From wonderful models to Coxeter categories
(joint work with Andrea Appel)

Valerio Toledano Laredo

Northeastern University

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Overview

- $\mathfrak{g}$ symmetrisable Kac–Moody algebra
- $U_\hbar\mathfrak{g}$ quantum group corresponding to $\mathfrak{g}/\mathbb{C}[[\hbar]]$
- Goal: establish a good equivalence

$$\text{representations of } U_\hbar\mathfrak{g} \longleftrightarrow \text{representations of } \mathfrak{g} \; (/\mathbb{C}[[\hbar]])$$

Known equivalences

Theorem (Drinfeld–Kohno, Kazhdan–Lusztig) If $\dim \mathfrak{g} < \infty$, there is an equivalence of braided tensor categories

$$(\text{Reps. of } U_\hbar\mathfrak{g}, \text{R}) \leftrightarrow (\text{Reps. of } \mathfrak{g}, \text{monodromy of the KZ equations})$$

Remark If $\dim \mathfrak{g} = \infty$, $\mathfrak{g}$ and $U_\hbar\mathfrak{g}$ have **different** abelian categories of representations $\Rightarrow$ DKKL equivalence cannot hold as stated. However,

Theorem (Etingof–Kazhdan '96–'08) For any symmetrisable Kac–Moody algebra $\mathfrak{g}$, there is an equivalence of braided tensor categories

$$F^{\text{EK}} : (\text{Cat. } \mathcal{O} \text{ for } U_\hbar\mathfrak{g}, \text{R}) \leftrightarrow (\text{Cat } \mathcal{O} \text{ for } \mathfrak{g}, \text{monodromy of KZ equations})$$

Corollary If $V_1, \ldots, V_n \in \mathcal{O}_\mathfrak{g}$, the action of the braid group B_n by monodromy of the KZ equations on $V_1 \otimes \cdots \otimes V_n$ is equivalent to its R–matrix action on $F^{\text{EK}}(V_1) \otimes \cdots \otimes F^{\text{EK}}(V_n)$.

An extended equivalence?

- W Weyl group of $\mathfrak{g}$
- B_W corresponding generalised braid group, with generators $\{S_i\}_{i\in\mathbf{I}}$ and relations

$$\underbrace{S_iS_j\cdots}_{m_{ij}} = \underbrace{S_jS_i\cdots}_{m_{ij}}$$

  for any $i \neq j$, $m_{ij} =$ order of s_is_j in W
- B_W acts on any $\mathcal{V}$ integrable representation of $U_\hbar\mathfrak{g}$ by Lusztig's quantum Weyl group operators
- B_W acts on any V integrable representation of $\mathfrak{g}$ by monodromy of the Casimir connection
- Goal find an equivalence which is equivariant for these actions
- Remark Neither action of B_W is built out of the braided tensor structure $\Rightarrow$ need to extend rather than modify the DKKL equivalence.

The quantum Weyl group action

- $\mathcal{V}$ integrable repr. of $U_\hbar\mathfrak{g}$
- Thm. (Lusztig) $\exists\{S_i\}_{i\in\mathbf{I}} \subset \mathrm{Aut}(\mathcal{V})$ satisfying the braid relations

$$\underbrace{S_i S_j \cdots}_{m_{ij}} = \underbrace{S_j S_i \cdots}_{m_{ij}}$$

- The corresponding action $\lambda_\hbar : B_W \to \mathrm{Aut}(\mathcal{V})$ is s.t. $\lambda_\hbar|_{\hbar=0}$ is the action of (a finite extension $\widetilde{W}$ of) W on the integrable $\mathfrak{g}$–module $\mathcal{V}/\hbar\mathcal{V}$.

