Venezia, September 16-19, 2019
for Corrado's 70th birthday

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The $K(\pi, 1)$ conjecture for affine Artin groups

joint work with Giovanni Paolini [AWS Laboratory, Los Angeles]
Proof of the $K(\pi, 1)$ conjecture for affine Artin groups, arkiv: 1907.11795

Princeton, le 3 août 2019

Dear Paolini, dear Salvetti,

Beautiful theorem!

Best

P. [illegible]

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Artin group of type W:

$$\mathbf{G_W} =< g_s, \ s \in S : \ g_s g_t g_s \dots = g_t g_s g_t \dots \ , \ s \neq t \ \ (m(s,t) \text{ factors}) >$$

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Reflection arrangement:

$\mathcal{A} = \{H :\ H$ is conjugate to some coordinate hyperplane $\ x_s = 0\}$

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$$\mathbf{Y} = V_{\mathbb{C}} \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}$$

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$(V_{\mathbb{C}} = V \oplus \mathbb{R}^{|S|},\ H_{\mathbb{C}} = H \oplus H)$

Orbit configuration space: $\mathbf{Y_W} = \mathbf{Y}/\mathbf{W}$

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Remark: when $\mathbf{W}$ is finite then $V = \mathbb{R}^{|S|}$; when $\mathbf{W}$ is affine then V is an half-space and one reduces to an action of $\mathbf{W}$ on the complexification of an affine space of dimension $|S| - 1$ through affine reflections.

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Theorem

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Known for $\mathbf{W}$ finite since Brieskorn, etc., '71; in general it derives from the PhD thesis of [Van Der Lek, '80] (see also [Sal, 94], [DeCon-Sal, 96]).

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Theorem (Paolini, S.)

The $K(\pi, 1)$ conjecture holds for all affine Artin groups.

It was known for type $\tilde{A}_n$, $\tilde{C}_n$ (Okonek '79), $\tilde{B}_n$ (Callegaro, S. JEMS, 2010)

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Configuration spaces of finite complex reflection groups (proved by Bessis '15).

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which use the theory of *dual* Artin groups.
They find finite dimensional classifying spaces (but with infinite number of cells) for affine Artin groups, but they do not relate them with the orbit spaces.

We get a much stronger result obtaining finite classifying spaces

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(we produce finite complexes whose structure is based on the "dual" structure of Artin groups, we simultaneously prove that well-known finite complexes (Sal. complex), whose structure is based on the standard structure, are $K(\pi, 1)$).

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First we need to define *dual* Artin groups.

We give some general definition.

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The group G becomes a poset setting

$$x \leq y \iff l(x) + l(x^{-1}y) = l(y)$$

i.e. if there is a minimal length factorization of y that starts with a minimal length factorization of x.

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The interval $[1, g]^G$ is *balanced* if: $\forall x \in G$, we have $l(x) + l(x^{-1}g) = l(g)$ if and only if $l(gx^{-1}) + l(x) = l(g)$.

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Theorem

If the interval $[1, g]^G$ is a balanced lattice, then the group G_g is a Garside group.

A Garside group is the fraction group of a *Garside monoid*:

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it is a lattice with respect to left and right divisibility, with left and right cancellation and with an element Δ (the Garside element) whose (left and right) divisors generate the group.

A Garside group has an explicit classifying space.

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For example, the classifying space of the Garside group G_g of a balanced interval $[1, g]^G$ is a Δ-complex whose d-simplices correspond to the sequences

$$x_1, \dots, x_d$$

where $x_i \in [1, g]^G$ and the product $x_1 \dots x_d$ is the left part of a minimal factorization of g.

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So generators are all reflections $R_0 = R \cap [1, w]^W$ and relations all visible paths inside the interval.

Remark

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2) For $\mathbf{W}$ *finite or* $\mathbf{W}$ *affine* j *is an isomorphism (we derive another proof in the affine case)*

3) When $\mathbf{W}$ *is finite the interval* $[1, w]^W$ *is a lattice so* $\mathbf{W_w}$ *is a Garside group.*

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This is an affine subspace of V, and let $\mu \in \mathrm{M{\scriptsize OV}}(u)$ be the unique vector of minimal norm.

