*Representation Theory In Venice*
A conference in honour of Corrado De Concini

# Valuations and Standard Monomial Theory
(work in progress)

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THE CLASSICAL CASE $G_{k,n} \hookrightarrow \mathbb{P}(\Lambda^k \mathbb{K}^n)$

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# Grassmann variety

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$\{p_{\underline{i}} \mid \underline{i} \in \mathrm{I}_{k,n}\}$ Plücker coordinates $\subset R_1 = (\Lambda^k \mathbb{K}^n)^*$, dual basis:

$\Lambda^k \mathbb{K}^n$: $\{e_{\underline{i}} = e_{i_1} \wedge \ldots \wedge e_{i_k} \mid \underline{i} \in \mathrm{I}_{k,n}\}$, $\mathbb{K}^n$: $\{e_1, \ldots e_n\}$,

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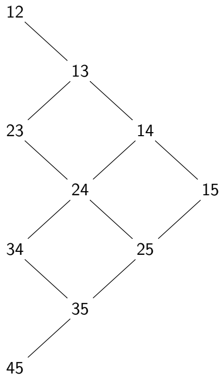
**Definition**

Standard monomial: $p_{\underline{i}} p_{\underline{j}} \cdots p_{\underline{\ell}}$ *standard* $\Leftrightarrow \underline{i} \leq \underline{j} \leq \ldots \leq \underline{\ell}$

# Example: $Gr_{2,5}$

$\mathrm{I}_{2,5}$




some standard monomials of degree 2

$p_{12}p_{12}, p_{12}p_{13}, p_{12}p_{14}, \ldots$

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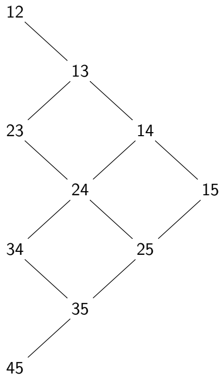


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straightening relations

$p_{23}p_{14} = p_{13}p_{24} - p_{12}p_{34}$

$p_{23}p_{15} = p_{13}p_{25} - p_{12}p_{35}$

$\ldots$

# Standard monomial theory

## Theorem

*(Hodge, Seshadri)*
$R = \bigoplus_{i \geq 0} R_i$ = *homogeneous coordinate ring of*

$$G_{k,n} \hookrightarrow \mathbb{P}(\Lambda^k \mathbb{K}^n)$$

- *the standard monomials of degree* $m$ *form a basis of* $R_m$
- *straightening relations of degree two (= express non-standard monomials as sum of standard monomials) generate the vanishing ideal of* $G_{k,n} \subset \mathbb{P}(\Lambda^k \mathbb{K}^n)$.
- *flat degeneration of* $G_{k,n}$ *into a union of projective spaces, the number of irreducible components equals the number of maximal chains in* $\mathrm{I}_{k,n}$.

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*We try to get a new approach via valuation theory and Newton-Okounkov bodies*

# An example

A family of subvarieties and a family of functions - (affine picture):

$X = \mathbb{A}^3 = \langle e_1, e_2, e_3 \rangle$, $\mathbb{K}[X] = \mathbb{K}[x_1, x_2, x_3]$

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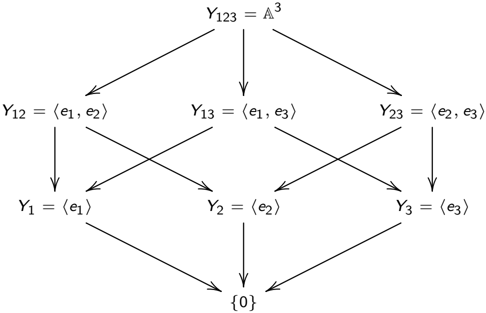