The Casimir connection ∇_C

- $\dim \mathfrak{g} < \infty$ for now
- $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra, $\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha\in R} \mathrm{Ker}(\alpha)$
- V integrable $\mathfrak{g}$–module
- ∇_C is a meromorphic connection on $V \times \mathfrak{h}_{\text{reg}} \to \mathfrak{h}_{\text{reg}}$,

$$\nabla_C = d - \frac{h}{2} \sum_{\alpha\in R^+} \frac{d\alpha}{\alpha} \mathcal{K}_\alpha$$

- $h \in \mathbb{C}$ deformation parameter
- $\mathcal{K}_\alpha = x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha$ (truncated) Casimir operator of $\mathfrak{sl}_2^\alpha \subset \mathfrak{g}$

Theorem (De Concini, Millson–TL, Felder–Markov–Tarasov–Varchenko)
The connection ∇_C is flat, and $\widetilde{W}$–equivariant for any $h \in \mathbb{C}$.

Monodromy $\mu_V^h : B_W = \pi_1(\mathfrak{h}_{\text{reg}}/W) \longrightarrow GL(V)$ deforms $\widetilde{W} \circlearrowright V$.

Why study ∇_C?

The Casimir connection is related to

1. Quantum integrable systems of Gaudin type related to $\mathfrak{g}$ (Rybnikov, Feigin–Frenkel–TL)
2. Wess–Zumino–Witten model corresponding to $\mathfrak{g}$ (Fedorov, Feigin–Frenkel–TL)
3. Isomonodromic deformations of irregular connections on $\mathbb{P}^1$ (Boalch, Xu–TL)
4. Wall–crossing & stability conditions (Joyce, Bridgeland–TL)
5. Enumerative geometry (q. cohomology) of Nakajima quiver varieties (Maulik–Okounkov)

Monodromy theorem

$B_W \xrightarrow{\mu_V} GL(V[[\mathsf{h}]])$ $\quad V \in \mathsf{Rep}_{\mathsf{fd}}(\mathfrak{g})$, formal Taylor series of μ_V^{h} at $\mathsf{h} = 0$

$B_W \xrightarrow{\lambda_{\mathcal{V}}} GL(\mathcal{V})$ $\quad \mathcal{V} \in \mathsf{Rep}_{\mathsf{fd}}(U_\hbar \mathfrak{g})$, qWeyl group action

Theorem 1 (TL, Conj. De Concini, TL)
Assume $\dim \mathfrak{g} < \infty$. Set $\hbar = 2\pi\iota\mathsf{h}$, and assume that $\mathcal{V}/\hbar\mathcal{V} \cong V$.

1. The representations μ_V and $\lambda_{\mathcal{V}}$ are equivalent.
2. The monodromy of ∇_C is defined over $\mathbb{Q}[[\hbar]]$.

Theorem 2 (Appel–TL, 2019) A similar result holds for an arbitrary symmetrisable Kac–Moody algebra.

Remark The statement of Thm. 2 is conceptually simpler, and much stronger than Thm. 1, even for $\dim \mathfrak{g} < \infty$.

Strategy of proof

- Both μ_V and $\lambda_{\mathcal{V}}$ deform $\widetilde{W} \circlearrowright V$.
- Look for an appropriate rigidity result (cf. Drinfeld's computation of the monodromy of the KZ equations in terms of the R–matrix of $U_\hbar\mathfrak{g}$).
- Problem find an algebraic structure which
  1. accomodates both μ_V and $\lambda_{\mathcal{V}}$
  2. has trivial deformation theory
- 1st attempt Look at actions of B_W on a fixed vector space $\mathcal{V}/\mathbb{C}[[\hbar]]$ which deform a given action of $\widetilde{W}$. This satisfies 1), but not 2) ($H^1(B_W, V)$ is very big).

Definition/Theorem (Appel-TL)

1. $\mathcal{O}^{\text{int}}_{U_\hbar\mathfrak{g}}$ is a braided **Coxeter** category.
2. $\mathcal{O}^{\text{int}}_{\mathfrak{g}}$ is a braided **Coxeter** category.
3. Braided Coxeter category structures on $\mathcal{O}^{\text{int}}_{\mathfrak{g}}$ are rigid.