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- $\textsc{Min}(u) = \{a \in E \mid u(a) = a + \mu\} \subseteq E$. This is an affine subspace of E.

There is an orthogonal decomposition

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If u is elliptic, then $\textsc{Mov}(u)$ is a linear subspace, $\mu = 0$, and $\textsc{Min}(u)$ coincides with the set of fixed points of u, which we denote by $\textsc{Fix}(u)$.

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For example: choose one Coxeter element $w \in W$, where $\mathbf{W}$ is an irreducible affine Coxeter group acting as a reflection group on an n-dimensional affine space E, where n is the rank of W.

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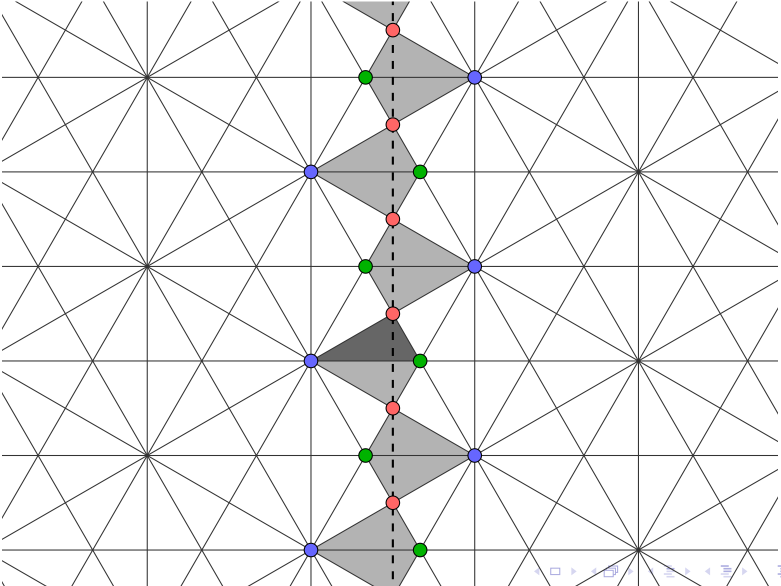
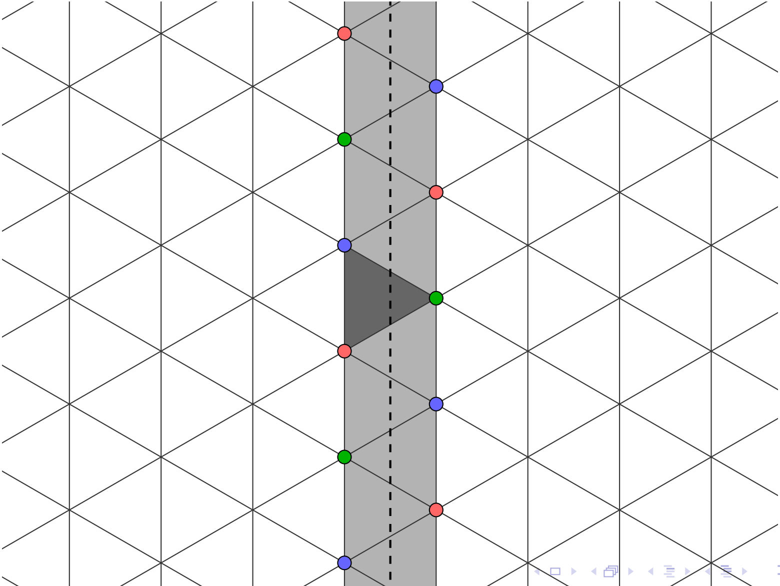
w is a hyperbolic isometry of reflection length $n+1$, and its min-set is a line ℓ called the *Coxeter axis*.

See the example $\tilde{G}_2,\ \tilde{A}_2$.





