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Axions, Gravitational Waves and Phase Transitions in the very Early Universe

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List of Pubblications

- *Dark Matter with Stückelberg Axions* Claudio Corianò, Paul H. Frampton, Nikos Irges, Alessandro Tatullo Front.in Phys. 7 (2019) 36 e-Print: 1811.05792 [hep-ph]
- *Exact Correlators from Conformal Ward Identities in Momentum Space and Perturbative Realizations* Claudio Corianò, Matteo Maria Maglio, Alessandro Tatullo, Dimosthenis Theofilopoulos.

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Introduction

Despite all the great successes of high energy physics, there is still a general sense of dissatisfaction regarding many unsolved fundamental issues in particle phenomenology, that don't find an answer within the Standard Model (SM) of the elementary particles. This is formulated as a non-abelian gauge theory, which is is quite success-full in explaining a large quantity of experimental data, from hadronic physics up to the electroweak scale, but that leaves several questions unanswered. Examples of these are: the origin of the neutrino masses, various anomalies in the flavour sector, the gauge hierarchy problem, the origin of dark matter.
From the theoretical point of view, one notices the flourishing of many models that extend the Standard Model in various directions, but the difficulties persist. From our viewpoint, one of the main shortcoming of the SM is the lack of a connection of this model with gravity, for being essentially a low energy theory, compared to more general formulations where the gauge structure is far wider and more unifying. Examples of such theories are those based on Grand Unification, involving larger symmetries, such as the Left-Right symmetric Model or models based on $SO(10)$ or trinification ($SU(3)^3$), or even based on exceptional symmetries such as E_6. In all these models, we need to match the theoretical predictions with all the experimental data, and these include those coming from the current cosmological observations, which indicate that in our universe, about a quarter of it energy budget, should be attributed to dark matter. This thesis work is centred around the study of a specific candidate for dark matter that emerges from extensions of the Standard Model as we get closer to the Planck scale.

In this thesis we will be discussing two important aspects of the physics of the early Universe, concerning, separately, axions and gravitational waves. The two topics,as just mentioned, are treated separately although it is possible to investigate the impact of the propagation of gravitational waves on an axion condensate. Axions have been introduced in order to solve the strong CP problem, by Peccei and Quinn, long ago ([22, 23, 24])and searches for their detection are ongoing at experimental level. Along the years, the theory has undergone several modifications and extentions. The axion-based model that we will discuss, deals with a specific extension of the theory derived from the theory of branes. The effective action derived for this model requires a so called 'Stueckelberg axion', in the form of a Nambu-Goldstone mode that couples to a gauge anomaly, in order to restore the gauge symmetry. The underlying assumption is that a gauge anomaly, produced by a string theory sector, is cancelled at field theory level by the exchange of a pseudo-scalar field. The result, at field theory level is that an Abelian anomaly in the spectrum of the effective field theory that is derived from string, is erased by the exchange of such pseudo-scalar. We will not discuss string/brane origin of this model but we will describe its structure and its differences with respect to the ordinary PQ theory in Chapter 6. For this reason the generation of such particle can be directly related to the possibility of having an anomalous Abelian gauge theory in the early Universe. This possibility is at the core of our proposal.

In this context one of the most important and promising ingredients of today theoretical high energy physics is the particle called axion. The interest in axions has arisen in a decisive way because it allows to solve at once two big puzzles: the strong CP problem and the dark matter problem. Moreover the formal introduction of axion turns out to be extremely simple and elegant and this has always aroused great excitement within the high energy community. The oldest origin of the axion dates back to the 70's when they were looking for a solution to the so-called $U(1)_A$ problem. The strong interaction theory in fact predicts the breaking of the group $U(1)_A$ by a chiral condensate operator, in the same way in which the symmetry of $SU(2)$ is broken giving rise to pseudo-bosons (the pions). The problem was that for $U(1)_A$ no particle was associated. The problem was solved by G. 't Hooft in 1976 when he realized that $U(1)_A$ is not a QCD symmetry, in fact the non Abelian nature of $SU(3)_c$ leads to particular properties of QCD able to violate the symmetries at a quantum level. The resolution of this problem brought another one. It was observed that the Lagrangian of QCD admits a topological term, the so-called θ-term. Through experimental tests it was found that $|\theta| < 1.3 \times 10^{-10}$. The extremely small value posed the following problem: while the introduction of the θ-term was completely justified on formal grounds, one could not explain such a small value. A fully satisfactory explanation of the problem has not yet been found. The most famous attempt to explain the dilemma dates back to the 1977 Peccei-Quinn proposal. The θ-term could be erased from the Lagrangian of QCD simply by introducing a new extra symmetry. But as soon as this solution was introduced Weinberg and Wilczek pointed out in 1978 that if Peccei-Quinn symmetry was embedded, then another particle, named axion, would arise. Since then there have been models that try to incorporate axion and unify it with the field content of the Standard Model(SM). The limit of 10 KeV as mass of the new particle has discouraged all the models that foresaw a visible axion in favor of the rise of models of invisible axions. Among these we can distinguish KSVZ and DFSZ models. Both models involve the introduction of a Higgs singlet equipped with a VEV that defines an energy scale directly related to the properties of the axion. The identification of the axion with a likely dark matter candidate, comes from the very light mass of the particle and its very weak interaction with ordinary matter. Since the theorization of this particle there have been incredible efforts to detect its elusive properties. An important contribution to axion hunting came from Sikivie who in 1983 proposed an axion helioscope to probe the presence of the axions coming from the Sun and the axion haloscope to verify the origin of axions from DM galactic halos. In recent years experiments have been designed to produce axions in the laboratory, such as the light shining through walls approach (LSW) and the vacuum magnetic birefringence (VMB). The measurements have been carried out over the years and have led to define constraints on the mass of the axion which is found to have a lower limit $m_a < 0.8eV$ and as an upper limit $m_a > 10^{-10}eV$.

Regarding the limit on the interactions, in particular on the coupling constant with the photon we find that $g_{a\gamma} < 6.6 \times 10^{-11} GeV^{-1}$. Another goal of theoretical research is to incorporate the axion particle within the context of the Standard Model and to relate the properties of axion to the properties of other fields contained in beyond Standard Models (BMS). PQ axions are connected with a global $U(1)_{PQ}$ symmetry. This symmetry is attached to the fermions of the Standard Model and although it is an anomalous one, it does not have any negative implications for the consistency of the theory. We recall that anomalous global symmetries do not destabilize the inner consistency of a gauge theory.

This thesis work investigates two aspects of the physics of the early Universe. One of them, as mentioned, deals with axions generated by Stueckelberg models. In particular, one of the contribution to this topic is contained in Chapter 8 when we entertain the possibility that the Stueckelberg scale, present in the model can be raised close to the GUT scale. The methodology, in this case, is quite similar to that of the ordinary two-Higgs doublet model discussed in [134], but in a more complex scenario based on SO(10) theory. An interesting result of this analysis is that, by selecting M_{Stueck}, the Stueckelberg scale, close to the Planck scale, we predict a mass for the Stueckelberg axi-Higgs particle that is ultralight (10^{-18}eV).This mass value, as we explain, has been considered in the astrophysical community as an interesting one. In order to solve some problems related to the matter distributions of the sub-galactic scale. A second topic that we will be addressed concerns the detection of gravitational waves in the phase transitions that accompany the spontaneous breaking of the gauge symmetry in the early Universe.

Chapter 1

Standard Model of Cosmology

In this chapter we will present a brief review of the standard model of Cosmology. The discussion will be mostly based on the works [1, 2]for the standard model and [1] and [3] for the inflationary model. We present a brief outline of the chapter. The starting point will be the Friedman-Lemaî̂tre-Robinson-Walker (FLRW) metric, as a solution of Einstein's equations of General Relativity . We will first discuss the field equations containing the scaling factor a(t). We will present some basic concepts of Cosmology as the particle horizon, the event horizon and the red-shift parameter. Then we will derive the form of Einstein equations for a metric of FLRW and we will find the form of the energy-momentum tensor. This in order to extract the 00 component of the Einstein equation, the so-called Friedman equation and the ij component, also known as the acceleration equation. These equations will be studied in order to obtain information about the scale factor that plays an important role in the evolution of the Universe, for example we will derive the time dependence of this factor. Then the critical density and the equations of state for a Universe both dominated by radiation and matter will be found. The role of cosmological constant and its connection with dark energy will be also discussed. Finally, at the end of the chapter we will present some unsolved problems, typical of the standard model as the flatness problem or the horizon problem, and we will present a possible solution constituted by the inflationary model.

Under the hypothesis of homogeneity and isotropy of the Universe, we can write the FLRW metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega_r\right] \tag{1.0.1}$$

where

$$d\Omega_r = d\theta^2 + sin^2\theta d\phi^2 \tag{1.0.2}$$

curvature

$$\begin{cases} k = -1 & \text{negative} \\ k = 0 & \text{flat} \\ k = +1 & \text{positive} \end{cases}$$

Particle horizon If at t=0 a photon is emitted, at time t has travelled a distance $d_H(T)$, which is called particle horizon. In the case of a beam of light

$$ds^2 = 0 \Rightarrow dt^2 = \frac{a^2(t)dr^2}{1 - kr^2} \tag{1.0.3}$$

$$\int_0^t \frac{dt'^2}{a^2(t')} = \int_0^r \frac{dr'}{(1 - kr'^2)^{1/2}} \Rightarrow \tag{1.0.4}$$

$$d_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} = a(t) \int_0^r \frac{dr'}{(1 - kr'^2)^{1/2}}. \tag{1.0.5}$$

Equation 1.0.5 is called particle horizon. It defines a surface whose points are causally connected.

Consider several dependences of a(t). For matter dominance:

$$a(t) \propto t^{2/3} \tag{1.0.6}$$

then

$$\begin{aligned} d_H(t) &= a(t) \int_0^t \frac{dt'}{a(t')} \\ &= t^{2/3} \int_0^t \frac{dt'}{t'^{2/3}} \\ &\approx t^{2/3} \frac{t^{-2/3+1}}{-2/3+1} \\ &= t^{2/3} \frac{t^{1/3}}{-2/3+1} = 3t. \end{aligned} \tag{1.0.7}$$

For a radiation-dominated Universe we should consider the dependence

$$a(t) \propto \sqrt{t} \tag{1.0.8}$$

$$
\begin{aligned}
d_H(t) &= a(t)\int_0^t \frac{dt'}{a(t')} \\
&= \sqrt{t}\int_0^t \frac{dt'}{\sqrt{t'}} \\
&= t^{1/2}\frac{t^{-1/2+1}}{-1/2+1} \\
&= 2t.
\end{aligned}
\tag{1.0.9}
$$

So, the particle horizon in a matter-dominated Universe is larger than in the radiation dominated one.

For an inflationary Universe the dependence to be considered is

$$
a(t) \propto e^{Ht} \tag{1.0.10}
$$

then

$$
\begin{aligned}
d_H(t) &= a(t)\int_0^t \frac{dt'}{a(t')} \\
&= e^{Ht'}\int_0^t \frac{dt'}{e^{Ht'}} \\
&= e^{Ht}\frac{1}{H}[e^{-Ht}+1] \\
&= \frac{1}{H}[e^{Ht}-1].
\end{aligned}
\tag{1.0.11}
$$

The FLRW metric allows us to calculate the red-shifting of light from distant objects. Let us consider two crests emitted respectively at $r = r_1$ at times $t = t_1$ and $t = t_1 + \Delta t_1$. They are received at times t_0 and $t = t_0 + \Delta t_0$. Then we have

$$
\int_{t_1}^{t_0} \frac{dt'}{a(t')} = \int_{t_1}^{t_0} \frac{dt'}{a(t')} = \int_0^{r_1} \frac{dr'}{\sqrt{1-kr'^2}} \tag{1.0.12}
$$

and

$$
\int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt'}{a(t')} = \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt'}{a(t')} = \int_0^{r_1} \frac{dr'}{\sqrt{1-kr'^2}}. \tag{1.0.13}
$$

Now subtract the two contributions

$$
\int_{t_1}^{t_0} \frac{dt'}{a(t')} - \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt'}{a(t')} = 0. \tag{1.0.14}
$$

So

$$
\int_{t_0+\Delta t_0}^{t_0} \frac{dt'}{a(t')} + \int_{t_1}^{t_1+\Delta t_1} \frac{dt'}{a(t')} = 0. \tag{1.0.15}
$$

giving

$$\int_{t_0}^{t_0+\Delta t_0} \frac{dt'}{a(t')} = \int_{t_1}^{t_1+\Delta t_1} \frac{dt'}{a(t')} \tag{1.0.16}$$

from which

$$\frac{\Delta t_0}{a(t_0)} = \frac{\Delta t_1}{a(t_1)} \tag{1.0.17}$$

but Δt_0 and Δt_1 are the two periods at emission and reception. Let us define the frequencies $\nu_0 = \frac{1}{\Delta t_0}$ and $\nu_1 = \frac{1}{\Delta t_1}$ and wave-lengths $\lambda_0 = \Delta t_0$ and $\lambda_1 = \Delta t_1$ (with c=1, in natural units). Define

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1}. \tag{1.0.18}$$

From 1.0.17 it follows that

$$\frac{\lambda_0}{a(t_0)} = \frac{\lambda_1}{a(t_1)} \tag{1.0.19}$$

$$\lambda_1 = \lambda_0 \frac{a(t_1}{a(t_0)}. \tag{1.0.20}$$

Replacing in z

$$z = \frac{\left(\lambda_0 - \lambda_0 \frac{a(t_1)}{a(t_0)}\right)}{\lambda_0 \frac{a(t_1)}{a(t_0)}} \tag{1.0.21}$$

from which, eliminating λ_0 one gets

$$\frac{a(t_0) - a(t_1)}{a(t_1)} = z \tag{1.0.22}$$

$$1 + z = \frac{a(t_0}{a(t_1}, \tag{1.0.23}$$

notice that for an expanding Universe $a(t_0) > a(t_1)$, so that

$$1 + z = \frac{a(t_0)}{a(t_1)}. \tag{1.0.24}$$

This means that a photon emitted at time t_1 undergoes a red-shift as the Universe expands. Consequently the wavelength will be increased by a factor $\frac{a(t_0)}{a(t_1)}$ and therefore its momentum decreases by the same factor. If $\mid t_0 - t_1 \mid$ is small, we can expand

$$\begin{aligned} a(t_1) &= a(t_0) + (t_1 - t_0)a(\dot{t}_0) + \frac{1}{2}(t_1 - t_0)^2 \ddot{a}(t_0) + ... \\ &= a(t_0)(1 + H_0(t_1 - t_0) - \frac{1}{2} q_0 H_0^2 (t_1 - t_0)^2 + ...) \end{aligned} \tag{1.0.25}$$

with

$$H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)} \tag{1.0.26}$$

is the Hubble parameter and

$$q_0 \equiv -\frac{\ddot{a}(t_0)}{a(t_0)H_0^2} = -\frac{\ddot{a}(t_0)a(t_0)}{\dot{a}^2(t_0)} \tag{1.0.27}$$

is called deceleration parameter. It can be introduced in the red-shift expansion

$$1 + z = (1 + H_0(t_1 - t_0) - \frac{1}{2}q_0 H_0^2 (t_0 - t_1)^2 + ...)^{-1} \tag{1.0.28}$$

from which

$$z = H_0(t_0 - t_1) + (1 + \frac{1}{2}q_0)H_0^2(t_1 - t_0)^2 + ... \tag{1.0.29}$$

inverting the geometrical series one gets

$$(t_0 - t_1) = \frac{1}{H_0}\left[z - \left(1 + \frac{1}{2}q_0\right)z^2 + ...\right]. \tag{1.0.30}$$

In general we make measurements of red-shift. From cosmological model we can infer the distance from the red-shift.

1.1 Einstein equations in FLRW metric

Now consider a FLRM metric. This is an ansatz from which we can derive the equations of motion for the arbitrary functions present in the ansatz. Let us calculate the coefficients of the affine connection1.0.1

$$\Gamma^0_{ij} = -\frac{\dot{R}}{R}g_{ij} \tag{1.1.1}$$

$$\Gamma^i_{j0} = \frac{\dot{R}}{R}\delta ij = \Gamma^i_{0j} \tag{1.1.2}$$

$$\Gamma^i_{jk} = (\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk}). \tag{1.1.3}$$

For the Ricci tensor we have

$$R_{00} = -3\frac{\ddot{R}}{R} \tag{1.1.4}$$

$$R_{ij} = -\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + \frac{2k}{R^2}\right)g_{ij} \tag{1.1.5}$$

which are the only non-null components. The curvature scalar is

$$\mathcal{R} \equiv g^{\mu\nu} R_{\mu\nu} = -6 \left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}. \right) \tag{1.1.6}$$

The Einstein field equations assume the form

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1.1.7}$$

where G is the Newtonian gravitational constant, $T_{\mu\nu}$ is the energy-momentum tensor and Λ is the cosmological constant. If we assume for the Universe the behaviour of a perfect fluid with pressure p and energy density ρ, then

$$T_{00} = \rho \tag{1.1.8}$$

$$T_{ij} = -p\delta_{ij}. \tag{1.1.9}$$

Combining all these ingredients together one gets the 00-component of Einstein equations for a perfect fluid also known as Friedman equation

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G \rho}{3} + \frac{\Lambda}{3} \tag{1.1.10}$$

and the ij-component, also known as acceleration equation

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = -8\pi G p + \Lambda \tag{1.1.11}$$

Subtracting 1.1.11 from 1.1.10 gives

$$\left(\frac{\ddot{a}}{a} \right) = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{1.1.12}$$

If we neglect Λ, then $\ddot{a}$ is negative, so if at the present time $\dot{a}$ is positive, it means that $\dot{a}$ was always positive and consequently R was increasing for any time .

From 1.1.11, for a vanishing Λ and vanishing curvature (k=0) one gets

$$\rho = \rho_c \equiv \frac{3H^2}{8\pi G} = 3M_p^2 H^2 \tag{1.1.13}$$

with ρ_c which stands for critical density, H is the Hubble parameter and M_p is the Planck mass, defined as follows

$$M_p^2 = \frac{1}{8\pi G}. \tag{1.1.14}$$

The Hubble parameter changes with time so does ρ_c. It is necessary to define another parameter which take into account these changes

$$\Omega \equiv \frac{\rho(t)}{\rho_c(t)}. \tag{1.1.15}$$

1.2 The energy density depends on the scale factor

Let us consider the conservation of the energy-momentum tensor

$$D_\nu T^{\mu\nu} = 0 \tag{1.2.1}$$

with

$$D_\lambda V^\mu = \partial_\lambda V^\mu + \Gamma^\mu_{\lambda\rho} V^\rho \tag{1.2.2}$$

is the covariant derivative of a vector V^μ. If one considers the $\mu = 0$ component of 1.2.1, then

$$\dot{\rho} + 3(\rho + p)H = 0. \tag{1.2.3}$$

The last equation describes the adiabatic evolution of a fluid coupled to the FLRM geometry.

In fact we consider an eigen-volume V with total energy $E = \rho V$. The volume will have a trend like ([4])

$$V(t) \approx a^3(t) \tag{1.2.4}$$

where a(t) is the scale factor. So

$$\frac{\dot{V}}{V} = 3\frac{a^2\dot{a}}{a^3} = 3\frac{\dot{a}}{a} = 3H \tag{1.2.5}$$

replacing $E = \rho V$

$$\left(\frac{\dot{E}}{V}\right) + 3H\left(\frac{E}{V} + p\right) = 0 \tag{1.2.6}$$

$$\frac{\dot{E}}{V} - \frac{E}{V}\frac{\dot{V}}{V} + \frac{\dot{V}}{V}\left(\frac{E}{V} + p\right) = 0 \tag{1.2.7}$$

which is exactly the condition to have an adiabatic evolution

$$dE + pdV = 0 \tag{1.2.8}$$

in fact, from 1.2.8 we infer that the entropy of the fluid remains constant if we consider a fluid evolving in a geometry like that described by 1.0.1. Now consider the equation of state for the fluid, expressed in the form

$$p = w\rho \tag{1.2.9}$$

replacing in 1.2.3 and remembering the definition of the Hubble parameter 6.4.11

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}(t)}{a(t)} = 0, \tag{1.2.10}$$

which can be rewritten using the Hubble parameter

$$\dot{\rho} + 3(\rho + p)H = 0. \tag{1.2.11}$$

In general the energy density and pressure are related by the equation of state 1.2.9. Insert 1.2.9 into 1.2.10 to obtain

$$\dot{\rho} + 3(\rho + wp)H = 0 \tag{1.2.12}$$

$$\dot{\rho} + 3\rho(1 + w)H = 0 \tag{1.2.13}$$

$$\tag{1.2.14}$$

from which separating the variables one gets

$$\rho \sim a^{-3(1+w)}. \tag{1.2.15}$$

Replacing this one as an ansatz in 1.2.10 gives

$$-3(1+w)a^{-3(1+w)-1}a + 3a^{-3(1+w)}(1+w)\frac{\dot{a}}{a} = 0 \tag{1.2.16}$$

giving

$$3\dot{a}\left[-(1+w)a^{-3(1+w)-1} + a^{-3(1+w-1)}(1+w)\right] = 0 \tag{1.2.17}$$

which is indeed a solution. We can distinguish various trends for ρ depending on the value of w. In the case of radiation we have $w = \frac{1}{3}$, so

$$p = \frac{1}{3}\rho \qquad\qquad \rho \sim a^{-4} \tag{1.2.18}$$

$$\tag{1.2.19}$$

while for matter w=0

$$p = 0 \qquad\qquad \rho \sim a^{-3}. \tag{1.2.20}$$

$$\tag{1.2.21}$$

In the case of radiation ρ decreases more rapidly, so we infer that matter will dominate at a later stage.

It is also interesting to note the case of equation of state with w=-1 which corresponds, (see section

1.5) to the equation of state associated with the cosmological constant. For w=-1, we will have the contribution given from vacuum energy and it plays the role of a cosmological constant. Replacing into 1.2.10

$$\dot{\rho} = 0 \Rightarrow \rho = constant. \tag{1.2.22}$$

1.3 Scale factor time trend

In the previous section we calculated the Friedman equation considering a null curvature k and null cosmological constant. These two approximations become all the better the closer we get to early times. In this case in the 1.1.10 the term $\frac{8\pi G\rho}{3}$ becomes dominant over the curvature term and over the cosmological constant. Substituting the 1.2.15 into the 1.1.10 gives the trend for a(t):

$$a(t) \sim t^{-\frac{3}{2}(1+w)} \tag{1.3.1}$$

which again gives, for radiation and matter, respectively

$$a \sim t^{\frac{1}{2}} \qquad H = \frac{1}{2}t^{-1} \tag{1.3.2}$$

$$a \sim t^{\frac{2}{3}} \qquad H = \frac{2}{3}t^{-1}. \tag{1.3.3}$$

As for the case of Λ domination, we assume the state equation

$$p = -\rho$$

and the term of 1.1.10 which contains Λ dominates over all the other, we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} \tag{1.3.4}$$

from which separating the variables and integrating one gets

$$a(t) = e^{\sqrt{\frac{\Lambda}{3}}t}. \tag{1.3.5}$$

1.4 Age of the Universe

It is possible to estimate the age of the Universe in a special case with $\Lambda = 0$. We also consider a phase of the Universe dominated by matter. To do this, we rewrite the 1.1.10 in terms of ρ_0, i.e., the value of ρ at the current era

$$\left(\frac{\dot{a}^2}{a}\right) + \frac{k}{a^2} = \frac{8\pi G\rho}{3}\rho_0\frac{a_0}{a} \tag{1.4.1}$$

which can be rewritten in terms of the relative present value of energy density

$$\Omega_0 = \frac{\rho_0}{\frac{3}{8}\pi G H_0^2} \tag{1.4.2}$$

giving

$$\left(\frac{\dot{a}}{a_0}\right) + H_0^2(\Omega_0 - 1) = \Omega_0 H_0^2 \frac{a_0}{a}. \tag{1.4.3}$$

We may change the variable

$$x \equiv \frac{a}{a_0} \tag{1.4.4}$$

so1.4.3 becomes

$$\dot{x}^2 + H_=^2(\Omega_0 - 1) = \Omega_0 H_0^2 x^{-1} \tag{1.4.5}$$

which can be solved separating the variables and integrating

$$t = \frac{1}{H_0}\int_0^1 \frac{dx}{\sqrt{\Omega_0(x^{-1} - 1) + 1}} \tag{1.4.6}$$

since $x \equiv \frac{a}{a_0}$, taking $a = a_0 \rightarrow x = 1$ at $t = t_0$.

Let us evaluate t_0 for an exactly flat Universe. We shall consider

$$\frac{k}{a_0^2} = H_0^2(\Omega_0 - 1) \tag{1.4.7}$$

if we require a flat Universe, i.e. $\Omega_0 = 1$, it means that $k = 0$. Then

$$\begin{aligned} t_0 &= \int_0^1 \frac{1}{H_0} \frac{dx'}{\sqrt{(x'^{-1} - 1) + 1}} \\ &= \frac{1}{H_0}\int_0^1 x'^{\frac{1}{2}} dx' = \frac{2}{3H_0}. \end{aligned} \tag{1.4.8}$$

According to the latest measurements

$$H_0^{-1} \approx h^{-1} 9.78 \times 10^9 yr \tag{1.4.9}$$

with $h \approx 0.72 \pm 0.05$, so that the age of the Universe is about

$$t_0 \approx 10^{10} yr. \tag{1.4.10}$$

1.5 The cosmological constant Λ

Equation 1.1.10 is the Friedman equation, while 1.1.11 is the acceleration equation, with ρ_c representing the critical density defined in 1.1.13 and M_p is the Planck mass. It is possible to rewrite equation 1.1.11in a different way, using the definition

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \tag{1.5.1}$$

so that one gets

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{8\pi G}{3}(\rho + \rho_\Lambda). \tag{1.5.2}$$

Let us take a look at the dimensions

$$[\Lambda] = \left[\frac{k}{a^2}\right] = \frac{1}{[L^2]} \tag{1.5.3}$$

which is the inverse of a squared length. As for the Newtonian constant

$$\frac{GM^2}{r^2} = force \tag{1.5.4}$$

so that in natural units

$$[G] = \frac{[M][L][T^{-2}][L^2]}{[M^2]} = \frac{1}{[M^2]}. \tag{1.5.5}$$

Recalling Einstein action

$$\mathcal{S} = 16\pi G \int d^4 x R \tag{1.5.6}$$

where R is the scalar curvature. Dimensional analysis gives

$$[\mathcal{S}] = 1 = [G][d^4x][R] = [G]\frac{1}{[M]^4}[M]^2 \to [G] = [M]^2. \tag{1.5.7}$$

If we consider this and recall the dimensions of Λ, one finds that

$$[\rho_\Lambda] = \left[\frac{\Lambda}{8\pi G}\right] = [M]^2 \times [M]^2 = [M]^4 \tag{1.5.8}$$

which is what we expect for a potential; recall indeed that

$$\left[\int d^4 x V(\phi)\right] = [1] \to [V] = [M]^4. \tag{1.5.9}$$

We could think to split Λ into pressure and density terms. In order to do so consider equation 1.1.12

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{1.5.10}$$

This equation has a static solution given by

$$\frac{\Lambda}{3} = -\frac{4}{3}\pi G[\rho_\Lambda + 3p_\Lambda] \tag{1.5.11}$$

which, if we use the definition of ρ_Λ to rewrite the term $\frac{\Lambda}{3}$, can be rearranged as follows

$$\rho_\Lambda \frac{8}{3}\pi G = -\frac{4}{3}\pi G \rho_\Lambda - \frac{4}{3}\pi G 3 p_\Lambda \tag{1.5.12}$$

giving

$$4\pi G \rho_\Lambda = -4\pi G p_\Lambda \tag{1.5.13}$$

from which we find the state equation for the cosmological constant

$$p_\Lambda = -\rho_\Lambda. \tag{1.5.14}$$

Equation 1.5.14 is a particular case of the general state equation $p = w\rho$ and is useful if we want to treat Λ as a fluid. It is possible to introduce a source term for the cosmological constant in the Einstein equations. We need to define $T_\Lambda \mu\nu$,and we set

$$\Lambda g_{\mu\nu} = 8\pi G T_\Lambda \mu\nu \Rightarrow T_\Lambda \mu\nu = \frac{\Lambda g_{\mu\nu}}{8\pi G} \tag{1.5.15}$$

It is possible to define Ω_k and Ω_Λ as we did for Ω_0. Reconsider Friedman equation at present time

$$H_0^2 + \frac{k}{a_0^2} = \frac{8}{3}\pi G\rho + \frac{\Lambda}{3}. \tag{1.5.16}$$

To determine ρ_c we required a flat (k=0) Universe without Λ, then

$$\rho_c = \frac{3H^2}{8\pi G} \tag{1.5.17}$$

and

$$\Omega_0 = \frac{\rho_0}{\rho_c}. \tag{1.5.18}$$

Similarly we can replace the curvature and cosmological term respectively with Ω_k and Ω_Λ in the Friedman equation,

$$H_0^2 + H_0^2\Omega_k = \frac{8}{3}\pi G\rho_0 + H_0^2\Omega_\Lambda \tag{1.5.19}$$

having defined

$$\begin{cases} \Omega_k = \frac{k}{a_0^2 H_0^2} \\ \Omega_\Lambda = \frac{\Lambda}{3H_0^2} \end{cases}$$

If we then replace ρ_0 using the definition of Ω_0 we can rewrite the Friedman equation as

$$H_0^2 - H_0^2\Omega_k = H_0^2\Omega_0 + H_0^2\Omega_\Lambda \tag{1.5.20}$$

from which follows the dimensionless form of the Friedman equation, which describes the cosmic dynamics

$$\Omega_k + \Omega_\Lambda + \Omega_0 = 1. \tag{1.5.21}$$

1.6 Inflation model

1.6.1 Flatness problem

The standard cosmological model presents some problems, since some initial conditions should be placed by hand and since it does not give an explanation for some problems such as flatness or the problem of the horizon. A. Guth proposed in [5] a short accelerated expansion phase started 10^{-36} seconds after the Big Bang and ended 10^{-32} seconds after the Big Bang. Let us consider the Friedman equation, with vanishing cosmological constant 1.1.10

$$H^2(t) = \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a^2(t)} \tag{1.6.1}$$

and introduce the variables $\Omega(t)$ defined 1.1.15. Equation 1.6.1 becomes

$$\Omega(t) - 1 = \frac{k}{a^2(t)H^2(t)} \sim a^{-\dot{2}}(t). \tag{1.6.2}$$

The last measurements ([6]), tell us that

$$|\Omega(t_0) - 1| = 0.000 \pm 0.005 \tag{1.6.3}$$

this is equivalent to saying that the Universe is essentially flat. Taking the 1.2.15, we substitute $w = p/\rho$, obtaining

$$\rho(a) \sim a^{-3\left(1+\frac{p}{\rho}\right)}. \tag{1.6.4}$$

Insert this one in 1.6.1 , with k=0, one gets

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 \sim \rho = a^{-3(1+w)} \Rightarrow \quad (1.6.5)$$

$$\Rightarrow \frac{1}{a}\frac{da}{dt} \sim a^{-3\frac{1+w}{2}} \quad (1.6.6)$$

$$\Rightarrow da\, a^{\frac{3w+1}{2}} \sim dt, \quad (1.6.7)$$

which for $w \neq -1$ gives

$$a(t) \sim t^{\frac{2}{3}(w+1)} \sim t^{\beta}. \quad (1.6.8)$$

In particular

$$\beta = \frac{2}{3(w+1)} = \begin{cases} \frac{2}{3} & \text{matter } w = 0 \\ \frac{1}{2} & \text{radiation } w = \frac{1}{3}, \end{cases} \quad (1.6.9)$$

while for w=-1

$$a(t) \sim e^{H(t)t}. \quad (1.6.10)$$

Let us get back to 1.6.2 and let us compute the evolution of $|\Omega(t_0) - 1|$:

$$|\Omega(t_0) - 1| \sim \dot{a}^{-2}(t) \sim \frac{t^{2(1-\beta)}}{\beta^2}. \quad (1.6.11)$$

If we consider only a matter or radiation-dominated Universe then $\beta < 1$, so 1.6.11 is a function that grows with time.
The ratio of the density of the Universe calculated at Planck time and at the present era gives

$$\frac{|\Omega(t_0) - 1|}{|\Omega(t_{Pl}) - 1|} \sim \frac{10^{17}}{10^{-43}} = 10^{60}. \quad (1.6.12)$$

Since we know that $|\Omega(t_0) - 1| \lesssim 10^{-3}$,i.e. the current density differs from the critical one by one part over a thousand, one finds that

$$|\Omega(t_p l) - 1| \lesssim 10^{-63} \quad (1.6.13)$$

namely, the density of the Universe after the Big Bang differs from the critical one by one part over 10^{-63}. This is equivalent to say that the Universe is almost flat. This requires this value to be fixed *ad hoc*. A phase of the accelerated expansion of the primordial Universe that causes the critical density value to fall would solve the problem. From equation 1.6.11, you can see that for $\beta > 1$, you get a situation like that:

$$|\Omega(t_0) - 1| \Rightarrow 0 \quad (1.6.14)$$

independently on the initial value of $|\Omega(t) - 1|$.

So, the ideal situation requires $\beta > 1$.

$$\ddot{a} = \beta(\beta - 1)t^{\beta-2} \Rightarrow \ddot{a} > 0 \tag{1.6.15}$$

which corresponds to an accelerating expanding phase of the Universe.

1.6.2 Horizon problem

This problem has to do with causality and signal propagation speed. According to WMAP and PLANCK ([7],[8]) data, the visible points in the sky are at the same temperature unless there are 10^{-3} anisotropies. Be $d_H(t, t_i)$ the particle horizon of a photon emitted at t_i and received at t time. By definition $d_H(t, t_i)$ is the maximum distance that allows a causal relationship between two points

$$d_H(t, t_i) = a(t) \int_{t_i}^{t} \frac{dt'}{a(t')} = \frac{a(t)}{1-\beta}(t^{1-\beta} - t_i^{1-\beta}) \tag{1.6.16}$$

having combined 1.0.5 and 1.6.8

For a radiation or matter-dominated Universe $\beta < 1$; in the limit $t - 1 \to 0$

$$d_H(t, t_i) \longrightarrow \frac{a(t)t^{1-\beta}}{1-\beta} = \frac{\beta}{1-\beta} \frac{1}{H(t)}. \tag{1.6.17}$$

Which is a finite value. So the signals can't propagate beyond d_H. Therefore some regions of the cosmic background radiation that cannot be causally connected. It therefore becomes difficult to explain the small Cosmic Microwave Background (CMB) anisotropies. If we assume an accelerated expansion phase, i.e. $\beta > 1$, namely in terms of scale factor [4]

$$\ddot{a} > 0 \tag{1.6.18}$$

$$d_H(t, t_i) \longrightarrow \infty \tag{1.6.19}$$

1.6.3 De Sitter solution

Let's try to find the simplest example of an inflationary model. We consider as a source a perfect fluid to which we associate the state equation

$$p = -\rho. \tag{1.6.20}$$

The equation of motion 1.2.10 becomes

$$\dot{\rho} = 0 \tag{1.6.21}$$

namely

$$p = -\rho = -\Lambda. \tag{1.6.22}$$

The energy-momentum tensor assumes the form

$$T^{\nu}_{\mu} = -p\delta^{\nu}_{\mu} = \Lambda\delta^{\nu}_{\mu}. \tag{1.6.23}$$

It is immediate to note that the tensor energy-momentum has the same form that it would assume if one considers a contribution given by the cosmological constant, that is, vacuum energy. Therefore we can interpret the cosmological constant in this way.