Remark The definition (to follow) of Coxeter category is inspired by the De Concini–Procesi wonderful model of a hyperplane complement.

Coxeter categories

- What is a braided tensor category $\mathcal{C}$ good for?
  - For any $V \in \mathrm{Ob}(\mathcal{C})$, $n \geq 1$, there is an action

    $$\rho_b : B_n \to \mathrm{Aut}(V_b^{\otimes n})$$

    which depends on the **choice of a bracketing** $b \in \mathcal{B}_n$ on the (non–associative) monomial $x_1 \cdots x_n$.
  - Example $b = ((x_1 x_2) x_3) \in \mathcal{B}_3$, $V_b^{\otimes 3} = ((V \otimes V) \otimes V)$.
  - For any $b, b' \in \mathcal{B}_n$, $V_b^{\otimes n}$ and $V_{b'}^{\otimes n}$ are isomorphic as B_n–modules, via an associativity constraint: $\Phi_{b'b} : V_b^{\otimes n} \to V_{b'}^{\otimes n}$.

- What is a Coxeter category $\mathcal{Q}$ good for?
  - For any $V \in \mathrm{Ob}(\mathcal{Q})$, there is an action

    $$\lambda_{\mathcal{F}} : B_W \to \mathrm{Aut}(V_{\mathcal{F}})$$

    which depends on the **choice of a 'W–bracketing'** $\mathcal{F}$.
  - (A $\mathfrak{S}_n$–bracketing is the same as an element of $\mathcal{B}_n$.)
  - For any W–bracketings $\mathcal{F}, \mathcal{G}$, $V_{\mathcal{F}}$ and $V_{\mathcal{G}}$ are isomorphic as B_W–modules, via a prescribed isomorphism $\Phi_{\mathcal{G}\mathcal{F}} : V_{\mathcal{F}} \to V_{\mathcal{G}}$.

Bracketings revisited: D = Dynkin diagram of type A_{n-1}

- pair of parentheses on $x_1 \cdots x_n \longleftrightarrow$ connected subdiagram of D.
- $p = x_1 \cdots x_{i-1}(x_i \cdots x_j)x_{j+1} \cdots x_n \longleftrightarrow B = [i, j-1] \subset D$
- Example $(((x_1 x_2)x_3)x_4) \longleftrightarrow [1,1], [1,2], [1,3] \subseteq [1,3]$.
- p, p' are consistent parentheses $\iff B, B' \subseteq D$ are *compatible*, *i.e.*,
  - $B \subset B'$ or $B' \subset B$, or
  - $B \perp B'$: $B \cap B' = \emptyset$, and no vertex in B is linked to a vertex in B' by an edge of D.
- Examples
  1. $(x_1 x_2)(x_3 x_4) \longleftrightarrow [1,1] \perp [3,3] \subseteq [1,3]$.
  2. $(x_1(x_2)x_3 x_4) \longleftrightarrow [1,1] \not\perp [2,3] \subseteq [1,3]$.

Definition (De Concini–Procesi)/Proposition

1. A nested set on $D = [1, n-1]$ is a collection of pairwise compatible, connected subdiagrams of D.
2. There is a bijection

$$\{\text{bracketings on } x_1 \cdots x_n\} \longleftrightarrow \{\text{maximal nested sets on } [1, n-1]\}$$

W–bracketings (=nested sets)

D diagram (unoriented graph, no loops, no multiple edges)

Example D=Dynkin diagram of W

Definition (De Concini–Procesi) A nested set on D is a collection $\mathcal{F} = \{B\}$ of pairwise compatible, connected subdiagrams of D.

