Cayley Hamilton algebras

Claudio Procesi

Claudio Procesi Roma, La Sapienza

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The embedding problem

The embedding problem

When is that a non commutative ring R can be embedded into the ring $M_n(A)$ of $n \times n$ matrices over a commutative ring A?

This question, addressed for the first time by Malcev, may have several possible answers.

The heart of the question is related to the existence of:
polynomial identities satisfied by matrices over a commutative ring.

The theorem of Amitsur–Levitzki

Amitsur and Levitzki discovered that the ring of $n \times n$ matrices over a commutative ring A satisfies a kind of higher order commutative law, the *standard identity*

$$St_{2n}(x_1, x_2, \ldots, x_{2n}) := \sum_{\sigma \in S_{2n}} \epsilon_\sigma x_{\sigma(1)} \ldots x_{\sigma(2n)}$$

where S_{2n} denotes the symmetric group on $2n$ elements and ϵ_σ the sign of a permutation σ.

Necessary and sufficient conditions

In fact there are other polynomial identities of $n \times n$ matrices which are independent of the standard identity.

Thus, the first condition for a ring R to be embeddable in $M_n(A)$ with A commutative is that it should satisfy all the polynomial identities satisfied by $n \times n$ matrices over a commutative ring.

Several examples show that in general this is not sufficient.

A universal map

Notice that, given any ring R and integer n there is always a *universal map*

$$\boxed{j_{n,R} : R \to M_n(A_{n,R})}$$

with $A_{n,R}$ commutative.
For the free algebra in m variables over F the algebra $A_{n,R}$ is the polynomial ring over $M_n(F)^m$ and x_i maps to the corresponding *generic matrix*.

A universal map

In fact $A_{n,R}$ represents the set valued functor on the category of commutative rings which to a commutative ring B associates the set $\hom(R, M_n(B))$ of homomorphisms. That is

$$\hom(A_{n,R}, B) \simeq \hom(R, M_n(B)), \quad \forall B \text{ commutative ring.}$$

So the embedding problem may be reformulated to

find conditions under which $j_{n,R}$ is an embedding.

The *projective group* is the representable group valued functor which to a commutative ring B associates the group of B automorphisms of $M_n(B)$.

By general facts the projective group PGL_n acts on $M_n(A_{n,R})$ and $j_{n,R}$ maps R to the invariants $M_n(A_{n,R})^{PGL_n}$ which contain all coefficients of the characteristic polynomials of elements of $j_{n,R}(R)$.

Finally the spectrum of the invariants $A_{n,R}^{PGL_n}$ parametrizes equivalence classes of semisimple representations of R.

Trace algebras

This fact suggested a new paradigm

in 1987 I proposed to change the problem by adding to the structure of a ring a *trace* and using instead of polynomial identities *trace identities*.

Surprisingly this makes the theory work smoothly.

Trace algebras

Trace algebras

Trace algebras

Definition

An associative algebra with trace, over a commutative ring A is an associative algebra R with a 1-ary operation

$$t : R \to R$$

which is assumed to satisfy the following axioms:

1. t is A–linear.
2. $t(a)b = b\,t(a), \quad \forall a, b \in R.$
3. $t(ab) = t(ba), \quad \forall a, b \in R.$
4. $t(t(a)b) = t(a)t(b), \quad \forall a, b \in R.$

This operation is called a *formal trace*.

Trace algebras

We denote $t(R) := \{t(a),\ a \in R\}$ the image of t. From the axioms it follows that $t(R)$ is a commutative algebra which we call the *trace algebra of R*.

Remark

We have the following implications:
Axiom 1) implies that $t(R)$ *is an A–submodule.*
Axiom 2) implies that $t(R)$ *is in the center of R.*
Axiom 3) implies that t is 0 on the space of commutators $[R, R]$.
Axiom 4) implies that $t(R)$ *is an A–subalgebra and that t is* $t(R)$*–linear.*

Free algebras

The definition is in the spirit of *universal algebra*, thus clearly there exist *the free algebra with trace in variables X and trace identities for a given trace algebra*.

Free algebras with trace

The free A algebra with trace in variables $X = \{x_i\}_{i \in I}$ is obtained from the usual free algebra $A\langle X \rangle$, by adding as polynomial variables, the classes of cyclic equivalence of monomials M, which we formally denote $t(M)$.

$$A\langle X \rangle[t(M)].$$

The Cayley–Hamilton identity

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The Cayley–Hamilton identity

We now assume to be in characteristic 0, that is on $\mathbb{Q}$ algebras.

