</u>. (also even).

Ex. Short, not even: $A = \mathbb{C}[x, p]$

$a * b = m\left(\exp\left(\alpha \cdot (\partial_p \otimes \partial_x) - (1-\alpha)(\partial_x \otimes \partial_p)\right)(a \otimes b)\right)$

$\alpha = \frac{1}{2}$ even, but otherwise not $\alpha \in \mathbb{C}$.

Ex. Minimal orbit: $\mathfrak{g}$ simple LA

X min. nilp. orbit, $A \cong \mathbb{C}[X]$.

$A = \bigoplus_{i \geq 0} A_{2i}$, $A_{2i} = V_{i\theta}$ (e.g. $A_2 = \mathfrak{g} = V_\theta$)

$\Rightarrow \exists!\ \phi$ $\mathfrak{g}$-inv. If $V_{k\theta} \subset V_{m\theta} \otimes V_{n\theta}$

then $k \geq |m-n| \Longrightarrow$ $*$ short (also even).

$\mathcal{A} = U(\mathfrak{g})/\mathcal{J}$ $\leftarrow$ Joseph ideal

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Conj. (BPR) Let X be a hyperKähler cone (symplectic singularity). Then

1. Short (even) $*$ exist for generic (even) quantization parameters.

2. They depend on finitely many parameters.

<u>Thm.</u> (2) holds (under nondeg. cond.)

(1) holds in many cases.

Key idea: (Kontsevich) $g: \mathcal{A} \to \mathcal{A}$ linear, filtr. pres.

<u>Def.</u> A g-twisted trace on $\mathcal{A}$ is a linear function $T: \mathcal{A} \to \mathbb{C}$ s.t. $T(\alpha\beta) = T(g(\beta)\alpha)$.

T is called <u>nondegenerate</u> if the bil. form $(\alpha, \beta) \longmapsto B_T(\alpha, \beta) = T(\alpha\beta)$ is nondeg. on each $F_i\mathcal{A}$.

<u>Lemma.</u> If T is a nondeg. g-twisted trace ($\mathbb{Z}/2$-equiv), then g is an algebra autom.

Can define $\mathcal{A}_i = (F_{i-1}\mathcal{A})^\perp$ inside $F_i\mathcal{A}$

$$\mathcal{A}_i \cong F_i\mathcal{A}/F_{i-1}\mathcal{A} = A_i$$

$\phi: A \to \mathcal{A}$ $\quad \phi = \bigoplus_i \phi_i$, $\quad \phi_i: A_i \to \mathcal{A}_i$

Prop. The corr. $*$-product is short.
Also if $*$ is short, have
$B_*(a,b) = CT(a*b)$ (A_i are orthogonal)
$T(a) = a_0 = CT(a)$ g defined uniquely.
Def. short $*$ is nondeg. if B_* is
nondeg.
Prop. We have a bij between
nondeg short $*$ and nondeg g-twist. traces.

To construct nondeg short $*$:
1) Fix a quantiz. par. λ $\mathcal{A} = \mathcal{A}_\lambda$.
2) Fix $g \in Aut(\mathcal{A})$ - alg. group.
3) $T \in HH_0(\mathcal{A}, \mathcal{A}_g)^*$ s
Claim. For symplectic sing. this space is finite dim.
Pf. Ex. $g = 1$. $\operatorname{gr} HH_0(\mathcal{A}, \mathcal{A}) \twoheadleftarrow HP_0(A, A)$
$HH_0(\mathcal{A}, \mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ $HP_0(A, A) = A/\{A, A\}$

Thm. (Kaledin) X has finitely many sympl. leaves

Thm. (E - Schedler) If X has finitely many sympl. leaves then $\mathcal{O}(X)/\{\mathcal{O}(X),\mathcal{O}(X)\}$ is fin. dim.

Existence: 1) $X = V/G$, V f.d. space G = finite subgr. of $Sp(V)$ ($\mathcal{A}$ - symplectic refl. algebras)

2) X = Nilpotent cone of $\mathfrak{g}$.

X - Poisson affine variety

D_X - repr. f-r of global sect.

$$\mathrm{Hom}(\mathcal{D}_X, M) = \Gamma(X, M).$$

$(X \subset V)$

$$D_X = D_V / I_X D_V \qquad M_X = \frac{D_X}{\mathrm{Ham}(X) \cdot D_X}$$

Thm. (with Schedler)

$H^{top}(M_X) \quad \mathcal{O}_X/\ldots$

1) $H^1(M_X) = \{O_X, O_X\}$

2) If X has f. many leaves then M_X is holom.