Nested sets and chains

A chain from $B \subseteq D$ to $\emptyset$ is a sequence of (not necessarily connected) subdiagrams

$$B = B_1 \supsetneq B_2 \supsetneq \cdots \supsetneq B_m = \emptyset$$

Lemma There is a surjection $\imath : \{\text{chains } B \to \emptyset\} \longrightarrow \mathsf{Ns}(B)$ given by

$$\imath(B_1 \supsetneq B_2 \supsetneq \cdots \supsetneq B_m) = \bigcup_{i=1}^{m-1} \text{connected components of } B_i$$

Examples

1. $[1,3] \supset [1,2] \supset [1,1] \longrightarrow \{[1,3],[1,2],[1,1]\}$
2. $[1,3] \supset ([1,1] \sqcup [3,3]) \supset [1,1] \longrightarrow \{[1,3],[1,1],[3,3]\}$
3. $[1,3] \supset ([1,1] \sqcup [3,3]) \supset [3,3] \longrightarrow \{[1,3],[1,1],[3,3]\}$

Nested sets on B/B' $(B' \subseteq B)$ correspond similarly to chains

$$B = B_1 \supsetneq B_2 \supsetneq \cdots \supsetneq B_m = B'$$

Topological & Geometric interlude

$$\{\text{bracketings on } x_1 \cdots x_n\} \longleftrightarrow \text{Stasheff associahedron } \mathcal{A}_n$$
$$\longleftrightarrow \text{exceptional divisor in } \overline{\mathcal{M}}_{0,n+3}$$

D=Dynkin diagram of $\mathfrak{g}$

$$\{\text{maximal nested sets on } D\} \longleftrightarrow \text{De Concini–Procesi associahedron } \mathcal{A}_D$$
$$\longleftrightarrow \text{divisor in the DCP wonderful model of } \mathfrak{h}_{\text{reg}}$$

Coxeter categories: fiber functors

One crucial difference between braided and Coxeter categories

- In a braided tensor category $\mathcal{C}$, B_n acts by morphisms in $\mathcal{C}$.
- In a Coxeter category $\mathcal{Q}$, B_W does **not** act by morphisms in $\mathcal{Q}$.

Toy example

- The Weyl group action of $\mathfrak{S}_n$ on a $GL_n(\mathbb{C})$–module is not through morphisms in $\mathcal{Q} = \mathsf{Rep}(GL_n(\mathbb{C}))$, but through morphisms of the underlying vector space. In other words, there is a forgetful functor

$$F : \mathcal{Q} \to \mathsf{Vec} = \mathcal{Q}_\emptyset$$

  and a map $\mathfrak{S}_n \to \mathsf{Aut}(F)$.

In general, in a Coxeter category $\mathcal{Q}$

1. There is a family of forgetful functors $F_\mathcal{F} : \mathcal{Q} \to \mathcal{Q}_\emptyset$ ($\mathcal{Q}_\emptyset = \mathsf{Vec}$ in examples), labelled by maximal nested sets $\mathcal{F}$ on D.
2. B_W acts on each $F_\mathcal{F}$. In other words, for any $V \in \mathcal{Q}$, $\mathcal{F} \in \mathsf{Mns}(D)$,

$$V_\mathcal{F} := F_\mathcal{F}(V) \rightsquigarrow \lambda_\mathcal{F} : B_W \to \mathsf{Aut}_{\mathcal{Q}_\emptyset}(V_\mathcal{F})$$

Tensor categories with many fiber functors

Algebra Tensor category $\mathcal{C}$ with one fiber functor $\mathsf{f} : \mathcal{C} \to \mathsf{Vec}$

Example $\mathcal{C} = \mathsf{Rep}(A)$, A a Hopf algebra, $\mathsf{f} =$ forgetful functor

Topology Tensor category $\mathcal{C}$ with **many** fiber functors $\mathcal{C} \to \mathsf{Vec}$

Example

- $X =$ topological space
- $X_0 \subseteq X$ given collection of basepoints
- $\pi_1(X; X_0)$ fundamental groupoid based at X_0
- $\mathcal{C} = \mathsf{Rep}(\pi_1(X; X_0)) = \mathsf{Fun}(\pi_1(X; X_0), \mathsf{Vec})$
- $\{\mathsf{f}_x\}_{x \in X_0} : \mathcal{C} \to \mathsf{Vec}$ collection of fiber functors, $\mathsf{f}_x(\mathbb{V}) = \mathbb{V}_x$.
- $\gamma \in \pi_1(X; X_0) \rightsquigarrow \Phi_\gamma \in \mathsf{Hom}(\mathsf{f}_{\gamma(0)}, \mathsf{f}_{\gamma(1)})$, natural transformation.