From the theory of symmetric functions it follows that the Cayley–Hamilton theorem for $n \times n$ matrices may be viewed as a *trace identity* in one variable x denoted by $CH_n(x)$, as example:

$$CH_2(x) = x^2 - tr(x)x + det(x) = x^2 - tr(x)x + \frac{1}{2}(tr(x)^2 - tr(x^2)).$$

Two theorems

A theorem of Razmyslov and Procesi states that:

Theorem

All trace identities of $M_n(\mathbb{Q})$ (in any number of variables) can be deduced from $CH_n(x)$.

Two theorems

A theorem of Procesi states that:

Theorem

The free algebra in the variables X, modulo the ideal of trace identities of $M_n(\mathbb{Q})$ is the algebra of polynomial maps of the space $M_n(\mathbb{Q})^X$ to $M_n(\mathbb{Q})$ which are equivariant under conjugation.

This is a *relatively free algebra* in the variety of trace algebras satisfying $CH_n(x)$.

To be concrete

if X has m elements, let $A_{m,n}$ denote the polynomial functions on the space $M_n(\mathbb{Q})^m$ (polynomials in mn^2 variables).
On this space, and hence on $A_{m,n}$, acts the group $PGL(n,\mathbb{Q})$ by conjugation.
The space of polynomial maps from $M_n(\mathbb{Q})^m$ to $M_n(\mathbb{Q})$ is

$$M_n(A_{m,n}) = M_n(\mathbb{Q}) \otimes A_{m,n}.$$

On this space acts diagonally $PGL(n,\mathbb{Q})$ and the invariants

$$M_n(A_{m,n})^{PGL(n,\mathbb{Q})} = (M_n(\mathbb{Q}) \otimes A_{m,n})^{PGL(n,\mathbb{Q})}$$

give the relatively free algebra in m variables in the variety of trace algebras satisfying $CH_n(x)$.

Cayley–Hamilton algebras

Definition

An n Cayley–Hamilton algebra, is a $\mathbb{Q}$ trace algebra R which satisfies the trace identity $CH_n(x)$.

Examples of n Cayley–Hamilton algebras are $\mathbb{Q}$ subalgebras, closed under trace of an algebra $M_n(A)$ with A a commutative $\mathbb{Q}$ algebra.

Surprise, these are all the n Cayley–Hamilton algebras!

The main Theorem

Let R be a trace algebra which satisfies the trace identity $CH_n(x)$ and $tr(1) = n$. Let $j_{n,R} : R \to M_n(A_{n,R})$ be the universal map (trace compatible) into $n \times n$ matrices over a commutative $\mathbb{Q}$ algebra.

Theorem

The map

$$j_{n,R} : R \xrightarrow{\simeq} M_n(A_{n,R})^{PGL(n,\mathbb{Q})}$$

is an isomorphism.

A corollary

The category of n Cayley–Hamilton algebras is equivalent to a full subcategory of the category of commutative $\mathbb{Q}$ algebras equipped with a $PGL(n, \mathbb{Q})$ action.

This category is only partially studied

A program would be to establish a full theory of this category, describing the interplay between the commutative algebra and associated invariant theory with the structure and representation theory of the corresponding non commutative algebras.

A generalization

The question is, can we extend the theory for a general algebra over a field of positive characteristic or even over the integers?

In this case rather than introducing an abstract trace one should introduce an abstract *determinant*, that is a *Norm*.
In order to do this let us recall the theory of *multiplicative polynomial maps*.

Norm algebras

Norm algebras

Norm algebras

Definition

An associative algebra with an n norm, over a commutative ring A is an associative algebra R with a 1-ary operation

$$N : R \to R$$

which is assumed to satisfy the following axioms:

1. N is an A–polynomial map homogeneous of degree n.
2. $N(a)b = b\,N(a), \quad \forall a, b \in R.$
3. $N(ab) = N(a)N(b), N(1) = 1, \quad \forall a, b \in R.$
4. $N(N(a)b) = N(a)^n N(b), \quad \forall a, b \in R.$

This operation is called a *formal norm*.

Norm algebras

We denote $N(R)$ the subalgebra of R generated by the (polarizations of the) elements $N(a),\ a \in R$. It follows that $N(R)$ is a commutative algebra which we call the *norm algebra of R*.

A polynomial map $F : R \to S$ between two associative algebras which satisfies $F(ab) = F(a)F(b), F(1) = 1, \quad \forall a, b \in R$ is called *multiplicative*.

It can be treated by the theory of *divided powers* $\Gamma_n(R)$.

Multiplicative maps

Under mild conditions the n–divided power of R equals the symmetric tensors of $R^{\otimes n}$:

$$\Gamma_n(R) = [R^{\otimes n}]^{S_n}.$$

If R is an algebra also $[R^{\otimes n}]^{S_n}$ is an algebra, called the *n–Schur algebra of R*.