Let us call a reflection $r \in [1, w]^W$ *horizontal* if its fixed set is parallel to ℓ, otherwise it is called *vertical*.

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In general, an isometry $u \in [1, w]^W$ is horizontal if it moves all points in a direction orthogonal to ℓ (in other words $\textsc{Dir}\,\textsc{Mov}(u)$ is orthogonal to $\textsc{Dir}(\ell)$) otherwise it is *vertical*.

Coarse combinatorial structure of the interval $[1, w]^W$:

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- (top row) u is hyperbolic and v is horizontal elliptic.
- (middle row) both u and v are vertical elliptic;
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- (top row) u is hyperbolic and v is horizontal elliptic.
- (middle row) both u and v are vertical elliptic;
- (bottom row) u is horizontal elliptic and v is hyperbolic;

The bottom and the top rows contain a finite number of elements, whereas the middle row contains infinitely many elements.

The roots corresponding to horizontal reflections form a root system $\Phi_h \subseteq \Phi$, called the *horizontal root system* associated with the Coxeter element $w \in W$.

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The number k of irreducible components varies from 1 to 3.

Type	Horizontal root system
$\tilde{A}_n$	$\Phi_{A_{p-1}} \sqcup \Phi_{A_{q-1}}$
$\tilde{C}_n$	$\Phi_{A_{n-1}}$
$\tilde{B}_n$	$\Phi_{A_1} \sqcup \Phi_{A_{n-2}}$
$\tilde{D}_n$	$\Phi_{A_1} \sqcup \Phi_{A_1} \sqcup \Phi_{A_{n-3}}$
$\tilde{G}_2$	Φ_{A_1}
$\tilde{F}_4$	$\Phi_{A_1} \sqcup \Phi_{A_2}$
$\tilde{E}_6$	$\Phi_{A_1} \sqcup \Phi_{A_2} \sqcup \Phi_{A_2}$
$\tilde{E}_7$	$\Phi_{A_1} \sqcup \Phi_{A_2} \sqcup \Phi_{A_3}$
$\tilde{E}_8$	$\Phi_{A_1} \sqcup \Phi_{A_2} \sqcup \Phi_{A_4}$

Table: Horizontal root systems. In the case $\tilde{A}_n$, the horizontal root system depends on the (p, q)-bigon Coxeter element.

Fact: Let W be an irreducible affine Coxeter group, and w one of its Coxeter elements. The interval $[1, w]^W$ is a lattice (and thus W_w is a Garside group) if and only if the horizontal root system associated with w is irreducible. This happens in the cases $\tilde{C}_n$, $\tilde{G}_2$, and $\tilde{A}_n$ if w is a $(n, 1)$-bigon Coxeter element.

Since the interval $[1, w]^W$ is not a lattice in general, in [mccammond2017] a new group of isometries $C \supseteq W$ is constructed, with the property that $[1, w]^C$ is a balanced lattice and $[1, w]^W \subseteq [1, w]^C$.

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The corresponding interval group C_w (called *braided crystallographic group*) is a Garside group, and there is a natural inclusion $W_w \subseteq C_w$.

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In our proof of the $K(\pi, 1)$ conjecture, one of the key points is to show that K_W is a already a classifying space for W_w, for every affine Coxeter group W, even when $[1, w]$ is not a lattice.
This can come as a surprise since the standard argument to show that K_W is a classifying space heavily relies on the lattice property.

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This completes the proof of the $K(\pi, 1)$ conjecture, and at the same time, it gives a new proof that the dual Artin group W_w is naturally isomorphic to the Artin group G_W (in the affine case).

Among the several technical intermediate steps, may be one of the most important to our proof of the deformation retraction $K_W \simeq X'_W$, is to construct an EL-labeling of the poset $[1, w]^W$.

The group enlargement $C \supset \mathbf{W}$ is obtained by enlarging the set T of translations contained in $[1, w]^W$: for each translation $t \in T$ one gets a finite number of extra translations $t_1, \ldots, t_k$ which factorize t.