Coxeter categories

Definition (ATL, Selecta 2019)

A braided Coxeter category of type D consists of 5 pieces of data.

1. Diagrammatic categories.

For any subdiagram $\emptyset \subseteq B \subseteq D$, a braided tensor category $\mathcal{Q}_B$.

Examples

1. $\mathcal{Q}_B = (\mathrm{Rep}\, U_\hbar\mathfrak{g}_B, R_B)$, $\mathfrak{g}_B = \langle e_i, f_i, h_i \rangle_{i \in B}$.
2. $\mathcal{Q}_B = (\mathrm{Rep}\, U\mathfrak{g}_B$, monodromy of the KZ equations for $\mathfrak{g}_B)$.

2. Restriction functors.

For any $B' \subseteq B$, and $\mathcal{F} \in \mathrm{Mns}(B, B')$, a (not necessarily braided) monoidal functor $F_\mathcal{F} : \mathcal{Q}_B \to \mathcal{Q}_{B'}$

Examples

1. $\mathcal{Q}_B = \mathrm{Rep}\, U_\hbar\mathfrak{g}_B$, $F_\mathcal{F} =$ (naive) restriction (independent of $\mathcal{F}$).
2. $\mathcal{Q}_B = (\mathrm{Rep}\, U\mathfrak{g}_B, e^{\hbar/2\Omega_{\mathfrak{g}_B}}, \Phi_{\mathrm{KZ}}^{\mathfrak{g}_B})$
   $F_\mathcal{F}$ needs to be constructed ($\Phi_B^{KZ} \neq \Phi_{B'}^{KZ}$).

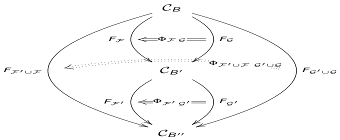
Coxeter categories

3. Associators. For any $B' \subseteq B$ and $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, an isomorphism of monoidal functors $\Phi_{\mathcal{G}\mathcal{F}} : F_{\mathcal{F}} \Rightarrow F_{\mathcal{G}}$ such that

$$\Phi_{\mathcal{H}\mathcal{G}} \cdot \Phi_{\mathcal{G}\mathcal{F}} = \Phi_{\mathcal{H}\mathcal{F}}$$

4. Joins. For any $B'' \overset{\mathcal{F}'}{\subseteq} B' \overset{\mathcal{F}}{\subseteq} B$ an isomorphism $\mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} : F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}' \cup \mathcal{F}}$ of monoidal functors $\mathcal{Q}_B \to \mathcal{Q}_{B''}$ satisfying

1. Vertical factorisation



2. Associativity For any $B''' \overset{\mathcal{F}''}{\subseteq} B'' \overset{\mathcal{F}'}{\subseteq} B' \overset{\mathcal{F}}{\subseteq} B$,

$$\mathsf{a}^{\mathcal{F}'\cup\mathcal{F}}_{\mathcal{F}''} \circ \mathsf{a}^{\mathcal{F}}_{\mathcal{F}'} = \mathsf{a}^{\mathcal{F}}_{\mathcal{F}''\cup\mathcal{F}'} \circ \mathsf{a}^{\mathcal{F}'}_{\mathcal{F}''}$$

as isomorphisms $F_{\mathcal{F}''} \circ F_{\mathcal{F}'} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}''\cup\mathcal{F}'\cup\mathcal{F}}$.