Multiplicative maps

The map

$$i : R \to [R^{\otimes n}]^{S_n}, \quad i : r \mapsto r^{\otimes n}$$

is a *universal multiplicative map*, that is, by a Theorem of Roby a multiplicative polynomial map $F : R \to S$ between two associative algebras, homogeneous of degree n factors as

$$\begin{array}{ccc} R & \xrightarrow{\ i\ } & [R^{\otimes n}]^{S_n} \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle \bar{F}} \\ & & S \end{array}$$

with $\bar{F}$ a homomorphism of algebras.

Norm algebras

Remark

The axioms of a norm imply that the norm N factors through a homomorphism $\bar{N} : [R^{\otimes n}]^{S_n} \to R$ with image the norm algebra. Moreover N is a polynomial map with respect also to the norm algebra.

By the theory of polynomial maps one can then define a formal characteristic polynomial

$$\chi_a(t) := N(t - a), \quad \forall a \in R.$$

Cayley–Hamilton algebras

Cayley–Hamilton algebras

Cayley–Hamilton algebras

Definition

An algebra R with an n–norm is a n–Cayley–Hamilton algebra if every element $a \in R$ satisfies its characteristic polynomial.

This definition is equivalent to the one given by trace in characteristic 0.

Cayley–Hamilton algebras, what can we say?

The problem is to see what holds of the theory developed in characteristic 0 in this general case.
Clearly we still have the universal map into matrices over a commutative algebra, compatible with the norms (the second is the determinant).

There is also a free n–Cayley–Hamilton algebra.

In order to identify this, we need a generalization of the approach of Ziplies and Vaccarino to the invariant theory of $n \times n$ matrices.

Matrix invariants

Given a commutative ring A and an integer n one can construct for each m the ring of m generic $n \times n$ matrices

$$A\langle \xi_1, \dots, \xi_m \rangle$$

and then the commutative algebra $B_{n,m}$ generated by all coefficients of the characteristic polynomials of elements of $A\langle \xi_1, \dots, \xi_m \rangle$.

Matrix invariants

If A is a field or the integers, by a theorem of Donkin, one has that $B_{n,m}$ is the ring of invariants of m–tuples of $n \times n$ matrices under simultaneous conjugation.
Finally the ring

$$A\langle \xi_1, \ldots, \xi_m \rangle B_{n,m}$$

is the ring of polynomial maps from m–tuples of $n \times n$ matrices to $n \times n$ matrices, which are equivariant under simultaneous conjugation.

Matrix invariants, Zubkov, Ziplies and Vaccarino.

Consider the multiplicative polynomial map

$$A\langle x_1, \dots, x_m\rangle \to A\langle \xi_1, \dots, \xi_m\rangle \stackrel{\det}{\to} B_{n,m}$$

from the free algebra to the matrix invariants. It factors through a homomorphism

$$D : [A\langle x_1, \dots, x_m\rangle^{\otimes n}]^{S_n} \to B_{n,m}$$

Matrix invariants, Zubkov, Ziplies and Vaccarino.

Putting together, a Theorem of Zubkov on the relations between matrix invariants with the approach of Ziplies and Vaccarino one finally has:

Theorem

The homomorphism D is surjective and its kernel is the ideal generated by the commutators.

Corollary

The algebra $A\langle \xi_1, \dots, \xi_m \rangle B_{n,m}$ of equivariant maps is the free n–Cayley–Hamilton algebra in m variables.

A reference:

This last material can be found in a recent book with Corrado:

C. De Concini, C. Procesi, *The invariant theory of matrices*
A.M.S. University Lecture Series v. **69**, 151 pp. (2017).

An open problem

What can one say about the universal map into matrices $j_{n,R} : R \to M_n(A_{n,R})^{PGL_n}$ for an n–Cayley–Hamilton algebra?

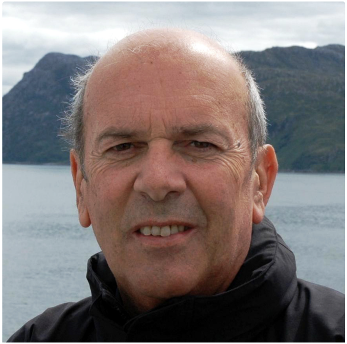
For R free we have seen that this is isomorphism, as well for all R containing $\mathbb{Q}$.

This last fact is due to the property that $PGL_n(\mathbb{Q})$ is linearly reductive. But if F is a field of positive characteristic $PGL_n(F)$ is NOT linearly reductive so the methods used in characteristic 0 fail.

Conclusions

I suspect that there may be examples of R where $j_{n,R}$ is neither injective nor surjective but I have not tried hard enough to find them!

THE END



AUGURI CORRADO