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So C is generated by $R \cup T_F$.

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- C generated by R_{hor}, R_{ver}, T_F
- $\mathbf{W}$ generated by R_{hor}, R_{ver}
- F generated by R_{hor}, T_F
- D generated by R_{hor}, T

The interval groups are related as follows:

$$[1, w]^C = [1, w]^W \cup [1, w]^F$$
$$[1, w]^D = [1, w]^W \cap [1, w]^F.$$

The intervals $[1,w]^D$ and $[1,w]^F$ are finite, whereas $[1,w]^W$ and $[1,w]^C$ are infinite.

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On the other hand, the intervals $[1,w]^D$ and $[1,w]^W$ are lattices if and only if the horizontal root system Φ_h is irreducible, in which case $D = F$ and $W = C$.

Construct the interval groups D_w, F_w, and C_w.

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Since the intervals $[1, w]^F$ and $[1, w]^C$ are lattices, the interval groups F_w and C_w are Garside groups and the corresponding interval complexes K_F and K_C are classifying spaces.

A consequence of the relations between the four intervals is that

$$K_C = K_W \cup K_F$$

and

$$K_D = K_W \cap K_F$$

Lemma (P.S.)

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We show that K_H decompose as a product $K_1 \times \cdots \times K_k$ of subcomplexes, each of them being a classifying space of a group of type $\tilde{A}_{k_i}$, according to the decomposition into irreducible components of the horizontal root system.

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We show that K_H decompose as a product $K_1 \times \cdots \times K_k$ of subcomplexes, each of them being a classifying space of a group of type $\tilde{A}_{k_i}$, according to the decomposition into irreducible components of the horizontal root system. Therefore K_H is a $K(\pi, 1)$-space.

Theorem (P.S.)

Let W be an irreducible affine Coxeter group, and w one of its Coxeter elements. The interval complex K_W is a classifying space for the dual Artin group W_w.

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This is obtained by a Mayer-Vietoris argument applied to the universal covering and using that K_C, K_F and K_D are $K(\pi, 1)$ spaces.

Now remind that d-simplices in K_W are sequences $[x_1|\ldots|x_d]$ such that the product $x_1\ldots x_d$ appears as a left factor of a minimal factorization of w.

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For every $T \in \Delta_W$, denote by w_T the product of the elements of T in the same relative order as in the list $s_1, s_2, \dots, s_n$.

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For every $T \in \Delta_W$, denote by w_T the product of the elements of T in the same relative order as in the list $s_1, s_2, \ldots, s_n$.
Then w_T is a Coxeter element of the parabolic subgroup W_T, and it belongs to $[1, w]^W$.

One can see that for every $T \subseteq S$ we have $[1, w_T]^{W_T} = [1, w_T]^W$, and the length functions of W_T and W agree on these intervals.

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Definition

Let X'_W be the finite subcomplex of K_W consisting of the simplices $[x_1|x_2|\cdots|x_d] \in K_W$ such that $x_1x_2\cdots x_d \in [1, w_T]$ for some $T \in \Delta_W$.

Remark that if $\mathbf{W}$ is finite, then $S \in \Delta_W$ and therefore $X'_W = K_W$.

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In this case, the interval complex K_W is a classifying space for the dual Artin group W_w , which is naturally isomorphic to the Artin group G_W.

For every $T \in \Delta_W$, the complex X'_W has a subcomplex consisting of the simplices $[x_1|x_2|\cdots|x_d]$ such that $x_1 x_2 \cdots x_d \in [1, w_T] = [1, w_T]^{W_T}$.

For every $T \in \Delta_W$, the complex X'_W has a subcomplex consisting of the simplices $[x_1|x_2|\cdots|x_d]$ such that $x_1 x_2 \cdots x_d \in [1, w_T] = [1, w_T]^{W_T}$.

This is exactly the interval complex associated with $[1, w_T]^{W_T}$, which coincides with X'_{W_T} and is a classifying space for the Artin group G_{W_T}.