Definition

1. A *labelling* on D is the data of $m_{ij} \in \{2, \dots, \infty\}$, for any $i \neq j \in \mathbf{I} = V(D)$, such that $m_{ij} = m_{ji}$ and $m_{ij} = 2 \quad$ if $i \perp j$.
2. The Artin braid group corresponding to D and its labelling is

$$B_D = \langle S_i \rangle_{i \in \mathbf{I}} / \underbrace{S_i S_j \; S_i \cdots}_{m_{ij}} = \underbrace{S_j S_i S_j \cdots}_{m_{ij}}$$

5. Local monodromies.

Elements $S_i^{\mathcal{Q}} \in \mathrm{Aut}(F_{\emptyset i})$, $i \in \mathbf{I}$, satisfying

1. Braid relations. For any $i \neq j \in \mathbf{I}$,

$$S_i^{\mathcal{Q}} S_j^{\mathcal{Q}} \; S_i^{\mathcal{Q}} \cdots = S_j^{\mathcal{Q}} S_i^{\mathcal{Q}} S_j^{\mathcal{Q}} \cdots$$

Coxeter categories

Definition

1. A *labelling* on D is the data of $m_{ij} \in \{2, \dots, \infty\}$, for any $i \neq j \in \mathbf{I} = V(D)$, such that $m_{ij} = m_{ji}$ and $m_{ij} = 2 \quad$ if $i \perp j$.
2. The Artin braid group corresponding to D and its labelling is

$$B_D = \langle S_i \rangle_{i \in \mathbf{I}} / \underbrace{S_i S_j \ S_i \cdots}_{m_{ij}} = \underbrace{S_j S_i S_j \cdots}_{m_{ij}}$$

4. Local monodromies.

Elements $S_i^{\mathcal{Q}} \in \mathrm{Aut}(F_{\emptyset i})$, $i \in \mathbf{I}$, satisfying

1. Braid relations. For any $i \neq j \in \mathbf{I}$,

~~$$S_i^{\mathcal{Q}} S_j^{\mathcal{Q}} \ S_i^{\mathcal{Q}} \cdots = S_j^{\mathcal{Q}} S_i^{\mathcal{Q}} S_j^{\mathcal{Q}} \cdots$$~~

and any $\mathcal{F} \ni \{i\}, \mathcal{G} \ni \{j\}$, the following holds in $\mathrm{Aut}(F_{\emptyset D})$

$$\mathrm{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i^{\mathcal{Q}}) \cdot S_j^{\mathcal{Q}} \cdot \mathrm{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i^{\mathcal{Q}}) \cdots = S_j^{\mathcal{Q}} \cdot \mathrm{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i^{\mathcal{Q}}) \cdot S_j^{\mathcal{Q}} \cdots$$

4. Local monodromies ctd.

2. Coproduct identity (compatibility of B_W and B_n actions).
For any $i \in \mathbf{I}$, and $U, V \in \mathcal{Q}_i$, the following is commutative

$$\begin{array}{ccc}
F_{\emptyset i}(U) \otimes F_{\emptyset i}(V) & \xrightarrow{J_{\emptyset i}} & F_{\emptyset i}(U \otimes V) \\
\downarrow {\scriptstyle S_i^{\mathcal{Q}} \otimes S_i^{\mathcal{Q}}} & & \downarrow {\scriptstyle S_i^{\mathcal{Q}}} \\
F_{\emptyset i}(U) \otimes F_{\emptyset i}(V) & & F_{\emptyset i}(U \otimes V) \\
\downarrow {\scriptstyle c_\emptyset} & & \downarrow {\scriptstyle F_{\emptyset i}(c_i)} \\
F_{\emptyset i}(V) \otimes F_{\emptyset i}(U) & \xrightarrow[J_{\emptyset i}]{} & F_{\emptyset i}(V \otimes U)
\end{array}$$

(analogue of $\Delta(S_i) = R_i^{-1} \cdot S_i \otimes S_i$).