So let us consider a scalar field ϕ and its potential. If we consider a stationary point ϕ_0 of the potential V

$$\left(\frac{\partial V}{\partial \phi}\right)\bigg|_{\phi=\phi_0} = 0. \tag{1.6.24}$$

This way the energy-momentum tensor assumes the form given in equation 1.6.23 and the Friedman equation in 1.1.10 becomes

$$\dot{a}^2 = \frac{8}{3}\pi G\Lambda^2 - k \tag{1.6.25}$$

which for k=0 is straightforwardly integrated

$$H_\Lambda (\left(\frac{8}{3}\pi G\Lambda\right)^{1/2} \equiv \left(\frac{\Lambda}{3M^2_{Pl}}\right)^{1/2} \tag{1.6.26}$$

$$a(t) = e^{H_\Lambda t}. \tag{1.6.27}$$

This solution is based on a geometry given by the following parameters

$$H = \frac{\dot{a}}{a} = H_\Lambda \tag{1.6.28}$$

$$\dot{H} = 0 \tag{1.6.29}$$

$$\frac{\ddot{a}}{a} = H^2_\Lambda \tag{1.6.30}$$

which doesn't have ant particle horizon as one can immediately verify applying the definition, but has an event horizon given by

$$d_e(t) = e^{H_\Lambda t}\int_t^\infty e^{-H_\Lambda t'}dt' = \frac{1}{H_\Lambda} \tag{1.6.31}$$

The form of a(t) in 1.6.28 is sufficient to satisfy the condition of inflation. Indeed the exponential form in the scale factor means that it can increase of may orders of magnitude.
It is now necessary to define a parameter that tells us how much inflation is needed, i.e. how long a phase of de Sitter's inflationary expansion must last. So let us define the so called e-folding number

$$N = \ln\left(\frac{a_f}{a_i}\right) \tag{1.6.32}$$

To quantify how much inflation is needed, let's consider the e-folding number and the problem

of flatness. Recall that radiation domination characterized by $\beta = 1/2$ began at the end of inflation, namely $t \sim 10^{34}$ and ended at $10^{12}s$; while matter domination characterized by $\beta = 2/3$, began at the end of the radiation domination and lasts nowadays namely, 10^{17} s.
We can now assume that the density of the Universe at the beginning of inflation is of order 1. Recalling the relation

$$|\Omega(t) - 1| \sim t^{2(1-\beta)} \tag{1.6.33}$$

we can go backward in time and find $\left|\Omega(t_f) - 1\right|$ at the end of inflation, where t_f os the time at the end of the inflationary expansion.

$$\left|\Omega(t_f) - 1\right| \sim 10^{-53}. \tag{1.6.34}$$

Now recalling that during the inflationary expansion H(t) is a constant, then

$$|\Omega(t) - 1| \sim a^{-2} \tag{1.6.35}$$

so

$$\frac{a(t_f)}{a(t_i)} \sim 10{-}26 \tag{1.6.36}$$

which in terms of N can be recast as

$$N = \ln\left(\frac{a_f}{a_i}\right) \sim 60. \tag{1.6.37}$$

This means that the scale factor should has increased during inflation of 10^{26} orders of magnitude.

1.6.4 Slow roll models

De Sitter's solution provides a proposal to achieve inflation. It has not yet been specified how inflation ends. In fact, this phase of accelerated expansion of the Universe must somehow find an exit, such as to induce a phase transition in the primitive Universe, in order to land in the Universe described by the standard cosmological model. To do this it is necessary to introduce a field, so called inflatonic and replace the energy density with a potential . A field that is governed by a potential that makes it "roll" slowly towards a minimum, corresponding to the reheating phase of the Universe, realizes an ideal situation of exit from inflation. From here the models called "slow roll" are born. The simplest model that realizes this scenario is that of a scalar field coupled with gravity.
This scalar field , called inflaton ,is coupled to gravity, and the action that governs this field is

$$S = \frac{M_P^2}{2} \int d^4x \left(\sqrt{-g}R + \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + V(\phi)\right). \tag{1.6.38}$$

ϕ has dimensions of a mass

$$[\phi] = [M]. \tag{1.6.39}$$

Varying the action S with respect to the metric , one gets the field equation

$$G_\mu \nu = \frac{1}{M_P^2}\left[\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right)\right]. \tag{1.6.40}$$

Notice that the r.h.s. member of the last equation is the energy-momentum tensor of the scalar field

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right) \tag{1.6.41}$$

and S_M is the action associated to the scalar field

$$S_\phi = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + V(\phi)\right). \tag{1.6.42}$$

Varying S with respect to the scalar field ϕ gives the equation of motion for the field

$$\nabla^\mu\nabla_\nu\phi - V_\phi = 0 \tag{1.6.43}$$

where

$$V_\phi = \frac{\partial V}{\partial\phi}. \tag{1.6.44}$$

Recall that in a curved space-time

$$\nabla^\mu\nabla_\nu\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi). \tag{1.6.45}$$

We now want to find solutions for these equations, restricting ourselves to the case of a geometry described by FLRW metrics. We consider the synchronous gauge, in which the FLRW metric is

$$ds^2 = dt^2 - a^2(t)\left|d^2x\right|. \tag{1.6.46}$$

In this metric equation 1.6.43 assumes the form

$$\ddot{\phi} + 3H\dot{\phi}^2 + V_\phi = 0. \tag{1.6.47}$$

While the non-vanishing components of the energy-momentum tensor are

$$T^0_0 = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1.6.48}$$

and

$$T^\theta_\theta = p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{1.6.49}$$

giving the state equation

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{\frac{1}{2}\dot{\phi}^2 - V(\phi)}. \tag{1.6.50}$$

In this context the Friedman equation 1.1.10 becomes

$$H^2 = \frac{1}{3M_P^2}\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right]. \tag{1.6.51}$$

These two combined together give

$$\frac{\ddot{a}}{a} = \frac{1}{3M_P^2}\left(\dot{\phi}^2 - V(\phi)\right). \tag{1.6.52}$$

According to the last equation $\ddot{a} > 0$, which stands for an accelerated expansion of the Universe,if $V(\phi)$ dominates over $\dot{\phi}^2$.

If we have the condition

$$V(\phi) >> \frac{1}{2}\dot{\phi}^2 \tag{1.6.53}$$

then pressure and energy density for the inflaton become respectively

$$\rho \approx V(\phi) \tag{1.6.54}$$

and

$$p \approx -V(\phi). \tag{1.6.55}$$

Under these conditions the behaviour of the inflaton field is the same as that assumed by the cosmological constant. In this way we obtain de Sitter's solution that implies a phase of inflation, which must be sufficiently long. Therefore this model must make sure that the cynical energy of the field does not change so that the potential prevails over it.

Let us consider equation 1.6.47. The second term can be considered to play the role of a damping term, as if there was a frictional force acting against the motion. If we want to keep the conditions to have a sufficiently long inflation we need to require that the acceleration $\ddot{\phi}$ is negligible with respect to the damping term, namely

$$|\ddot{\phi}| << 3H\,|\dot{\phi}|\,. \tag{1.6.56}$$

In this way the field \$*phi* reaches the minimum very slowly, thus generating a sort of very slow rolling, hence the name slow-roll for these models. Under these approximations we get

$$3H\dot{\phi} = -V_\phi \tag{1.6.57}$$

and

$$H^2 = \frac{1}{3M_P^2}V(\phi). \tag{1.6.58}$$

The requirement that the scalar field does not reach the minimum potential too quickly can be formalized as follows: let us square equation 1.6.57 and replace it in equation 1.6.58. One finds

$$\dot{\phi}^2 = \frac{V_\phi^2}{9H^2} = \frac{M_P^2}{3}\left(\frac{V_\phi}{V}\right)^2 \tag{1.6.59}$$

so that the condition 1.6.53 becomes

$$\left(\frac{V_\phi}{V}\right)^2 << 1. \tag{1.6.60}$$

We then define the first slow roll parameter

$$\epsilon = \frac{M_P^2}{3}\left(\frac{V_\phi}{V}\right)^2 \tag{1.6.61}$$

This parameter is necessary to define the first condition of slow roll inflation: the expanding Universe is dominated by the energy of the inflatonic potential. The previous condition is recast this way

$$\epsilon \ll 1. \tag{1.6.62}$$

It is now necessary to parametrize the damping condition of the inflation phase. Let us therefore consider equation 1.6.57 and let us consider its the derivative with respect to time and we combine it with 1.6.58, getting

$$\ddot{\phi} = -\left(\epsilon + M_P^2 \frac{V_{\phi\phi}}{V}\right) H\dot{\phi} \tag{1.6.63}$$

so that the condition 1.6.56 is satisfied if , defined

$$\eta = M_P^2 \frac{V_{\phi\phi}}{V} \tag{1.6.64}$$

we have

$$|\eta| \ll 1. \tag{1.6.65}$$

The conditions on these parameters set constraints on the form of the potential. ϵ and η are called slow-roll parameters.

It is also possible to express the number of e-foldings in terms of potential. In fact, the definition

$$N = \ln \frac{a(t_f)}{a(t_i)}. \tag{1.6.66}$$

can be manipulated to give

$$N = \int_{t_i}^{t_f} \frac{\dot{a}}{a} dt = \int_{t_i}^{t_f} H dt = -\frac{1}{M_P^2}\int_{t_i}^{t_f} \frac{V(\phi)}{V_\phi(\phi)}\dot{\phi} dt \tag{1.6.67}$$

which after a change of variable can be rewritten as

$$N = -\frac{1}{M_P^2}\int_{\phi_i}^{\phi_f}\frac{V(\phi)}{V_\phi(\phi)}d\phi. \tag{1.6.68}$$

This form of the definition of the e-folding number will be useful to analyze explicit forms for the potential $V(\phi)$.

1.6.5 Polynomial potential

A general class of suitable potential sis given by those ones given in the polynomial form

$$V(\phi) \sim \phi^n, \tag{1.6.69}$$

with n>0. The other parameters are straightforwardly calculated

$$\frac{V_\phi}{V} = \frac{n}{V} \tag{1.6.70}$$

and

$$\frac{V_{\phi\phi}}{V} = \frac{n(n-1)}{\phi^2}. \tag{1.6.71}$$

The needed conditions $\epsilon, |\eta| < 1$ for slow-roll inflation are

$$\frac{\phi^2}{M_P^2} \gg 1. \tag{1.6.72}$$

This class of model provides a sufficient amount of inflation, in fact the condition on the e-folding number N»1, is re-expressed as

$$N = \ln\left(\frac{a_f}{a_i}\right) = \frac{1}{M_P^2}\int_{\phi_i}^{\phi_f}\frac{V(\phi)}{V_\phi(\phi)}d\phi = \frac{1}{2n}\left(\frac{{\phi_i}^2}{M_P^2} - \frac{{\phi_f}^2}{M_P^2}\right). \tag{1.6.73}$$

and automatically implies that

$$\phi^2 >> M_P^2 \tag{1.6.74}$$

as stated before.

In addition, this class of models offers an exit from inflation, provided by the very dynamics of the parameters that control the inflation phase itself. In fact ϵ and η depend on ϕ, in particular

$$\epsilon \sim \eta \sim \frac{1}{\phi^2}$$

so as ϕ decreases, the slow-roll conditions for ϵ and η are no longer met. This provides an exit from the inflation phase.

The simplest potential model one can imagine is the one given by the term mass

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{1.6.75}$$

We calculate the slow-roll parameters

$$\epsilon = \eta = 2\left(\frac{M_P}{\phi}\right)^2 \tag{1.6.76}$$

and the conditions $\epsilon, |\eta| < 1$ to have a slow-roll inflation are translated

$$\phi > \sqrt{2}M_P \equiv \phi_f. \tag{1.6.77}$$

The e-foldings number is given by

$$N = \left(\frac{\phi}{2M_P}\right)^2 - \frac{1}{2}. \tag{1.6.78}$$

1.7 Reheating

Once the conditions for slow-roll inflation are met, the field goes out of the dampened regime and starts to swing around the minimum potential. The field thus disperses its energy and if there are couplings of inflaton with other fields its oscillations are damped by the creation of particles that further disperse the energy. This is the so called reheating phase([9, 10, 11, 12]). Let Γ_ϕ be the decay rate of the inflaton field and let H_{osc} be the value of the Hubble parameter at the end of the inflationary period. The process of energy-loss due to particle production which causes the damping can be described by the equation

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi + V_\phi(\phi) = 0. \tag{1.7.1}$$

Multiplying the last equation by $\frac{\dot{\phi}}{2}$ and using equation 1.6.48 we can rewrite

$$\dot{\rho_\phi} + (3H + \Gamma_\phi)\dot{\phi}^2 = 0. \tag{1.7.2}$$

We know that for a simple harmonic oscillator the average of the kinetic energy is the same as the average of the potential energy, so we can write

$$\frac{1}{2}\langle\dot{\phi}^2\rangle = \langle V(\phi)\rangle = \frac{1}{2}\langle\rho_\phi\rangle. \tag{1.7.3}$$

so equation 1.7.1 can be rewritten

$$\langle\dot{\rho_\phi}\rangle + (3H + \Gamma_\phi)\langle\rho_\phi\rangle = 0. \tag{1.7.4}$$

From now on we will denote $\langle\rho_\phi\rangle$ as ρ_ϕ. Now we can distinguish several cases

- $t \lesssim \Gamma_\phi^{-1}$, there is no sufficient time to create new particles so the previous equation can be reduced to
$$\dot{\rho_\phi} + 3H\rho_\phi = 0 \tag{1.7.5}$$
which is exactly the same equation valid for energy density conservation during matter dominance

- $t \sim \Gamma_\phi^{-1}$, now particle production is possible, starting the reheating phase of the Universe and we have
$$\rho_\phi = \frac{\pi^2}{30}\left(N_B + \frac{7}{8}N_F\right)T_R^4 \tag{1.7.6}$$
where obviously T_R is the reheating temperature N_B and N_F and are respectively the number of boson and fermion species.

One can try to calculate T_R. So let us define a mass scale M such that
$$\rho_\phi \equiv M^4 \tag{1.7.7}$$
at the beginning of inflation. Since during inflation ρ_ϕ doesn't change, because $V(\phi)$ stays rather flat, we can write that
$$H_{osc}^2 = \frac{8\pi M^4}{3M_P^2} \tag{1.7.8}$$
so the period of oscillation is
$$t_{osc} \sim \frac{1}{H_{osc}} = \sqrt{\frac{3}{8\pi}}M_P\frac{1}{M^2}. \tag{1.7.9}$$
Recalling that ρ_ϕ varies with time in matter-dominated phase we are allowed to write
$$\frac{\rho_\phi(\Gamma_\phi^{-1})}{\rho_\phi(t_{osc})} = \frac{\Gamma_\phi^2 M_P^2}{8\pi t_{osc}} \tag{1.7.10}$$
and finally
$$T_R = \left(\frac{45}{a\pi^3(N_B + \frac{7}{8}N_F}\right)^{\frac{1}{4}}\sqrt{\Gamma_\phi M_P}. \tag{1.7.11}$$
Another important example of inflation potential comes from the hypothesis that he inflaton is an axion. In this case
$$V(\phi) = V_0\left[cos\left(\frac{\phi}{f}\right)\right] + 1. \tag{1.7.12}$$

Chapter 2

Brief review of thermal field theory

Before we begin we present the outline of this chapter. We will briefly present a review of quantum statistical mechanics. We will first focus on the treatment of quantum statistics for bosons and fermions, for this we will follow the treatment in [13]. Next we will expound briefly on thermal field theory and how it is possible to consider a thermal bath for a system of non-interacting particles through the path integral formalism, until we arrive at the definition of the free energy function for a scalar field and derive from it all relevant thermodynamic observables. The same procedure, discussed in detail from the mathematical point of view, will also be treated for the case of gauge bosons and fermions. Later we will see how this formalism can be traced back to the more general formalism of Green's functions, and we will follow the discussion in [14]. Finally we will apply the tools of thermal quantum field theory to the case of a specific effective potential of a ϕ^4 theory. We will evaluate, in this application, the thermal contributions given by the potential and the conditions that must occur for the system to have a phase transition,as discussed in detail in [1]. The chapter concludes with a brief discussion of some formal arguments that will be applied to the phenomenon of Bose-Einstein condensation of a neutral scalar field, for which we will follow [13] . Let us consider a grand canonical ensemble for a system in thermo-dynamical equilibrium and define the fundamental object

$$\hat{\rho} = \exp\left[-\beta(H - i\mu_i \hat{N}_i)\right] \tag{2.0.1}$$

called density matrix operator. $\hat{N}_i$ is the particle number operator, and μ_i is the chemical potential. The index i runs over several possible particle species. The density matrix $\hat{\rho}$ is used to compute the average of physical observables. So let us suppose we have an operator $\hat{A}$, then

$$A = \langle \hat{A} \rangle = \frac{Tr\hat{A}\hat{\rho}}{Tr\hat{\rho}} \tag{2.0.2}$$

where of course Tr means that we must perform the trace. The partition function is so defined

$$Z = Z(V, T, \mu_1, \mu_2, ...) = Tr\hat{\rho} \tag{2.0.3}$$

and it is used to calculate all the other thermo-dynamical variables. So we have for pressure, number of particles, entropy and energy respectively

$$P = \frac{\partial(T \ln Z)}{\partial V} \tag{2.0.4}$$

$$N_i = \frac{\partial(T \ln Z)}{\partial \mu_i} \tag{2.0.5}$$

$$S = \frac{\partial(T \ln Z)}{\partial T} \tag{2.0.6}$$

$$E = -PV + TS + \mu_i N_i. \tag{2.0.7}$$

2.0.1 Single-particle boson system

Let us consider a system consisting of bosons occupying various states; each state has energy ω each state can be occupied by an arbitrary number of particles. Let's assume that there are no interactions between these particles. We can think of this system as a collection of harmonic oscillators. Let us denote the system with n particles with $|n\rangle$ and let $|0\rangle$ be the vacuum state. The orthogonality relation

$$\langle n | n' \rangle = \delta_{nn'} \tag{2.0.8}$$

and the completeness

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = 1. \tag{2.0.9}$$

ust be satisfied. Let us now introduce the creation and annihilation operators, whose action on the eigenstates of the systems are respectively

$$a^{\dagger} |n\rangle = \sqrt{n+1}\, |n+1\rangle \tag{2.0.10}$$

and

$$a |n\rangle = \sqrt{n}\, |n-1\rangle . \tag{2.0.11}$$

If n=0 then we have

$$a |0\rangle = 0. \tag{2.0.12}$$

The number operator $\hat{N}$ is defined

$$\hat{N} |n\rangle = a^{\dagger} a |n\rangle = n |n\rangle \tag{2.0.13}$$

so the commutation relation between a and $a^{\dagger}$ is

$$[a, a^{\dagger}] = aa^{\dagger} - a^{\dagger} a = 1. \tag{2.0.14}$$

Any other state can be created by repeatedly applying the creation operator

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^{\dagger})^{n}|0\rangle . \tag{2.0.15}$$

Now we need to write the Hamiltonian for this system so that we can replace it in the definition of the partition function and then all the other thermo-dynamical observables. From non-relativistic quantum mechanics, we can write the Hamiltonian for this system of bosons.

$$H = \frac{1}{2}\omega\left(aa^{\dagger} + a^{\dagger}a\right) = \omega\left(a^{\dagger}a + \frac{1}{2}\right) = \omega\left(\hat{N} + \frac{1}{2}\right). \tag{2.0.16}$$

The last term can be neglected since it represents the zero-point energy. Replacing 2.0.16 in 2.0.3 one finds

$$\begin{aligned} Z &= Tre^{-\beta(H-\mu\hat{N})} = Tre^{-\beta(\omega-\mu)\hat{N}} \\ &= \sum_{n=0}^{\infty}\langle n| e^{-\beta(\omega-\mu)\hat{N}}|n\rangle = \sum_{n=0}^{\infty} e^{-\beta(\omega-\mu)n} \\ &= \frac{1}{1-e^{-\beta(\omega-\mu)}} \end{aligned} \tag{2.0.17}$$

so, the mean number of particles can be written as

$$N = \frac{1}{e^{\beta(\omega-\mu)}-1}. \tag{2.0.18}$$

From 2.0.18 it is possible to find the classical limit for

$$N << 1$$

which means that

$$T << \omega - \mu$$

, and then

$$N = e^{-\beta(\omega-\mu)}. \tag{2.0.19}$$

2.0.2 Single-particle fermion system

The previous discussion can also be extended to the fermionic case. Such systems are called Fermi gas. Fermi-Dirac statistics prevent more than one fermion from occupying the same state, so the only states allowed are $\langle 0|$ and $|1\rangle$, so denoting with $\alpha^{\dagger}$ and with α the creation and annihilation operators we have

$$\alpha^{\dagger}|0\rangle = |1\rangle \quad (2.0.20)$$

$$\alpha^{\dagger}|1\rangle = |0\rangle \quad (2.0.21)$$

$$\alpha|1\rangle = |0\rangle \quad (2.0.22)$$

$$\alpha|0\rangle = |0\rangle \quad (2.0.23)$$

and the number operator is defined

$$\hat{N} = \alpha^{\dagger}\alpha \quad (2.0.24)$$

so that

$$\hat{N}|n\rangle = n|n\rangle . \quad (2.0.25)$$

Creation and annihilation operator for fermions satisfy the anti-commutation relation

$$\{\alpha, \alpha^{\dagger}\} = \alpha\alpha^{\dagger} + \alpha^{\dagger}\alpha = 1 \quad (2.0.26)$$

and the Hamiltonian assumes the form

$$H = \frac{1}{2}\omega(\alpha^{\dagger}\alpha - \alpha\alpha^{\dagger}) = \omega\left(\hat{N} - \frac{1}{2}\right). \quad (2.0.27)$$

The same consideration made previously for bosons can also be made in the fermionic case. Again we can now replace the Hamiltonian 2.0.27 in 2.0.3 and one finds

$$\begin{aligned} Z &= Tr e^{-\beta(H-\mu\hat{N})} = Tr e^{-\beta(\omega-\mu)\hat{N}} \\ &= \sum_{n=0}^{1} \langle n| e^{-\beta(\omega-\mu)\hat{N}} |n\rangle \\ &= \sum_{n=0}^{1} e^{-\beta(\omega-\mu)n} = 1 + e^{-\beta(\omega-\mu)} \end{aligned} \quad (2.0.28)$$

so the mean number of particles in the fermion case is given by

$$N = \frac{1}{e^{-\beta(\omega-\mu)}} + 1. \quad (2.0.29)$$

Again, it is possible to find the classical limit for

$$N << 1$$

which means that

$$T << \omega - \mu$$

, and then

$$N = e^{-\beta(\omega-\mu)}. \tag{2.0.30}$$

2.0.3 Gases in a box

Let us now consider a gas in a cubic box with side length L. Let us wait for the thermodynamic equilibrium to be reached. At this point we can treat the gas as if its particles do not interact with each other. We now impose the boundary conditions: the wave function of the particle vanishes on the boundary of the box and, called λ the wave length, we have

$$\lambda_x = \frac{2L}{j_x} \tag{2.0.31}$$

$$\lambda_x = \frac{2L}{j_y} \tag{2.0.32}$$

$$\lambda_x = \frac{2L}{j_z} \tag{2.0.33}$$

where j_x, j_y, j_z are integer numbers, and for the momenta

$$p_x = \frac{2\pi}{\lambda_x} \tag{2.0.34}$$

$$p_y = \frac{2\pi}{\lambda_y} \tag{2.0.35}$$

$$p_z = \frac{2\pi}{\lambda_z}. \tag{2.0.36}$$

For each mode we have an Hamiltonian, so indicating with **j**

$$\mathbf{i} = (j_x, j_y, j_z) \tag{2.0.37}$$

the total Hamiltonian is

$$H = \sum_{\mathbf{j}} H_{\mathbf{j}} \tag{2.0.38}$$

and the number operator

$$\hat{N} = \sum_{\mathbf{j}} \hat{N}_{\mathbf{j}}. \tag{2.0.39}$$

and the partition function

$$Z = Tre^{-\beta(H-\mu\hat{N})} = \prod_{\mathbf{j}} Tre^{\beta(H_j-\mu\hat{N}_j)} = \prod_{\mathbf{j}} Z_{\mathbf{j}}. \tag{2.0.40}$$

If we want to calculate all the thermo-dynamical observables we nee the ln Z

$$\ln Z = \sum_{j_x=1}^{\infty} \sum_{j_y=1}^{\infty} \sum_{j_z=1}^{\infty} \ln Z_{j_x,j_y,j_z}. \tag{2.0.41}$$

If we consider the limit $L \Rightarrow \infty$, we can replace the discrete sum with an integral

$$\ln Z = \frac{L^3}{\pi^3} \int_0^\infty d\,|p_x| \int_0^\infty d\,|p_y| \int_0^\infty d\,|p_z| \ln Z(\mathbf{p}). \tag{2.0.42}$$

If we extend the integration interval from $-\infty$ to ∞ and we divide by two, one gets

$$\ln\ = V \int \frac{d^3p}{(2\pi)^3} \ln Z(p). \tag{2.0.43}$$

Replacing the form of the partition function Z one gets

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \ln \left(1 \pm e^{-\beta(\omega-\mu)}\right)^{\pm 1} \tag{2.0.44}$$

with the plus sign for fermions and the minus sign for bosons. So the computation of thermo-dynamical observables is straightforward

$$P = \frac{T}{V} \ln Z \tag{2.0.45}$$

$$N = V \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta(\omega-\mu)} \pm 1} \tag{2.0.46}$$

$$E = V \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta(\omega-\mu)} \pm 1}. \tag{2.0.47}$$

Explicit expressions for P in some limit-cases will be given in section 2.1.

2.1 Brief review of thermal field theory- Path integral approach

In order to consider possible sequences of symmetry breaking leading from the GUT group to the current Standard Model structure, it is necessary to consider finite temperature effects. In fact the minima developed in the context of a null temperature field theory could be deeper than those developed by a null temperature theory. We consider the hypothesis that the Universe then goes through a sequence of symmetry breaks as it cools down, through phase transitions. The succession of these phase transitions greatly influences the structure of the primitive Universe.

Let us consider one of the most important tool of statistical mechanics, the partition function Z

$$Z = Tre^{-\beta \hat{H}} \tag{2.1.1}$$

where $\hat{H}$ is the Hamiltonian operator and $\beta = (k_B T)-1$, with T temperature and k_B is the Boltzmann constant. From now on we shall consider natural units for k_B, so

$$\beta = \frac{1}{T}. \tag{2.1.2}$$

The operation of trace means that we compute the sum over all independent states. From Z we can define other thermo-dynamical potentials like the free energy F

$$Z = e^{-\beta F}. \tag{2.1.3}$$

Recall the usual relations between several thermo-dynamical quantities

$$F = E - TS \tag{2.1.4}$$

$$P = -\frac{\partial F}{\partial V} S = -\frac{\partial F}{\partial T}\bigg|_V \tag{2.1.5}$$

$$\tag{2.1.6}$$

where E, S, P and V are respectively internal energy, entropy, pressure and volume .

If we divide 2.1.5 by the volume we get

$$\rho = \mathcal{F} + Ts \tag{2.1.7}$$

where ρ, $\mathcal{F}$ and s are respectively, energy density, free energy density and entropy density with the definition

$$E = \int d^3x\rho. \tag{2.1.8}$$

The simplest case to consider is the Lagrangian for a scalar field, (which could for example play the role of an inflaton field). This field could contribute to the partition function. So let us consider the lagrangian density for such a field

$$\mathcal{L}(\phi, \partial_\mu\phi) = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{2.1.9}$$

As usual in the formalism of finite field theory, scalar fields are replaced by periodic fields $\phi(\tau, \mathbf{x})$, where

$$\tau = it \tag{2.1.10}$$

and we impose the boundary or periodic conditions

$$\phi(\tau = 0, \mathbf{x}) = \phi(\tau = \beta, \mathbf{x}). \tag{2.1.11}$$

The partition function assumes the form

$$Z = \tilde{N}(\beta) \int_{periodic} \mathcal{D}\phi \left[\int_0^\beta d\tau \int d^3x \mathcal{L}(\phi, \bar{\partial}_\mu \phi) \right] \tag{2.1.12}$$

recall that in this context

$$\bar{\partial}_\mu \phi \equiv \left(i\frac{\partial \phi}{\partial \tau}, \nabla \phi \right) \tag{2.1.13}$$

and that $\tilde{N}(\beta)$ is a temperature-dependent normalization constant. The integral in 2.1.12 is Gaussian, therefore it can be computed exactly, so completing the square,

$$Z = \tilde{N}(\beta) \int_{periodic} \mathcal{D}\phi \exp \left(-\frac{1}{2} \int_0^\beta d\tau' \int d^3x' \int_0^\beta d\tau \int d^3x \phi(\bar{x}') \mathbf{A}(\bar{x}', \bar{x}) \phi(\bar{x}) \right) \tag{2.1.14}$$

where we put

$$\mathbf{A}(\bar{x}', \bar{x}) = (-\bar{\partial}'_\mu \bar{\partial}^\mu + m^2) \delta(\bar{x}' - \bar{x}). \tag{2.1.15}$$

From gaussian integral

$$Z = \tilde{N}(\beta) \exp \left(-\frac{1}{2} Tr \ \ln \mathbf{A} \right). \tag{2.1.16}$$

The periodicity conditions 2.1.11 allows us to express the field $\phi(\bar{x})$ through Fourier transform

$$\phi(\bar{x}) = \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} e^{-i\omega_n \tau} e^{i\mathbf{p}\cdot\mathbf{x}} \tilde{\phi}(\omega_n, \mathbf{p}) \tag{2.1.17}$$

where ω_n are the Matsubara frequencies for bosons and are defined by

$$\omega_n = \frac{2\pi n}{\beta} \tag{2.1.18}$$

with integer n. Now we simplify the notation using

$$\bar{p} \equiv (i\omega_n, \mathbf{p}) \tag{2.1.19}$$

$$\bar{p} \cdot \bar{x} \equiv \omega_n \tau - \mathbf{p} \cdot \mathbf{x}. \tag{2.1.20}$$

$$\bar{p}^2 \equiv -(\omega_n^2 + p^2). \tag{2.1.21}$$

We also expand the $\delta(\bar{x}' - \bar{x})$ distribution and the operator $\mathbf{A}(\bar{x}', \bar{x})$:

$$\delta(\bar{x}' - \bar{x}) = \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} e^{-i\bar{p}\cdot(\bar{x}' - \bar{x})} \tag{2.1.22}$$

$$\mathbf{A}(\bar{x}', \bar{x}) = \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} e^{-i\bar{p}\cdot(\bar{x}' - \bar{x})} (-\bar{p}^2 + m^2). \tag{2.1.23}$$

Thus we can calculate

$$
\begin{aligned}
Tr\ \ln\ \mathbf{A} &= \int_0^\beta d\tau \int d^3x \frac{1}{\beta}\sum_n \int \frac{d^3p}{(2\pi)^3} \ln\left(-\bar{p}^2+m^2\right) \\
&= \int d^3x \sum_n \int \frac{d^3p}{(2\pi)^3} \ln\left(\omega_n^2+\mathbf{p}^2+m^2\right).
\end{aligned}
\tag{2.1.24}
$$

Now we still need to calculate the sum over the Matsubara frequencies, the result gives

$$
\begin{aligned}
Tr\ \ln\ \mathbf{A} = \int d^3x \int \frac{d^3p}{(2\pi)^3}\{\beta\sqrt{\mathbf{p}+m^2} + 2\ \ln[1-\exp\left(-\beta\sqrt{\mathbf{p}^2+m^2}\right)] \\
+\sqrt{\mathbf{p}^2+m^2}+constant\}
\end{aligned}
\tag{2.1.25}
$$

And finally we can use this result in 2.1.16

$$
-\beta F = \ln\ Z = -\int d^3x \int \frac{d^3p}{(2\pi)^3}\left(\frac{\beta}{2}\sqrt{\mathbf{p}^2+m^2} + \ln[1-\exp\left(-\beta\sqrt{\mathbf{p}^2+m^2}\right)]\right). \tag{2.1.26}
$$

If the mass parameter in 2.1.26 is small when compared to the temperature (namely β^{-1}),which correspond to an ultra relativistic Bose gas, the previous integral is very easy to compute, so one gets all the thermo-dynamical functions, such as the free energy density

$$
\mathcal{F} = -\frac{\pi^2}{90\beta^4} = -\frac{\pi^2T^4}{90} \tag{2.1.27}
$$

when $T >> m$.