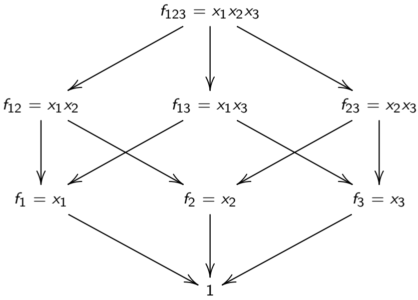

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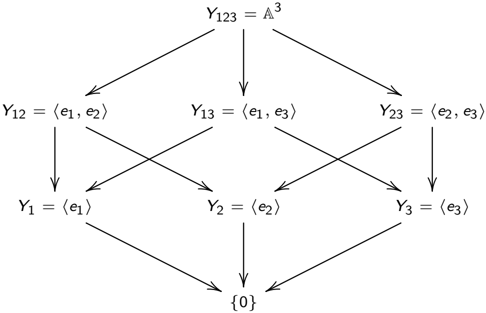


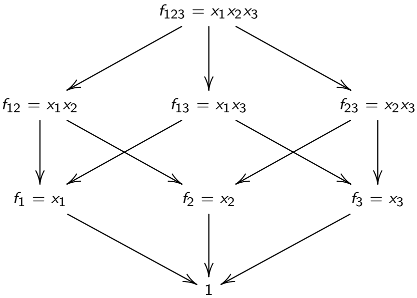


family of functions defining (set theoretically) family of subvarieties.

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- $\{Y_p\}_{p\in A}$ family of projective subvarieties of $X$
  $Y_{p_{min}} = pt,\ Y_{p_{max}} = X,\ Y_p \supseteq Y_q \Leftrightarrow p \geq q$
- $\{f_p\}_{p\in A}$ family of homogeneous functions (on $V$) such that
  - $f_p|_{Y_p} \not\equiv 0$
  - $Y_p = \{x \in X \mid f_q(x) = 0 \forall q \not\leq p\}$ (set theoretically)
  - $H_p = \{[v] \in \mathbb{P}(V) \mid f_p(v) = 0\}$
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- to make presentation more consistent, we assume in the following the $Y_p$ are projectively normal, in applications we do not need it

**Example**

$X = G_{k,n}$ *Grassmann variety,*
$A = \mathrm{I}_{k,n} = \{\underline{i} = (i_1, \ldots, i_k) \mid 1 \leq i_1 < \ldots < i_k \leq n\}.$
$Y_p$*'s* $= \{X(\underline{i}) \mid \underline{i} \in \mathrm{I}_{k,n}\}$ *Schubert varieties*
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# Examples

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**Example**

$X = G/B \subset \mathbb{P}(V(\lambda)).$
$A = W$ *Weyl group, Bruhat order.*
$Y_p$*'s* $= X(\tau)$ *Schubert varieties,* $\tau \in W$.
$f_p$*'s* $= \{p_\tau\}_{\tau \in W}$ *duals of extremal weight vectors* $\tau(v_\lambda)$

# A graph

**Hasse graph** $\mathcal{G}_A$ of $A$ with weights: assume $p > q$ and $p$ covers $q$:

$$p \xrightarrow{b} q \quad \text{where } b = \text{ vanishing multiplictity of } f_p|Y_p \text{ in } Y_q$$

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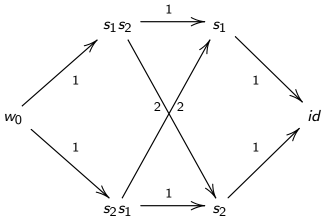
**Example**

*$X = G/B$: $\mathcal{G}_A$ = Bruhat graph, weights = Pieri-Chevalley formula*

$SL_3/B \hookrightarrow \mathbb{P}(\mathfrak{sl}_3)$ :



# Valuations

In the following: $N = lcm(weights\ in\ \mathcal{G}_A)$.