Coxeter categories: representations of B_W

Proposition. Let $\mathcal{Q}$ be a braided Coxeter category of type D.

1. There is a collection of homomorphisms

$$\lambda_{\mathcal{F}} : B_W \to \mathrm{Aut}(F_{\mathcal{F}})$$

   labelled by maximal nested sets on D, such that for any $\mathcal{F}, \mathcal{G} \in \mathrm{Mns}(D)$, $\lambda_{\mathcal{G}} = \mathrm{Ad}(\Phi_{\mathcal{G}\mathcal{F}}) \circ \lambda_{\mathcal{F}}$ $(\star)$

2. The collection $\{\lambda_{\mathcal{F}}\}$ is uniquely determined by $(\star)$, and the following normalisation condition: if $\mathcal{F}$ contains a one vertex diagram $\{i\}$, then

$$\lambda_{\mathcal{F}}(S_i) = S_i^{\mathcal{Q}}$$

Remark The normalisation condition is analogous to the fact that, in a braided tensor category, the generator T_i of B_n only acts on the i and $i+1$ tensor copies in $V_b^{\otimes n}$ if b contains $\cdots(x_i x_{i+1})\cdots$

Main results I: (Quantum) reality check

Proposition (Appel–TL, Selecta 2018) There is a braided Coxeter category $\mathbb{O}_\hbar^{\text{int}}$ with

- Diagrammatic categories $\left(\mathcal{O}^{\text{int}}_{U_\hbar\mathfrak{g}_B}, R_{U_\hbar\mathfrak{g}_B}\right)$, $B \subseteq D$.
- (standard) Restriction functors $F_{\mathcal{F}} : \mathcal{O}^{\text{int}}_{U_\hbar\mathfrak{g}_B} \to \mathcal{O}^{\text{int}}_{U_\hbar\mathfrak{g}_{B'}}$
- (trivial) Associators $\Phi_{\mathcal{G}\mathcal{F}} = \mathbf{1}_{\text{Res}_{U_\hbar\mathfrak{g}_{B'}, U_\hbar\mathfrak{g}_B}}$
- (trivial) Joins $\mathrm{a}^{\mathcal{F}}_{\mathcal{F}'} : \text{Res}_{U_\hbar\mathfrak{g}_{B''}, U_\hbar\mathfrak{g}_{B'}} \circ \text{Res}_{U_\hbar\mathfrak{g}_{B'}, U_\hbar\mathfrak{g}_B} = \text{Res}_{U_\hbar\mathfrak{g}_{B''}, U_\hbar\mathfrak{g}_B}$.
- Local monodromies: $S_i^{\mathbb{O}_\hbar^{\text{int}}} = S_i^\hbar$, qWeyl group element.

Main results II: Transfer to $\mathcal{O}_{\mathfrak{g}}$

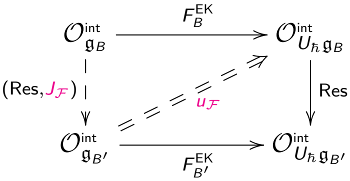
Theorem (Appel–TL, Selecta 2019)

$\mathbb{O}_{\hbar}^{\text{int}}$ is equivalent to a braided Coxeter category $\mathbb{O}_{\text{trans}}^{\text{int}}$ with

- Diagrammatic categories $\left(\mathcal{O}_{\mathfrak{g}_B}^{\text{int}}, e^{\hbar/2\Omega_{\mathfrak{g}_B}}, \Phi_{\text{KZ}}^{\mathfrak{g}_B}\right)$.
- Restriction functors $F_{\mathcal{F}} = \left(\text{Res}_{\mathfrak{g}_{B'},\mathfrak{g}_B}, J_{\mathcal{F}}\right)$
  Res is standard restriction, $J_{\mathcal{F}}$ some $\otimes$ structure.