Consequently the free energy is

$$
F = \int d^3x \mathcal{F} \tag{2.1.28}
$$

the pressure, for ultra-relativistic bosons

$$
P = \frac{\pi^2T^4}{90} \tag{2.1.29}
$$

the entropy density is given by

$$
\mathcal{S} = \frac{2\pi^2T^3}{45} \tag{2.1.30}
$$

and the energy density is

$$
\rho = \frac{\pi^2T^4}{30} \tag{2.1.31}
$$

2.2 Partition function for gauge bosons

We continue the discussion by considering the partition function also for the case of gauge bosons, for which we have the function Z

$$Z = [\tilde{N}(\beta)]^{2d_G} \int_{periodic} \mathcal{D}A^\mu \int_{periodic} \mathcal{D}\eta^* \mathcal{D}\eta \times \exp \int_0^\beta d\tau \int d^3x \mathcal{L}(A_a^\mu, \eta_a) \tag{2.2.1}$$

where η represents the ghost and d_G is the number of gauge fields. We consider an Abelian case, and we consider a free-field limit, i.e. $g \to 0$

$$Z = [\tilde{N}(\beta)]^{2d_G} \int_{periodic} \mathcal{D}A^\mu \int d\bar{x} \left(-\frac{1}{4} \bar{F}^{\mu\nu} \bar{F}_{\mu\nu} - \frac{1}{2\xi} (\bar{\partial}_\mu A^\mu)^2 \right) \int_{periodic} \mathcal{D}\eta^* \mathcal{D}\eta \times \exp \int \exp \int d\bar{x} \bar{\partial}_\mu^* \bar{\partial}^\mu \eta \tag{2.2.2}$$

with

$$\bar{F}^{\mu\nu} \equiv \bar{\partial}^\mu A^\nu - \bar{\partial}^\nu A^\mu \tag{2.2.3}$$

and

$$\int d\bar{x} \equiv \int_0^\beta d\tau \int d^3x. \tag{2.2.4}$$

So, inserting the explicit form of $\bar{F}^{\mu\nu}$

$$\begin{aligned} Z = [\tilde{N}(\beta)]^{2d_G} \int_{periodic} \mathcal{D}A^\mu \exp -\frac{1}{2} \int d\bar{x}' \int d\bar{x} A_\mu(\bar{x}) B^{\mu\nu}(\bar{x}', \bar{x}) A_\nu(\bar{x}) \times \int_{periodic} \\ \mathcal{D}\eta^* \mathcal{D}\eta \times \exp - \int d\bar{x}' d\bar{x} \eta^*(\bar{x}') C(\bar{x}', \bar{x}) \eta(\bar{x}) \end{aligned} \tag{2.2.5}$$

where

$$B^{\mu\nu}(\bar{x}', \bar{x}) = (g^{\mu\nu} \bar{\partial}^\rho_{x'} \bar{\partial}_{\rho x}) \delta(\bar{x}' - \bar{x}) \tag{2.2.6}$$

and

$$C(\bar{x}', \bar{x}) = \bar{\partial}^\rho_{x'} \partial_{\rho x} \delta(\bar{x}' - \bar{x}). \tag{2.2.7}$$

After the Gaussian integration the result is

$$Z = [\tilde{N}(\beta)]^2 \exp\left(-\frac{1}{2} Tr \ln \mathbf{B} \right) \exp(Tr \, \ln \mathbf{C}) \tag{2.2.8}$$

In order to find the ln B it is convenient to Fourier transform

$$B^{\mu\nu}(\bar{x}', \bar{x}) = \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} e^{-i\bar{p}\cdot(\bar{x}' - \bar{x})} \times [\bar{p}^2 (g^{\mu\nu} - \bar{p}^{-2} \bar{p}^\mu \bar{p}^\nu) + \bar{p}^2 \xi^{-1} \bar{p}^{-2} \bar{p}^\mu \bar{p}^\nu] \tag{2.2.9}$$

and now we take the trace of the ln

$$Tr \, \ln \mathbf{B} = \frac{1}{\beta} \int d\bar{x} \sum_n \int \frac{d^3p}{(2\pi)^3} [3 \ln \bar{p}^2 + \ln\left(\xi^{-1} p^{-2}\right)] = \int d^3x \sum_n \int \frac{d^3p}{(2\pi)^3} 4 \ln(\omega_n^2 + \mathbf{p}^2). \tag{2.2.10}$$

The same procedure can be applied to C

$$C(\bar{x}', \bar{x}) = \frac{1}{\beta} \int d\bar{x} \sum_n \int \frac{d^3p}{(2\pi)^3} e^{-i\bar{p}\cdot(\bar{x}'-\bar{x})\bar{p}^2} \tag{2.2.11}$$

taking the trace

$$Tr \, \ln \mathbf{C} = \int d\bar{x} \sum_n \int \frac{d^3p}{(2\pi)^3} \ln(\omega_n^2 + \mathbf{p}^2). \tag{2.2.12}$$

Replacing all these results in the expression of 2.2.2 gives

$$Z = [\tilde{N}(\beta)]^2 \exp\left(-\frac{1}{2} \int d^3x \sum_n \int \frac{d^3p}{(2\pi)^3)} 2\ln({\omega_n}^2 + \mathbf{p}^2)\right). \tag{2.2.13}$$

In the end we find the free energy

$$\mathcal{F} = -2\pi^2 \frac{T^4}{90}. \tag{2.2.14}$$

2.3 Partition function for fermions

We can extend the previous discussion to the case of fermions and calculate their partition function. In the case of fermions we must remember that they possess the property of being antisymmetric on the period of the wave function ψ, that is

$$\psi(\tau = 0, \mathbf{x}) = -\psi(\tau = \beta, \mathbf{x}). \tag{2.3.1}$$

Let us start from the partition function expressed in the form

$$Z = N'(\beta) \int_{antiperiodic} \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp \int_0^\beta d\tau \int d^3x \mathcal{L}(\psi). \tag{2.3.2}$$

A fermionic wave function can be expanded

$$\psi(\bar{x}) = \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi^3)} e^{-i\bar{p}\cdot\bar{x}}(\bar{p}). \tag{2.3.3}$$

In the fermion case the Matsubara frequencies ω_n are given by

$$\omega_n = \frac{2n + 1\pi}{\beta}. \tag{2.3.4}$$

The Lagrangian used to derive the partition function is

$$\mathcal{L}(\psi) = \bar{\psi}(x)(i\gamma^\mu \bar{\partial}_\mu - m)\psi(\bar{x}) \tag{2.3.5}$$

so

$$Z = N'(\beta) = \int_{antiperiodic} \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left(-\int d\bar{x} \int d\bar{x}\bar{\psi}(\bar{x'})D(\bar{x'},\bar{x})\psi(\bar{x})\right) \tag{2.3.6}$$

where

$$D(\bar{x'},\bar{x}) = (i\gamma^\mu\bar{\partial}_\mu + m)\delta(\bar{x'}-\bar{x}). \tag{2.3.7}$$

Using the Gaussian integration one finds

$$Z = N'(\beta)\exp(Tr\,\ln\mathbf{D}). \tag{2.3.8}$$

Again we Fourier transform

$$D(\bar{x'},\bar{x}) = \frac{1}{\beta}\sum_n \int \frac{d^3p}{(2\pi)^3} e^{i\bar{p}\cdot(\bar{x}'-\bar{x}}(-\bar{\not{p}}+m), \tag{2.3.9}$$

so taking the trace of the ln

$$\begin{aligned} Tr\,\ln\,\mathbf{D} &= \int_0^\beta d\tau \int d^3x \frac{1}{\beta}\sum_n \int \frac{d^3p}{(2\pi)^3} 2\ln\left(m^2-\bar{p}^2\right) \\ &= 2\int d^3x \sum_n \int \frac{d^3p}{(2\pi)^3}\ln\left(\omega_n^2+\mathbf{p}^2+m^2\right) \\ &= 2\int d^3x \sum_n \int \frac{d^3p}{(2\pi)^3}\{\beta\sqrt{\mathbf{p}^2+m^2}+2\ln[1+\exp\left(-\beta\sqrt{\mathbf{p}^2+m^2}\right)]+ \\ &+ (\sqrt{\mathbf{p}^2+m^2}) - independent\ constant\}. \end{aligned} \tag{2.3.10}$$

Replacing this in the expression the the partition function one finds

$$-\beta F = \ln\,Z = 2\int d^3 \int \frac{d^3p}{(2\pi)^3}\{\beta\sqrt{\mathbf{p}^2+m^2}+2\,\ln[1+\exp\left(-\beta\sqrt{\mathbf{p}^2+m^2}\right)]\} \tag{2.3.11}$$

which, for $T \gg m$, gives

$$\mathcal{F} = -7\pi^2\frac{T^4}{180}. \tag{2.3.12}$$

Finally we can summarize the result of the previous sections: for an ideal ultra-relativistic gas($T \gg m$ the free energy density is

$$\mathcal{F} = -\pi^2 T^4\frac{(N_B+\frac{7}{8}N_F)}{90} \tag{2.3.13}$$

with N_B and N_F respectively number of bosons and number of fermions, the pressure is

$$P = \pi^2 T^4\frac{(N_B+\frac{7}{8}N_F)}{90}, \tag{2.3.14}$$

the entropy density is

$$\mathcal{S} = 2\pi^2 T^3 \frac{(N_B + \frac{7}{8}N_F)}{45} \tag{2.3.15}$$

and the energy density is

$$\rho = \pi^2 T^4 \frac{(N_B + \frac{7}{8}N_F)}{30}. \tag{2.3.16}$$

2.4 Temperature-dependent Green functions and generating functionals

In this section we will try to extend the concept of Green's function to the case of systems immersed in a thermal bath. We then consider the following general Green's function expressed via the field operators $\hat{\phi}$, functions of τ

$$\mathcal{G}^N(\bar{x}_1, ..., \bar{x}_N) = \langle T_\tau(\hat{\phi}(\hat{x}_1)...\hat{\phi}(\bar{x}_N))\rangle \tag{2.4.1}$$

where as usual

$$\bar{x} \equiv (-i\tau, x) \tag{2.4.2}$$

and T_τ stand for ordered product according to increasing values of τ.

The expectation value $\langle ... \rangle$ is calculated

$$\langle T_\tau(\hat{\phi}(\hat{x}_1)...\hat{\phi}(\bar{x}_N))\rangle = \frac{Tr[e^{-\beta\hat{H}}T_\tau(\hat{\phi}(\hat{x}_1)...\hat{\phi}(\bar{x}_N))]}{Tr[e^{-\beta\hat{H}}]}. \tag{2.4.3}$$

Let us consider the specific case of $\mathcal{G}^{(\mathcal{N})}(\bar{x}_1, \bar{x}_2)$ and let us consider an operator $\hat{A}$ expressed in the form

$$\hat{A} = \int d^3x \hat{\phi}^2(t=0, x). \tag{2.4.4}$$

Its expectation value is

$$\langle \hat{A} \rangle = \frac{Tr[e^{-\beta\hat{H}} \int d^3x \hat{\phi}^2(t=0, x)]}{Tr[e^{-\beta\hat{H}}]}. \tag{2.4.5}$$

Let us now consider now

$$\begin{aligned} \lim_{\tau' \to \tau^+, x' \to x} \int d^3x \mathcal{G}^2(\bar{x}', \bar{x}) &= \frac{\int d^3x Tr[e^{\beta\hat{H}}\hat{\phi}(\tau, x)\hat{\phi}(\tau, x)]}{Tr[e^{-\beta\hat{H}}]} \\ &= \frac{\int d^3x Tr[e^{-\beta\hat{H}}\hat{\phi}^2(0, x)]}{Tr[e^{-\beta\hat{H}}]} \end{aligned} \tag{2.4.6}$$

so we have found that

$$\hat{A} = \lim_{\tau' \to \tau^+, x' \to x} \int d^3x \mathcal{G}^2(\bar{x}', \bar{x}). \tag{2.4.7}$$

In the field theory language the Green functions can be expressed as

$$\mathcal{G}^N(\bar{x}_1,...,\bar{x}_N) = \frac{\int_{periodic}\mathcal{D}\phi\int\mathcal{D}\pi\phi(\bar{x}_1)...\phi(\bar{x}_N)\ \exp\int_0^\beta d\tau\int d^3x\left(i\pi\frac{\partial\phi}{\partial\tau}-\mathcal{H}\right)}{\int_{periodic}\mathcal{D}\phi\int\mathcal{D}\pi\ \exp\int_0^\beta d\tau\int d^3x\left(i\pi\frac{\partial\phi}{\partial\tau}-\mathcal{H}\right)}. \tag{2.4.8}$$

We can also introduce the generation functional

$$\bar{W}[J] = \frac{\int_{periodic}\mathcal{D}\phi\left(\int_0^\beta\int d^3x(\mathcal{L}(\phi,\bar{\partial}_\mu\phi)+J\phi\right)}{\int_{periodic}\mathcal{D}\phi\ \exp\left(\int_0^\beta d\tau\int d^3x\mathcal{L}(\phi,\bar{\partial}_\mu\phi)\right)} \tag{2.4.9}$$

which can be used to generate all the temperature-dependent Green functions through functional differentiation

$$\mathcal{G}^N(\bar{x}_1,...,\bar{x}_N) = \frac{\delta^N\bar{W}[J]}{\delta J(\bar{x}_N)...\partial J(\bar{x}_1)})\Bigg|_{J=0} \tag{2.4.10}$$

As regards the generating functional for connected temperature dependent Green functions, it can be defined as

$$\bar{W}[J] = e^{\bar{X}[J]} \tag{2.4.11}$$

and the Green functions are calculated according to

$$\mathcal{G}^N(\bar{x}_1,...,\bar{x}_N) = \frac{\delta^N\bar{X}[J]}{\delta J(\bar{x}_N)...\delta J(\bar{x}_1)}\Bigg|_{J=0}. \tag{2.4.12}$$

2.4.1 Scalar field case

We can now consider the specific case of a field theory containing scalar fields

$$\bar{W}_0[J] = \frac{\int\mathcal{D}\phi\ \exp\left(-\frac{1}{2}\int d\bar{x}'\int d\bar{x}\phi\phi(\bar{x}')A(\bar{x}',\bar{x}\phi(\bar{x})+\int d\bar{x}J(\bar{x})\phi(\bar{x}))\right)}{\int_{periodic}\mathcal{D}\phi\left(-\frac{1}{2}d\bar{x}'\int d\bar{x}\phi(\bar{x}')A(\bar{x}',\bar{x})\phi(\bar{x})\right)} \tag{2.4.13}$$

where

$$A(\bar{x}',\bar{x}) = (-\bar{\partial}'_\mu\bar{\partial}^\mu+m^2)\delta(\bar{x}'-\bar{x}). \tag{2.4.14}$$

Using path integral integration we get

$$\bar{W}_0[J] = \exp\left(-\frac{1}{2}\int d\bar{x}'\int d\bar{x}J(\bar{x}')\bar{\Delta}_F(\bar{x}'-\bar{x})J(\bar{x})\right) \tag{2.4.15}$$

where $\bar{\Delta}_F(\bar{x}'-\bar{x})$ is defined as

$$\bar{\Delta}_F(\bar{x}'-\bar{x}) = -A^{-1}(\bar{x}',\bar{x}) \tag{2.4.16}$$

which can be found by Fourier transforming

$$\bar{\Delta}_F(\bar{x}' - \bar{x}) = \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} e^{-i\bar{p}\cdot(\bar{x}',\bar{x})} D\bar{e}lta_F(\bar{p}). \tag{2.4.17}$$

2.4.2 Effective potential at finite temperature

We now consider a classical field $\phi_c(\bar{x})$ that can be extracted from the generating functional of connected Green's functions in the following way

$$\phi_c(\bar{x}) = \frac{\delta \bar{X}[J]}{\delta J(\bar{x})} \tag{2.4.18}$$

The thermal average, (i.e. the expectation value) of the field $\phi(\bar{x})$ in the presence of the source term J is calculated as

$$\frac{\delta \bar{W}[J]}{\delta J(\bar{x})} = \langle \hat{\phi}(\bar{x}) \rangle_J \tag{2.4.19}$$

Using the definition of the generating functional for connected Green functions one finds the calssical field $\phi_c(\bar{x})$ expressed as

$$\phi_c(\bar{x}) = \langle \hat{\phi}(\bar{x}) \rangle_J / \bar{W}[J]. \tag{2.4.20}$$

If the source is switched off

$$\phi_c(\bar{x}) = \langle \phi(\bar{x}) \rangle \qquad\qquad for\, J = 0 \tag{2.4.21}$$

and

$$\begin{aligned} \langle \phi(\bar{x}) \rangle &= Tr[e^{\beta \hat{H}} \hat{\phi}(\tau, x)] / Tr[e^{-\beta \hat{H}}] \\ &\sim Tr[e^{\beta \hat{H}} \hat{\phi}(0, x)] / Tr[e^{-\beta \hat{H}}] \end{aligned} \tag{2.4.22}$$

and combining the last two equations one finds

$$\phi_c(\bar{x}) = \langle \phi(0, x) \rangle. \tag{2.4.23}$$

So it can be concluded that for zero source term the classical field $\phi_c(\bar{x})$ is the thermal average value $\langle \phi(0, x) \rangle$.
The effective action is given by

$$\bar{\Gamma}[\phi_c] = \bar{X}[J] - \int d\bar{x} J(\bar{x}) \phi_c(\bar{x}) \tag{2.4.24}$$

and we could perform an expansion getting

$$\bar{\Gamma}[\phi_c] == \int d\bar{x} \left(-\bar{V}(\phi)_c + \frac{\bar{A}(\phi)_c}{2} \bar{\partial}_\mu \phi_c \bar{\partial}^\mu \phi_c + \ldots \right) \tag{2.4.25}$$

where the first term in the expansion is interpreted as the effective finite temperature potential $\bar{V}(\phi_c)$. But if the classical field has no dependence on space or on τ then the source term is reduced to

$$\frac{d\bar{V}}{d\phi_c} = J. \tag{2.4.26}$$

which for switched off source gives

$$\frac{d\bar{V}}{d\phi_c} = J = 0 \tag{2.4.27}$$

which means that if we want the expectation value of the field operator we just need to minimize the temperature-dependent effective potential.

2.5 First order phase transitions

We now try to give a more explicit form to the temperature-dependent effective potential, considering both the contributions given at tree-level and at one loop. First we write the temperature-dependent effective potential as the sum of the tree-level and one-loop contributions

$$\bar{V}(\phi_c) = \bar{V}_0(\phi_c) + \bar{V}_1(\phi_c) \tag{2.5.1}$$

and we could further separate the one-loop contribution into two terms, respectively at zero temperature and at finite temperature

$$\bar{V}_= \bar{V}_1^0 + \bar{V}_1^T. \tag{2.5.2}$$

The complete Lagrangian with scalar fields, fermions and gauge bosons is

$$\begin{aligned} \mathcal{L}_{quad}(\phi_c, \tilde{\phi}) = &-\frac{1}{2}[\hat{M}_S^2(\phi_c)]_{ij}\tilde{\phi}_i\tilde{\phi}_j + \frac{1}{2}[\hat{M}_V^2(\phi_c)]_{ab}A_a^\mu A_{b\mu} \\ &- [\hat{M}_F^2(\phi_c)]_{rs}\bar{\psi}_r\bar{\psi}_s + \frac{1}{2}\bar{\partial}_\mu\tilde{\phi}_i\bar{\partial}^\mu\tilde{\phi}_i \\ &- \frac{1}{4}(\bar{\partial}^\mu A_a^\nu - \bar{\partial}^\nu A_a^\mu)(\bar{\partial}_\mu A_{\nu a} - \bar{\partial}_\nu A^{a\mu}) \\ &- \frac{1}{2\xi}(\bar{\partial}_\mu A_a^\mu)^2 + \bar{\partial}_\mu\eta_a^*\bar{\partial}^\mu\eta_a. \end{aligned} \tag{2.5.3}$$

where ϕ_c, $\tilde{\phi}_i$ and η_a indicate respectively the set of average values of the scalar fields, the shifted scalar fields and the Fadeev-Popov ghost fields.
Using a matrix notation we indicate the mass-squared matrices with $\hat{M}_S^2$,$\hat{M}_V^2$ and $\hat{M}_F^2$ whose eigenvalues are respectively $(M_S^2)_i$,$(M_V^2)_a$ and $(M_F^2)_r$. We can now give the expression for the one-loop

temperature-dependent term of the effective potential

$$\bar{V}_1^T(\phi_c) = \frac{T^4}{2\pi^2}\int_0^\infty dyy^2\Big\{\sum_i \ln\left[1-\exp\left(-\sqrt{y^2+T^{-2}(M_S^2)_i}\right)\right] + \sum_a\left(3\ln\left[1-\exp\left(-\sqrt{y^2+T^{-2}(M_V^2)_a}\right)\right] - \ln(1-e^{-y})\right) - 4\sum_r \ln\left[1+\exp\left(-\sqrt{y^2+T^{-2}(M_F^2)_r}\right)\right]\Big\}. \tag{2.5.4}$$

This expression could be simplified. We could first consider the limit in which all the mass matrices eigenvalues are much bigger than T^2; in this case $\bar{V}_1^T$ goes to zero exponentially because all of its terms goes to zero. The other approximation one could use is the high temperature limit in which T^2 is much larger than the eigenvalues ; so we first notice that

$$\begin{aligned}&\frac{T^4}{2\pi^2}\int_0^\infty dyy^2\ln\left[1-\exp\left(-\sqrt{y^2+RT^{-2}}\right)\right] =\\ &= -\frac{\pi^2T^4}{90}+\frac{RT^2}{24}-\frac{R^{3/2}T}{12\pi}-\frac{R^2}{64\pi^2}\ln\left(\frac{R}{a_bT^2}\right)\\ &+\frac{R^2}{16\pi^{5/2}}\sum_{l=1}^\infty(-1)^l\frac{\zeta(2l+1)}{(l+1)!}\left(\frac{R}{4\pi^2T^2}\right)^l\end{aligned} \tag{2.5.5}$$

with

$$a_b = 16\pi^2\ln\left(\frac{3}{2}-2\gamma_E\right) \tag{2.5.6}$$

and

$$\ln\, a_b = 5.4076. \tag{2.5.7}$$

Similarly

$$\begin{aligned}&\frac{T^4}{2\pi^2}\int_0^\infty dyy^2\ln\left[1+\exp\left(-\sqrt{y^2+RT^{-2}}\right)\right] =\\ &= -\frac{7\pi^2T^4}{720}+\frac{RT^2}{48}-\frac{R^2}{64\pi^2}\ln\left(\frac{R}{a_fT^2}\right)\\ &-\frac{R^2}{16\pi^{5/2}}\sum_{l=1}^\infty(-1)^l\frac{\zeta(2l+1)}{(l+1)!}(1-2^{-2l-1})\Gamma\left(l+\frac{1}{2}\right)\left(\frac{R}{4\pi^2T^2}\right)^l\end{aligned} \tag{2.5.8}$$

with

$$a_f = \pi^2\ln\left(\frac{3}{2}-2\gamma_E\right) = \frac{a_b}{16} \tag{2.5.9}$$

and

$$\ln\, a_b = 2.6351. \tag{2.5.10}$$

So finally the finite temperature potential can be written

$$\begin{aligned}
\bar{V}_1^T(\phi_c) \simeq & -\frac{\pi^2 T^4}{90}\left(N_B + \frac{7}{8}N_F\right) \\
& + \frac{T^2}{24}\left[\sum_i (M_S^2)_i + 3\sum_i a(M_V^2)_a + 2\sum_i (M_F^2)_r\right] \\
& - \frac{T}{12\pi}\left[\sum_i (M_S^3)_i + 3\sum_a (M_V^2)_a\right] + ... = \\
& = -\frac{\pi^2 T^4}{90}\left(N_B + \frac{7}{8}N_F\right) \\
& + \frac{T^2}{24}[tr\hat{M}_S^2(\phi_c) + 3tr\hat{M}_V^2(\phi_c) + 2tr\hat{M}_F^2(\phi_c)] \\
& - \frac{T}{12\pi}[\{tr\hat{M}_S^2(\phi_c)\}^{3/2} + 3\{tr\hat{M}_V^2(\phi_c)\}^{3/2}] + ...
\end{aligned} \tag{2.5.11}$$

It is easy to notice that the T^4 term in the previous expression is just the free energy.

2.6 Temperature-dependent Higgs model

We now exploit the formalism developed in the previous sections to discuss a temperature-dependent Higgs model containing scalar and vector fields. Let us begin with the finite temperature Lagrangian

$$\begin{aligned}
\mathcal{L} = & \bar{D}_\mu \phi \bar{D}^\mu \phi^* - m^2 \phi^* \phi - (\lambda/4)(\phi^*\phi)^2 \\
& - \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - \frac{1}{2\xi}(\bar{\partial}_\mu A^\mu)^2 + \bar{\partial}_\mu \eta^* \bar{\partial}^\mu \eta
\end{aligned} \tag{2.6.1}$$

with $m^2 < 0$,and the covariant derivatives defined as

$$\bar{D}_\mu \phi \equiv (\bar{\partial}_\mu + ieA_\mu)\phi \tag{2.6.2}$$

$$\bar{D}_\mu \phi^* \equiv (\bar{\partial}_\mu - ieA_\mu)\phi^*. \tag{2.6.3}$$

The Lagrangian in 2.6.1 contains the ghost terms in η in order to cancel the contribution to free energy given by the un-physical degrees of freedom, whose presence is caused by the gauge bosons. We know shift the scalar field by

$$\frac{\phi_c}{\sqrt{2}} \equiv \langle \hat{\phi}(\bar{x}) \rangle \tag{2.6.4}$$

and we expand according to

$$\phi = \frac{1}{\sqrt{2}}(\phi_c + \phi_1 + i\phi_2) \tag{2.6.5}$$

where ϕ_1 and ϕ_2 are the shifted fields. We rewrite the Lagrangian retaining only the quadratic terms

$$\begin{aligned}\mathcal{L}_{quad} = & \frac{1}{2}(\bar{\partial}_\mu \phi_1)^2 + \frac{1}{2}(\bar{\partial}_\mu \phi_2)^2 - \frac{1}{2}\left(m^2 + \frac{3\lambda}{4}\phi_c^2\right)\phi_1^2 \\ & - \frac{1}{2}\left(m^2 + \frac{\lambda}{4}\phi_c^2\right)\phi_2^2 - \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} + \frac{e^2}{2}\phi_c^2 A_\mu A^\mu \\ & - \frac{1}{2\xi}(\partial_\mu \bar{A}^\mu)^2 - \bar{\partial}_\mu \eta^* \bar{\partial}^\mu \eta\end{aligned} \tag{2.6.6}$$

and for the quadratic terms in ϕ_1 and ϕ_2 we can use the notation

$$\hat{M}_S^2(\phi_c) = diag\left\{m^2 + \frac{3\lambda}{4}\phi_c^2, m^2 + \frac{\lambda}{4}\phi_c^4\right\} \tag{2.6.7}$$

$$\hat{M}_V^2(\phi_c) = e^2\phi_c^2. \tag{2.6.8}$$

where e is the gauge coupling constant.

2.6.1 $e^4 \ll \lambda$ case

We consider the one-loop zero temperature potential term, which is

$$\bar{V}_0(\phi_c) = \frac{m^2}{2}\phi_c^2 + \frac{\lambda}{16}\phi_c^4 \tag{2.6.9}$$

while the one-loop zero temperature term can be neglected as long as $e^4 \ll \lambda$. In addition for high -temperatures we have

$$T^2 \gg \lambda\phi_c^2, e^2\phi_c^2, -m^2. \tag{2.6.10}$$

so the one-loop temperature dependent correction can be written as

$$\bar{V}_1^T(\phi_c) = -\frac{4\pi T^4}{90} + \frac{(\lambda + 3e^2)T^2}{24}\phi_c^2 - \frac{CT}{3}\phi_c^3 + ... \tag{2.6.11}$$

where C is defined through

$$
\begin{aligned}
4\pi C &= \{tr\hat{M}_S^2(\phi_c)\}^{3/2} + 3\{tr\hat{M}_V^2(\phi_c)\}^{3/2} \\
&= \left(m^2\phi_c^{-2} + \frac{3\lambda}{4}\right)^{3/2} + \left(m^2\phi_c^{-2} + \frac{\lambda}{4}\right)^{3/2} + 3e^3 \\
&\simeq \left(\frac{3\lambda}{4}\right)^{3/2} + \left(\frac{\lambda}{4}\right)^{3/2} + 3e^3
\end{aligned}
\tag{2.6.12}
$$

for $\lambda\phi_c \gg m^2$. Adding this term to the one-loop zero temperature term the final potential is

$$
\bar{V}^T(\phi_c) = -\frac{4\pi^2 T^4}{90} + \frac{1}{2}m^2(T)\phi_c^2 + \frac{(\lambda+3e^2)T^2}{12}\phi_c^2 - \frac{CT}{3}\phi_c^3 + \frac{\lambda}{16}\phi_c^4 \tag{2.6.13}
$$

but we can recast this form defining the temperature dependent mass

$$
m^2(T) = m^2 + \frac{(\lambda+3e^2)T^2}{12}. \tag{2.6.14}
$$

We can minimize the potential

$$
\frac{\partial \bar{V}}{\partial \phi_c} = 0 \tag{2.6.15}
$$

and find the solution

$$
\phi_c = 0 \tag{2.6.16}
$$

and this solution is valid as long as the temperature satisfies

$$
T^2 > T_0^2 \equiv -12m^2/(\lambda+3e^2). \tag{2.6.17}
$$

But for $T < T_1$ where

$$
T_1^2 \equiv \frac{T_0^2}{1 + C^2T_0^2/\lambda m^2} = \frac{-12\lambda m^2}{\lambda(\lambda+3e^2) - 12C^2} > T_0^2 \tag{2.6.18}
$$

a second minimum arises at

$$
\phi_c = v(T) \equiv v\left[\frac{CT}{\sqrt{\lambda}|m|} + \left(1 - \frac{T^2}{T_1^2}\right)^{1/2}\right] \tag{2.6.19}
$$

with

$$
v \equiv \frac{2|m|}{\sqrt{\lambda}}. \tag{2.6.20}
$$

We could also define the mass of the Higgs particle, related to the fluctuations around the minimum

$$m_H^2(T) = \equiv \frac{\partial < 2\bar{V}}{\partial \phi_c^2}\bigg|_{\phi_c = v(T)} = CTv(T) - 2m^2(T) \tag{2.6.21}$$

and in a similar way we could define the temperature-dependent vector boson mass

$$m_V(T) \equiv e\, v(T). \tag{2.6.22}$$

Now for $T < T_1$ we have two minima, one at $\phi_c = 0$ and the other at $\phi_c = v(T) \equiv v\left[\frac{CT}{\sqrt{\lambda}|m|} + \left(1 - \frac{T^2}{T_1^2}\right)^{1/2}\right]$. But if the temperature decreases further the minimum at $\phi_c = 0$ becomes degenerate. This situation is illustrated in Fig.2.1

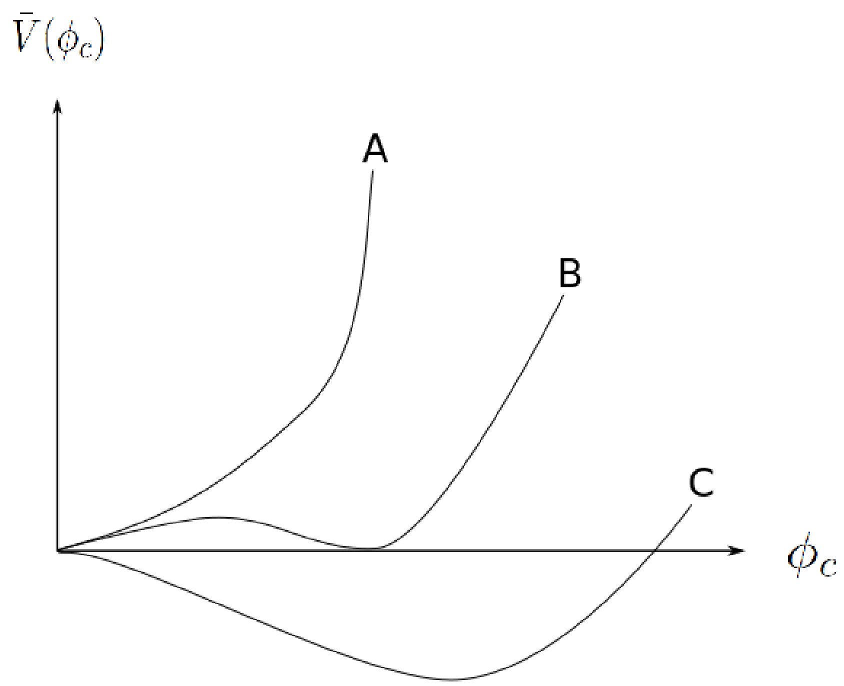

Figure 2.1 This is the potential for the model we are discussing for $e^4 \ll \lambda$. The curves A, B,C represent respectively the cases for $T > T_1$,$T > T_c$,$T < T_0$.

The critical temperature at which this can be found imposing

$$\frac{8}{9}C^2T^2 = \lambda m^2(T) \tag{2.6.23}$$

and then one finds

$$T_c^2 = \frac{-12\lambda m^2}{\lambda(\lambda+3e^2) - 32C^2/3}. \tag{2.6.24}$$

If the temperature drops below the critical temperature, the non-zero minimum is global and the system enters the phase characterized by the spontaneous symmetry breaking. In particular it can be seen that the value of the field ϕ_c passes, at $T = T_c$, in a discontinuous way from the null value to the value

$$\phi_c = v(T_c) = \frac{8CT_c}{3\lambda} \tag{2.6.25}$$

and this is the signature of a first-order phase transition. Then if we imagine to lower the temperature , it would fall below $T = T_0$ and then $m^2(T)$ is now negative and the minimum at $\phi_c = 0$ is just a local minimum. But if C is so small to be neglected, the three temperatures previously defined will assume the same value, so the system goes through a second order phase transition at $T = T_c$.

2.6.2 $e^4 \gg \lambda$ case

Let us now consider the case where the gauge coupling constant is much larger than the ϕ^4 coupling constant. In this case the zero-temperature one loop correction is not negligible and furthermore the high -temperature approximation is no longer allowed. So let us first write the effective potential, including the zero-temperature correction

$$\bar{V}(\phi_c) = \frac{m^2}{2}\phi_c^2 + \frac{\lambda}{16}\phi_c^4 + B\phi_c^4\left[\ln\left(\frac{\phi_c^2}{M^2}\right)\right] + \bar{V}_1^T(\phi_c) \tag{2.6.26}$$

where M is the renormalization scale and B is defined as

$$B = \frac{1}{64\pi^2}\left(\frac{5}{8}\lambda^2 + 3e^4\right) \tag{2.6.27}$$

but if $\lambda < e^4$ then we could simplify

$$B \simeq \frac{3e^4}{64\pi^2}. \tag{2.6.28}$$

If we do not use any high temperature approximation, the one-loop temperature dependent correction to the potential can be written

$$\begin{aligned}\bar{V}_1^T(\phi_c) = \frac{T^4}{2\pi^2}\int_0^\infty dy\, y^2 \Big\{ &\ln\left[1 - \exp\left(-\sqrt{y^2 + T^{-2}(m^2 + 3\lambda\phi_c^2/4)}\right)\right] \\ &+ \ln\left[1 - \exp\left(-\sqrt{y^2 + T^{-2}(m^2 + \lambda\phi_c^2/4)}\right)\right] \\ &+ 3\ln\left[1 - \exp\left(-\sqrt{y^2 + T^{-2}e^2\phi_c^2}\right)\right] - \ln\left(1 - e^{-y}\right)\Big\}.\end{aligned} \tag{2.6.29}$$

We now want to find a criterion for whether it is still possible to use the high-temperature approximation. So, let us first consider the definition of C in 2.6.12, now for $e^4 \gg \lambda$ we have

$$C \simeq \frac{3e^3}{4\pi}. \tag{2.6.30}$$

We insert this in 2.6.24 and we find

$$T_c^2 \simeq -\frac{4m^2}{e^2}. \tag{2.6.31}$$

Recalling the definition of the asymmetric minimum in 2.6.25 we now find

$$\phi_c^2 = v^2 = \frac{-4m^2}{\lambda} \tag{2.6.32}$$

We can calculate the vector boson mass

$$m_v^2 = e^2v^2 = \frac{-4m^2}{e^2}\frac{e^4}{\lambda}. \tag{2.6.33}$$

So in the end we find that

$$T_c^2 \ll m_V^2 = e^2v^2 = \frac{-4m^2}{e^2}\frac{e^4}{\lambda}, \tag{2.6.34}$$

which means that we can no longer use the high-temperature approximation when we consider $e^4 \gg \lambda$. We now compute the value of the potential at the symmetric and asymmetric minima. the symmetric one is $\phi_c = 0$, so that

$$\bar{V}(\phi_c = 0) = \bar{V}_1^T(\phi_c = 0) \simeq -\frac{4\pi^2 T^4}{90}. \tag{2.6.35}$$

The asymmetric minimum is at $\phi_c = v$. In this case the contribution to $\bar{V}_1^T(\phi_c)$ coming from the gauge mass field $e\phi_c$ can be neglected since they are exponentially suppressed. So one finds

$$\bar{V}_1^T(\phi_c = v) \simeq -\frac{2\pi^2 T^4}{90} + \frac{T^2}{24}(-2m^2) - \frac{T}{12\pi}(-2m^2)^{\frac{3}{2}}. \tag{2.6.36}$$

The potential assumes the following form

$$\bar{V}(\phi_c = v) = -\frac{m^4}{\lambda} - \frac{\pi^2 T^4}{45} - \frac{m^2T^2}{12} - \frac{T}{12\pi}(-2m^2)^{\frac{3}{2}}. \tag{2.6.37}$$

We can neglect the terms in T^2 and m^3 because negligible compared to the terms in T^4, so one finds that the symmetric minimum gives a lower value for the potential than the one given by the asymmetric minimum and this happens for

$$T > \left(\frac{45}{\pi^2\lambda}\right)^{\frac{1}{4}} |m| \equiv T_{c1}. \tag{2.6.38}$$

This situation is illustrated in Fig.2.2

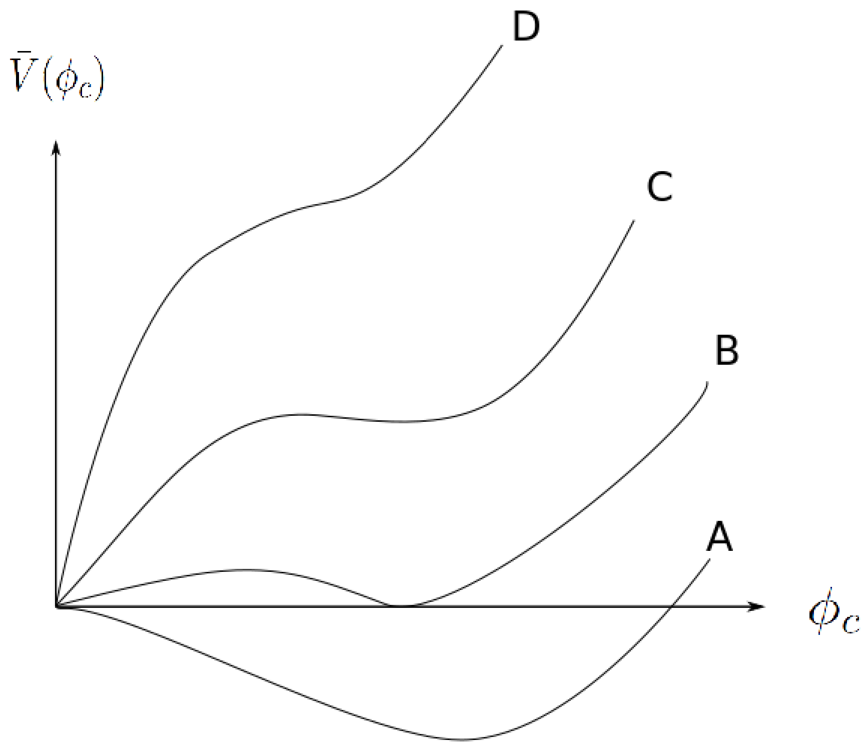

Figure 2.2 Development of asymmetric minima. Curves A corresponds to the zero temperature case, B to the case at T_c and curves C and D correspond to higher temperatures.