Fix a maximal chain $\mathfrak{C}$ in $A$: (maximally linearly ordered subset of $A$)

$$\mathfrak{C} \quad : \qquad p_r \qquad > p_{r-1} \quad > \ldots > p_1 \quad > p_0$$

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Idea: use $\nu_j$ and $f_{p_j}$ to define a $\mathbb{Q}^{r+1}$-valued valuation on $R$

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Fixed maximal chain $\mathfrak{C} \to$ affine cones:

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**Definition**

$$h_{j-1} := \left. \frac{h_j^N}{f_{p_j}^{N\nu_j(h_j)/b_j}} \right|_{\hat{Y}_{p_{j-1}}}$$

# Valuations

Forget about the numbers, but keep in mind: by Nagata, Rees and Samuel on asymptotic theory of ideals:

**Lemma**

*Given $h$ homogeneous, there exists always a maximal chain such that $\forall j = 0, \ldots, r$: $h_j$ is a regular homogeneous function on $\hat{Y}_{p_j}$.*

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**Definition**

*Let $\mathcal{V}_{\mathfrak{C}} : R - \{0\} \to \mathbb{Q}^{r+1}$ be defined by*

$$h \mapsto (c_r \nu_r(h_r), c_{r-1}\nu_r(h_{r-1}) \dots, c_0\nu_0(h_0))$$

*where $\nu_0(h_0)$ is the vanishing order of $h_0$ in the origin of $\hat{Y}_0$.*

$c_r, \dots, c_0$ are renormalization factors. †

# Valuations

## Remark

The renormalization factors $c_r, \ldots, c_0$ are chosen such that the functions $f_{p_r}, \ldots, f_{p_0}$ are mapped onto the corners of the standard simplex:

$$\mathcal{V}_{\mathfrak{C}}(f_{p_j}) = (0, \ldots, 0, \underbrace{1, 0, \ldots, 0}_{j+1})$$

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## Theorem

*$\mathcal{V}_{\mathfrak{C}} : R - \{0\} \to \mathbb{Q}^{r+1}$ is a valuation with at most one-dimensional leaves.*

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**A quasi-valuation:**

$$h \mapsto \min\{\mathcal{V}_{\mathfrak{C}}(h) \mid \mathfrak{C} \text{ maximal chain}\}$$

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non-negativity: Rees

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- *The irreducible component associated to a maximal chain $\mathfrak{C}$ is the toric variety associated to the semigroup*

$$\Gamma_{\mathfrak{C}} := \{\mathcal{V}(h) \mid h \in R \textit{ homogeneous}, \mathcal{V}_{\mathfrak{C}}(h) \textit{ is minimal}\} \subset \mathbb{Q}_{\geq 0}^{r+1}$$

## Remark

- *If $g$ is homogeneous and $\mathcal{V}_{\mathfrak{C}}(h) = (a_r, \ldots, a_0)$ is minimal, then*

$$\deg g = a_0 \deg f_{p_0} + a_1 \deg f_{p_1} + \ldots + a_r \deg f_{p_r}.$$

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## Remark

*Grassmann variety, $G_{k,n}$, $p_{\underline{i}}$ Plücker coordinate:*
*$\mathcal{V}_{\mathfrak{C}}(p_{\underline{i}})$ is minimal if and only if $\underline{i} \in \mathfrak{C}$.*
*So $\bar{p}_{\underline{i}}\bar{p}_{\underline{j}} = 0$ in $gr_{\mathcal{V}} R \Leftrightarrow \underline{i}$ and $\underline{j}$ are not comparable.*
*Further $N = 1$, so all elements in $gr_{\mathcal{V}} R$ are standard monomials.*

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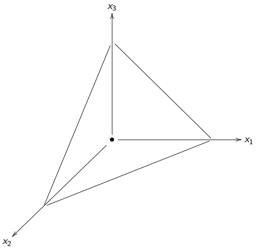
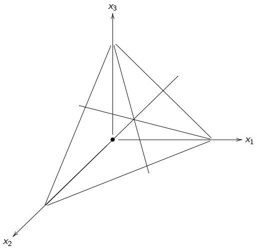
$X = \mathbb{A}^3 = \langle e_1, e_2, e_3 \rangle$, $\mathbb{K}[X] = \mathbb{K}[x_1, x_2, x_3]$
Applying the machinery to this example = cutting a cone into 6 pieces:




$\{x_1^{a_1} x_2^{a_2} x_3^{a_3} \mid a_1, a_2, a_3 \in \mathbb{N}\}$

$R$

$\bigcup_{\sigma \in S_3} \left\{ x_1^{a_1} x_2^{a_2} x_3^{a_3} \mid \begin{array}{c} a_1, a_2, a_3 \in \mathbb{N}; \\ a_{\sigma(1)} \leq a_{\sigma(2)} \leq a_{\sigma(3)} \end{array} \right\}$

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We assume in the following: our familiy of projective subvarieties and the functions $\{f_p\}_{p\in A}$ satisfies in addition the following condition:

- all $f_p$ have the same degree (not really necessary)
- for every $p \rightarrow^b q$, one can extract a root, i.e $\exists\, \eta \in \mathbb{K}(Y_p)$, such:

$$\eta^b = \frac{f_q}{f_p}|_{Y_p}.$$

**Lemma**

*The functions $f_p, \eta f_p, \eta^2 f_p, \ldots, \eta^b f_p = f_q$ regular homogeneous functions of the same degree on $\hat{Y}_p$.*

# A kind of root operator

**Lemma**

*Let $g \in R$ be a homogeneous function. Let $\mathfrak{C} = (p_r, \ldots, p_0)$ be a maximal chain in $A$ such that $\mathcal{V}_{\mathfrak{C}}(g) = (a_r, \ldots, a_0)$ is minimimal. Set $\ell = a_r b$ where $p_r \to^b p_{r-1}$).*

- *the functions below are homogeneous regular functions on $Y_{p_r}$, of the same degree as $g$:*

$$g, \eta g, \eta^2 g, \ldots, \eta^\ell g,$$

- *the last function does not vanish on $Y_{p_{r-1}}$.*
- $\mathcal{V}(\eta^j g) = \mathcal{V}(g) - \frac{j}{b_r}(e_r - e_{r-1})$ *for* $j \leq \ell$

# The semigroup

Using an inductive procedure....

**Proposition**

*The semigroup $\Gamma_{\mathfrak{C}}$ is contained in*

$$\Gamma_{\mathfrak{C}} \subseteq \left\{ v = \begin{pmatrix} a_r \\ \vdots \\ a_0 \end{pmatrix} \in \mathbb{Q}^{r+1}_{\geq 0} \;\middle|\; \begin{array}{r} b_r a_r \in \mathbb{Z} \\ b_{r-1}(a_r + a_{r-1}) \in \mathbb{Z} \\ \ldots \\ b_1(a_r + a_{r-1} + \ldots + a_1) \in \mathbb{Z} \\ a_0 \deg f_{p_0} + a_1 \deg f_{p_1} + \ldots + a_r \deg f_{p_r} \in \mathbb{N} \end{array} \right\}$$

# Some conjectures

**Conjecture**

*Equality holds!*

$$\Gamma_{\mathfrak{C}} = \left\{ v = \begin{pmatrix} a_r \\ \vdots \\ a_0 \end{pmatrix} \in \mathbb{Q}_{\geq 0}^{r+1} \;\middle|\; \begin{array}{r} b_r a_r \in \mathbb{Z} \\ b_{r-1}(a_r + a_{r-1}) \in \mathbb{Z} \\ \ldots \\ b_1(a_r + a_{r-1} + \ldots + a_1) \in \mathbb{Z} \\ a_0 \deg f_{p_0} + a_1 \deg f_{p_1} + \ldots + a_r \deg f_{p_r} \in \mathbb{N} \end{array} \right\}$$

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- *the degree of $X \subseteq \mathbb{P}(V)$ is equal to*

$$\sum_{\text{maximal chains}} \prod (\text{weights on the chain})$$

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# To do

**Remark**

- *further candidates for theory: Richardson varieties, Bott-Samelson varieties, complete symmetric spaces, . . . Most of them are known to have a standard monomial theory. Uniform construction?*
- *are the "algebraic geometric root operators" invertible?*
- *connection with cluster varieties? Even not clear for Grassmann varieties.*

!! Happy Birthday Corrado !!

!! Best wishes for Elisabetta !!