Remarks

1. The tensor structure $J_{\mathcal{F}}$ is not trivial: $\Phi_{\text{KZ}}^{\mathfrak{g}_B} \neq \Phi_{\text{KZ}}^{\mathfrak{g}_{B'}}$.
2. Main ingredients needed (ATL, Selecta 2018)



Main results III: Rigidity

Theorem (Appel–TL, Advances 2019) Braided Coxeter structures with

1. Diagrammatic categories $\left(\mathcal{O}^{\text{int}}_{\mathfrak{g}_B}, e^{\hbar/2\Omega_{\mathfrak{g}_B}}, \Phi^{\mathfrak{g}_B}_{\text{KZ}}\right)$.
2. Restriction functors $F_{\mathcal{F}} = \left(\text{Res}_{\mathfrak{g}_{B'},\mathfrak{g}_B}, J_{\mathcal{F}}\right)$.

are unique (up to a unique equivalence) **provided** they are of PROPic origin.

Theorem (Appel–TL, Selecta 2019) The transferred braided Coxeter structure $\mathbb{O}^{\text{int}}_{\text{trans}}$ coming from $U_\hbar\mathfrak{g}$ is PROPic.

Main results IV: The Casimir connection

Theorem (TL, arXiv:1601.04076 for $\dim \mathfrak{g} < \infty$, Appel–TL for general $\mathfrak{g}$)
There is a braided Coxeter category $\mathbb{O}_{\nabla}^{\text{int}}$ with

1. Diagrammatic categories $\left(\mathcal{O}_{\mathfrak{g}_B}^{\text{int}}, e^{\hbar/2\Omega_{\mathfrak{g}_B}}, \Phi_{\text{KZ}}^{\mathfrak{g}_B}\right)$.
2. Restriction functors $F_{\mathcal{F}} = \left(\text{Res}_{\mathfrak{g}_{B'},\mathfrak{g}_B}, J_{\mathcal{F}}\right)$.

which accounts for

1. $B_n \circlearrowright V^{\otimes n}[[\hbar]]$, $V \in \text{Rep}(U\mathfrak{g}_B)$, monodromy of KZ equations for $\mathfrak{g}_B$.
2. $B_W \circlearrowright V[[\hbar]]$, monodromy of the Casimir equations for $\mathfrak{g}$.

Ingredients

- The tensor structure $J_{\mathcal{F}}$ arises from an ODE on $\mathbb{P}^1$ with irregular singularities (dynamical KZ equations).
- The associators $\Phi_{\mathcal{G}\mathcal{F}}$ are constructed from the Casimir connection by work of De Concini–Procesi.
- W–equivariant resummation of the Casimir connection for $\dim \mathfrak{g} = \infty$ ($\sum_{\alpha \in \mathsf{R}_+} d\alpha/\alpha \cdot \mathcal{K}_\alpha$ is an ∞ sum).

Proposition (Appel–TL) The braided Coxeter structure $\mathbb{O}_{\nabla}^{\text{int}}$ is PROPic.

Summary

Theorem (Appel–TL) For any symmetrisable KM algebra $\mathfrak{g}$, there is an equivalence between

1. the braided Coxeter category $\mathbb{O}_{\hbar}^{\text{int}}$ underlying
   - $B_n \circlearrowright \mathcal{V}^{\otimes n}$, R–matrix action.
   - $B_W \circlearrowright \mathcal{V}$, quantum Weyl group action.
2. the braided Coxeter category $\mathbb{O}_{\nabla}^{\text{int}}$ underlying
   - $B_n \circlearrowright V^{\otimes n}[[\hbar]]$, monodromy of KZ equations for $\mathfrak{g}$.
   - $B_W \circlearrowright V[[\hbar]]$, monodromy of the Casimir equations for $\mathfrak{g}$.

Corollary The monodromy of the Casimir connection on $V \in \mathcal{O}_{\mathfrak{g}}^{\text{int}}$ is equivalent to the quantum Weyl group action of B_W on $F^{\text{EK}}(V) \in \mathcal{O}_{U_\hbar \mathfrak{g}}^{\text{int}}$.