The phase transition takes occurs at T_{c1} and it is a first order phase transition since the expectation value changes discontinuously from 0 to v. If the temperature of the system is $T \ll T_{c1}$, a phase transition cannot occur since it is necessary to break the potential barrier that separates the symmetric minimum from the asymmetric one.
But, if one considers the radiative correction, it can provide a symmetry breaking as explained by Coleman and Weinberg. In this case the potential can be written as

$$\bar{V}(\phi_c) = B\left(\frac{\alpha}{2}v^2\phi_c^2 - \frac{\alpha+2}{4}\phi_c^4 + \phi_c^4 \ln\frac{\phi_c^2}{v^2}\right) + \bar{V}_1^T(\phi_c) \tag{2.6.39}$$

$$\alpha = 2B^{-1}\left(\frac{22}{3}B - \frac{\lambda}{8}\right). \tag{2.6.40}$$

The mass of the Higgs boson could be extracted using

$$m_H^2 = \frac{d^2\bar{V}}{d\phi_c^2}\bigg|_{\phi_c=v} = 2Bv^2(4-\alpha). \tag{2.6.41}$$

In the case of the Coleman-Weinberg model $\alpha = 0$.

2.7 Bose-Einstein condensation

2.7.1 Neutral scalar field

Before considering the phenomenon of Bose-Einstein condensation, it is necessary to develop some formal tools needed later. We consider for this purpose the most general Lagrangian for a neutral scalar field ϕ

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - U(\phi) \tag{2.7.1}$$

where $U(\phi)$ is the potential

$$U(\phi) = g\phi^3 + \lambda\phi^4. \tag{2.7.2}$$

We define the conjugate momentum

$$\pi = \frac{\mathcal{L}}{\partial(\partial_0\phi)} = \frac{\partial\phi}{\partial t}. \tag{2.7.3}$$

We also need the Hamiltonian defined as the Legendre transform of the Lagrangian

$$\mathcal{H} = \pi\frac{\partial\phi}{\partial t} - \mathcal{L} = \frac{1}{2}\pi^2\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi). \tag{2.7.4}$$

The partition function of the system is defined using the usual methods

$$\begin{aligned} Z = \lim_{N\to\infty}&\left(\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\frac{d\pi_i}{2\pi}\int_{periodic}d\phi_i\right) \\ &\exp\left(\sum_{j=1}^{N}\int d^3x\Bigg\{i\pi_j(\phi_{j+1}-\phi_j)\right. \\ &\left.-\Delta\tau\left[\frac{1}{2}\pi_j^2 + \frac{1}{2}(\nabla\phi_j)^2 + \frac{1}{2}m^2\phi_j^2 + U(\phi)\right]\Bigg\}\right). \end{aligned} \tag{2.7.5}$$

The integrations on the momentum can be carried on using Gaussian integration, and we divide the space into M^3 cubes with $V = L^3$, so

$$Z = \lim_{M,N\to\infty} (2\pi)^{M^3N/2} \int \left(\prod_i = 1^N d\phi_i \right)$$
$$\times \exp\left\{ \Delta\tau \sum_{j=1}^{N} \int d^3x \left[-\frac{1}{2} \left(\frac{\phi_{j+1} - \phi_j}{\Delta\tau} \right)^2 - \frac{1}{2}(\nabla\phi_j)^2 - \frac{1}{2} m^2 \phi_j^2 - U(\phi_j) \right] \right\} \tag{2.7.6}$$

now, taking the continuum limit we find

$$Z = N' \int_{periodic} \mathcal{D}\phi \left(\int_0^\beta d\tau \int d^3x \mathcal{L} \right) \tag{2.7.7}$$

where N' is just a normalization constant. The last equation is the the partition function for a system of neutral scalar field expressed as a functional integral. If we consider a system of non-interacting bosons, it is as if we switch the potential off, so $U(\phi) = 0$, so the action is

$$S = \int_0^\beta d\tau \int d^3x \mathcal{L} = -\frac{1}{2} \int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial\phi}{\partial\tau} \right)^2 + (\nabla\phi)^2 + m^2\phi^2 \right]. \tag{2.7.8}$$

After integration by parts the final form is

$$S = -\frac{1}{2} \int_0^\beta d\tau \int d^3x \left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2 \right) \phi. \tag{2.7.9}$$

We can expand the field ϕ by Fourier expansion

$$\phi(\mathbf{x}, \tau) = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x} + \omega_n\tau)} \phi_n(\mathbf{p}) \tag{2.7.10}$$

with $\omega_n = 2\pi n T$, because of the periodic condition $\phi(\mathbf{x}, \beta) = \phi(\mathbf{x}, 0)$. If we now replace the previous expansion in the expression of the action and imposing that the scalar field is real we find

$$S = -\frac{1}{2}\beta^2 \sum_n \sum_{\mathbf{p}} (\omega_n^2 + \omega^2) \phi_n(\mathbf{p}) \phi_n^*(\mathbf{p}) \tag{2.7.11}$$

where $\omega = \sqrt{\mathbf{p}^2 + m^2}$.. We also notice that the dependence of the integrand is governed by the magnitude of the field, so it is useful to define $A_n(\mathbf{p}) = |\phi_n(\mathbf{p})|$.

Replacing all this in the partition function and integrating out the phase we find

$$Z = N' \prod_n \prod_{\mathbf{p}} \Big\{ \int_{-\infty}^{\infty} dA_n(\mathbf{p}) \exp\Big[-\frac{1}{2}\beta^2(\omega_n^2 + \omega^2) A_n^2(\mathbf{p}) \Big] \Big\}$$
$$= N' \prod_n \prod_{\mathbf{p}} (2\pi)^{1/2} [\beta^2(\omega_n^2 + \omega^2)]^{-1/2}. \tag{2.7.12}$$

The partition function could also be reformulated as

$$Z = N' \int \mathcal{D}\phi \exp\Big[-\frac{1}{2}(\phi, D\phi) \Big] = N''(det\, D)^{-1/2} \tag{2.7.13}$$

where as usual N'' is a normalization constant. D is the following operator:

$$D = \beta^2(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2)$$

. The expression in 2.7.13 is derived using the relation

$$\int_{\infty}^{\infty} dx_1 ... dx_n e^{-x_i D_{ij} x_j} = \pi^{n/2}(det\, D)^{-1/2}. \tag{2.7.14}$$

We can extract the logarithm from 2.7.12, ignoring the overall constant

$$\ln\, Z = -\frac{1}{2} \sum_n \sum_{\mathbf{p}} \ln[(\beta^2(\omega_n^2 + \omega^2)]. \tag{2.7.15}$$

Problems arise when we want to deal with sums of logarithms. We can use the identity

$$\ln[(2\pi n)^2 + \beta^2\omega^2] = \int_1^{\beta^2\omega^2} \frac{d\theta^2}{\theta^2 + (2\pi n)^2} + \ln[1 + (2\pi n)^2] \tag{2.7.16}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + (\frac{\theta}{2\pi})^2} = \frac{2\pi^2}{\theta}\Big(1 + \frac{2}{e^\theta - 1}\Big). \tag{2.7.17}$$

Using the last two equations in 2.7.12 and discarding the temperature-independent term we find

$$\ln Z = -\sum_{\mathbf{p}} \int_1^{\beta\omega} d\theta \Big(\frac{1}{2} + \frac{1}{e^\theta - 1} \Big). \tag{2.7.18}$$

Performing the integration we find

$$\ln Z = V \int \frac{d^3p}{(2\pi)^2} \Big[-\frac{1}{2}\beta\omega - \ln\Big(1 - e^{-\beta\omega}\Big) \Big] \tag{2.7.19}$$

which is exactly the same result given by equation 2.0.44 with $\mu = 0$. But the last results also include the zero-point energy.

2.7.2 Condensation

Let us consider a system of charged bosons, described by a complex scalar field Φ. Consider the Lagrangian of the system

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \lambda(\Phi * \Phi)^2. \tag{2.7.20}$$

This Lagrangian is invariant under a general local U(1) transformation

$$\Phi \to \Phi' = \Phi e^{i\alpha} \tag{2.7.21}$$

with α a constant. From Noether's theorem we know that each symmetry corresponds to a conserved current, so let us compute this current. We apply the transformation but we now consider α as a local factor, i.e. dependent on a space coordinate

$$\begin{aligned} \mathcal{L} \to \mathcal{L}' &= \partial_\mu (\Phi^* e^{i\alpha(x)}) \partial^\mu (\Phi e^{-i\alpha(x)}) - m^2 \Phi^* \Phi - \lambda(\Phi^* \Phi)^2 \\ &= \mathcal{L} + \Phi^* \Phi \partial_\mu \alpha \partial^\mu \alpha + i \partial_\mu \alpha (\Phi^* \partial^\mu \Phi - \Phi \partial^\mu \Phi^*). \end{aligned} \tag{2.7.22}$$

The equation of motion for $\alpha(x)$ is

$$\partial^\mu \frac{\partial \mathcal{L}'}{\partial(\partial^\mu \alpha)} = \frac{\partial \mathcal{L}'}{\partial \alpha}. \tag{2.7.23}$$

But

$$\frac{\partial \mathcal{L}'}{\partial \alpha} = 0 \tag{2.7.24}$$

so the current

$$\frac{\partial \mathcal{L}'}{\partial(\partial^\mu \alpha)} = \Phi^* \Phi \partial_\mu \alpha + i \Phi^* \partial_\mu \Phi - i \Phi \partial_\mu \Phi^* \tag{2.7.25}$$

is a conserved current. If we restore $\alpha(x)$ as a constant the current density is now written

$$j_\mu = i(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) \tag{2.7.26}$$

so,

$$\partial^\mu j_\mu = 0. \tag{2.7.27}$$

The full current and the conserved charge associated are written respectively

$$J_\mu = \int d^3x j_\mu(x) \tag{2.7.28}$$

$$Q = \int d^3x j_0(x). \tag{2.7.29}$$

We make the complex nature of the fieldΦ explicit, by decomposing

$$\Phi = \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}. \tag{2.7.30}$$

We compute the conjugate momenta

$$\pi_1 = \frac{\partial\phi_1}{\partial t} \tag{2.7.31}$$

$$\pi_2 = \frac{\partial\phi_2}{\partial t}. \tag{2.7.32}$$

We can now write the Hamiltonian of the system

$$\mathcal{H} = \frac{1}{2}\left[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2\phi_1^2 + m^2\phi_2^2\right] + \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2 \tag{2.7.33}$$

and the conserved charge is

$$Q = \int d^3x(\phi_2\pi_1 - \phi_1\pi_2). \tag{2.7.34}$$

Let us define the partition function of the system

$$\begin{aligned} Z = \int \mathcal{D}\pi_1\mathcal{D}\pi_2 \int_{periodic} \mathcal{D}\phi_1\mathcal{D}\phi_2 \times \exp\Bigg[\int_0^\beta d\tau \int d^3x \\ \times \left(i\pi_1\frac{\partial\phi_2}{\partial\tau} + i\pi_2\frac{\partial\phi_1}{\partial\tau} - \mathcal{H}(\pi_1,\pi_2,\phi_1,\phi_2) + \mu(\phi_2\pi_1 - \phi_1\pi_2)\right)\Bigg] \end{aligned} \tag{2.7.35}$$

where μ is the chemical potential.
We can now integrate out the fields associated with the conjugate momenta and find

$$\begin{aligned} Z = (N')^2 \int_{periodic} \mathcal{D}\phi_1\mathcal{D}\phi_2 \\ \times \exp\Bigg\{\int_0^\beta d\tau \int d^3x\Bigg[-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial\tau} - i\mu\phi_2\right)^2 - \frac{1}{2}\left(\frac{\partial\phi_2}{\partial\tau} - i\mu\phi_1\right)^2 \\ -\frac{1}{2}(\nabla\phi_1)^2 - \frac{1}{2}(\nabla\phi_2)^2 - \frac{1}{2}m^2\phi_1 - \frac{1}{2}m^2\phi_2 - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2\Bigg]\Bigg\} \end{aligned} \tag{2.7.36}$$

where N' a normalization factor. The last integral, due to the presence of $\lambda \neq 0$ cannot be evaluated by the classical method of Gaussian integration. But let us consider $\lambda = 0$ and Fourier-expand the components of Φ

$$\phi_1 = \sqrt{2}\zeta cos\theta + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} e^{\mathbf{p}\cdot\mathbf{x}+\omega_n\tau}\phi_{1;n}(\mathbf{p}) \tag{2.7.37}$$

$$\phi_2 = \sqrt{2}\zeta sin\theta + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} e^{\mathbf{p}\cdot\mathbf{x}+\omega_n\tau}\phi_{2;n}(\mathbf{p}), \tag{2.7.38}$$

where ζ and θ are functions necessary regulate the infra-red behaviour of the Φ field, i.e. $\phi_{1;0}(\mathbf{p} = 0)$ and $\phi_{2;0}(\mathbf{p} = 0)$.(In addition ζ will be considered the so called order parameter of the system and its behaviour will give us information about the order of the phase transition we are discussing.)

In this way, the behaviour of the field is consistently determined and this allows for the possibility of condensation of particles, a fraction of which are in the state at $\mathbf{p} = 0$.
So let us consider the case for $\lambda = 0$, if we replace the Fourier-expansion of the fields into the 2.7.36 and integrate by parts, we obtain

$$Z = (N')^2 \left(\prod_n \prod_{\mathbf{p}} \int d\phi_{1;n}(\mathbf{p}) d\phi_{2;n}(\mathbf{p}) \right) e^S \tag{2.7.39}$$

where S is the action

$$\beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2} \sum_n \sum_{\mathbf{p}} (\phi_{1;-n}(-\mathbf{p}), \phi_{2;-n}(-\mathbf{p}))\, D \begin{pmatrix} \phi_{1;n}(\mathbf{p}) \\ \phi_{2;n}(\mathbf{p}) \end{pmatrix} \tag{2.7.40}$$

where

$$D = \beta^2 \begin{pmatrix} \omega_n^2 + \omega^2 - \mu^2 & -2\mu\omega_n \\ 2\mu\omega_n & \omega_n^2 + \omega^2 - \mu^2 \end{pmatrix}. \tag{2.7.41}$$

After integration,

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 + \ln(\det\, D)^{\frac{1}{2}} \tag{2.7.42}$$

and the second term could be recast in the form

$$\begin{aligned} \ln det\, D &= \ln\left\{ \prod_n \prod_{\mathbf{p}} \beta^4 [(\omega_n^2 + \omega^2 - \mu^2)^2 + 4\mu^2\omega_n^2] \right\} \\ &= \ln\left\{ \prod_n \prod_{\mathbf{p}} \beta^2 [(\omega_n^2 + (\omega - \mu)^2] \right\} + \ln\left\{ \prod_n \prod_{\mathbf{p}} \beta^2 [(\omega_n^2 + (\omega + \mu)^2] \right\}. \end{aligned} \tag{2.7.43}$$

Replacing the $\ln det\, D$ in 2.7.42

$$\begin{aligned} \ln Z = \beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2} \sum_n \sum_{\mathbf{p}} \ln\{\beta^2 [\omega_n^2 + (\omega - \mu)^2]\} \\ - \frac{1}{2} \sum_n \sum_{\mathbf{p}} \ln\{\beta^2 [\omega_n^2 + (\omega + \mu)^2]\}. \end{aligned} \tag{2.7.44}$$

Recalling the result in the previous equation we find, after the substitutions $\omega \to \omega - \mu$ and $\omega \to \omega + \mu$

respectively in the previous equation

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - V \int \frac{d^3p}{(2\pi)^2} \times \left[\beta\omega + \ln\left(1 - e^{-\beta(\omega-\mu)}\right) + \ln\left(1 - e^{-\beta(\omega+\mu)}\right)\right]. \tag{2.7.45}$$

We first note that the integral is convergent if $|\mu| \leqslant m$. Furthermore the parameter θ does not appear in the final expression of the logarithm of the partition function, in accordance with the gauge invariance of U(1) of the Lagrangian. Instead, the parameter θ does appear and can be treated as a variational parameter. With respect to this parameter $\ln Z$ is an extremum, therefore

$$\frac{\partial \ln Z}{\partial \zeta} = 2\beta V(\mu^2 - m^2)\zeta = 0 \tag{2.7.46}$$

and this determines $\zeta = 0$, unless $|\mu| = m$. In this case we cannot find ζ through variational conditions. Instead we note that the charge density is

$$\rho = \frac{Q}{V} = \frac{T}{V}\left(\frac{\partial \ln Z}{\partial \mu}\right)\Big|_{\mu=m} = 2m\zeta^2 + \rho^*(\beta, \mu = m) \tag{2.7.47}$$

where

$$\rho^* = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{e^{\beta(\omega-m)} - 1} - \frac{1}{e^{\beta(\omega+m)} - 1}\right). \tag{2.7.48}$$

We must now make a few observations: If we lower the temperature to a fixed density ρ, then μ decreases until it reaches the value $\mu = m$. If we lower the temperature further then ρ^* will reach a value less than ρ

$$\zeta^2 = \frac{\rho - \rho^*(\beta, \mu = m)}{2m}. \tag{2.7.49}$$

The critical temperature T_c is given by imposing

$$\rho = \rho^*(\beta_c, \mu = m) \tag{2.7.50}$$

obtaining , for the non-relativistic particles

$$T_c = \frac{2\pi}{m}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3} \quad \text{per } \rho \ll m^3. \tag{2.7.51}$$

While for the relativistic particles we have

$$T_c = \left(\frac{3\rho}{m}\right)^{1/2} \quad \text{per } \rho \gg m^3. \tag{2.7.52}$$

The existence of such a temperature tells us that the system undergoes a phase transition. If we consider the limit $m \to 0$ then $|\mu| \to 0$ so $T_c \to \infty$ which means that all the particles are in the condensate at all temperatures.

Chapter 3

Bubble nucleation

In the chapter previous we considered the possibility of having within a field theory, a phase transition considering the characteristics of the temperature dependent potential. In this chapter we will better study the characteristics of particular field configurations that admit phase transition. A first-order phase transition requires the transition of a solution of a field theory from a metastable minimum to a stable minimum. In a classical theory in fact it is possible to have two equilibrium states associated with different energy densities. We will show that quantum theory instead predicts that the state with higher energy density is unstable and this determines the transition through a potential barrier, that is the achievement of the true minimum of the theory, that is the so-called true vacuum. We will calculate in detail the rate of formation of the true vacuum, that is a phenomenon known as bubble nucleation, starting from a Lagrangian of a theory with a scalar field. In order to do this we will follow the treatment exposed in [13], [1], [15] and especially in [16] and [17] where the authors consider the first quantum correction to the theory. In [16] and [17] is developed the theory at zero temperature, later extended to non-zero temperatures by [18] and [19].

In the second part of the chapter we will consider the modification of the theory of bubble nucleation considering a dependence on temperature and we will discuss in detail the mathematical aspects that characterize the treatment, in particular we will show how to perform the calculation of the nucleation rate through the approach of the path integral[15].

The results of the discussion of this chapter will then be used in the following, talking about phase transition from false to true vacuum in the cosmological field with consequent formation of gravitational waves.

3.0.1 Vacuum decay

Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi). \tag{3.0.1}$$

We consider a potential characterization by the presence of two unequal minima that we will call $\phi = \phi_{tv}$ and $\phi = \phi_{fv}$ where the former is the lowest (true vacuum) and the latter is the higher (false vacuum). Likewise, the potential values at these minima will be indicated respectively with V_{tv} and V_{fv}. In the section we will see that from a quantum point of view the false vacuum state can decay through a tunnel effect. This can also be read from a cosmological point of view. The Universe may have gone through a phase transition of the first order that brought it from the false to the true state of vacuum. The goal is now to describe under what conditions the process may have occurred. A typical potential with a false vacuum is shown in Fig.3.1

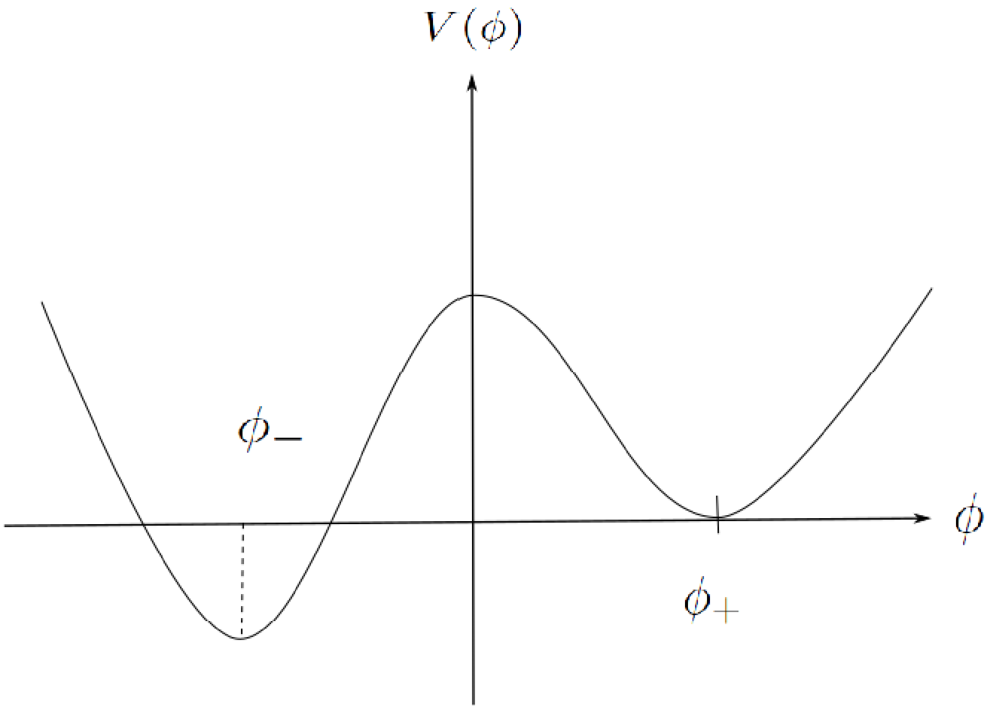

Figure 3.1 Potential with false and true vacuum

Consider the following potential energy

$$U[\phi(x)] = \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]. \tag{3.0.2}$$

We have seen how the tunnel effect can determine the decay of the false vacuum state through quantum corrections. At this point a bubble of true vacuum is formed, through the nucleation process, whose parameters have been dictated in the previous section. The true vacuum bubble can form at any point of the space, the decay rate is proportional to the volume and therefore we will consider the quantity defined as the bubble nucleation rate

$$\Gamma/\mathcal{V}. \tag{3.0.3}$$

The tunnelling process is dominated by the path that minimizes the integral penetration of barrier B. This path can be found solving the equation of motion coming from the action

$$S = \int d\tau d^3x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right) + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right] \tag{3.0.4}$$

with equation of motion

$$\frac{d^2\phi}{d\tau^2} + \nabla^2\phi = \frac{dV}{d\phi} \tag{3.0.5}$$

and finding the bounce solution indicated by ϕ_b. The bubble nucleation rate can be summarized as follows

$$\frac{\Gamma}{\mathcal{V}} = Ae^{-B} \tag{3.0.6}$$

where the coefficient A has been found in the last section and the exponent B is given by

$$B = S(q_b) - S(\phi_{fv}) \tag{3.0.7}$$

where $S(q_b)$ is the action for the bounce solution and

$$S(\phi_{fv}) = \int d\tau d^3x V_{fv}. \tag{3.0.8}$$

We must require that for $\tau_{init} = +\infty$ and $\tau_{fin} = -\infty$ we have

$$\phi(x, \pm\infty) = \phi_{fv}. \tag{3.0.9}$$

This requirement means that the bounce solution finds its beginning and end in the false vacuum. We also require that

$$\phi(|x|, \tau) = \phi_{fv}. \tag{3.0.10}$$

which means that the bounce solution is localized in the region of the true vacuum and around it one finds the false vacuum. Both the motion equations and these constraint conditions exhibit a symmetry of O(4)so it makes sense to look for solutions with this symmetry in mind. To do this, it is convenient to change the variable s defined as

$$\rho = \sqrt{\tau^2 + r^2}. \tag{3.0.11}$$

The equation of motion becomes

$$\phi'' + \frac{3}{\rho}\phi' = \frac{dV}{d\phi} \tag{3.0.12}$$

with primes denoting differentiation with respect to s. The boundary conditions assume the form

$$\phi(\infty) = \phi_{fv} \tag{3.0.13}$$

and if we want to prevent the solution from being singular we must require

$$\phi'(0) = 0. \tag{3.0.14}$$

Now let us examine some conditions on the form of the potential that realize some calculable approximations. Let us begin with a symmetrical potential

$$V_1(\phi) = V_1(-\phi) \tag{3.0.15}$$

whose minima are set at $\pm a$. A possible form for this kind of potential is

$$V_1 = \frac{\lambda}{8}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 \tag{3.0.16}$$

with $a^2 = \frac{\mu^2}{\lambda}$. We could now consider the breaking of the previous symmetry in the case where

$$\epsilon \equiv V_{fv} - V_{tv} \tag{3.0.17}$$

is very small if compared to the value of the barrier of $V(\phi)$. This situation is known as thin-wall approximation.
We could write this new potential as

$$V = V_1 + \frac{2\epsilon}{a}(\phi - a). \tag{3.0.18}$$

We can use an analogy from classical mechanics to understand what happens [16]. The idea is to consider equation 3.0.12 as the equation of motion of a particle subject to a potential $-V(\phi)$ as shown in Fig. 3.2.

Let us consider a sphere that is in the vicinity of $\phi_{fv} = \phi_-$. After a big time, let us say for $\rho = R$ the sphere will start to move and descend along the valley of potential and will reach ϕ_+ infinitely in time. Using this mechanical analogy we can consider the bounce solution as a sphere of radius R sliding along the potential with a thin wall separating the two minima. For s near R therefore we can neglect in the motion equation the term associated with the viscous force and we can also neglect the part of the potential V that depends on r; the motion equation is therefore reduced to

$$\frac{d^2\phi}{d^2r} = V'(\phi). \tag{3.0.19}$$

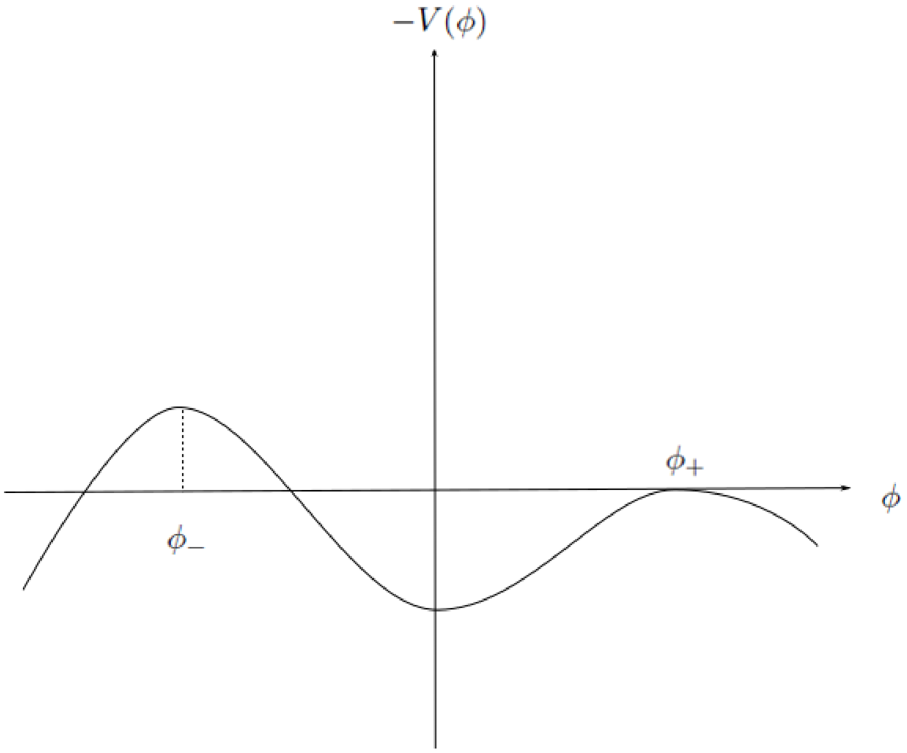

Figure 3.2 Potential of the equation of motion 3.0.12

This last equation is well known and represents the soliton equation, whose solution is

$$r = \int_0^{\phi_1} \frac{d\phi}{[2V_+(\phi)]^{1/2}} \tag{3.0.20}$$

and in terms of ϕ

$$\phi_1 = a\, tanh\left(\frac{1}{2}\mu x\right). \tag{3.0.21}$$

So we can summarize the shape of ϕ as follows

$$\phi = \begin{cases} -a & \rho << R \\ \phi_1(r-R) & \rho \simeq R \\ +a & \rho >> RR \end{cases} \tag{3.0.22}$$

The contribution given by the snapshot is found only in the second line, that is for $\rho \simeq R$, because we are on the dividing edge between the false and the true void and we know that the instant solution is the one that connects in a non trivial way the true and the false vacuum. The total Euclidean action is

$$S_E = 2\pi^2 R^3 \sigma - \frac{1}{2}\pi^2 R^4 \epsilon \tag{3.0.23}$$

where σ is defined as the soliton's energy

$$\sigma = \int dx\{\frac{1}{2}(\phi')^2 + [V(\phi) - V_{fv}]\} = \int_{\phi_{fv}}^{\phi_{tv}} \sqrt{2[V(\phi) - V_{fv}]}. \tag{3.0.24}$$

The first contribution in the Euclidean action S_E comes from the wall while the second refers to the bubble interior. As stated before R is the radius of the bubble and it can be found varying the action and requiring that it is a stationary point:

$$\frac{dS_E}{dR} = 6\pi^2 R^2 \sigma - 2\pi^2 R^3 \epsilon \tag{3.0.25}$$

which gives

$$R_b = \frac{3\sigma}{\epsilon}. \tag{3.0.26}$$

Notice that in 3.0.25 the first term could be considered as a surface tension term. Replacing in the action S_E

$$B = S_E = \frac{27\pi^2\sigma^4}{2\epsilon^3}. \tag{3.0.27}$$

If we calculate the second derivative of the action we find

$$\left.\frac{d^2 S}{dR^2}\right|_{R_b} = -\frac{18\pi^2\sigma^2}{\epsilon} < 0 \tag{3.0.28}$$

and this tells us that R_b actually maximizes the action.

We have seen how the bounce solution determines the characteristics of the nucleation rate. It also contains information about the evolution of the bubble after nucleation. We indicate with (x, t) the bounce solution in Euclidean space and with (x, t) the solution in Minkowskian space; respectively we will have

$$\frac{d^2\phi_E}{d\tau^2} + \nabla^2\phi_E - \frac{dV}{d\phi} = 0 \tag{3.0.29}$$

and

$$-\frac{d^2\phi_M}{d\tau^2} + \nabla^2\phi_M - \frac{dV}{d\phi} = 0 \tag{3.0.30}$$

with the constraints

$$\left.\frac{d\phi_E(x,\tau)}{d\tau}\right|_{\tau=0} = 0 \tag{3.0.31}$$

$$\left.\frac{d\phi_M(x,t)}{d\tau}\right|_{t=0} = 0. \tag{3.0.32}$$

The evolution equation is nothing but the analytical continuation from Euclidean to Minkowskian, and therefore also the bounce solution can be considered as an analytical continuation from Euclidean to Minkowskian. This tells us that the growth of the bubble after nucleation is the same whatever the Lorentzian observer is. The O(4) invariance for the bounce solution becomes O(3,1)invariance for the Minkowskian. As the bubble expands it describes the paraboloid

$$|r^2| - R^2 = t^2 \tag{3.0.33}$$

The expansion of the bubble also involves a transport of energy density. We can try to estimate it. We just have to calculate the Lorentz-boosting of the surface tension term of the wall at rest. From this we get

$$E_{wall} = 4\pi R^2 \frac{\sigma}{\sqrt{1-v^2}} = \frac{4\pi}{3} R^3 \epsilon. \tag{3.0.34}$$

This result also has an interesting interpretation. It can be considered as the latent heat released following the conversion of the false vacuum into a true vacuum. This energy has contributed to increase the kinetic energy of the wall, and this will have fundamental implications when we talk about gravitational waves generated by phase transitions of this type and especially with regard to the definition of the calculable physical parameters that characterize the phase transition and measurable thanks to the detection of the gravitational wave spectrum.

We can ask ourselves the problem of considering our system immersed in a thermal bath and then consider the effects of the temperature $T = \frac{1}{\beta}$. We have to imagine the system divided into two levels, one of false overpopulated vacuum and one of true underpopulated vacuum. The goal is to detect what the effects of temperature are in terms of decay rates. Therefore we can adopt a path integral approach. Let us therefore consider the partition function

$$Z = e^{-\beta F} = \sum_j e^{-\beta E_j} = \int \mathcal{D}\phi e^{-S_E(\phi)} \tag{3.0.35}$$

As we are considering being placed on the false vacuum the E_j weights and the free energy F will acquire an imaginary part, which will make its contribution in the decay rate

$$\Gamma = -2ImF = -2Z^{-1} = -2Z^{-1} \sum_j e^{-\beta E_j} Im\, E_J. \tag{3.0.36}$$

The introduction of temperature into the system has the effect of returning a decay rate as the thermally weighted average([19, 18]).