By definition, X'_W is the union of all subcomplexes X'_{W_T} for $T \in \Delta_W$.

There is a well known complex X_W whose cells are indexed by the simplicial complex Δ_W, and which is known to be homotopy equivalent to the orbit configuration space $\mathbf{Y_W}$ of $\mathbf{W}$.

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Similarly to X'_W, the complex X_W is the union of the complexes X_{W_T} for $T \in \Delta_W$.

Each X_{W_T} is a classifying space for G_{W_T}, because the $K(\pi, 1)$ conjecture holds for spherical Artin groups .

Our second main step is:

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Theorem

For every Coxeter group W, the complex X'_W is homotopy equivalent to the complex X_W and so to the orbit configuration space $\mathbf{Y_W}$.

As an alternative description of X'_W we have

Remark

Let W be an irreducible affine Coxeter group, with a set S of simple reflections and a Coxeter element w obtained as a product of the elements of S. Denote by C_0 the chamber of the Coxeter complex associated with S. A simplex $[x_1|x_2|\cdots|x_d] \in K_W$ belongs to X'_W if and only if $x_1 x_2 \cdots x_d$ is an elliptic element that fixes at least one vertex of C_0.

Now we come to the last step of our proof: we show that the complex K_W contracts to the finite subcomplex X'_W.

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This is done by using *discrete Morse theory*: this is a combinatorial version of classical Morse theory, mainly Morse theory for CW-complexes K, which consists essentially in assigning a coherent sequence of contractions which reduce the complex to a smaller one.

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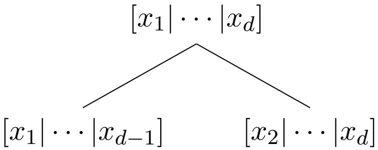
For every d-simplex $\sigma = [x_1|\ldots|x_d] \subset K_W$ such that $x_1 \ldots x_d = w$, we consider the left and right boundary faces $[x_1|\ldots|x_{d-1}]$ and $[x_2|\ldots|x_d]$


$[x_1|\cdots|x_d]$
$[x_1|\cdots|x_{d-1}]$
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Let $\varphi\colon [1,w]^C \to [1,w]^C$ be the conjugation by the Coxeter element w: $\varphi(u) = w^{-1}uw$.

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Then we get factorizations:

$$w = x_1 \dots x_d = x_2 \dots x_d \varphi(x_1) = x_3 \dots x_d \varphi(x_1)\varphi(x_2) = \dots$$

and

$$w = x_1 \dots x_d = \varphi^{-1}(x_d) x_1 \dots x_{d-1} = \varphi^{-1}(x_{d-1})\varphi^{-1}(x_d) x_1 \dots x_{d-2} = \dots$$

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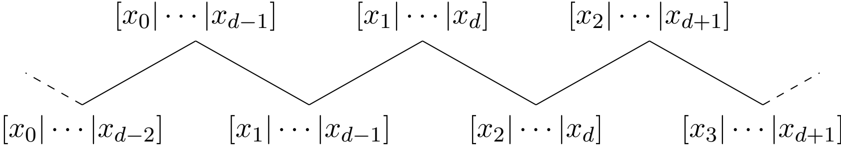
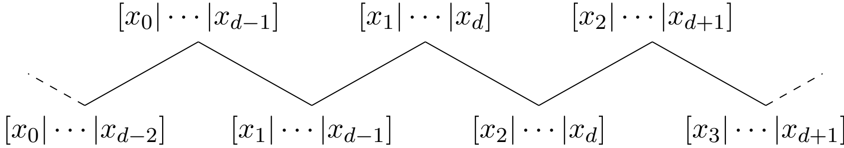
so a piece of the Hasse diagram is given by



where $x_{i+d} = \varphi(x_i)$.



where $x_{i+d} = \varphi(x_i)$.
We define this as *the component* containing $[x_1|\ldots|x_d]$.