In the previous section we calculated the exponential factor of the false vacuum decay rate. Now we set ourselves the goal of casting the pre-factor. The treatment is initially referred to a system with only one degree of freedom, then it will be possible to extend it to the case of a field theory. We use the path integral approach to find the probability amplitude to find a particle at position x_f at time T/2 which started from position x_i at time -T/2. This amplitude is given by

$$\langle x_f | e^{-\frac{HT}{\hbar}} |x_i\rangle = \int \mathcal{D}q(\tau) e^{-\frac{S_E(q)}{\hbar}}. \tag{3.0.37}$$

Now we can insert the closure relation of a complete set of eigenstates

$$H|n\rangle = E_n |n\rangle \tag{3.0.38}$$

so we have

$$\langle x_f | e^{-\frac{HT}{\hbar}} |x_i\rangle = \sum_n e^{-\frac{E_n T}{\hbar}} \langle x_f|n\rangle \langle n|x_i\rangle. \tag{3.0.39}$$

We are interestied in finding the lowest level of energy which is the only one that survives in the limit of large T, so

$$\langle x_f | e^{-\frac{HT}{\hbar}} |x_i\rangle = \sum_n e^{-\frac{E_n T}{\hbar}} \langle x_f|n\rangle \langle n|x_i\rangle \xrightarrow[T\to\infty]{} e^{-\frac{E_0 T}{\hbar}} \langle x_f|0\rangle \langle 0|x_i\rangle \tag{3.0.40}$$

where $|0\rangle$ is the lowest state of the system. We can now approximate the path integrals using Gaussian integrals about the stationary point. If we consider a configuration

$$q(\tau) = \bar{q(\tau)} + \sum_n c_n \psi_n(\tau) \tag{3.0.41}$$

with τ eigenfunction, with eigenvalue λ_n, of the operator

$$\left.\frac{\partial^2 S}{\partial q(\tau)\partial q(\tau')}\right|_{q=\bar{q(\tau)}} = -\frac{d^2}{d^2\tau} + V''(\bar{q}(\tau)) \equiv S''(\bar{q}) \tag{3.0.42}$$

and then we perform a change of variable

$$\mathcal{D}q = \prod_n \frac{dc_n}{\sqrt{2\pi}}. \tag{3.0.43}$$

Let us assume for now that all the eigenvalues λ_n are positive; in the following we will fix this statement. Using this expansion we can now perform the Gaussian integration and find the contributions to the path integral

$$\begin{aligned} I &= \int \prod_n \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{S(\bar{q})+\frac{1}{2}\sum_k \lambda_k c_k^2+\dots}{\hbar}} \\ &= e^{-S(\bar{q})}[detS''(\bar{q})]^{-1/2}(1+\mathcal{O}(\hbar)). \end{aligned} \tag{3.0.44}$$

We can apply this result to two configurations:the instantons and the bounce solutions. In the first case we can consider a double well. Let us suppose we are to calculate the quantity

$$\langle a| e^{-\frac{HT}{\hbar}} |a\rangle \tag{3.0.45}$$

which corresponds to the constant trivial solution $q_0(\tau) = a$. Providing that V(q)=0 in correspondence of the minimum we have

$$I_0 = (detS''(q_0))^{1/2}. \tag{3.0.46}$$

The case for

$$\langle -a| e^{-\frac{HT}{\hbar}} |a\rangle$$

correspond to the instanton solution and the contribution to the path integral is

$$I = e^{-S_1}(detS''(q_1))^{1/2}. \tag{3.0.47}$$

where S_1 is the Euclidean action for the instanton configuration.

At this point you can see that we are faced with a problem: the operator3.0.42, has zero eigenvalues, in fact just consider

$$-\frac{d^2}{d^2\tau} + V'(q(\tau)) = 0 \tag{3.0.48}$$

and let us act with the operator

$$\partial_\tau$$

we find that

$$\psi_0(\tau) = (S_E(q_1(q))^{-1/2}\frac{dq_1}{d\tau} \tag{3.0.49}$$

is a zero mode which correspond the broken τ-translation symmetry. If we integrate over the coefficient c_0 corresponding to the zero mode, we would end up with a divergent quantity.// This problem can be solved replacing the integration over the zero mode coefficients with the integration over a collective coordinate z that encodes the information about the center of the instanton. // This change of variable implies that

$$dq = \frac{dq_1}{dz}dz = \psi_0 dc_0 \tag{3.0.50}$$

which, comparing with 3.0.49, gives

$$dc_0 = (S_E(q_1))^{-1/2}dq_1. \tag{3.0.51}$$

This means that we can trade the integration over the zero eiegen-modes with an integration over the collective coordinate z. Thus the total contribution to the path integral which takes into account also the zero modes is

$$I_1 = e^{-\frac{S_E(q_1)}{\hbar}} (det' S''_E(q_0))^{-1/2} KT \tag{3.0.52}$$

where det' indicates that only the non-zero modes are included, and K is defined as

$$K = \left(\frac{N}{2\pi}\right)^{1/2} \left[\frac{det' S''(q_b)}{det S''(q_0)}\right]^{-1/2} \tag{3.0.53}$$

There is still another configuration that must be considered. We can consider as approximate stationary points a set of n alternating instantons and anti-instantons separated by an interval τ. We can consider a configuration on n instantons and anti-instantons q_n. The corresponding action is

$$S(q_n) = nS_1 \tag{3.0.54}$$

that is n times the action of a configuration with one instanton.
Each of these instanton and anti-instanton has zero modes, so we perform the same change of variable. We introduce the variable z_j for each instanton so that we gain the Jacobian factor $\left(\frac{N}{2\pi}\right)^{n/2}$ and after integration over each of the z_j variables we have

$$\int_{-T/2}^{T/2} dz_1 \int_{z_1}^{T/2} dz_2 1 ... \int_{z_{n-1}}^{T/2} dz_n = \frac{T^n}{n!}. \tag{3.0.55}$$

The final result is

$$I_n = e^{-nS_1} [det S''(q_0)]^{-1/2} K^n \frac{T^n}{n!}. \tag{3.0.56}$$

Adding the contributions given by the stationary point and the approximated ones we can write the amplitude for the transitions

$$\begin{aligned} \langle a| e^{-HT} |a\rangle &= \sum_{even n} I_n \\ &= [det S''(q_0)]^{-1/2} \sum_{even n} \frac{[e^{-S_1} KT]^n}{n!} \\ &= [det S''(q_0)]^{-1/2} cosh[e^{-S_1} KT]. \end{aligned} \tag{3.0.57}$$

and

$$\begin{aligned} \langle -a| e^{-HT} |a\rangle &= \sum_{even\, n} I_n \\ &= [det S''(q_0)]^{-1/2} \sum_{odd\, n} \frac{[e^{-S_1} KT]^n}{n!} \\ &= [det S''(q_0)]^{-1/2} sinh[e^{-S_1} KT]. \end{aligned} \tag{3.0.58}$$

We can also give the expression for the splitting of the two lowest levels

$$e^{(E_- - E_+)T} = \exp\left[2KTe^{-S_1}\right] \tag{3.0.59}$$

and in particular

$$\Delta = E_- - E_+ = 2Ke^{-S_1}. \tag{3.0.60}$$

The other configuration is the bounce solution. Let us consider a double-hole potential where there are two minima, one local and the other absolute. If we start from an initial state $|\Psi(0)\rangle = |L\rangle$ and then we find the evolution $|\Psi(t)\rangle$, the result is an oscillating behaviour, because of the shape of the potential.
Let us now imagine to make the absolute minimum broader than the other and to go in the limit one of which the width of this minimum to infinity. In the case of a very wide minimum starting from a $|L\rangle$ state, we would have a different behaviour for $|\Psi(0)\rangle = |L\rangle$: we would have an exponentially decreasing behaviour. We are then allowed to think of $|L\rangle$ as a metastable state, so it is characterized by a complex energy which has the following relation with the decay width

$$ImE = -\frac{\Gamma}{2}. \tag{3.0.61}$$

The matrix element for the bounce solution, with metastable minimum set at q=a, is

$$\langle a| e^{-HT} |a\rangle = \sum_n e^{-E_n T} |\langle a|a\rangle|^2 = \int \mathcal{D}q(\tau) e^{-S_E(q)}. \tag{3.0.62}$$

For the trivial solution $q_0(\tau) = a$ we have the usual contribution

$$I_0 = (detS''_E(q_0))^{-1/2} \tag{3.0.63}$$

while for the bounce solution we have the usual result

$$I_1 = e^{-\frac{S_E(q_1)}{\eta bar}} (det' S''_E(q_0))^{-1/2} KT \tag{3.0.64}$$

with

$$K = \left(\frac{S_E(q_b)}{2\pi}\right)^{1/2} \left[\frac{det' S''_E(q_b)}{detS''_E(q_0)}\right]^{-1/2}. \tag{3.0.65}$$

We must remember the presence of a zero eigenvalue associated with the operator S''_E. Because of the presence of this zero mode, the auto-function has a node. S''_E has the form of a Schoedinger operator, so we know the general property that if a Schroedinger operator has a node then there must necessarily be another auto-function associated with a negative auto-value corresponding to a lower energy level. So the square root in the 3.0.64 will produce imaginary values. This is related to the existence of a decay rate for the false vacuum. Problems arise when we integrate over the negative eigenvalues. In order to fix this problem we can consider the following integral

$$J = \int_{+\infty}^{-\infty} dc \frac{1}{\sqrt{2\pi}} e^{-S(c)} \tag{3.0.66}$$

where c is the parameter that controls the field configurations $x(\tau)$. With the given contour the integral is divergent. We can make it well defined by deforming the path of integration; in this way the integral acquires an imaginary part that with saddle approximation gives:

$$ImJ = Im \int_{b}^{b+i\infty} \frac{dc}{\sqrt{2\pi}} e^{-\frac{[S_E(b)-\frac{1}{2}S_E''(b)(c-b)^2+...]}{\hbar}} = \frac{1}{2} e^{-\frac{S_E(b)}{\hbar}}. \tag{3.0.67}$$

Now we can give the correct form for the factor K

$$k = \frac{i}{2} \left(\frac{S_E(q_b)}{2\pi} \right)^{1/2} \left| \frac{det' S_E''(q_b)}{det S_E''(q_0)} \right|^{-\frac{1}{2}}. \tag{3.0.68}$$

and replacing in the decay rate

$$\Gamma = -2ImE_0 = \left(\frac{S_E(q_b)}{2\pi} \right)^{1/2} \left| \frac{det' S_E''(q_b)}{det S_E''(q_0)} \right|^{-\frac{1}{2}} e^{-S(q_b)}. \tag{3.0.69}$$

Chapter 4

Relic axions and neutrinos

In the first chapter we have assumed that during the earliest stages of the Universe's life, the various species that constituted it were in thermal equilibrium with each other because their rate of interaction was greater than the H rate of expansion of the Universe. However, there were phases in which this equilibrium was disturbed. The departure from equilibrium is what determines the abundance of the relics of some species of constituents. For this chapter we will follow mainly the discussion in[1] and we will consider the calculation of relic abundances for various species in particular for the neutrino and for the axion. We will focus in the discussion of the CP-strong problem closely related to axion which is its solution. We will present a brief review of the context in which the axion field is placed in particular we will refer to how the calculation of the anomaly has to do with this hypothetical particle. We will also present at the end of the chapter some cosmological implications of the existence of axions and we will derive some constraints on their mass and on the value of the coupling constant. Let us consider any species X; its number density at thermodynamic equilibrium will be given by

$$n_{X,eq} = \frac{g}{(2\pi)^3} \int d^3p \frac{1}{e^{E(p)/T} \pm 1} \tag{4.0.1}$$

con g number of degrees of freedom, $E(p) = \sqrt{p^2 + m^2}$ and as usual the sign + is for fermions and the sign - is for bosons. We can consider the relativistic limit, i.e. $T >> m_X$, and we will have for fermions and bosons respectively

$$n_{X,eq} = \frac{3\zeta(3)}{4\pi^2} g T^3 \tag{4.0.2}$$

and

$$n_{X,eq} = \frac{\zeta(3)}{\pi^2} g T^3. \tag{4.0.3}$$

where $\zeta(3)$ is the Riemann zeta function. We can also calculate the energy density and pressure associated with the thermal bath of particles of species X

$$\rho X, eq = \frac{g}{(2\pi)^3} \int d^3p E(p) \frac{1}{e^{E(p)/T} \pm 1} \tag{4.0.4}$$

and

$$pX,eq = \frac{g}{(2\pi)^3}\int d^3p \frac{|p|^2}{3E(p)}\frac{1}{e^{E(p)/T}\pm 1}. \tag{4.0.5}$$

In the relativistic limit, the above relations are greatly simplified, in fact

$$\rho = 3p = \frac{\pi^2}{30} g_{*,T} T^4 \tag{4.0.6}$$

with

$$g_{*,T} = \sum_{bosons} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{fermions} g_i \left(\frac{T_i}{T}\right)^4 \tag{4.0.7}$$

where the index i refers to the i-th species present at a specific temperature T_i. The entropy at equilibrium per co-moving volume is

$$s \equiv \frac{S}{V}\frac{\rho + p}{T}. \tag{4.0.8}$$

As before we are interested in the relativistic limit. In this the entropy density has the following trend

$$s = \frac{2\pi^2}{45} g_{*S,T} T^3 \tag{4.0.9}$$

but in this case

$$g_{*S,T} = \sum_{bosons} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{fermions} g_i \left(\frac{T_i}{T}\right)^3 \tag{4.0.10}$$

As mentioned previously in the chapter (cosmology) the co-moving entropy is conserved, so it is convenient to define the abundance of a certain species X, rescaling the number density with the co-moving entropy

$$Y_X \equiv \frac{n_X}{s}. \tag{4.0.11}$$

Combining the expression 4.0.9 with the definition of the number densities for fermions and bosons we obtain respectively

$$Y_{X,eq,T} = \frac{45\zeta(3)}{2\pi^4}\frac{\frac{3g}{4}}{g_{*S,T}} = 0.278\frac{\frac{3g}{4}}{g_{*S,T}} \tag{4.0.12}$$

and

$$Y_{X,eq,T} = \frac{45\zeta(3)}{2\pi^4}\frac{g}{g_{*S,T}} = 0.278\frac{g}{g_{*S,T}}. \tag{4.0.13}$$

The relics abundances also contribute to the total energy density of the Universe ρ_0. These are calculated relative to the critical energy density defined as

$$\rho_c = \frac{3H_0^2}{8\pi G_N} = 10.54 h^2 keV cm^{-3} \tag{4.0.14}$$

with $H_0 = 100hkms^{-1}Mpc^{-1}$ and h encodes the uncertainty associated with the Hubble parameter measurement

$$h = 0.71^{+0.04}_{-0.03}. \tag{4.0.15}$$

Therefore we define the total energy density in a dimensionless version

$$\Omega_0 \equiv \frac{\rho_0}{\rho_c}. \tag{4.0.16}$$

The same can be done for all species X, having defined by $\rho_{X,0}$ the energy density associated with species X at the current time. The dimensionless density of species X will therefore be defined as.

$$\Omega_X \equiv \frac{\rho_{X,0}}{\rho_c}. \tag{4.0.17}$$

Their sum will give

$$\sum_X \Omega_{X,0} = \Omega_0. \tag{4.0.18}$$

According to WMAP ([7])

$$\Omega_0 = 1.02 \pm 0.002. \tag{4.0.19}$$

We now derive a lower bound on the mass of a given species X. We consider the energy density associated with species X, expressed in terms of the mass m_X

$$\rho_{X,0} = n_{X,0}m_X = Y_{X,0}s_0m_X \tag{4.0.20}$$

in this way, since

$$\Omega_{X,0} < \Omega_0 \tag{4.0.21}$$

we obtain

$$m_X < \frac{\rho_c\Omega_0}{s_0Y_{X,0}}. \tag{4.0.22}$$

4.1 Relic neutrino abundances

We now attempt to calculate relic abundances for primordial neutrinos. During the early life of the Universe neutrinos were held in thermodynamic equilibrium by processes of the type

$$\nu\, e \to \nu\, e \tag{4.1.1}$$

and

$$\nu\bar{\nu} \to e\bar{e}. \tag{4.1.2}$$

The total interaction rate for neutrinos is expressed as ([20])

$$\Gamma_\nu = \frac{2}{\hbar}\frac{G_F^2 T^5}{(\hbar c)^6}. \tag{4.1.3}$$

The expansion rate of the Universe is given by

$$H = \sqrt{\frac{8\pi G_N \rho}{3}} = \sqrt{\frac{4\pi^3 G_N g_{*,T}}{45}} T \tag{4.1.4}$$

and if $\Gamma_\nu > H$ then there is thermodynamic equilibrium.

If however $T \simeq 1MeV$ then $\Gamma_{int} \simeq H$ and below this temperature the neutrinos decouple as the expansion becomes dominant and this gives rise to a primordial relic of decoupled neutrinos. This discussion is analogous to the case of the decoupling of photons from matter that gave rise to the CMB. We then define $T_{dec} \simeq 1MeV$ as the decoupling temperature of neutrinos, and we also define the current abundance of decoupled neutrinos as follows

$$Y_{\nu,T_{dec}} = \frac{n_\nu}{s} = 0.278\frac{g_\nu}{g_{*S,T_{dec}}} \tag{4.1.5}$$

where $g_n u = \frac{3}{2}$. We now try to calculate a lower bound on the neutrino mass by exploiting the 4.0.22 constraint. We first calculate s_0

$$s_0 = \frac{2\pi^2}{45} g_{*S,T_{dec}} T_0^3 \tag{4.1.6}$$

and as contributions we consider photons having g=2 and three neutrino families having g=2 Consider a chiral neutrino left, we will have

$$g_{*S,T_{dec}} = 2 + \frac{7}{8} \times 3 \times 2 \left(\frac{T_\nu}{T_0}\right)^3. \tag{4.1.7}$$

Of course since we are considering decoupled neutrinos it will be $T_0 \neq T_\nu$. We therefore need an estimate of the ratio $\frac{T_\nu}{T_0}$. We consider that the electron-positron annihilation processes do not affect neutrinos in the sense that these processes transfer entropy only to photons and not to neutrinos because the latter are decoupled. Therefore for $T << m_e$ only photons will be in thermodynamic equilibrium so $g_* = 2$. The comoving entropy must be conserved and this implies that the temperature of photons increase by a factor $\left(\frac{11}{4}\right)^{1/3}$, so to compensate it must be

$$\frac{T_\nu}{T_0} = \left(\frac{4}{11}\right)^{1/3} \tag{4.1.8}$$

and

$$g_{*S,T_0} = \frac{43}{11} \tag{4.1.9}$$

so that setting $T_0 = 2.725\,K$

$$s_0 = 2889 cm^{-1}. \tag{4.1.10}$$

Now using WMAP data ([7])

$$\Omega_0 h^2 = 0.51 \pm 0.04 \tag{4.1.11a}$$

$$\Omega_{\nu\bar{\nu}} H_0^2 = \frac{8\zeta(3)}{3\pi} \frac{g_\nu g_{*S,T_0}}{g_{*S,T_{dec}}} G_N T_0^3 m_\nu \tag{4.1.11b}$$

where

$$\Omega_{\nu\bar{\nu}} \equiv \frac{\rho_{\nu,0}}{\rho_c} \tag{4.1.12}$$

we reach, using the constraint 4.0.22

$$m_\nu < 48 eV \tag{4.1.13}$$

or

$$m_\nu < 12.7 eV \tag{4.1.14}$$

depending on whether one uses the stronger constraint on the uncertainty h (see4.0.15).

4.2 Axions

A non-trivial homotopy class can be associated with both electro-weak and strong forces. Therefore there is a non-trivial homotopy group associated with SU(3)

$$\pi_3(SU(3)) = Z. \tag{4.2.1}$$

The homotopy group is associated with solutions of non-perturbative type, known as instantons[21]. The instanton interpolating the different vacua in the SU(3) gauge theory or QCD satisfies the action in the Euclidean space-time

$$S_E = \frac{8\pi^2 |q|}{g_3^2} \tag{4.2.2}$$

where $|q|$ is the so called Pontryagin index, defined as

$$\frac{g_3^2}{16\pi} \int d^4x tr(G^{\mu\nu} G_{\mu\nu}). \tag{4.2.3}$$

g_3 is the SU(3) gauge coupling constant and $G^{\mu\nu}$ is the field strength associated with SU(3). In this context, there are thus multiple topologically distinct vacuum states, and the Pontryagin index q classifies these states. An instanton can be viewed as a solution to the classical equations of motion that interpolates these states, so one can represent a state called a $\theta - vacuum$ state as follows

$$|\theta\rangle = \sum_q e^{-iq\theta} |q\rangle . \tag{4.2.4}$$

It turns out to be possible, therefore, to include an additional term in the Lagrangian of QCD that takes into account these field configurations:

$$\mathcal{L}_{eff} = \mathcal{L}_{QCD} + \frac{\theta g_3^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}. \tag{4.2.5}$$

which is CP-violating.
A term similar to the θ-term appears if redefinitions of quark fields are made under axial U(1)

$$U(1)_A : q \to e^{i\alpha\gamma_5} q. \tag{4.2.6}$$

The axial current associated with the transformation

$$j^5_\mu = \sum_q \bar{q}\gamma_\mu\gamma_5 q = \sum_q (\bar{q}_R\gamma_\mu q_R - \bar{q}_L\gamma_\mu q_L) \tag{4.2.7}$$

is said to be anomalous since it is not conserved and its 4-divergence can be written in a form analogous to the θ-term

$$\partial^\mu j^5_\mu = \frac{N_f g_3^2}{16\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu} \tag{4.2.8}$$

with N_f as number of flavours.

Thus if one were to redefine the quark fields by an axial U(1) transformation setting the $\alpha = -\frac{\theta}{2N_f}$-parameter. The θ-term would be successfully eliminated if the quarks were massless. Actually just because of the presence of the mass terms in the Lagrangian

$$-\bar{q}_{Li} M_{ij} q_{Rj} - \bar{q}_{Ri} M^\dagger_{ij} q_{Lj} \tag{4.2.9}$$

redefinition by axial transformation also on these mass terms

$$U(1)_A : M \to e^{2i\alpha} M M^\dagger \to e^{-2i\alpha} M^\dagger. \tag{4.2.10a}$$

In the end the effect of the transformation of the mass terms is to give

$$\bar{\theta} = \theta + 2N_f \, arg(detM) \tag{4.2.11}$$

instead of removing the θ-term. $\bar{\theta}$ is the effective theta-term of the theory, also called QCD vacuum angle. The presence of this term which cannot be a priori excluded from the Lagrangian produces physical effects. If this term is not null then it should contribute to the electric dipole of the neutron d_n, and the upper bound has been measured, so as to fix

$$\bar{\theta} \lesssim 10^{-10}. \tag{4.2.12}$$

One of the biggest unsolved problems in modern physics is why Nature chose this small value: it is the so-called 'strong CP problem'. Each value of *thēta* would lead to a different version of QCD, so we cannot exclude any a priori.

It has been observed that even an electro-weak theory can admit terms similar to the θ-term such as

$$\mathcal{L}_\theta = \frac{\theta g_2^2}{32\pi^2} W^a_{\mu\nu} \tilde{W}^{a\mu\nu} + \frac{\theta_1 g_1^2}{32\pi^2} B_{\mu\nu} \tilde{B}^{\mu\nu}. \tag{4.2.13}$$

The term associated with θ_1 can be rewritten as a surface term and because of the trivial topology it describes can be neglected. Differently for the topological term associated with θ_2. It describes a topological structure that cannot be neglected. The associated surface term cannot be neglected even if it is written as a total divergence.

We also know that the symmetries associated with U(1) are conserved except for one, i.e. $\frac{1}{3}B - L_l$ where B is the baryonic number and L_l the leptonic one. The latter symmetry is anomalous so we can think of applying a $U(1)_B$ transformation on the quarks so as to eliminate the term associated with θ_2. This procedure inspired a possible solution to the CP strong problem. Peccei and Quinn ([22, 23, 24]) proposed to introduce an extra symmetry of type U(1), henceforth denoted $U(1)_{PQ}$, which can compensate for the term arg det M and thus adds to the algebraic structure of the Standard Model (SM). However, this procedure cannot be applied with the minimal field content of the Standard Model, since redefining, for example, the Higgs doublet, one would have, say, a $e^{i\delta}$ phase for the down component and a $e^{-i\delta}$ phase for the up component, which compensate each other by not changing arg det M. Thus the introduction of this new symmetry $U(1)_{PQ}$ requires the introduction of new scalar fields. This new field has been called axion: a(x). On it acts the transformation under $U(1)_{PQ}$:

$$\frac{a(x)}{v_{PQ}} \to \frac{a(x)}{v_{PQ}} + \alpha \tag{4.2.14}$$

where $\mathcal{L}_{PQ}$ is the VEV involved in the spontaneous breaking of $U(1)_{PQ}$ and α is the transformation parameter. The transformation under $U(1)_{PQ}$ also acts on chiral fermions

$$f(x) \to e^{-ix_f \alpha} f(x) \tag{4.2.15}$$

where x_f is the charge associated to the PQ group. The current generated by this transformation is

$$j^{(PQ)\mu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \alpha)} = v_{PQ} \partial^\mu a + \sum_f x_f \bar{f} \gamma^\mu f. \tag{4.2.16}$$

Classically this current is conserved, but being chiral, just as in the case of the current given by the transformation $U(1)_A$, it is anomalous, in fact

$$\partial^\mu j_\mu^{(PQ)} = \xi_3 \frac{g_3^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \xi_2 \frac{g_2^2}{32\pi^2} W^a_{\mu\nu}\tilde{W}^{a\mu\nu} + \xi_i \frac{g_1^2}{32\pi^2} B_{\mu\nu}\tilde{B}^{\mu\nu} \tag{4.2.17}$$

where ξ $(i = 1, 2, 3)$ is a constants that depends on $U(1)_{PQ}$ charges of the fermionic fields. The axion a(x) plays a role in the coupling of fermionic fields with scalar fields, i.e., in Yukawa couplings. A local transformation of a fermionic field involves the following:

$$f(x) \to \exp\left[\frac{-ia(x)x_f}{v_{PQ}}\right] \tag{4.2.18}$$

in this way axion is removed from the Yukawa terms. Given the nature of the group $U(1)_{PQ}$ new interactions are naturally introduced. First of all by operating a local transformation interactions, containing derivatives, are generated with the axion

$$\bar{f}\gamma^\mu i\partial_\mu f \to \bar{f}\gamma^\mu i\partial_\mu f + \frac{x_f}{v_{PQ}}(\partial_\mu a)\bar{f}\gamma^\mu f \tag{4.2.19}$$

and, since $U(1)_{PQ}$ is anomalous, the following interactions are generated.

$$\mathcal{L}_{anom} = \frac{a(x)}{v_{PQ}}\left[\xi_3 \frac{g_3^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu} \xi_2 \frac{g_2^2}{32\pi^2} W^a_{\mu\nu}\tilde{W}^{a\mu\nu} + \xi_1 \frac{g_1^2}{32\pi^2} B^a_{\mu\nu}\tilde{B}^{a\mu\nu}\right]. \tag{4.2.20}$$

Summing up all these observations, the effective Lagrangian that is obtained is

$$\begin{aligned}\mathcal{L}_{eff} = \mathcal{L}_{sm} + &\left[\bar{\theta} + \frac{\xi_3}{v_{PQ}}a(x)\right]\frac{g_3^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \\ &+ \left[\theta_2 + \frac{\xi_2}{v_{PQ}}a(x)\right]\frac{g_2^2}{32\pi^2} W^a_{\mu\nu}\tilde{W}^{a\mu\nu} + \left[\theta_1 + \frac{\xi_1}{v_{PQ}}a(x)\right]\frac{g_1^2}{32\pi^2} B^a_{\mu\nu}\tilde{B}^{a\mu\nu} + \\ &\frac{1}{2}(\partial_\mu a)^2 + \frac{1}{v_{PQ}}(\partial_\mu a)^2[j^{PQ\mu} - v_{PQ}(\partial_\mu a)]\end{aligned} \tag{4.2.21}$$

The anomaly acts as a potential for the axion, and thus its VEV is no longer arbitrary. In this way it is possible to make the axion a(x) acquire a particular VEV, such that it cancels the θ-term, as shown by Peccei and Quinn

$$\bar{\theta} + \frac{\xi_3}{v_{PQ}}\left\langle\bar{\theta}\middle|a(x)\middle|\bar{\theta}\right\rangle = 0. \tag{4.2.22}$$

Therefore, the real axion turns out to be the physical one

$$\hat{a}(x) \equiv a(x) - \left\langle\bar{\theta}\middle|a(x)\middle|\bar{\theta}\right\rangle. \tag{4.2.23}$$

The Lagrangian can therefore be rewritten in terms of the physical axion

$$\begin{aligned}\mathcal{L}_{eff} = \mathcal{L}_{sm} + \frac{\xi_3 g_3^2}{32\pi^2 v_{PQ}}\hat{a}(x)G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \frac{\xi_2 g_2^2}{32\pi^2 v_{PQ}}\hat{a}(x)W^a_{\mu\nu}\tilde{W}^{a\mu\nu} + \\ + \frac{\xi_1 g_1^2}{32\pi^2 v_{PQ}}\hat{a}(x)B^a_{\mu\nu}\tilde{B}^{a\mu\nu} + \frac{1}{2}(\partial_\mu \hat{a})^2 + \\ + \frac{1}{v_{PQ}}(\partial_\mu \hat{a})^2[j^{PQ\mu} - v_{PQ}(\partial_\mu \hat{a})].\end{aligned} \tag{4.2.24}$$

We are now interested in investigating the physical consequences of introducing an axion field constructed as in 4.2.23. We begin by making some considerations about the mass of the axion. As shown in ([25, 26, 27, 28, 29]), the mass of the axion results to be

$$m_a \simeq 0.62eV \left(\frac{10GeV}{f_a}\right) \tag{4.2.25}$$

where

$$f_a = \frac{v_{PQ}}{\xi_3} \tag{4.2.26}$$

is the parameter that controls the intensity of the coupling between the axion and the gluon in 4.2.24, i.e. its decay constant. In fact considering the effective potential V_{eff} generated by this coupling we have

$$\begin{aligned} m_a^2 = \langle \frac{\partial^2 V_{eff}}{\partial a^2} \rangle &= -\frac{1}{f_a}\frac{g_3^2}{32\pi^2}\frac{\partial}{\partial a}\langle G^a_{\mu\nu}\tilde{G}^{a\mu\nu}\rangle \\ &\sim \frac{\Lambda^4_{QCD}}{f_a^2}\end{aligned} \tag{4.2.27}$$

A scale of the order of magnitude of the electroweak rupture scale associated with f_a, i.e.

$$f_a \sim v \sim 250GeV \tag{4.2.28}$$

would give a very tiny mass

$$m_a \sim 24keV. \tag{4.2.29}$$

This tells us that axions while not mass-less are extremely light. Furthermore, the Lagrangian 4.2.24 predicts a decay for axion of the type

$$a \to \gamma\gamma \tag{4.2.30}$$

because it contains a term like

$$\mathcal{L}_{a\gamma\gamma} = -g_\gamma \frac{\alpha_{em}}{\pi}\frac{a(x)}{f_a}\mathbf{E} * \mathbf{B}. \tag{4.2.31}$$

where g_γ is completely defined by the Peccei-Quinn charges

$$g_\gamma = \frac{\xi_1 + \xi_2}{2\xi_3}. \tag{4.2.32}$$

The Lagrangian 4.2.24 also contains the coupling of axion with fermionic fields

$$\mathcal{L}_f = -\frac{1}{v_{PQ}}(\partial_\mu a) \sum_{\chi=R,L} x_{f_\chi} \bar{f} \gamma_\mu a_\chi f \tag{4.2.33}$$

with a_χ defined as

$$a_\chi = \frac{1}{2}(1 \pm \gamma_5) \tag{4.2.34}$$

where the sign is determined depending on whether $\chi = R, L$ and g_f is defined in terms of right and left PQ charges

$$g_f = x_{f_R} - x_{f_L}. \tag{4.2.35}$$

We know that this axion as just described cannot be observed [29]. The fundamental problem is that in this model both the mass of the axion and its decay constant are constrained to the same quantity, namely the electroweak breaking scale. Generally it is assumed that $f_a \sim v_{EW}$. However in some models we assume that $f_a >> v_{EW}$ and this results in an extremely light axion since

$$m_a \sim f_a$$

is weakly interacting. Several proposals have been made to untie these two quantities and make the PQ symmetry breaking scale independent of the electroweak one. A new singlet under $SU(2) \times U(1)$,σ is introduced and axion emerges as the phase of this electroweak singlet:

$$\sigma = \frac{f_a}{\sqrt{2}} e^{i\frac{a}{f_a}} \tag{4.2.36}$$

This singlet gives rise to two different types of axion depending on whether there is a direct coupling with ordinary leptons and quarks.two cases can be distinguished The first one is the axion known as KSZV axion.. We assume the existence of a new quark X endowed with PQ charge electroweak doublet endowed with electric charge q_X coupled to the complex scalar field σ also endowed with charge under PQ. X couples to σ according to the Lagrangian

$$\mathcal{L}_{KSVZ} = -h\bar{X}_L \sigma X_R - h^* \bar{X} \sigma^\dagger X_L\,. \tag{4.2.37}$$

Coupling with other ordinary quarks emerges indirectly from the anomalous Lagrangian

$$\mathcal{L}_{anomaly} = \frac{a}{f_a}\left[\frac{\alpha_s}{8\pi} G_a^{\mu\nu} \tilde{G}_{a\mu\nu} + 3q_X^2 \frac{\alpha_s}{4\pi} F_{\mu\nu} \tilde{F}^{\mu\nu}\right]. \tag{4.2.38}$$

The second case is the so-called DFSZ axion. In this case quarks and leptons couple directly with the axion (but still via σ), because quarks and leptons in this case are charged under $U(1)_{PQ}$. Now it is necessary to consider two Higgs fields Φ_1 and Φ_2

$$\mathcal{L}_{axion} = k\Phi_1^T C\Phi_2(\sigma^\dagger)^2 + h.c. \tag{4.2.39}$$

4.3 Constraints on axions from astrophysics

We can derive some upper bounds on the mass of the m_a axion from some astrophysical considerations. In fact the emission of particles by stars contributes to their cooling. In particular, the larger f_a the smaller the mass and the less the axion emission will affect the history of the star. By exploiting models of a star' s evolution related to the production processes of various particle species and combining these models with experimental observations, it is possible to derive constraints on the coupling constants. Consider for example the following process

$$\gamma e \to ae \tag{4.3.1}$$

Which is proportional to the constant g_{aee}^2. It is defined as

$$g_{aee} = g_e \frac{m_e}{v_{PQ}} = \frac{x_{e_R} - x_{e_L} m_a m_e}{\xi_3(0.62 \times 10^{16} eV^2)} \tag{4.3.2}$$

From the observations we find that

$$|g_{aee}| \lesssim 0.5 \times 10^{-12}. \tag{4.3.3}$$

from which we derive that

$$\left|\frac{x_{e_R} - x_{e_L}}{\xi_3}\right| m_a \lesssim 0.62 \times 10^{-12} eV \tag{4.3.4}$$

and hence

$$m_a \lesssim 10^{-2} eV \tag{4.3.5}$$

in the case of a DFSZ axion. Another process that can be considered is the Primakoff process

$$\gamma \leftrightarrow a. \tag{4.3.6}$$

In this case, the cross section of the process is proportional to $g_{a\gamma\gamma}^2$

$$g_{a\gamma\gamma} = g_\gamma \frac{\alpha_{em}}{\pi f_a} = \frac{m_a g_\gamma \alpha_e m}{\pi(0.62 \times 10^{16} eV^2)} \tag{4.3.7}$$

from which, thanks to observations we get [30]

$$|g_{a\gamma\gamma}| \lesssim 0.6 \times 10^{-10} GeV^{-1} \tag{4.3.8}$$

so that

$$|g_\gamma| \lesssim 0.16\, eV \tag{4.3.9}$$

from which we can find the minimum value for the mass of the axion (both DFSZ and KSVZ)

$$m_a \lesssim 0.4\, eV. \tag{4.3.10}$$

4.4 Role of the axion in Cosmology

If axion exists it has been produced in the early stages of life of the Universe and therefore its presence should have significant implications on observations. One of the possible methods of production of axions is that of creation and destruction by photo-production or gluon-production

$$\begin{aligned} \gamma q &\leftrightarrow aq \\ gq &\leftrightarrow aq \end{aligned} \tag{4.4.1}$$

These processes are characterized by an absorption rate Γ^T_{abs}, given by

$$\Gamma^T_{abs} = n_T \langle \sigma |v| \rangle_{abs} \tag{4.4.2}$$

where n_T is the number density, σ is the scattering cross section, $|v|$ is the relative velocity between the axion and the target T and the symbol $\langle ... \rangle$ denotes a thermal average. If the expansion rate of the Universe is slow compared to Γ^T_{abs} then these processes reach thermal equilibrium with the axion density given by 4.0.3.

However, if the axions interact with each other too weakly, Γ^T_{abs} is too slow for thermal equilibrium to be reached and so their number density will settle at a lower value than that given at equilibrium. We consider how the abundance of axions, given by the equation 4.0.13, varies via the Boltzmann equation. We choose $g_{a,eff} = 1$

$$\frac{dY_a}{dt} = -\Gamma_{abs}(Y_a - Y_{a,eq}) \tag{4.4.3}$$

integrating out we have

$$Y_a(t) - Y_{a,eq} = (Y_a(0) - Y_{a,eq}) \exp\left(- \int_0^t \Gamma_{abs} dt' \right) \tag{4.4.4}$$

which is the difference between the abundance at time t and its value at thermal equilibrium. It is useful to rewrite it with the change of variable

$$x \equiv \frac{m_N}{T}. \tag{4.4.5}$$

During the radiation dominance the scale factor is

$$a(t) \propto t^{1/2} \tag{4.4.6}$$

so that we notice, for the Hubble parameter

$$H = \frac{1}{2t} \propto T^2 \propto x^{-2} \tag{4.4.7}$$

in the end the abundance of axions can be recast in the form

(4.4.8)

We now analyze the product $\langle\sigma|v|\rangle_{abs}$. If we consider N nucleons as the target T for axions and consider a phase below the quark-hadrons transition, in the non-relativistic limit we would have

$$n_N = g_N \left(\frac{m_N^2}{2\pi x}\right)^{3/2} e^{-x} \tag{4.4.9}$$

La cross-section è

$$\langle\sigma|v|\rangle_{abs} = g_{aNN}^2 x^{-2} m_\pi^{-2} \tag{4.4.10}$$

and we choose

$$g_{aNN} \approx \frac{m_N}{f_{PQ}}. \tag{4.4.11}$$

The explicit form for H in the radiation-domination phase is

$$H = \left(\frac{8\pi\rho}{3m_P^2}\right)^{1/2} = \frac{2\pi}{3}\left(\frac{\pi g_*}{5}\right)^{1/2} \frac{m_N^2}{x^2 m_P} \tag{4.4.12}$$

Combining all this information we have

$$\begin{aligned}\frac{\Gamma_{abs}(x)}{H(x)} &= \frac{3g_N}{2\pi^3}\left(\frac{5}{8g_*}\right)^{1/2}\left(\frac{m_N^{3/2} m_P^{1/2}}{f_{PQ} m_\pi}\right)^2 x^{3/2} e^{-x} \\ &\approx \left(\frac{10}{g_*}\right)^{1/2}\left(\frac{m_a}{1.2\times 10^{-3} eV}\right)^2 x^{3/2} e^{-x}.\end{aligned} \tag{4.4.13}$$

We can also estimate the final value of relic abundance, above the quark-hadron phase transition

$$\begin{aligned} Y_a(\infty) &= Y_{a,eq}\Bigg\{1 - \left(1 - \frac{Y_a(0)}{Y_{a,eq}}\right) \\ &\times \exp\left[-\frac{3g_N}{2\pi^3}\left(\frac{5}{8g_*}^{1/2}\left(\frac{m_N^{3/2}m_P^{1/2}}{f_{PQ}m_\pi}\right)^2\right)I(x_{qh})\right]\Bigg\} \\ &= \frac{0.278}{g_{*,dec}}\Bigg\{1-(1-3.6g_{*,dec}Y_a(0)) \\ &= \exp\left[-\left(\frac{10}{g_*}\right)^{1/2}\left(\frac{m_a}{1.2\times 10^{-3}eV}\right)^2 I(x_{qh}\right]\Bigg\} \end{aligned} \tag{4.4.14}$$

where we put

$$\begin{aligned} I(x_{qh}) &\equiv \int_{x_{qh}}^{\infty} x\prime^{-5/2}e^{-x\prime}dx\prime \\ &= -\frac{2}{3}[x_{qh}^{-3/2}e^{-x_{qh}}(2x_{qh}-1)+2\sqrt{\pi}(erf\sqrt{x_{qh}})-1)] \end{aligned} \tag{4.4.15}$$

where $g_{*,dec}$ denotes the value of g_* during the decoupling process. x_{qh} is approximately 5 at the quark-hadron transition, therefore $I \sim 10^{-4}$.

Everything we have discussed is within a framework in which the classical axion field is assumed to be a constant, i.e., that value necessary to eliminate the strong $\bar{\theta}$-term. However, when the temperature $T \sim f_a \gg \Lambda_{QCD}$, it is no longer possible to set a minimum value for the potential generated by the instantaneous effects, which gives mass only for $T \sim \Lambda_{QCD}$. Above this temperature it is not possible to think of the axion field being associated with a particle value $\bar{\theta} = 0$. The axion field will therefore slip towards its minimum and therefore it does not make sense to talk about a constant value for the axion field. This mechanism, known as vacuum misalignment is at the basis of assigning to the axion field a non-zero value and a non-zero energy density. To calculate it we take the action for axion

$$\begin{aligned} S &= \int d^4x\sqrt{g}(\frac{1}{2}\dot{a}^2 - \frac{1}{2}m_a^2a^2 + \Gamma_a\dot{a}) \\ &= \int d^4xR^3(t)(\frac{1}{2}\dot{a}^2 - \frac{1}{2}m_a^2a^2 + \Gamma_a\dot{a}) \end{aligned} \tag{4.4.16}$$

where we have changed notation for the scale factor R(t).We have also dropped the terms higher than the quadratic ones. From this we calculate the equation of motion

$$\frac{d}{dt}[R^3(\dot{a}+\Gamma_a]+R^3m_a^2(T)=0 \tag{4.4.17}$$

We can now manipulate the previous equation by considering that Γ_a is small, so it can be neglected.

At $T \gg \Lambda_{QCD}$ the axion is massless so it assumes the constant value

$$a(t) = a_i = const. \tag{4.4.18}$$

then as the temperature decreases $m_a^2(T)$ increases, and the equation of motion now becomes

$$\ddot{a} + 3H\dot{a} + m_a^2(T) = 0 \tag{4.4.19}$$

where we have set $H \equiv \dot{R}/R$, as usual for the Hubble parameter. When the temperature reaches the value T_i

$$m_a(T_i) = 3H(T_i) \tag{4.4.20}$$

so a(t) begin to oscillate with frequency $m_a(T)$. The energy density of the axion field is

$$\rho_a = \frac{1}{2} + \frac{1}{2}m_a^2 a^2 \tag{4.4.21}$$

and recalling equation 4.4.19

$$\dot{\rho_a} = \dot{m}_a m_a a^2 - 3H\dot{a}. \tag{4.4.22}$$

If we average over one oscillation we get

$$\langle \dot{\rho_a} \rangle = \left(\frac{\dot{m}_a}{m_a} - 3H \right) \langle \rho_a \rangle \tag{4.4.23}$$

and its solution is

$$\langle \rho_a \rangle R^3(t) \propto m_a(T). \tag{4.4.24}$$

This means that the number density of the axion $n_a = \langle \rho_a / m_a(T)$ scales with $R^{-3}(t)$. Also the entropy density s scales like that if we assume that there is no entropy production during the misalignment process; we conclude that $\frac{n_a}{s}$ is conserved, i.e.

$$\left.\frac{n_a}{s}\right|_{T_i} = \frac{45 m_a(T_i) a_i^2}{4\pi^2 g_* T_i^3} = \frac{45 a_i^2}{2\sqrt{5\pi g_*} T_i m_P}. \tag{4.4.25}$$

The misaligned axion energy density is given by

$$\Omega_a^{mis} = \frac{\rho_{a_0}^m is}{\rho_c} = \left.\frac{n_a}{s}\right|_{T_i} m_a \frac{s_0}{\rho_c}. \tag{4.4.26}$$

Chapter 5

General Relativity in a nutshell

5.0.1 Some conventions

In this chapter we will present a very brief review of some fundamental results of the theory of General Relativity. First we will take up some concepts of Riemannian differential geometry. We will give the definition of the Christoffel symbol and relate it to the definition of the Riemann tensor. Next we will show the Einstein field equations derived from the Hilbert-Einstein action variation. Next we will present the linear approximation of Einstein's equations in order to introduce the solutions represented by gravitational waves. The characteristics of gravitational waves will be illustrated. Finally the physical effects of gravitational wave interactions with test masses will be introduced. In this chapter we will use the following conventions. We indicate the space-time coordinates of a point with the Greek indexes

$$x^\mu \in \{x^0, x^1, x^2, x^3\} = \{ct, x, y, z\} \tag{5.0.1}$$

while we use the Latin indexes to denote the three-dimensional coordinates.

$$x^i \in \{x^1, x^2, x^3\} = \{x, y, z\}. \tag{5.0.2}$$

Instead we indicate the scalar product as follows

$$\mathbf{A} \prod \mathbf{B} = \sum_{i=1}^{3} A^i B^i \tag{5.0.3}$$

and the vector product as

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A^j B^k \tag{5.0.4}$$

where ϵ_{ijk} is the Levi-Civita symbol.
The partial derivatives is defined as

$$\partial_\mu = \frac{\partial}{\partial_\mu} = \left(\frac{1}{c}\partial_t, \partial_i\right). \tag{5.0.5}$$

The metric for a flat(Minkowskian) space time is

$$\eta_{\mu\nu} = (-,+,+,+) \tag{5.0.6}$$

while the metric for a curved space-time is denoted by

$$g_{\mu\nu} \tag{5.0.7}$$

whose determinant is g.

The invariant length element is defined

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2 d\tau^2. \tag{5.0.8}$$

In order to define the product between two four-vectors we use Einstein convention, according to which repeated upper or lower indexes are summed over

$$A^\mu B_\mu = \sum_{\mu=0}^{3} A^\mu B_\mu. \tag{5.0.9}$$

In the following we will need some elements borrowed from Riemannian differential geometry, such as the Christoffel symbols, defined as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} + \partial_\sigma g_{\mu\nu}) \tag{5.0.10}$$

through which it is possible to define the Riemann tensor

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha \nu\sigma - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}. \tag{5.0.11}$$

The Ricci tensor is obtained by contracting the Riemann tensor

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} \tag{5.0.12}$$

and the Ricci scalar is

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{5.0.13}$$

Another important ingredient of General Relativity is the energy-momentum tensor $T^{\mu\nu}$ which is defined through the variation of the matter action, namely S_M with respect to the metric transformation $g_{\mu\nu} \to g\mu\nu + \delta g_{\mu\nu}$,

$$\partial S_M = \frac{1}{2c}\int d^4x \sqrt{-g} T^{\mu\nu}\delta g_{\mu\nu}. \tag{5.0.14}$$

The final result of General Relativity is the famous Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{5.0.15}$$

in this version without the cosmological constant.

5.0.2 General Relativity in Brief

In the context of General Relativity, space-time is seen as a four-dimensional Riemannian surface. The equation that governs General Relativity is Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{5.0.16}$$

whose right member represents the amount of mass-energy density, i.e. the energy-momentum tensor, while on the left we find the four-dimensional surface curvature expressed by the Ricci tensor. and the scalar curvature defined as

$$R = R^{\mu}_{\mu}. \tag{5.0.17}$$

The energy-momentum tensor contains information about the source of the gravitational field, i. e. the curvature of space-time. Therefore, the presence of non-vanishing mass energy-momentum density determines a non-nothing curvature of the Riemannian manifold. This curvature is responsible for the motion of matter subject to the gravitational field and governed by the equation of geodetics. One of the characteristics of General Relativity is to be formulated in covariant form, i.e. invariant under the symmetry group of all possible transformations of smooth coordination. In fact, defined the element of length

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{5.0.18}$$

you can make a coordinate transformation $x^{\mu} \to x'^{\mu}(x)$
such that the length element is invariant, i. e.

$$ds^2 = g_{\mu\nu}(x)dx^{\mu}dx^{\mu} = g\prime_{\mu\nu}dx'^{\mu}dx'^{\nu}. \tag{5.0.19}$$

The metric tensor transforms as

$$g_{\mu\nu} \to g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}g_{\rho\sigma}(x). \tag{5.0.20}$$

This transformation represents the gauge symmetry of General relativity.
The motion of a test particle immersed in a gravitational field is governed by the so called geodetic equation, that is the path that minimizes the distance between two points

$$\frac{d^2x^{\rho}}{d\tau^2} + \Gamma^{\rho}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0 \tag{5.0.21}$$

Now we need to define the derivative of a v^μ vector along the geodesics

$$\frac{Dv^\mu}{D\tau} = v^\mu_{;\beta}\frac{dx^\beta}{d\tau} = \frac{dv^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}v^\alpha\frac{dx^\beta}{d\tau}. \tag{5.0.22}$$

The prevoious one is called covariant derivative. We now want to show how Riemann's tensor describes the tidal forces, i.e. the reciprocal acceleration between two free-falling particles.
The free-falling particle will follow the equation of geodesics 5.0.21, while an accelerated particle will deviate from it, so its universe line can express itself as

$$x^\mu(\tau) + \xi^\mu(\tau) \tag{5.0.23}$$

which will satisfy the equation

$$\frac{d^2(x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{d(x^\mu + \xi^\nu)}{d\tau}\frac{d(x^\rho + \xi^\rho)}{d\tau} = 0. \tag{5.0.24}$$

If we consider a deviation from the geodesics $|\xi^\mu(\tau)|$ small when compared to $g_{\mu\nu}$, we can think of subtracting member by member the 5.0.21 and the 5.0.24 and expand into the linear terms in ξ, getting

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{5.0.25}$$

Using the definition of covariant derivative and simplifying one finally gets the equation of the geodesics deviation

$$\frac{D^2\xi^\mu}{d\tau^2} = -R^\mu_{\nu\sigma\rho}\xi^\rho u^\nu u^\sigma \tag{5.0.26}$$

where we have defined the four-velocity as

$$u^\mu = \frac{dx^\mu}{d\tau}$$

.

5.1 Linearization of Einstein equations

Let us consider the weak field approximation. In order to do so we can consider the perturbed Minkowskian metric. The covariant metric tensor will then assume the form

$$g_{\mu\nu}(t, \mathbf{x}) = \eta\mu\nu + h_{\mu\nu}(t, \mathbf{x}) + \mathcal{O}(h^2) \tag{5.1.1}$$

while for the contravariant metric tensor we have

$$g^{\mu\nu}(t, \mathbf{x}) = \eta^{\mu\nu} - h^{\mu\nu}(t, \mathbf{x}) + \mathcal{O}(h^2) \tag{5.1.2}$$

. In fact one can verify that $g^{\mu\nu}g_{\mu\lambda} = \delta^{\nu}_{\lambda}$. The weak field approximation holds since we require that

$$|h_{\mu\nu}| \ll 1. \tag{5.1.3}$$

The $\mathcal{O}(h^2)$ means that we can neglect the quadratic terms in the expansion of the perturbed metric. If we want to get a linearized version of Einstein's field equations we have to develop Ricci's $R_{\mu\nu}$ tensor in series of $h_{\mu\nu}$ powers and omit all the terms higher than the first order. The indexes will now be lowered (raised) using the unperturbed metric tensors $\eta_{\mu\nu}$ ($\eta^{\mu\nu}$).

Before moving on to Ricci's tensor, we need to linearize all the ingredients it contains. Let's start with the linearized version of the affine connection, at the first order in $h_{\mu\nu}$

$$\Gamma^{\lambda}_{\mu\nu} \approx \frac{1}{2}\eta^{\sigma\lambda}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) \tag{5.1.4}$$

so the linearized Ricci tensor is given by

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\mu h^{\sigma}_{\nu} + \partial_\sigma\partial_\nu h^{\sigma}_{\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}). \tag{5.1.5}$$

where $h = h^{\lambda}_{\lambda} = \eta^{\mu\lambda}h_{\mu\lambda}$ is the trace of the perturbation of the metric.

So the linearized Einstein equation in the vacuum is

$$R_{\mu\nu} = \frac{1}{2}(\Box^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial^\lambda\partial_\nu h_{\mu\lambda} - \partial_\mu\partial^\lambda h_{\nu\lambda}) = 0. \tag{5.1.6}$$

Our goal now is to modify the previous expression in order to obtain a more compact version of Einstein's equation in the vacuum. This process leads to an analogy that the expression of Maxwell's equations in covariant form. $\Box^2 A_\beta = 0$. Just as in electromagnetism, where we chose the Lorentz gauge $\partial_\alpha A^\alpha = 0$ we can take advantage of the gauge freedom also for the gravitational field and fix a specific gauge choice called harmonic gauge

$$\partial_\mu h^{\mu}_{\nu} = \frac{1}{2}\partial_\nu h^{\mu}_{\mu}. \tag{5.1.7}$$

If we exploit this gauge choice in the linearized expression of the Ricci tensor

$$\begin{aligned}
\partial_\mu\partial_\nu h - \partial^\lambda\partial_\nu h_{\mu\lambda} - \partial_\mu\partial^\lambda h_{\nu\lambda} &= \partial_\mu(2\partial_\lambda h^{\lambda}_{\nu}) - \partial_\nu\partial_\lambda h^{\lambda}_{\mu} - \partial_\lambda\partial_\mu h^{\lambda}_{\nu} \\
&= \partial_\mu\partial_\lambda h^{\lambda}_{\nu} - \partial_\nu\partial_\lambda h^{\lambda}_{\mu} \\
&= \partial_\mu\left(\frac{1}{2}\partial_\nu h^{\lambda}_{\lambda}\right) - \partial_\nu\left(\frac{1}{2}\partial_\mu h^{\lambda}_{\lambda}\right) = 0.
\end{aligned} \tag{5.1.8}$$

So finally the form of the linearized Einstein equation in the vacuum is

$$\Box^2 h_{\mu\nu} = 0. \tag{5.1.9}$$

We would like to show that it is always possible to choose a reference system in which the harmonic gauge is valid. Let x_μ be a coordinate system in which the harmonic gauge is not valid. We can then perform a transformation of coordinates

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{5.1.10}$$

with $|\epsilon^\mu(x)| \ll 1$. After this transformation the metric tensor will be

$$g'^{\mu\nu} = g^{\lambda\rho} \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\rho} \tag{5.1.11}$$

$$\begin{aligned} g'^{\mu\nu} &= (\eta^{\lambda\rho} - h^{\lambda\rho})(\delta^\mu_\lambda + \partial_\lambda \epsilon^\mu)(\delta^\nu_\rho + \partial_\rho \epsilon^\nu) \\ &= (\eta^{\mu\rho} - h^{\mu\rho} + \eta^{\lambda\rho}\partial_\lambda \epsilon^\mu - \underbrace{h^{\lambda\rho}\partial_\lambda \epsilon^\mu}_{\mathcal{O}(h\epsilon)})(\delta^\nu_\rho + \partial_\rho \epsilon^\nu) \\ &\approx \eta^{\mu\nu} - h^{\mu\nu} + \eta^{\lambda\nu}\partial_\lambda \epsilon^\mu + \eta^{\mu\rho}\partial_\rho \epsilon^\nu - \underbrace{h^{\mu\rho}\partial_\rho \epsilon^\nu}_{\mathcal{O}(h\epsilon)} + \underbrace{\partial^\rho \epsilon^\mu \partial_\rho \epsilon^\nu}_{\mathcal{O}(\epsilon^2)} \\ &\approx \eta^{\mu\nu} - h^{\mu\nu} + \partial^\nu \epsilon^\mu + \partial^\mu \epsilon^\nu. \end{aligned} \tag{5.1.12}$$

The last line gives the form of the new metric tensor after the coordinate transformation. We have neglected the $\mathcal{O}(\epsilon^2)$,$\mathcal{O}(h\epsilon)$ and $\mathcal{O}(h^2)$ terms. So in the end

$$h'^{\mu\nu} = h^{\mu\nu} - \partial^\nu \epsilon^\mu - \partial^\mu \epsilon^\nu. \tag{5.1.13}$$

gives the form of the perturbation of the metric in the new coordinate frame. We can now compute both terms of 5.1.7 in the new coordinate frame

$$\begin{aligned} &\partial_\mu h'^\mu_\nu = \frac{1}{2}\partial_\nu h'^\mu_\mu \iff \\ &\partial_\mu h^\mu_\nu - \partial_\mu \partial_\nu \epsilon^\mu - \partial_\mu \partial^\mu \epsilon_\nu = -\frac{1}{2}\partial_\nu \partial_\mu \epsilon^\mu - \frac{1}{2}\partial_\nu \partial_\mu \epsilon^\mu + \frac{1}{2}\partial_\nu h^\mu_\mu \iff \\ &\partial_\mu h^\mu_\nu - \partial_\mu \partial^\mu \epsilon_\nu = \frac{1}{2}\partial_\nu h^\mu_\mu. \end{aligned} \tag{5.1.14}$$

that is

$$\Box^2 \epsilon_\nu = \partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\mu_\mu \tag{5.1.15}$$

but this vanishes because of the initial gauge choice. This means that one can always choose a reference frame in which the harmonic gauge is valid, as long as ϵ_ν is a solution of $\Box^2 \epsilon_\nu = \partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\mu_\mu = 0$.

Now we can reconsider the equation5.1.9 which is the linearized Einstein equation in the vacuum

$$\Box^2 h_{\mu\nu} = 0. \tag{5.1.16}$$

We now know that it is always possible to enforce the harmonic gauge condition 5.1.7 and also in this case remains analogy with electromagnetism, so the solution of the 5.1.9 can be given as linear superposition of plane waves

$$h_{\mu\nu}(t, \mathbf{x}) = \mathcal{R}e(e_{\mu\nu} e^{ik^\lambda x_\lambda}). \tag{5.1.17}$$

The solution given by 5.1.17 represents the gravitational wave itself. In fact, by replacing in 5.1.9 one gets

$$0 = \Box^2 h_{\mu\nu} = ik^\lambda ik_\lambda e_{\mu\nu} e^{ik^\lambda x_\lambda} \implies k^\lambda k_\lambda = 0 \rightarrow \lambda\nu = 1 \tag{5.1.18}$$

and from this we deduce that the gravitational wave of propagates at the speed of light in a vacuum.

In addition, using the harmonic gauge condition we find that

$$ik_\lambda e^\lambda_\nu e^{ik^\lambda x_\lambda} = \frac{i}{2} k_\nu e^\lambda_\lambda e^{ik^\lambda x_\lambda} \iff k_\lambda e^\lambda_\nu = \frac{1}{2} k_\nu e^\lambda_\lambda, \tag{5.1.19}$$

this implies that as in the case of electromagnetic waves also gravitational waves are transverse waves, i.e. the component along the propagation direction is null.

The $e_{\mu\nu}$ tensor is the polarization tensor and has 10 independent components. However, by applying the harmonic gauge the independent components are reduced to 4. Moreover, with a proper choice of the coordinates system it is possible to reduce the independent components to 2. They correspond to the two possible polarization states of the gravitational wave. Let's show how this can happen. First we choose the z-axis as the propagation direction of the wave. The wave vector is therefore

$$k^\mu = (k, 0, 0, k). \tag{5.1.20}$$

For $\nu = 1$ we have

$$k_\mu e^\mu_1 = \frac{1}{2} k_1 e^\mu_\mu \equiv 0. \tag{5.1.21}$$

Recalling that $k_\nu \equiv \eta_{\mu\nu} k^\mu = (-k, 0, 0, k)$ we can write explicitly the l.h.s of the previous equation

$$0 = k_0 e^0_1 + k_3 e^3_1 = -ke^0_1 + ke^3_1 \implies e^0_1 = e^3_1 \implies e_{01} = -e_{31}. \tag{5.1.22}$$

The same is done for $\nu = 2$

$$k_\mu e^\mu_2 = \frac{1}{2} k_2 e^\mu_\mu = 0 \implies e_{02} = -e_{32}. \tag{5.1.23}$$

and for $\nu = 0$

$$
\begin{aligned}
k_\mu e_0^\mu &= \frac{1}{2} k_0 e_\mu^\mu = -\frac{1}{2} k^0 e_\mu^\mu \implies \\
k^0 e_{00} + k^3 e_{30} &= -\frac{1}{2} k^0 (e_1^1 + e_2^2 + e_3^3 + e_0^0)
\end{aligned}
\tag{5.1.24}
$$

From which

$$
e_{00} + e_{30} = -\frac{1}{2}(e_{11} + e_{22} + e_{33} - e_{00}). \tag{5.1.25}
$$

Similarly for $\nu = 3$

$$
e_{03} + e_{33} = \frac{1}{2}(e_{11} + e_{22} + e_{33} - e_{00}). \tag{5.1.26}
$$

and summing respectively the l.h.s and the r.h.s of the previous two equations one finds

$$
e_{30} = e_{03} = -\frac{1}{2}(e_{33} + e_{00}) \tag{5.1.27}
$$

and replacing this in the equation for $\nu = 3$ one gets

$$
e_{22} = -e_{11}. \tag{5.1.28}
$$

Summarizing, up to this point, using only the harmonic gauge condition, we have found the following relationship between the components of the wave polarization vector

$$
e_{01} = -e_{31}, \tag{5.1.29a}
$$

$$
e_{02} = -e_{32}, \tag{5.1.29b}
$$

$$
e_{03} = -\frac{1}{2}(e_{33} + e_{00}), \tag{5.1.29c}
$$

$$
e_{22} = -e_{11}. \tag{5.1.29d}
$$

Now, as mentioned earlier, we can make a coordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$ and if we want the harmonic gauge choice to be valid we require that ϵ^μ satisfied the equation

$$
\Box^2 \epsilon^\mu = 0 \tag{5.1.30}
$$

and its solutions are given in the form

$$
\epsilon^\mu = \mathcal{R}e\{i\tilde{\epsilon}^\mu e^{ik^\lambda x_\lambda}\}. \tag{5.1.31}
$$

Recalling equation 5.1.13 we can write the perturbation in the new frame of reference

$$
h'_{\mu\nu} = \mathcal{R}e\{e'_{\mu\nu} e^{ik^\lambda x_\lambda}\} = h_{\mu\nu} - \partial_\nu \epsilon_\mu - \partial_\mu \epsilon_\nu. \tag{5.1.32}
$$

since $\partial_\nu \epsilon_\mu = -k_\nu \tilde{\epsilon}_\mu e^{ik^\lambda x_\lambda}$ we have

$$e'_{\mu\nu} = e_{\mu\nu} + k_\nu \tilde{\epsilon}_\mu + k_\mu \tilde{\epsilon}_\nu. \tag{5.1.33}$$

So that, in terms of the components of the polarization tensor we have

$$e'_{11} = e_{11} + k_1\tilde{\epsilon}_1 + k_1\tilde{\epsilon}_1 = e_{11}, \tag{5.1.34a}$$
$$e'_{12} = e_{12} + k_2\tilde{\epsilon}_1 + k_1\tilde{\epsilon}_2 = e_{12}, \tag{5.1.34b}$$
$$e'_{13} = e_{13} + k_3\tilde{\epsilon}_1 + k_1\tilde{\epsilon}_3 = e_{13} + k\tilde{\epsilon}_1 \tag{5.1.34c}$$
$$e'_{23} = e_{23} + k_3\tilde{\epsilon}_2 + k_2\tilde{\epsilon}_3 = e_{23} + k\tilde{\epsilon}_2 \tag{5.1.34d}$$
$$e'_{33} = e_{33} + 2k_3\tilde{\epsilon}_3 \tag{5.1.34e}$$
$$e'_{00} = e_{00} - 2k_0\tilde{\epsilon}_0. \tag{5.1.34f}$$

We can now choose $\tilde{\epsilon}_1 = -e_{13}/k$, $\tilde{\epsilon}_2 = -e_{23}/k$, $\tilde{\epsilon}_3 = -e_{33}/(2k)$, $\tilde{\epsilon}_0 = e_{00}/(2k)$ in order to let 4 components of the tensor vanish, i.e. $e'_{13} = e'_{23} = e'_{33} = e'_{00} = 0$. Furthermore imposing the conditions 5.1.29 we have $e'_{01} = -e'_{31} = 0$, $e'_{02} = -e'_{32} = 0$ and $e'_{03} = -(e'_{33} + e'_{00})/2 = 0$. So as previously mentioned, the gravitational wave that moves along the z direction has a polarization tensor whose non-null components are only $e_{22} = -e_{11}$ and $e_{12} = e_{21}$. In other words, the gravitational wave has only two polarization states.
We can observe that this particular choice of coordinates defines the so-called transverse trace-less gauge, in fact

$$e^\mu_\mu = re^0_0 + e^1_1 + e^2_2 + e^3_3 = e^1_1 + e^2_2 = 0. \tag{5.1.35}$$

So a gravitational wave is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & 0 \\ 0 & e_{12} & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot e^{ik^\lambda x_\lambda}. \tag{5.1.36}$$

Considering a quantum field theory, we can interpret a field described by a two index tensor as a spin 2 particle; therefore we can interpret the $h_{\mu\nu}$ field as that field that describes the mediating particle of the gravitational field, the graviton and propagates at the speed of light since $k^\mu k_\mu = 0$.

5.2 Detection of GW

Let us now see what are the physical effects given by the interaction of a gravitational wave with test masses. We'll get some conditions that may suggest how to set up an experimental apparatus able to detect the waves themselves.
Let's consider two massive particles A and B immersed in a gravitational field. They will be character-

ized by universe lines

$$x_A^\mu = x^\mu(\tau) \tag{5.2.1}$$

for particle A and

$$x_B^\mu = x^\mu(\tau) + \delta x^\mu(\tau) \tag{5.2.2}$$

for particle B. δx^μ is the distance between the two universe lines. We use the geodesics equations applied on this difference, which we will call geodesics deviation

$$\frac{D^2 \delta x^\lambda}{D\tau^2} = R^\lambda_{\nu\mu\rho} \delta x^\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}. \tag{5.2.3}$$

Let us now consider an inertial reference system $x'^a lpha$ centered on A . In this case we can rewrite the coordinates of the two particles

$$x'^{\,i}_A = 0 \tag{5.2.4}$$

for A, while for B

$$x'^{\,\mu}_B = \delta x'^\mu. \tag{5.2.5}$$

Furthermore we have

$$t_A = \tau, \tag{5.2.6a}$$

$$\left.\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau}\right|_A = (1, \mathbf{0}) \tag{5.2.6b}$$

$$g'_{\mu\nu}|_A = \eta'_{\mu\nu} \tag{5.2.6c}$$

$$g'_{\mu\nu,\alpha}|_A = 0 \implies \Gamma'^\alpha_{\mu\nu}|_A = 0. \tag{5.2.6d}$$

The above equations tell us that we can only consider the components imported for $\rho = 0$ and $\nu = 0$, so we can rewrite the equation of the geodesic deviation

$$\frac{\mathrm{d}^2 \delta x^j}{\mathrm{d}t^2} = R^j_{0k0} \delta x^k. \tag{5.2.7}$$

We also consider valid the Transverse-Traceless gauge (TT), so we can state that the components $h_{\mu 0} = h_{0\mu}$ vanish. So, in the weak field approximation we can say that the following relation holds

$$R^{(1)}_{j0k0} = R^{(1)}_{0j0k} = \frac{1}{2}(\partial_k \partial_j h_{00} - \partial_k \partial_0 h_{j0} - \partial_0 \partial_j h_{0k} + \partial_0 \partial_0 h_{jk}) = \frac{1}{2} h^{\mathrm{TT}}_{jk,00} \tag{5.2.8}$$

We are now replacing this result in 5.2.6b and one gets

$$\frac{\mathrm{d}^2 \delta x^j}{\mathrm{d}t^2} = \frac{1}{2} \frac{h^j_k}{\delta t^2} \delta x^k. \tag{5.2.9}$$

Before the arrival of the gravitational wave the two particles are at a constant position from each other $\delta x^j(0)$. Since we are considering the field approximation of this constant distance this constant distance will be perturbed by an infinitesimal amount

$$\delta x^j(t) = \delta x_0^j + \delta x_1^j(t). \tag{5.2.10}$$

We now replace the previous equation into equation 5.2.9 and we neglect all the terms of order higher the one

$$\frac{\mathrm{d}^2 \delta x_1^j}{\mathrm{d}t^2} = \frac{1}{2} \frac{h_k^j}{\delta t^2} \delta x_0^k. \tag{5.2.11}$$

this is the last form of the deviation geodesic equation. Its solution is given by

$$\delta x_1^j(t) = \delta x^j(t) - \delta x_0^j \approx \frac{1}{2} h_k^j(x^\mu(t)) \delta x_0^k. \tag{5.2.12}$$

which can be recast as

$$\delta x^j(t) \approx \delta x^k(0) \left(\delta_k^j + \frac{1}{2} h_k^j(x^\mu(t)) \right). \tag{5.2.13}$$

We will use this equation to understand what are the effects of the passage of a gravitational wave on the system constituted by the two particles. Let's remember that in the TT gauge the only non-zero components of the gravitational perturbation which propagates in the z direction, are

$$h_{11}^{\mathrm{TT}} = -h_{22}^{\mathrm{TT}} \tag{5.2.14a}$$
$$h_{12}^{\mathrm{TT}} = h_{21}^{\mathrm{TT}} \tag{5.2.14b}$$

Now we can distinguish two cases. Let us first consider the case where the two masses are aligned in the z-direction. In this case their initial distance is $\delta x^j(0) = (0.0, L_0)$. which when the wave passes, on the other hand, changes and turns out to be

$$\delta x^j(t) = \delta x^k(0) \left(\delta_k^j + \frac{1}{2} h_k^j \right) = \delta x^3(0) \left(\delta_3^j + \frac{1}{2} h_3^j \right) = \delta x^3(0) \delta_3^j, \tag{5.2.15}$$

according to equation 5.2.13. From this we can deduce that the displacement along any j direction remains constant: $\delta x^j(t) = \delta x^j(0)$. Let's suppose now to have the system of the two particles aligned along a direction perpendicular to the z-axis. Before the wave passes, their separation is $\delta x^j(0) = (0, L_0, 0)$. Once the wave arrives it turns to be

$$\delta x^j(t) = \delta x^k(0) \left(\delta_k^j + \frac{1}{2} h_k^j(x^\mu(t)) \right) = \delta x^2(0) \left(\delta_2^j + \frac{1}{2} h_2^j(x^\mu(t)) \right), \tag{5.2.16}$$

and therefore it is deduced that after the passage of the wave change the separations along y and along z. Therefore, the gravitational wave is not only transverse but produces physical effects that are detectable in the direction transverse to those of propagation.

5.3 Generation of GW

We now want to study under what conditions it is possible to have the generation of gravitational waves. We will exploit an analogy with electromagnetism, we will use the same procedure used to understand which are the relevant terms in the development in multi-poles of the energy emitted by a system of charges in motion, to obtain a gravitational analogue. In the electromagnetic case we had

$$I_{em} = \underbrace{\frac{2}{3c^3}(\ddot{\mathbf{d}}_{\rm em})^2}_{\text{charge dipole}} + \underbrace{\frac{2}{3c^3}(\ddot{\mathbf{m}}_{em})^2}_{\text{magnetic dipole}} + \underbrace{\frac{1}{180c^5}\dddot{Q}_{ij,em}\dddot{Q}^{ij,em}}_{\text{charge quadrupole}} + \cdots \tag{5.3.1}$$

where $\mathbf{d}_{em} = \sum_k q_k \mathbf{r}_k$ is the electric dipole, $\mathbf{m}_{em} = \sum_k q_k \mathbf{r}_k \times \mathbf{v}_k/(2c)$ is the magnetic dipole and $Q^{em}_{ij} = \sum_k q_k(3x_i x_j - \delta_{ij} r_k^2)$ is the charge quadrupole. Now, to switch to the analogous gravitational case just replace the charges with the masses. The terms involved in the multi-pole expansion are

$$\mathbf{d}_g = \sum_k m_k \mathbf{r}_k = M\mathbf{R}_{cm}, \tag{5.3.2a}$$

$$\mathbf{m}_g = \sum_k \mathbf{r}_k \times (m_k \mathbf{v}_k) = \mathbf{L}, \tag{5.3.2b}$$

$$Q_{ij,\mathrm{g}} = \sum_k m_k(3x_i x_j - \delta_{ij} r_k^2), \tag{5.3.2c}$$

where we have replaced q_k with m_k. We have done the following definitions

$$M = \sum_k m_k$$

is the total mass of the system,

$$\mathbf{R}_{\rm cm} = \sum_k m_k \mathbf{r}_k / M$$

is the position of the center of mass of the system, and

$$\mathbf{L}$$

is the total angular momentum.