One can show:

Lemma

- *The component $\mathcal{C}$ of $[x_1| \dots |x_d]$ is infinite iff one x_i is vertical elliptic (so all x_j are elliptic).*

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Lemma

- *The component $\mathcal{C}$ of $[x_1|\dots|x_d]$ is infinite iff one x_i is vertical elliptic (so all x_j are elliptic).*
- *Every component $\mathcal{C}$ intersects $\mathcal{F}(X'_W)$.*
- *There are a finite number of components.*

Now let $K' \subset K_W$ be the finite subcomplex such that:

- $\mathcal{F}(K')$ contains all the finite components of K;

- for every infinite component $\mathcal{C}$, one has that $\mathcal{F}(K') \cap \mathcal{C}$ is the path going from the leftmost to the rightmost element of $\mathcal{F}(X'_W) \cap \mathcal{C}$.

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So $K' \supset X$ is an approximation of X'_W but it is larger.

Theorem

K_W deformation retracts onto K'.

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It remains to see that K' deformation retracts onto X'_W.

This is also achieved by discrete Morse theory but it requires much more work.

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In order to find an acyclic matching in $K' \setminus X'_W$ we prove an intermediate (interesting) result.

Theorem

Let W be an irreducible affine Coxeter group, and w one of its Coxeter elements. There exists a total ordering on $R_0 = R \cap [1, w]^W$ (the axial ordering) which makes the poset $[1, w]^W$ EL-shellable.

Theorem

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The EL-shellability of $[1, w]^W$ for finite W was already known.

Recall that a poset $\mathcal{P}$ is EL-shellable (*edge-lexicographic*-shellable) if there exists a weight function $\lambda : \mathcal{E}(\mathcal{P}) \to \mathcal{Q}$ ($\mathcal{Q}$ a poset) such that:

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- first, there are the vertical reflections that fix a point of ℓ above C_0, and r comes before r' if $\mathrm{Fix}(r) \cap \ell$ is below $\mathrm{Fix}(r') \cap \ell$;

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- first, there are the vertical reflections that fix a point of ℓ above C_0, and r comes before r' if $\text{Fix}(r) \cap \ell$ is below $\text{Fix}(r') \cap \ell$;
- then, there are the horizontal reflections in R_{hor}, following any suitable total ordering $\prec_{hor}$ constructed separately;

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- then, there are the horizontal reflections in R_{hor}, following any suitable total ordering $\prec_{hor}$ constructed separately;
- finally, there are the vertical reflections that fix a point of ℓ below C_0, and again r comes before r' if $\textsc{Fix}(r) \cap \ell$ is below $\textsc{Fix}(r') \cap \ell$.

The relative order between vertical reflections that fix the same point of ℓ can be chosen arbitrarily, since one sees that such reflections commute.

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The ordering of the horizontal reflections is obtained by ordering separately each irreducible component: recall that Φ_{hor} decomposes in irreducible root systems of type $\tilde{A}_{n_i}, i = 1 \ldots, k$. The corresponding reflections are suitably ordered and then one takes a shuffle ordering of them.

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be the simplex which is immediately at the left (resp. right) of σ inside its component.

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Definition

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(i) $l(x_i) \geq 2$;

(ii) $l(x_i) = 1$, $i \leq d-1$, and $x_i \prec r$ for every reflection $r \leq x_{i+1}$ in $[1, w]$.

If no such i exists, let $\delta(\sigma) = \infty$.

Definition (Matching function)

Given $\sigma \in \mathcal{F}(K'_W) \setminus \mathcal{F}(X'_W)$, define $\mu(\sigma) \in \mathcal{F}(K_W)$ as follows.

(1) If $\pi(\sigma) \neq w$, let $\mu(\sigma) = \lambda(\sigma)$.

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Suppose now that $\pi(\sigma) = w$, and $\pi(\rho(\sigma))$ fixes a vertex of C_0. Let $\delta = \delta(\sigma)$. Notice that $\delta \neq \infty$.