If the system is isolated then

$$\ddot{\mathbf{d}}_g = M\ddot{\mathbf{R}}_{cm} = \dot{\mathbf{P}} = \mathbf{0}, \tag{5.3.3a}$$

$$\dot{\mathbf{m}}_g = \dot{\mathbf{L}} = \boldsymbol{\tau} = 0, \tag{5.3.3b}$$

where $\mathbf{T}$ is the total momentum and τ is the total torque. Under these considerations the terms analogous to the electric and magnetic moment are null and therefore in the development in multiples the dominant term will be the term of mass quadrupole i.e.

$$I_{\mathrm{og}} = \frac{1}{45c^5}\dddot{Q}_{ij}\dddot{Q}^{ij}. \tag{5.3.4}$$

where the factor $\frac{1}{45}$ is due to the fact that we have to take into account the degeneracy due to the spin 2 of the gravitational field. Let's consider the linearized Einstein equation with in the weak field approximation

$$\Box^2\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = -\frac{16\pi G}{c^4}\tau_{\mu\nu}. \tag{5.3.5}$$

and we define the tensor

$$\psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{5.3.6}$$

. In the TT gauge choice $h = 0$ so $\psi_{\mu\nu} = h_{\mu\nu}$.
In terms of the new tensor it can be rewritten as

$$\partial_\mu\psi^{\mu\nu} = 0 \tag{5.3.7}$$

and the Einstein equation is now translated in

$$\Box^2\psi_{\mu\nu} = -\frac{16\pi G}{c^4}\tau_{\mu\nu}. \tag{5.3.8}$$

Now we want to understand how to describe a wave generated at great distances from the point of detection. In particular we place ourselves at large distances R with respect to the wavelength λ of the generated wave. Formally the Einstein equation 5.3.8 is identical to the Maxwell equation

$$\Box^2 A_\beta = -\frac{4\pi}{c}J_\beta \tag{5.3.9}$$

Using the same solving methods the general solution of the 5.3.8 is given by

$$\psi_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int\frac{(\tau_{\mu\nu})|_{t-R/c}}{R}\mathrm{d}V', \tag{5.3.10}$$

and assuming that the velocities are much smaller than c we can approximate

$$\psi_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4 r}\int(\tau_{\mu\nu})|_{t-r/c}\mathrm{d}V'. \tag{5.3.11}$$

We use the property now

$$\partial_\mu\tau^{\mu\nu} = 0 \tag{5.3.12}$$

which replaces the general conservation property of the momentum-energy tensor. In addition we notice that $\tau_{\mu\mu}$ is symmetric with respect to the exchange of the indices, so

$$\partial^k \tau_{ik} + \partial^0 \tau_{i0} = 0, \tag{5.3.13a}$$

$$\partial^k \tau_{0k} + \partial^0 \tau_{00} = 0. \tag{5.3.13b}$$

(we are omitting the fact that these equations are all evaluated at t-r/c). We multiply equation 5.3.13a by x_j and we integrate over all space

$$\begin{aligned}\partial^0 \int \tau_{i0} x_j \mathrm{d}V &= -\int \partial^k \tau_{ik} x_j \mathrm{d}V = -\int \partial^k (\tau_{ik} x_j) \mathrm{d}V + \int \tau_{ik} \delta_j^k \mathrm{d}V = \\ &= -\int \partial^k (\tau_{ik} x_j) \mathrm{d}V + \int \tau_{ij} \mathrm{d}V.\end{aligned} \tag{5.3.14}$$

The first integral can be regarded as a surface term, so it can be neglected since it gives no contributions:

$$\int \tau_{ij} \mathrm{d}V = \partial^0 \int \tau_{i0} x_j \mathrm{d}V = \frac{1}{2} \partial^0 \int (\tau_{i0} x_j + \tau_{j0} x_i) \mathrm{d}V. \tag{5.3.15}$$

As for the second integral we multiply by $x_i x_j$ and we get

$$\begin{aligned}\partial^0 \int \tau_{00} x_i x_j \mathrm{d}V &= -\int \partial^k \tau_{0k} x_i x_j \mathrm{d}V \\ &= -\int \partial^k (\tau_{0k} x_i x_j) \mathrm{d}V + \int \tau_{0k} \delta_i^k x_j \mathrm{d}V + \int \tau_{0k} x_i \delta_j^k \mathrm{d}V \\ &= \int (\tau_{0j} x_i + \tau_{0i} x_j) \mathrm{d}V.\end{aligned} \tag{5.3.16}$$

Let's now combine the results obtained into a single equation and find

$$\begin{aligned}\int \tau_{ij} \mathrm{d}V &= \frac{1}{2} \partial^0 \int (\tau_{0j} x_i + \tau_{0i} x_j) \mathrm{d}V = \frac{1}{2} (\partial^0)^2 \int \tau_{00} x_i x_j \mathrm{d}V \\ &= \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int \tau_{00} x_i x_j \mathrm{d}V.\end{aligned} \tag{5.3.17}$$

This last equation says that volume integrals can be converted to integrals on the component $\tau_{00} = T_{00} + t_{00}$ approximate with T_{00} . So we can rewrite $\tau_{00} = \rho c^2$ where ρ is the mass density at rest. Replacing these results in 5.3.11

$$\psi_{ij} = \frac{2G}{c^4 r} \frac{\partial^2}{\partial t^2} \int \rho x_i x_j \mathrm{d}V. \tag{5.3.18}$$

We denote with a system size. Let's put ourselves in the condition in which $R >> \lambda >> a$, with λ as the wave length, so that we can describe the waves generated as plain waves if we put ourselves over long distances.
Adopting the TT gauge

$$\psi_{ij} = h_{ij} = \frac{2G}{c^4 r} \frac{\partial^2}{\partial t^2} \int \rho x_i x_j \mathrm{d}V = \frac{2G}{3c^4 r} \left(\ddot{Q}_{ij} + \delta_{ij} \frac{\partial^2}{\partial t^2} \int \rho r^2 \mathrm{d}V \right), \tag{5.3.19}$$

with

$$Q_{ij} = \int \rho(3x_i x_j - r^2\delta_{ij})\mathrm{d}V \tag{5.3.20}$$

as mass quadrupole. We are considering a wave propagating in direction say x^1. Since in the TT gauge the only non vanishing components are $h_{22} = -h_{33}$ and $h_{23} = h_{32}$ we have the flux of energy carried in the x^1 direction

$$ct^{01} = \frac{c^3}{16\pi G}\left((\dot{h}_{23})^2 + \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2\right). \tag{5.3.21}$$

Equation 5.3.19can be re-expressed explicitly as

$$h_{23} = \frac{2G}{3c^4 r}\ddot{Q}_{23}, \tag{5.3.22a}$$

$$h_{22} - h_{33} = \frac{2G}{3c^4 r}(\ddot{Q}_{22} - \ddot{Q}_{33}), \tag{5.3.22b}$$

and replacing in 5.3.21 we have

$$ct^{01} = \frac{G}{36\pi c^5 r}\left((\dddot{Q}_{23})^2 + \frac{1}{4}(\dddot{Q}_{22} - \dddot{Q}_{33})^2\right). \tag{5.3.23}$$

The total irradiated energy is given by

$$I = I = \frac{G}{45c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle \tag{5.3.24}$$

Chapter 6

Dark Matter with Stueckelberg Axions

Introduction

We review a class of models which generalize the traditional Peccei-Quinn (PQ) axion solution by a Stueckelberg pseudoscalar. Such axion models represent a significant variant with respect to earlier scenarios where axion fields were associated with global anomalies, because of the Stueckelberg field, which is essential for the cancellation of gauge anomalies in the presence of extra $U(1)$ symmetries. The extra neutral currents associated to these models have been investigated in the past in orientifold models with intersecting branes, under the assumption that the Stueckelberg scale was in the multi-TeV region. Such constructions, at the field theory level, are quite general and can be interpreted as the four-dimensional field theory realization of the Green-Schwarz mechanism of anomaly cancellation of string theory. We present an overview of models of this type in the TeV/multi TeV range in their original formulation and their recent embeddings into an ordinary GUT theory, presenting an $E_6 \times U(1)_X$ model as an example. In this case the model contains two axions, the first corresponding to a Peccei-Quinn axion, whose misalignment takes place at the QCD phase transition, with a mass in the meV region and which solves the strong CP problem. The second axion is ultralight, in the $10^{-20} - 10^{-22}$ eV region, due to a misalignment and a decoupling taking place at the GUT scale. The two scales introduced by the PQ solution, the PQ breaking scale and the misalignment scale at the QCD hadron transition, become the Planck and the GUT scales respectively, with a global anomaly replaced by a gauge anomaly. The periodic potential and the corresponding oscillations are related to a particle whose De Broglie wavelength can reach 10 kpc. Such a sub-galactic scale has been deemed necessary in order to resolve several dark matter issues at the astrophysical level.

It is by now well established that astrophysical and cosmological data coming either from measurements of the velocities of stars orbiting galaxies, in their rotation curves, or from the cosmic microwave background, indicate that about $\sim$ 80% of matter in the universe is in a unknown form, and the expectations for providing an answer to such a pressing question run high. These observational results are justified within the standard ΛCDM dark matter/dark energy model [31] which has been very successful in explaining the data. It predicts a dark energy component about $68 \pm 1\%$ of the total mass/density contributions of our universe in the form of a cosmological constant. The latter accounts for the dark energy dominance in the cosmological expansion at late times and provides the cosmological acceleration measured by Type Ia supernovae [32, 33], with ordinary baryonic dark matter contributing just a few percent of the total mass/energy content ($\sim 5\%$) and a smaller neutrino component. Cold dark matter with small density fluctuations, growing gravitationally and a spectral index of the perturbations $n_S \sim 1$ is compatible with an early inflationary stage and accounts for structure formation in most of the early universe eras. By now, data on the CMB, weak lensing and structure formation, covering redshifts from large $z \sim 10^3$ down to $z <\sim O(1)$ where the full nonlinear regime of matter dominance is at work, have been confronted with N-body gravitational simulations for quite some time, with comparisons which are in general agreement with ΛCDM. Such simulations, characterized by perturbations with the above value of the spectral index show the emergence of hierarchical, self-similar structures in the form of halos and sub-halos of singular density $(\rho(r) \sim 1/r$ in terms of the radius r) [34] in the nonlinear regime. However, while the agreement between ΛCDM and the observations is significant at most scales, at a small sub-galactic scale, corresponding to astrophysical distances relevant for the description of the stellar distributions ($\sim$ 10 kpc), cold dark matter models predict an abundance of low-mass halos in excess of observations [35]. Difficulties in characterizing this sub-galactic region have usually been attributed to inaccurate modeling of its baryonic content, connected with star formation, supernova explosions and black hole activity which take place in that region, causing a redistribution of matter.

There are various possibilities to solve this discrepancy, such as invoking the presence of warm dark matter (WDM), whose free streaming, especially for low mass WDM particles, could erase halos and sub-halos of low mass. At the same time they could remove the predicted dark matter cusps in $\rho(r)$, present in the simulations for $r \simeq 0$ [34] but not detected observationally. As observed in [35] and recently re-addressed in [36], these issues define a problem whose resolution may require a cold dark matter component which is ultralight, in the $10^{-20} - 10^{-22}$ eV range. Proposals for such component of dark matter find motivations mostly within string theory, where massless moduli in the form of scalar and pseudoscalar fields abound at low energy. They are introduced at the Planck scale and their flat potentials can be lifted by a small amount, giving rise to ultralight particles. However, the characterization of a well-defined gauge structure which may account for the generation of such ultralight particle(s) and which may eventually connect the speculative scenarios to the electroweak scale can be pursued in various ways. It has been recently proposed [37] that particles of this kind may emerge from grand unification in the presence of anomalous abelian symmetries, revisiting previous constructions.

The goal of this review is to summarize the gauge structure of these models which require an anomalous fermion spectrum with gauge invariance restored by a Wess-Zumino interaction, by the inclusion of a Stueckelberg axion. Such models can be thought as the field theory realization of the mechanism of anomaly cancellation derived from string theory. The models reviewed here are characterized by some distinctive key features that we are going to discuss, establishing their relation to the Peccei-Quinn model, of which they are an extension at a field theory level.

6.1 Anomalous U(1)'s

The Peccei-Quinn (PQ) mechanism, proposed in the 1970's to solve the strong CP problem [23, 38, 39] had been originally realized by assigning an additional abelian chiral charge to the fermion spectrum of the Standard Model (SM). Alternatively, a similar symmetry can be present in a natural way in specific gauge theories based on groups of higher rank with respect to the SM gauge group. This is the case, for instance of the $U(1)_{PQ}$ symmetry found in the E_6 GUT discussed in [40] (as well as in other realizations), naturally present in this theory and which can lead to a solution of the strong CP problem.

As we are going to discuss, the mass of the axion, either in the presence of global or local anomalies is connected to the instanton sector of a non-abelian theory and it is crucial for the mechanism of misalignment to be effective that the axion couples to the gauge sector of the same theory. In fact, the possibility that more than one axion is part of the spectrum of a certain gauge theory is not excluded, with the mass of each axion controlled by independent mechanism(s) of vacuum misalignment induced at several scales, if distinct gauge couplings for each of such particles with different gauge sectors are present [41, 42]. We will illustrate this point in the extended E_6 theory that we will overview in the next sections, where the inclusion of an extra anomalous $U(1)$ gauge symmetry realizes such a scenario. Different mechanisms of vacuum misalignment may be held responsible for the generation of axions of different masses, whose sizes may vary considerably.

6.1.1 Anomaly cancellation at field theory level with an axion

In the case of a Stueckelberg axion, as already mentioned, the PQ symmetry is generalized from global to a local gauge symmetry and the Wess Zumino interactions are needed for the restoration of gauge invariance of the effective action. Such generalizations, originally discussed in the context of low scale orientifold models [43], where anomalous abelian symmetries emerge from stacks of intersecting branes, have been proposed in the past as possible scenarios to be investigated at the LHC [44, 45, 46, 47, 48, 49], together with their supersymmetric extensions [50, 51, 42]. While anomalous abelian symmetries are interesting in their own right, especially in the search for extra neutral currents at the LHC [52, 53, 47] [54], one of the most significant aspects of such anomalous extensions is in fact the presence of an axion which is needed in order to restore the gauge invariance of the effective action. It was called the "axi-Higgs" in [43] [44] - for being generated by the mechanism of Higgs-Stueckelberg mixing in the CP-odd scalar sector, induced by a PQ-breaking periodic potential, later studied for its implications for dark matter in [41]. The appearance of such a potential is what allows one component of the Stueckelberg field to become physical. A periodic potential can be quickly recognized as being of instanton origin and related to the θ-vacuum of Yang-Mills theory and can be associated with phase transitions in non-abelian theories. Recent developments have taken into consideration the possibility that the origin of such a potential of this form can be set at a very large scale, such as the scale of grand unification (GUT). Its size is related to the value of the gauge coupling at the GUT scale, characterized by a typical instanton suppression, where the mechanism of vacuum misalignment takes place.

6.1.2 An ultralight axion

In the case of a misalignment generated at the GUT scale, the mass of the corresponding axion is strongly suppressed and can reach the far infrared, in the range of $10^{-20} - 10^{-22}$ eV, which is in the optimal range for a possible resolution of several astrophysical issues, such as those mentioned in the introduction [36]. Proposals for a fuzzy component of dark matter require a weakly interacting particle in that mass range. As in the PQ (invisible axion) case, also in this case two scales are needed in order to realize a similar scenario. In the PQ case the two scales correspond to f_a, the large PQ breaking scale and the hadronic scale which links the axion mass, f_a, the pion m_π and the light quarks masses m_u, m_d, in an expression that we will summarize below. In the case of Stueckelberg axions these fields can be introduced as duals of a 2-form ($B_{\mu\nu}$), defined at the Planck scale (M_P) and coupled to the field strength (F) of an anomalous gauge boson via a $B \wedge F$ interaction [37].
The mechanism of Higgs-axion mixing and the generation of the periodic potential can take place at a typical GUT scale. It is precisely the size of the potential at the GUT scale, which is controlled by the θ-vacuum of the corresponding GUT symmetry, which is responsible for the generation of an ultralight axion in the spectrum. As already mentioned, in the model discussed in [37] a second axion is present, specific to the E_6 part of the $E_6 \times U(1)_X$ symmetry, which is sensitive to the $SU(3)$ colour sector of the Standard Model after spontaneous symmetry breaking. This second field takes the role of an ordinary PQ axion and solves the strong CP problem. We will start by recalling the main features of the PQ solution, in particular the emergence of a mass/coupling relation in such a scenario which narrows the window for axion detection down and gets enlarged in the presence of a gauge anomaly in Stueckelberg models [49]. We will then turn, in the second part of this review, to a discussion of the Stueckelberg extension. We will describe the features of such models in their non-supersymmetric formulation. Their supersymmetric version requires a separate discussion, for predicting both an axion and a neutralino as possible dark matter relics [51, 42].

6.2 The invisible PQ axion

The theoretical prediction for the mass range in which to locate a PQ axion is currently below the eV region. The PQ solution to the strong CP problem has been formulated according to two main scenarios involving a light pseudoscalar $(a(x))$ which nowadays take the name from the initials of the proponents, the KSVZ axion (or hadronic axion) and the DFSZ [55, 56] axion, the latter introduced in a model which requires, in addition, a scalar sector with two Higgs doublets H_u and H_d, besides the PQ complex scalar Φ.
The small axion mass is attributed to a vacuum misalignment mechanism generated by the structure

of the QCD vacuum at the QCD phase transition, which causes a tilt in the otherwise flat PQ potential. The latter undergoes a symmetry breaking at a scale v_{PQ}, in general assumed to lay above the scales of inflation H_I and of reheating (T_R), and hence quite remote from the electroweak/confinement scales. Other possible locations of v_{PQ} with respect to H_I and T_R are also possible.

In both solutions the Peccei-Quinn scalar field Φ, displays an original symmetry which can be broken by gravitational effects, with a physical Goldstone mode $a(x)$ which remains such from the large v_{PQ} scale down to Λ_{QCD}, when axion oscillations occur. In the DFSZ solution, the axion emerges as a linear combination of the phases of the CP-odd sector and of Φ which are orthogonal to the hypercharge (Y) and are fixed by the normalization of the kinetic term of the axion field a. The solution to the strong CP problem is then achieved by rendering the parameter of the θ-vacuum dynamical, with the angle θ replaced by the axion field ($\theta \to a/f_a$), with f_a being the axion decay constant.

The computation of the axion mass m_a is then derived from the vacuum energy of the θ-vacuum $E(\theta)$ once this is re-expressed in terms of the QCD chiral Lagrangian, which in the two quark flavour (u,d) case describes the spontaneous breaking of the $SU(2)_L \times SU(2)_R$ flavour symmetry to a diagonal $SU(2)$ subgroup, with the 3 Goldstone modes $(\pi^{\pm}, \pi^0)$ being the dynamical field of the low energy dynamics. In this effective chiral description in which the θ parameter is present, the vacuum energy acquires a dependence both on neutral pseudoscalar π^0 and on θ of the form

$$E(\pi^0, \theta) = -m_\pi^2 f_\pi^2 \sqrt{\cos^2\frac{\theta}{2} + \left(\frac{m_d - m_u}{m_d + m_u}\right)^2 \sin^2\frac{\theta}{2}} \cos\left(\pi^0 - \phi(\theta)\right) \tag{6.2.1}$$

with

$$\phi(\theta) \equiv \frac{m_d - m_u}{m_d + m_u} \sin\frac{\theta}{2}. \tag{6.2.2}$$

At the minimum, when $\pi^0 = f_\pi \phi(\theta)$, the vacuum energy assumes the simpler form

$$E(\theta) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2\frac{\theta}{2}} \tag{6.2.3}$$

which expanded for small θ gives the well-know relation

$$E(\theta) = -m_\pi^2 f_\pi^2 + \frac{1}{2} m_\pi^2 f_\pi^2 \frac{m_u m_d}{(m_u + m_d)^2} \theta^2 + \dots \tag{6.2.4}$$

and the corresponding axion mass

$$m_a^2 = \frac{m_\pi^2 f_\pi^2}{f_a^2} \frac{m_u m_d}{(m_u + m_d)^2} \tag{6.2.5}$$

as $\theta \to a/f_a$. Before getting into a more detailed analysis of the various possible extensions of the traditional PQ scenarios, we briefly review the KSVZ (hadronic) and DFSZ (invisible) axion solutions.

6.2.1 KSVZ and DFSZ axions

In both the DFSZ and KSVZ scenarios a global anomalous $U(1)_{PQ}$ symmetry gets broken at some large scale v_{PQ}, with the generation of a Nambu-Goldstone mode from the CP-odd scalar sector. In the KSVZ case the theory includes a heavy quark Q which acquires a large mass by a Yukawa coupling with the scalar Φ. In this case the Lagrangian of Q takes the form

$$\mathcal{L} = |\partial\Phi|^2 + i\bar{Q}\not{D}\ Q + \lambda\Phi\bar{Q}_L Q_R + h.c. - V(\Phi) \tag{6.2.6}$$

with a global $U(1)_{PQ}$ chiral symmetry of the form

$$\Phi \rightarrow e^{i\alpha}\Phi \tag{6.2.7}$$

$$Q \rightarrow e^{-\frac{i}{2}\alpha\gamma_5}Q \tag{6.2.8}$$

with an $SU(3)_c$ covariant derivative (D) containing the QCD color charge of the heavy fermion Q. The scalar PQ potential can be taken of the usual Mexican-hat form and it is $U(1)_{PQ}$ symmetric. Parameterising the PQ field with respect to its broken vacuum

$$\Phi = \frac{\phi + v_{pq}}{\sqrt{2}} e^{i\frac{a(x)}{v_{PQ}}} + \ldots \tag{6.2.9}$$

the Yukawa coupling of the heavy quark Q to the CP-odd phase of Φ, $a(x)$, takes the form

$$\lambda\frac{v_{pq}}{\sqrt{2}} e^{i\frac{a(x)}{v_{PQ}}}\bar{Q}_L Q_R. \tag{6.2.10}$$

At this stage one assumes that there is a decoupling of the heavy quark from the low energy spectrum by assuming that v_{PQ} is very large. The standard procedure in order to extract the low energy interaction of the axion field is to first redefine the field Q on order to remove the exponential with the axion in the Yukawa coupling

$$e^{i\gamma_5\frac{a(x)}{2v_{PQ}}}Q_{L/R} \equiv Q'_{L/R}. \tag{6.2.11}$$

This amounts to a chiral transformation which leaves the fermionic measure non-invariant

$$\mathrm{D}\bar{Q}\mathrm{D}Q \rightarrow e^{i\int d^4x \frac{6a(x)}{32\pi^2 v_{PQ}} G(x)\tilde{G}(x)}\mathrm{D}\bar{Q}\mathrm{D}Q \tag{6.2.12}$$

and generates a direct coupling of the axion to the anomaly $G\tilde{G}$. Here the factor of 6 is related to the number of L/R components being rotated, which is 6 if Q is assigned to the triplet of $SU(3)_c$.

The kinetic term of Q is not invariant under this field redefinition and generates a derivative coupling of $a(x)$ to the axial vector current of Q. For n_f triplets, for instance, the effective action of the axion, up to dimension-5 takes the form

$$\mathcal{L}_{eff} = \frac{1}{2}\partial_\mu a(x)\partial^\mu a(x) + \frac{6n_f}{32\pi^2 v_{PQ}} a(x) G\tilde{G} + \frac{1}{v_{PQ}}\partial_\mu a \bar{Q}\gamma^\mu\gamma_5 Q + \dots \tag{6.2.13}$$

where we have neglected extra higher dimensional contributions, suppressed by v_{PQ}.
In the case of the DFSZ axion, the solution to the strong CP problem is found by introducing a scalar Φ together with two Higgs doublets H_u and H_d. In this case one writes down a general potential, function of these three fields, which is $SU(2) \times U(1)$ invariant and possesses a global symmetry

$$H_u \to e^{i\alpha X_u} H_u, \qquad H_d \to e^{i\alpha X_d} H_d, \qquad \Phi \to e^{i\alpha X_\Phi}\Phi \tag{6.2.14}$$

with $X_u + X_d = -2X_\Phi$. It is given by a combination of terms of the form

$$V = V(|H_u|^2, |H_d|^2, |\Phi|^2, |H_u H_d^\dagger|^2, |H_u \cdot H_d|^2, H_u \cdot H_d, \Phi^2) \tag{6.2.15}$$

where $H_u \cdot H_d$ denotes the $SU(2)$ invariant scalar product. The identification of the axion field is made by looking for a linear combination of the phases which is not absorbed by a gauge transformation. This can be done, for instance, by going to the unitary gauge and removing all the NG modes of the broken gauge symmetry. The corresponding phase, which is the candidate axion, is the result of a process of mixing of the PQ field with the Higgs sector at a scale where the symmetry of the potential is spontaneously broken by the two Higgs fields.

6.3 TeV scale: Stueckelberg axions in anomalous $U(1)$ extensions of the Standard Model

Intersecting D-brane models are one of those constructions where generalized axions appear [57, 58, 59, 60]. In the case in which several stacks of such branes are introduced, each stack being the domain in which fields with the gauge symmetry $U(N)$ live, several intersecting stacks generate at their common intersections, fields with the quantum numbers of all the unitary gauge groups of the construction, such as

$$U(N_1) \times U(N_2) \times ... \times U(N_k) = SU(N_1) \times U(1) \times SU(N_2) \times U(1) \times ... \times SU(N_k) \times U(1). \tag{6.3.1}$$

The phases of the extra $U(1)$'s are rearranged in terms of an anomaly-free generator, corresponding to an (anomaly free) hypercharge $U(1)$ (or $U(1)_Y$), times extra $U(1)$'s which are anomalous, carrying both their own anomalies and the mixed anomalies with all the gauge factors of the Standard Model. This general construction can be made phenomenologically interesting.

Using this approach, the Standard Model can be obtained by taking for example 3 stacks of branes: a first stack of 3 branes, yielding a $U(3)$ gauge symmetry, a second stack of 2 branes, yielding a symmetry $U(2)$ and an extra single $U(1)$ brane, giving a gauge structure of the form $SU(3) \times SU(2) \times U(1) \times U(1) \times U(1)$. Linear combinations of the generators of the three $U(1)$'s allow us to rewrite the entire abelian symmetry in the form $U(1)_Y \times U(1)' \times U(1)''$, with the remaining $U(1)' \times U(1)''$ factors carrying anomalies which need to be cancelled by extra operators. The simplest realization of the Standard Models (SM) is obtained by 2 stacks and a single brane at their intersections, giving a symmetry $U(3) \times U(2) \times U(1)$. In this case, in the hypercharge basis, the gauge structure of the model can be rewritten in the form $SU(3)_c \times SU(2)_w \times U(1)_Y \times U(1)' \times U(1)''$.

We consider the case of a single $U(1)' \equiv U(1)_B$ anomalous gauge symmetry, where the Stueckelberg field $b(x)$ couples to the gauge field B_μ by the gauge invariant term

$$\mathcal{L}_{St} = \frac{1}{2}\left(\partial_\mu b - M B_\mu\right)^2 \tag{6.3.2}$$

which is the well-known Stueckelberg form. M is the Stueckelberg mass. The Stueckelberg symmetry of the Lagrangian (6.3.2) is revealed by acting with gauge transformations of the gauge fields B_μ under which the axion b varies by a local shift

$$\delta_B B_\mu = \partial_\mu \theta_B \qquad\qquad \delta b = M\theta_B \tag{6.3.3}$$

parameterized by the local gauge parameters θ_B. Originally, the Stueckelberg symmetry was presented as a way to give a mass to an abelian gauge field while still preserving the gauge invariance of the theory. However, it is clear nowadays that its realization is the same one as obtained, for instance, in an abelian-Higgs model when one decouples the radial excitations of the Higgs fields from its phase [49]. The bilinear $\partial B b$ mixing present in Eq. (6.3.2) is an indication that the b field describes a Nambu-Goldstone mode which could, in principle, be removed by a unitary gauge condition. We will come back to this point later in this review. There is a natural way to motivate Eq. (6.3.2).

If we assume that the $U(1)_B$ gauge symmetry is generated within string theory and realized around the Planck scale, the massive anomalous gauge boson acquires a mass through the presence of an $A \wedge F$ coupling in the bosonic sector of a string-inspired effective action [61]. The starting Lagrangian of the effective theory involves, in this case, an antisymmetric rank-2 tensor $A_{\mu\nu}$ coupled to the field strength $F_{\mu\nu}$ of B_μ

$$\mathcal{L} \;=\; -\frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} - \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} + \frac{M}{4}\, \epsilon^{\mu\nu\rho\sigma} A_{\mu\nu}\, F_{\rho\sigma}, \tag{6.3.4}$$

where

$$H_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\rho A_{\mu\nu} + \partial_\nu A_{\rho\mu}, \qquad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{6.3.5}$$

is the kinetic term for the 2-form and g is an arbitrary constant. Besides the two kinetic terms for $A_{\mu\nu}$ and B_μ, the third contribution in Eq. (6.3.4) is the $A \wedge F$ interaction.

The Lagrangian is dualized by using a "first order" formalism, where H is treated independently from the antisymmetric field $A_{\mu\nu}$. This is obtained by introducing a constraint with a Lagrangian multiplier field $b(x)$ in order to enforce the condition $H = dA$ from the equations of motion of b, in the form

$$\mathcal{L}_0 = -\frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} - \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} - \frac{M}{6} \epsilon^{\mu\nu\rho\sigma} H_{\mu\nu\rho} B_\sigma + \frac{1}{6} b(x) \epsilon^{\mu\nu\rho\sigma} \partial_\mu H_{\nu\rho\sigma}. \tag{6.3.6}$$

The appearance of a scale M in this Lagrangian is crucial for the cosmological implications of such a theory [47], since it defines the energy region where the mechanism of anomaly cancellation comes into play [43]. The last term in (6.3.6) is necessary in order to reobtain (6.3.4) from (6.3.6). If, instead, we integrate by parts the last term of the Lagrangian given in (6.3.6) and solve trivially for H we find

$$H^{\mu\nu\rho} = -\epsilon^{\mu\nu\rho\sigma} \left(M B_\sigma - \partial_\sigma b \right), \tag{6.3.7}$$

and inserting this result back into (6.3.6) we obtain the expression

$$\mathcal{L}_A = -\frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \left(M B_\sigma - \partial_\sigma b \right)^2 \tag{6.3.8}$$

which is the Stueckelberg form for the mass term of B. This rearrangement of the degrees of freedom is an example of the connection between Lagrangians of antisymmetric tensor fields and their dual formulations which, in this specific case, is an abelian massive Yang-Mills theory in a Stueckelberg form.

The axion field, generated by the dualization mechanism, appears as a Nambu-Goldstone mode, which can be removed by a unitary gauge choice. However, as discussed in [43], the appearance, at a certain scale, of an extra potential which will mix this mode with the scalar sector, will allow to extract a physical component out of b, denoted by χ.

The origin of such a mixing potential is here assumed to be of non-perturbative origin and triggered at a scale below the Stueckelberg scale M. It is at this second scale where a physical axion appears in the spectrum of the theory. The local shift invariance of $b(x)$ is broken by the vev of the Higgs sector appearing in the part of the potential that couples the Stueckelberg field to the remaining scalars, causing a component of the Stueckelberg to become physical. The scale at which this second potential is generated and gets broken is the second scale controlling the mass of the axion, χ. Such a potential is by construction periodic in χ, as we are going to illustrate below and it is quite similar to the one discussed in Eq. (8.1.4). Its size is controlled by constants (λ_i) which are strongly suppressed by the exponential factor ($\sim e^{-S_{inst}}$, with S_{inst} the instanton action), determined by the value of the action in the instanton background.

In models with several $U(1)$'s this construction is slightly more involved, but the result of the mixing of the CP odd phases leaves as a remnant, also in this case, only one physical axion [43], whose mass is controlled by the size of the Higgs-axion mixing.

f	Q	u_R	d_R	L	e_R
q^B	q^B_Q	$q^B_{u_R}$	$q^B_{d_R}$	q^B_L	$q^B_{e_R}$

f	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_B$
Q	3	2	1/6	q^B_Q
u_R	3	1	2/3	$q^B_Q + q^B_u$
d_R	3	1	−1/3	$q^B_Q - q^B_d$
L	1	2	−1/2	q^B_L
e_R	1	1	−1	$q^B_L - q^B_d$
H_u	1	2	1/2	q^B_u
H_d	1	2	1/2	q^B_d

Table 6.1 Charges of the fermion and of the scalar fields

6.3.1 Stueckelberg models at the TeV scale with two-Higgs doublets

The type of models investigated in the past have been formulated around the TeV scale and discussed in detail in their various sectors [44, 45, 46, 47, 48, 49, 62] [63]. We offer a brief description of such realizations, which extend the symmetry of the SM minimally and as such are simpler than in other realizations involving larger gauge symmetries. They have the structure of effective actions where dimension-5 interactions are introduced in order to restore the gauge invariance of the Lagrangian in the presence of an anomalous gauge boson (and corresponding fermion spectrum). Therefore, they are quite different from ordinary anomaly-free versions of the same theories. They include one extra anomalous $U(1)_B$ symmetry, the Stueckelberg field and a set of scalars with a sufficiently wide CP odd sector in order to induce a mixing potential between the scalar fields and the Stueckelberg. Obviously, such models are of interest at the LHC for predicting anomalous gauge interactions in the form of extra neutral currents [47, 52] with respect to those of the electroweak sector.