3. If $l(x_\delta) \geq 2$, define $\mu(\sigma) = [x_1|\cdots|x_{\delta-1}|y|z|x_{\delta+1}|\cdots|x_d]$, where y is the $\prec$-smallest reflection of $R_0 \cap [1, x_\delta]$, and $yz = x_\delta$.

Definition (Matching function)

Given $\sigma \in \mathcal{F}(K'_W) \setminus \mathcal{F}(X'_W)$, define $\mu(\sigma) \in \mathcal{F}(K_W)$ as follows.

1. If $\pi(\sigma) \neq w$, let $\mu(\sigma) = \lambda(\sigma)$.
2. If $\pi(\sigma) = w$, and $\pi(\rho(\sigma))$ does not fix a vertex of C_0, let $\mu(\sigma) = \rho(\sigma)$.

Suppose now that $\pi(\sigma) = w$, and $\pi(\rho(\sigma))$ fixes a vertex of C_0. Let $\delta = \delta(\sigma)$. Notice that $\delta \neq \infty$.

3. If $l(x_\delta) \geq 2$, define $\mu(\sigma) = [x_1|\cdots|x_{\delta-1}|y|z|x_{\delta+1}|\cdots|x_d]$, where y is the $\prec$-smallest reflection of $R_0 \cap [1, x_\delta]$, and $yz = x_\delta$.
4. If $l(x_\delta) = 1$, define $\mu(\sigma) = [x_1|\cdots|x_{\delta-1}|x_\delta x_{\delta+1}|x_{\delta+2}|\cdots|x_d]$.

Definition (Matching function)

Given $\sigma \in \mathcal{F}(K'_W) \setminus \mathcal{F}(X'_W)$, define $\mu(\sigma) \in \mathcal{F}(K_W)$ as follows.

1. If $\pi(\sigma) \neq w$, let $\mu(\sigma) = \lambda(\sigma)$.
2. If $\pi(\sigma) = w$, and $\pi(\rho(\sigma))$ does not fix a vertex of C_0, let $\mu(\sigma) = \rho(\sigma)$.

Suppose now that $\pi(\sigma) = w$, and $\pi(\rho(\sigma))$ fixes a vertex of C_0. Let $\delta = \delta(\sigma)$. Notice that $\delta \neq \infty$.

3. If $l(x_\delta) \geq 2$, define $\mu(\sigma) = [x_1 | \cdots | x_{\delta-1} | y | z | x_{\delta+1} | \cdots | x_d]$, where y is the $\prec$-smallest reflection of $R_0 \cap [1, x_\delta]$, and $yz = x_\delta$.
4. If $l(x_\delta) = 1$, define $\mu(\sigma) = [x_1 | \cdots | x_{\delta-1} | x_\delta x_{\delta+1} | x_{\delta+2} | \cdots | x_d]$.

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Therefor μ gives a perfect matching in $\mathcal{F}(K') \setminus \mathcal{F}(X'_W)$.

It remains to show that such matching is acyclic.

Theorem

The matching $\mathcal{M}$ on $\mathcal{F}(K'_W)$ is acyclic.

The proof is technical and consists in finding a sort of "invariant" which decreases along an alternating closed path

$$\sigma_1 \gtrdot \tau_1 \lhd \sigma_2 \gtrdot \tau_2 \lhd \cdots \gtrdot \tau_m \lhd \sigma_{m+1} = \sigma_1$$

giving a contradiction.

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giving a contradiction.

So

Theorem

Let W be an irreducible affine Coxeter group, with a set of simple reflections $S = \{s_1, s_2, \ldots, s_{n+1}\}$ and a Coxeter element $w = s_1 s_2 \cdots s_{n+1}$. The interval complex K_W deformation retracts onto its subcomplex X'_W.

Theorem (P.S.)

Let W be an irreducible affine Coxeter group. The $K(\pi, 1)$ conjecture holds for the corresponding Artin group G_W.

Thank you

Thank you
and

Happy Birthday, Corrado!