The effective action has the structure given by

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{Yuk} + \mathcal{S}_{an} + \mathcal{S}_{WZ} \tag{6.3.9}$$

where $\mathcal{S}_0$ is the classical action. The same structure will characterize also other, more complex, realizations. It contains the usual gauge degrees of freedom of the Standard Model plus the extra anomalous gauge boson B which is already massive before electroweak symmetry breaking, via a Stueckelberg mass term, as it is clear from (6.3.8). We show the structure of the 1-particle irreducible effective action in Fig. 6.1. We consider a 2-Higgs doublet model for definiteness, which will set the ground for more complex extensions that we will address in the next sections. We consider an $SU(3)_c \times SU(2)_w \times U(1)_Y \times U(1)_B$ gauge symmetry model, characterized by an action $\mathcal{S}_0$, corresponding to the first contribution shown in Fig. 6.1, plus one loop corrections which are anomalous and break gauge invariance whenever there is an insertion of the anomalous gauge boson B_μ in the trilinear fermion

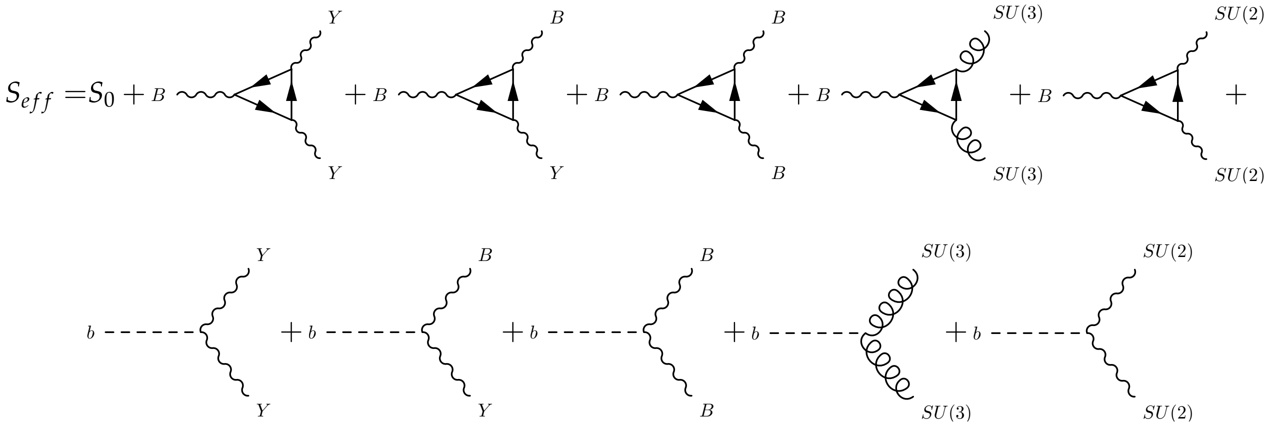

Figure 6.1 The 1PI effective action for a typical low scale model obtained by adding one extra anomalous $U(1)_B$ to the Standard Model action. Shown are the one-loop trilinear anomalous interactions and the corresponding counterterms, involving the b field.

vertices. In the last line of the same figure are shown the $(b/M)F \wedge F$ Wess-Zumino counterterms needed for restoring gauge invariance, which are suppressed by the Stueckelberg scale M. Table 6.1 shows the charge assignments of the fermion spectrum of the model, where we have indicated by q the charges for a single generation, having taken into account the conditions of gauge invariance of the Yukawa couplings. Notice that the two Higgs fields carry different charges under $U(1)_B$, which allow to extend the ordinary scalar potential of the two-Higgs doublet model by a certain extra contribution. This will be periodic in the axi-Higgs χ, after the two Higgses, here denoted as H_u and H_d, acquire a vev. Specifically, q_L^B, q_Q^B denote the charges of the left-handed lepton doublet (L) and of the quark doublet (Q) respectively, while $q_{u_r}^B, q_{d_r}^B, q_{e_R}^B$ are the charges of the right-handed $SU(2)$ singlets (quarks and leptons). We denote by $\Delta q^B = q_u^B - q_d^B$ the difference between the two charges of the up and down Higgses (q_u^B, q_d^B) respectively and from now on we will assume that it is non-zero. The trilinear anomalous gauge interactions induced by the anomalous $U(1)$ and the relative counterterms, which are all parts of the 1-loop effective action, are illustrated in Fig. 6.1.

6.3.2 Fermion/gauge field couplings

The models that we are discussing are characterized by one extra neutral current, mediated by a Z' gauge boson. The interaction of the fermions with the gauge fields is defined by the Lagrangian

$$\mathcal{L}_{int}^{quarks} = \begin{pmatrix} \bar{u}_{L\,i} & \bar{d}_{L\,i} \end{pmatrix} \gamma^{\mu} \left[-g_s T^a G^a_{\mu} - g_2 \tau^a W^a_{\mu} - \frac{1}{12} g_Y Y_{\mu} - \frac{1}{2} g_B q^B_Q B_{\mu} \right] \begin{pmatrix} u_{L\,i} \\ d_{L\,i} \end{pmatrix} +$$
$$+ \bar{u}_{R\,i} \gamma^{\mu} \left[-g_s T^a G^a_{\mu} - g_2 \tau^a W^a_{\mu} - \frac{1}{3} g_Y Y_{\mu} - \frac{1}{2} g_B q^B_{u_R} B_{\mu} \right] u_{R\,i}$$
$$+ \bar{d}_{R\,i}\, \gamma^{\mu} \left[-g_s T^a G^a_{\mu} - g_2 \tau^a W^a_{\mu} + \frac{1}{6} g_Y Y_{\mu} - \frac{1}{2} g_B q^B_{d_R} B_{\mu} \right] d_{R\,i}. \tag{6.3.10}$$

while the Higgs sector is characterized by the two Higgs doublets

$$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix} \qquad H_d = \begin{pmatrix} H_d^+ \\ H_d^0 \end{pmatrix} \tag{6.3.11}$$

where H_u^+, H_d^+ and H_u^0, H_d^0 are complex fields with (with some abuse of notation we rescale the fields by a factor of $1/\sqrt{2}$)

$$H_u^+ = \frac{\mathrm{Re}H_u^+ + i\mathrm{Im}H_u^+}{\sqrt{2}}, \qquad H_d^- = \frac{\mathrm{Re}H_d^- + i\mathrm{Im}H_d^-}{\sqrt{2}}, \qquad H_u^- = H_u^{+*}, \qquad H_d^+ = H_d^{-*}. \tag{6.3.12}$$

Expanding around the vacuum we get for the neutral components

$$H_u^0 = v_u + \frac{\mathrm{Re}H_u^0 + i\mathrm{Im}H_u^0}{\sqrt{2}}, \qquad H_d^0 = v_d + \frac{\mathrm{Re}H_d^0 + i\mathrm{Im}H_d^0}{\sqrt{2}}. \tag{6.3.13}$$

which will play a key role in determining the mixing of the Stueckelberg field in the periodic potential. The electroweak mixing angle is defined by $\cos\theta_W = g_2/g, \sin\theta_W = g_Y/g$, with $g^2 = g_Y^2 + g_2^2$. We also define $\cos\beta = v_d/v, \sin\beta = v_u/v$ with $v^2 = v_d^2 + v_u^2$. The matrix rotates the neutral gauge bosons from the interaction to the mass eigenstates after electroweak symmetry breaking and has elements which are $O(1)$, being expressed in terms of ratios of coupling constants, which correspond to mixing angles. It is given by

$$\begin{pmatrix} A_{\gamma} \\ Z \\ Z' \end{pmatrix} = O^A \begin{pmatrix} W_3 \\ A^Y \\ B \end{pmatrix} \tag{6.3.14}$$

which can be approximated to leading order as

$$O^A \simeq \begin{pmatrix} \frac{g_Y}{g} & \frac{g_2}{g} & 0 \\ \frac{g_2}{g} + O(\epsilon_1^2) & -\frac{g_Y}{g} + O(\epsilon_1^2) & \frac{g}{2}\epsilon_1 \\ -\frac{g_2}{2}\epsilon_1 & \frac{g_Y}{2}\epsilon_1 & 1 + O(\epsilon_1^2) \end{pmatrix} \tag{6.3.15}$$

where

$$\begin{aligned} \epsilon_1 &= \frac{x_B}{M^2}, \\ x_B &= \left(q_u^B v_u^2 + q_d^B v_d^2 \right). \end{aligned} \tag{6.3.16}$$

Once the WZ counterterms will be rotated into the gauge eigenstates and the b field into the physical χ field, there will be a direct coupling of the anomaly to the physical gauge bosons. This will involve both the neutral and the charged sectors. More details can be found in [46].

6.3.3 Counterterms

Fixing the values of the counterterms in simple single $U(1)$ models like the one we are reviewing, allows to gain some insight into the possible solutions of the gauge invariance conditions on the Lagrangian. The numerical values of the counterterms appearing in the second line of Fig. 6.1 are fixed by such conditions, giving

$$\begin{aligned} C_{BYY} &= -\frac{1}{6}q_Q^B + \frac{4}{3}q_{u_R}^B + \frac{1}{3}q_{d_R}^B - \frac{1}{2}q_L^B + q_{e_R}^B, \\ C_{YBB} &= -(q_Q^B)^2 + 2(q_{u_r}^B)^2 - (q_{d_R}^B)^2 + (q_L^B)^2 - (q_{e_R}^B)^2, \\ C_{BBB} &= -6(q_Q^B)^3 + 3(q_{u_R}^B)^3 + 3(q_{d_R}^B)^3 - 2(q_L^B)^3 + (q_{e_R}^B)^3, \end{aligned}$$

$$\begin{aligned} C_{Bgg} &= \frac{1}{2}(-2q_Q^B + q_{d_R}^B + q_{u_R}^B), \\ C_{BWW} &= \frac{1}{2}(-q_L^B - 3q_Q^B). \end{aligned} \tag{6.3.17}$$

They are, respectively, the counterterms for the cancellation of the mixed anomaly $U(1)_B U(1)_Y^2$ and $U(1)_Y U(1)_B^2$; the counterterm for the BBB anomaly vertex or $U(1)_B^3$ anomaly, and those of the $U(1)_B SU(3)^2$ and $U(1)_B SU(2)^2$ anomalies. They are defined in the Appendix. From the Yukawa couplings we get the following constraints on the $U(1)_B$ charges

$$q_Q^B - q_d^B - q_{d_R}^B = 0 \qquad q_Q^B + q_u^B - q_{u_R}^B = 0 \qquad q_L^B - q_d^B - q_{e_R}^B = 0. \tag{6.3.18}$$

Using the equations above, we can eliminate some of the charges in the expression of the counterterms, obtaining

$$
\begin{aligned}
C_{BYY} &= \frac{1}{6}(3q_L^B + 9q_Q^B + 8\Delta q^B), \\
C_{YBB} &= 2\left[q_d^B(q_L^B + 3q_Q^B) + 2\Delta q^B(q_d^B + q_Q^B) + (\Delta q^B)^2\right], \\
C_{BBB} &= (q_L^B - q_d^B)^3 + 3(q_d^B + q_Q^B + \Delta q^B)^3 + 3(q_Q^B - q_d^B)^3 - 2(q_L^B)^3 - 6(q_Q^B)^3, \\
C_{Bgg} &= \frac{\Delta q^B}{2}, \\
C_{BWW} &= \frac{1}{2}(-q_L^B - 3q_Q^B).
\end{aligned}
\tag{6.3.19}
$$

The equations above parametrize, in principle, an infinite class of models whose charge assignments under $U(1)_B$ are arbitrary, with the charges in the last column of Tab. (6.1) taken as their free parameters. The coupling of the axion to the corresponding gauge bosons can be fixed by a complete solution to the anomaly constraints, which may provide us with an insight into the possible mechanisms of misalignment that could take place at both the electroweak and at the QCD phase transitions.

6.3.4 Choice of the charges

Due to the presence, in general, of a nonvanishing mixed anomaly of the $U(1)_B$ gauge factor with both $SU(2)$ and $SU(3)$, the Stueckelberg axion of the model has interactions with both the strong and the weak sectors, which both support instanton solutions, and therefore could acquire a mass non-perturbatively both at the electroweak and at the QCD phase transitions. In this case we take into account the possibility of having sequential misalignments, with the largest contribution to the mass coming from the latter. Obviously, for a choice of charges characterized by $\Delta q = 0$, in which both doublets of the Higgs sector H_u and H_d carry the same charge under $U(1)_B$, the axion mass will not acquire any instanton correction at the QCD phase transition. In this case the potential responsible for Higgs-axion mixing would vanish. In this scenario a solution to the anomaly equations with a vanishing electroweak interaction of the Stueckelberg can be obtained by choosing $q_L^B = -3q_Q^B$.

If instead the charges are chosen in a way to have both non-vanishing weak (C_{BWW}) and strong (C_{Bgg}) counterterms, it is reasonable to expect that the misalignment of the axion potential will be sequential, with a tiny mass generated at the electroweak phase transition, followed by a second misalignment induced at the strong phase transition. The instanton configurations of the weak and strong sectors will be contributing differently to the mass of the physical axion. However, due to the presence of a coupling of this field with the strong sector, its mass will be significantly dominated by the QCD phase transition, as in the Peccei-Quinn case.

6.3.5 The scalar sector

The scalar sector of the anomalous abelian models is characterized, as already mentioned, by the ordinary electroweak potential of the SM involving, in the simplest formulation, two Higgs doublets $V_{PQ}(H_u, H_d)$ plus one extra contribution, denoted as $V_{\not{P}\not{Q}}(H_u, H_d, b)$ - or V' (PQ breaking) in [43] - which mixes the Higgs sector with the Stueckelberg axion b, needed for the restoration of the gauge invariance of the effective Lagrangian

$$V = V_{PQ}(H_u, H_d) + V_{\not{P}\not{Q}}(H_u, H_d, b). \tag{6.3.20}$$

The appearance of the physical axion in the spectrum of the model takes place after the phase-dependent terms - here assumed to be of non-perturbative origin and generated at a phase transition - find their way in the dynamics of the model and induce a curvature on the scalar potential. The mixing induced in the CP-odd sector determines the presence of a linear combination of the Stueckelberg field b and of the Goldstones of the CP-odd sector which acquires a tiny mass. From (6.3.20) we have a first term

$$V_{PQ} = \mu_u^2 H_u^\dagger H_u + \mu_d^2 H_d^\dagger H_d + \lambda_{uu}(H_u^\dagger H_u)^2 + \lambda_{dd}(H_d^\dagger H_d)^2 - 2\lambda_{ud}(H_u^\dagger H_u)(H_d^\dagger H_d) + 2\lambda'_{ud}|H_u^T \tau_2 H_d|^2 \tag{6.3.21}$$

typical of a two-Higgs doublet model, to which we add a second PQ breaking term

$$\begin{aligned} V_{\not{P}\not{Q}} = \; & \lambda_0 (H_u^\dagger H_d e^{-i g_B (q_u - q_d)\frac{b}{2M}}) + \lambda_1 (H_u^\dagger H_d e^{-i g_B (q_u - q_d)\frac{b}{2M}})^2 + \lambda_2 (H_u^\dagger H_u)(H_u^\dagger H_d e^{-i g_B (q_u - q_d)\frac{b}{2M}}) + \\ & \lambda_3 (H_d^\dagger H_d)(H_u^\dagger H_d e^{-i g_B (q_u - q_d)\frac{b}{2M}}) + \text{h.c.} \end{aligned} \tag{6.3.22}$$

These terms are allowed by the symmetry of the model and are parameterized by one dimensionful (λ_0) and three dimensionless couplings ($\lambda_1, \lambda_2, \lambda_3$). Their values are weighted by an exponential factor containing as a suppression the instanton action. In the equations below we will rescale λ_0 by the electroweak scale $v = \sqrt{v_u^2 + v_d^2}$ ($\lambda_0 \equiv \bar{\lambda}_0 v$) so as to obtain a homogeneous expression for the mass of χ as a function of the relevant scales of the model which are, besides the electroweak vev v the Stueckelberg mass M and the anomalous gauge coupling of the $U(1)_B$, g_B.

The gauging of an anomalous symmetry has some important effects on the properties of this pseudoscalar, first among all the appearance of independent mass and couplings to the gauge fields. This scenario allows then a wider region of parameter space in which one could look for such particles [44, 46, 49], rendering them "axion-like particles" rather than usual axions. We will still refer to them as axions for simplicity. So far only two complete models have been put forward for a consistent analysis of these types of particles, the first one non-supersymmetric [43] and a second one supersymmetric [51].

6.3.6 The potential for a generic Stueckelberg mass

The physical axion χ emerges as a linear combination of the phases of the various complex scalars appearing in combination with the b field. To illustrate the appearance of a physical direction in the phase of the extra potential, we focus our attention on just the CP-odd sector of the total potential, which is the only one that is relevant for our discussion. The expansion of this potential around the electroweak vacuum is given by the parameterization

$$H_u = \begin{pmatrix} H_u^+ \\ v_u + H_u^0 \end{pmatrix} \qquad H_d = \begin{pmatrix} H_d^+ \\ v_d + H_d^0 \end{pmatrix}. \tag{6.3.23}$$

where v_u and v_d are the two vevs of the Higgs fields. This potential is characterized by two null eigenvalues corresponding to two neutral Nambu-Goldstone modes (G_0^1, G_0^2) and an eigenvalue corresponding to a massive state with an axion component (χ). In the $(\mathrm{Im}H_d^0, \mathrm{Im}H_u^0, b)$ CP-odd basis we obtain the following normalized eigenstates

$$\begin{aligned}
G_0^1 &= \frac{1}{\sqrt{v_u^2+v_d^2}}(v_d, v_u, 0) \\
G_0^2 &= \frac{1}{\sqrt{g_B^2(q_d-q_u)^2 v_d^2 v_u^2 + 2M^2\left(v_d^2+v_u^2\right)}}\left(-\frac{g_B(q_d-q_u)v_d v_u^2}{\sqrt{v_u^2+v_d^2}}, \frac{g_B(q_d-q_u)v_d^2 v_u}{\sqrt{v_d^2+v_u^2}}, \sqrt{2}M\sqrt{v_u^2+v_d^2}\right) \\
\chi &= \frac{1}{\sqrt{g_B^2(q_d-q_u)^2 v_u^2 v_d^2 + 2M^2(v_d^2+v_u^2)}}\left(\sqrt{2}Mv_u, -\sqrt{2}Mv_d, g_B(q_d-q_u)v_d v_u\right)
\end{aligned} \tag{6.3.24}$$

and we indicate with O^χ the orthogonal matrix which allows to rotate them to the physical basis

$$\begin{pmatrix} G_0^1 \\ G_0^2 \\ \chi \end{pmatrix} = O^\chi \begin{pmatrix} \mathrm{Im}H_d^0 \\ \mathrm{Im}H_u^0 \\ b \end{pmatrix}, \tag{6.3.25}$$

which is given by

$$O^\chi = \begin{pmatrix} \frac{v_d}{v} & \frac{v_u}{v} & 0 \\ -\frac{g_B(q_d-q_u)v_d v_u^2}{v\sqrt{g_B^2(q_d-q_u)^2 v_d^2 v_u^2 + 2M^2 v^2}} & \frac{g_B(q_d-q_u)v_d^2 v_u}{v\sqrt{g_B^2(q_d-q_u)^2 v_d^2 v_u^2 + 2M^2 v^2}} & \frac{\sqrt{2}Mv}{\sqrt{g_B^2(q_d-q_u)^2 v_d^2 v_u^2 + 2M^2 v^2}} \\ \frac{\sqrt{2}Mv_u}{\sqrt{g_B^2(q_d-q_u)^2 v_u^2 v_d^2 + 2M^2 v^2}} & -\frac{\sqrt{2}Mv_d}{\sqrt{g_B^2(q_d-q_u)^2 v_u^2 v_d^2 + 2M^2 v^2}} & \frac{g_B(q_d-q_u)v_d v_u}{\sqrt{g_B^2(q_d-q_u)^2 v_u^2 v_d^2 + 2M^2 v^2}} \end{pmatrix} \tag{6.3.26}$$

where $v = \sqrt{v_u^2 + v_d^2}$.
χ inherits WZ interaction since b can be related to the physical axion χ and to the Nambu-Goldstone

modes via this matrix as

$$b = O^{\chi}_{13}G^1_0 + O^{\chi}_{23}G^2_0 + O^{\chi}_{33}\chi, \tag{6.3.27}$$

or, conversely,

$$\chi = O^{\chi}_{31}\mathrm{Im}H_d + O^{\chi}_{32}\mathrm{Im}H_u + O^{\chi}_{33}b. \tag{6.3.28}$$

Notice that the rotation of b into the physical axion χ involves a factor O^{χ}_{33} which is of order v/M. This implies that χ inherits from b an interaction with the gauge fields which is suppressed by a scale M^2/v. This scale is the product of two contributions: a $1/M$ suppression coming from the original Wess-Zumino counterterm of the Lagrangian ($b/MF\tilde{F}$) and a factor v/M obtained by the projection of b into χ due to O_{χ}.
The direct coupling of the axion to the physical gauge bosons via the Wess-Zumino counterterms is obtained by the usual rotation to the mass eigenstates which can be obtained from the rotation matrix O^A defined in (6.3.15). The final expression of the coupling of the axi-Higgs to the photon $g_{\chi\gamma\gamma}\chi F_{\gamma}\tilde{F}_{\gamma}$, is defined by a combination of matrix elements of the rotation matrices O^A and O^{χ}. Defining $g^2 = g_2^2 + g_Y^2$, the expression of this coefficient can be derived in the form

$$g^{\chi}_{\gamma\gamma} = \frac{g_B g_Y^2 g_2^2}{32\pi^2 M g^2} O^{\chi}_{33} \sum_f \left(-q^B_{f\,L} + q^B_{f\,R}\left(q^Y_{f\,R}\right)^2 - q^B_{f\,L}\left(q^Y_{f\,L}\right)^2 \right). \tag{6.3.29}$$

Notice that this expression is cubic in the gauge coupling constants, since factors such as g_2/g and g_Y/g are mixing angles while the factor $1/\pi^2$ originates from the anomaly. Therefore one obtains a general behaviour for $g^{\chi}_{\gamma\gamma}$ of $O(g^3 v/M^2)$, with charges which are, in general, of order unity.

6.3.7 Periodicity of the extra potential

Equivalently, it is possible to reobtain the results above by an analysis of the phases of the extra potential, which shows how this becomes periodic in χ, the axi-Higgs. This approach shows also quite directly the gauge invariance of χ as a physical pseudoscalar. In fact, if we opt for a polar parametrization of the neutral components in the broken phase

$$H^0_u = \frac{1}{\sqrt{2}}\left(\sqrt{2}v_u + \rho^0_u(x)\right) e^{i\frac{F^0_u(x)}{\sqrt{2}v_u}} \qquad H^0_d = \frac{1}{\sqrt{2}}\left(\sqrt{2}v_d + \rho^0_d(x)\right) e^{i\frac{F^0_d(x)}{\sqrt{2}v_d}}, \tag{6.3.30}$$

where we have introduced the two phases F_u and F_d of the two neutral Higgs fields, information on the periodicity is obtained by combining all the phases of V'

$$\theta(x) \equiv \frac{g_B(q_d - q_u)}{2M} b(x) - \frac{1}{\sqrt{2}v_u} F^0_u(x) + \frac{1}{\sqrt{2}v_d} F^0_d(x). \tag{6.3.31}$$

Using the matrix O^χ to rotate on the physical basis of the CP-odd scalar sector, the phase describing the periodicity of the potential turns out to be proportional to the physical axion χ, modulo a dimensionfull constant (σ_χ)

$$\theta(x) \equiv \frac{\chi(x)}{\sigma_\chi}, \tag{6.3.32}$$

where we have defined

$$\sigma_\chi \equiv \frac{2 v_u v_d M}{\sqrt{g_B^2 (q_d - q_u)^2 v_d^2 v_u^2 + 2M^2 (v_d^2 + v_u^2)}}. \tag{6.3.33}$$

Notice that σ_χ, in our case, takes the role of f_a of the PQ case, where the angle of misalignment is identified by the ratio a/f_a, with a the PQ axion.
As already mentioned, the re-analysis of the V' potential is particularly useful for proving the gauge invariance of χ under a $U(1)_B$ infinitesimal gauge transformation with gauge parameter $\alpha_B(x)$. In this case one gets

$$\begin{aligned}
\delta H_u &= -\frac{i}{2} q_u g_B \alpha_B H_u \\
\delta H_d &= -\frac{i}{2} q_d g_B \alpha_B H_d \\
\delta F_0^u &= -\frac{v_u}{\sqrt{2}} q_u g_B \alpha_B \\
\delta F_0^d &= -\frac{v_d}{\sqrt{2}} q_d g_B \alpha_B \\
\delta b &= -M - S\alpha_B
\end{aligned} \tag{6.3.34}$$

giving for (6.3.32) $\delta\theta = 0$. The gauge invariance under $U(1)_Y$ can also be easily proven using the invariance of the Stueckelberg field b under the same gauge group, sand the fact that the hypercharges of the two Higgses are equal. Finally, the invariance under $SU(2)$ is obvious since the linear combination of the phases that define $\theta(x)$ are not touched by the transformation.
From the Peccei-Quinn breaking potential we can extract the following periodic potential

$$V' = 4 v_u v_d \left(\lambda_2 v_d^2 + \lambda_3 v_u^2 + \lambda_0\right) \cos\left(\frac{\chi}{\sigma_\chi}\right) + 2\lambda_1 v_u^2 v_d^2 \cos\left(2\frac{\chi}{\sigma_\chi}\right), \tag{6.3.35}$$

with a mass for the physical axion χ given by

$$m_\chi^2 = \frac{2 v_u v_d}{\sigma_\chi^2}\left(\bar\lambda_0 v^2 + \lambda_2 v_d^2 + \lambda_3 v_u^2 + 4\lambda_1 v_u v_d\right) \approx \lambda v^2. \tag{6.3.36}$$

The size of the potential is driven by the combined product of non-perturbative effects, due to the exponentially small parameters $(\bar\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, with the electroweak vevs of the two Higgses. Notice also the irrelevance of the Stueckelberg scale M in determining the value of $\sigma_\chi \sim O(v)$ and of m_χ near the transition region, due to the large suppression factor λ in Eq. (6.5.16). One point that needs to be stressed is the fact that at the electroweak epoch the angle of misalignment generated by the extra potential is parameterized by χ/σ_χ, while the interaction of the physical axion with the gauge fields is suppressed by M^2/v. This feature is obviously unusual, since in the PQ case both scales reduce to a single scale, the axion decay constant f_a.

6.3.8 The Yukawa couplings and the axi-Higgs

The Yukawa couplings determine an interaction of the axi-Higgs to the fermions. This interaction is generated by the rotation in the CP-odd sector of the scalars potential, which mixes the CP-odd components, with the inclusion of the Stueckelberg b, via the matrix O_χ. The Yukawa couplings of the model are given by

$$\begin{aligned}
\mathcal{L}^{unit.}_{\rm Yuk} &= -\Gamma^d\,\bar{Q}H_d d_R - \Gamma^d\,\bar{d}_R H_d^\dagger Q - \Gamma^u\,\bar{Q}_L(i\sigma_2 H_u^*)u_R - \Gamma^u\,\bar{u}_R(i\sigma_2 H_u^*)^\dagger Q_L \\
&\quad -\Gamma^e\,\bar{L}H_d e_R - \Gamma^e\,\bar{e}_R H_d^\dagger L \\
&= -\Gamma^d\,\bar{d}H_d^0 P_R d - \Gamma^d\,\bar{d}H_d^{0*}P_L d - \Gamma^u\,\bar{u}H_u^{0*}P_R u - \Gamma^u\,\bar{u}H_u^0 P_L u \\
&\quad -\Gamma^e\,\bar{e}H_d^0 P_R e - \Gamma^e\,\bar{e}H_d^{0*}P_L e,
\end{aligned} \tag{6.3.37}$$

where the Yukawa coupling constants Γ^d, Γ^u and Γ^e run over the three generations, i.e. $u = \{u, c, t\}$, $d = \{d, s, b\}$ and $e = \{e,\ \mu,\ \tau\}$. Rotating the CP-odd and CP-even neutral sectors into the mass eigenstates and expanding around the vacuum one obtains

$$\begin{aligned}
H_u^0 &= v_u + \frac{ReH_u^0 + i\,ImH_u^0}{\sqrt{2}} \\
&= v_u + \frac{(h^0\sin\alpha - H^0\cos\alpha) + i\,\left(O^\chi_{11}G_0^1 + O^\chi_{21}G_0^2 + O^\chi_{31}\chi\right)}{\sqrt{2}}
\end{aligned} \tag{6.3.38}$$

$$\begin{aligned}
H_d^0 &= v_d + \frac{ReH_d^0 + i\,ImH_d^0}{\sqrt{2}} \\
&= v_d + \frac{(h^0\cos\alpha + H^0\sin\alpha) + i\,(O^\chi_{12}G_0^1 + O^\chi_{22}G_0^1 + O^\chi_{32}\chi)}{\sqrt{2}}
\end{aligned} \tag{6.3.39}$$

where the vevs of the two neutral Higgs bosons $v_u = v\sin\beta$ and $v_d = v\cos\beta$ satisfy

$$\tan\beta = \frac{v_u}{v_d}, \qquad v = \sqrt{v_u^2 + v_d^2}. \tag{6.3.40}$$

The fermion masses are given by

$$
\begin{aligned}
m_u &= v_u \Gamma^u, \qquad & m_\nu &= v_u \Gamma^\nu, \\
m_d &= v_d \Gamma^d, \qquad & m_e &= v_d \Gamma^e,
\end{aligned} \tag{6.3.41}
$$

where the generation index has been suppressed. The fermion masses, defined in terms of the two expectation values v_u, v_d of the model, show an enhancement of the down-type Yukawa couplings for large values of $\tan\beta$ while at the same time the up-type Yukawa couplings get a suppression. The couplings of the h^0 boson to fermions are given by

$$
\mathcal{L}_{\text{Yuk}}(h^0) = -\Gamma^d\, \bar{d}_L d_R \left(\frac{\cos\alpha}{\sqrt{2}} h^0\right) - \Gamma^u\, \bar{u}_L u_R \left(\frac{\sin\alpha}{\sqrt{2}} h^0\right) - \Gamma^e\, \bar{e}_L e_R \left(\frac{\cos\alpha}{\sqrt{2}} h^0\right) + c.c. \tag{6.3.42}
$$

The couplings of the H^0 boson to the fermions are

$$
\mathcal{L}_{\text{Yuk}}(H^0) = -\Gamma^d\, \bar{d}_L d_R \left(\frac{\sin\alpha}{\sqrt{2}} H^0\right) - \Gamma^u\, \bar{u}_L u_R \left(-\frac{\cos\alpha}{\sqrt{2}} H^0\right) - \Gamma^e\, \bar{e}_L e_R \left(\frac{\sin\alpha}{\sqrt{2}} H^0\right) + c.c. \tag{6.3.43}
$$

The interaction of χ with the fermions is proportional to the rotation matrix O^χ and to the mass of the fermion. The decay of the axi-Higgs is driven by two contributions, the direct point-like WZ interaction ($\chi/MF\tilde{F}$) and the fermion loop. The amplitude can be separated in the form corresponding to the two contributions from diagrams a) and b) of Fig. 6.2

$$
\mathcal{M}^{\mu\nu}(\chi \to \gamma\gamma) = \mathcal{M}^{\mu\nu}_{WZ} + \mathcal{M}^{\mu\nu}_{f}. \tag{6.3.44}
$$

The direct coupling related to the anomaly is given by the vertex shown in Fig. 6.2 a)

$$
\mathcal{M}^{\mu\nu}_{WZ}(\chi \to \gamma\gamma) = 4 g^\chi_{\gamma\gamma} \varepsilon[\mu, \nu, k_1, k_2] \tag{6.3.45}
$$

coming from the WZ counterterm $\chi F_\gamma \tilde{F}_\gamma$ which gives a decay rate of the form

$$
\Gamma_{WZ}(\chi \to \gamma\gamma) = \frac{m_\chi^3}{4\pi} (g^\chi_{\gamma\gamma})^2. \tag{6.3.46}
$$

We remark that $g^\chi_{\gamma\gamma}$ is of $O(g^3 v/M^2)$, as derived from Eq. (6.3.29), with charges that have been chosen of $O(1)$.
It is

Comparative studies of the decay rate into photons for the axi-Higgs with the ordinary PQ axion have been performed for a Stueckelberg scale confined in the TeV range and a mass of χ in the same range expected for the PQ axion. The analysis shows that the total decay rate of χ into photons is of the order $\Gamma_\chi \sim 10^{-50}$ GeV, which is larger than the decay rate of the PQ axion in the same channel (10^{-60}), but small enough to be long- lived, with a lifetime larger than the age of the universe. We show in Fig.

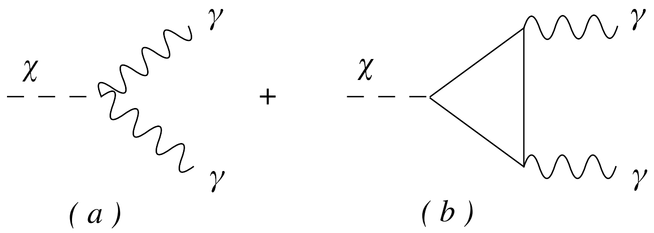

Figure 6.2 Contributions to the $\chi \to \gamma\gamma$ decay. describing the anomaly contribution (a) and the interaction mediated by the Yukava coupling in the fermion loop (b).

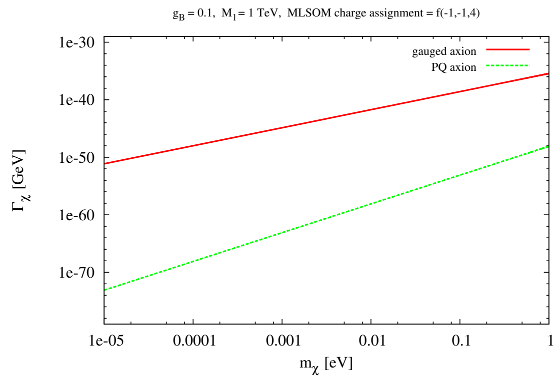

Figure 6.3 Total decay rate of the axi-Higgs for several mass values. Here, for the PQ axion, we have chosen $f_a = 10^{10}$ GeV.

6.3 the result of this study, where we compare predictions for the decay rate of the axi-Higgs into two photons to that of the ordinary PQ axion.

The charge assignment of the anomalous model have been denoted as $f(-1,1,4)$, where we have used the convention

$$f(q^B_{Q_L}, q^B_L, \Delta q^B) \equiv (q^B_{Q_L}, q^B_{u_R}; q^B_{d_R}, q^B_L, q^B_{e_R}, q^B_u, q^B_d). \tag{6.3.47}$$

These depend only upon the three free parameters $q^B_{Q_L}, q^B_L, \Delta q^B$. The parametric solution of the anomaly equations of the model $f(q^B_{Q_L}, q^B_L, \Delta q^B)$, for the particular choice $q^B_{Q_L} = -1, q^B_L = -1$, reproduces the entire charge assignment of a special class of intersecting brane models (see [58] and [61] and the discussion in [49])

$$f(-1,-1,4) = (-1,0,0,-1,0,+2,-2). \tag{6.3.48}$$

We refer to [41] for further details on these studies.

6.4 Relic density for a low (~ 1 TeV) Stueckelberg scale

The computation of the relic density for the Stueckelberg axi-Higgs can be performed as in [42], adopting a low scale scenario, where the extra V' (6.3.35) potential which causes the vacuum misalignment is generated around the electroweak scale.

One starts from the Lagrangian

$$\mathcal{S} = \int d^4x\sqrt{g}\left(\frac{1}{2}\dot{\chi}^2 - \frac{1}{2}m_\chi^2\Gamma_\chi\dot{\chi}\right), \tag{6.4.1}$$

where Γ_χ is the decay rate of the axion, where the potential has been expanded around its minimum up to quadratic terms. The same action can be derived from the quadratic approximation to the general expression

$$\mathcal{S} = \int d^4x R^3(t)\left(\frac{1}{2}\sigma_\chi^2\left(\partial_\alpha\theta\right)^2 - \mu^4\left(1-\cos\theta\right) - V_0\right) \tag{6.4.2}$$

which, as just mentioned, is constructed from the expression of V' given in Eq. (6.3.35). Here $\mu \sim v$, is the electroweak scale. We also set to zero other contributions to the vacuum potential ($V_0 = 0$). In a Friedmann-Robertson-Walker spacetime metric, with a scaling factor $R(t)$, this action gives the equation of motion

$$\frac{d}{dt}\left[\left(R^3(t)(\dot{\chi}+\Gamma_\chi\right)\right] + R^3 m_\chi^2(T) = 0. \tag{6.4.3}$$

We will neglect the decay rate of the axion in this case and set $\Gamma_\chi \approx 0$. At this point, we are free to set the scale at which the V' potential, which is of non-perturbative origin, is generated. Therefore it will be zero above the electroweak scale (or temperature T_{ew}), which will give $m_\chi = 0$ for $T \gg T_{ew}$. The general equation of motion derived from Eq. (6.4.3), introducing a temperature dependent mass, can be written as

$$\ddot{\chi} + 3H\dot{\chi} + m_\chi^2(T)\chi = 0, \tag{6.4.4}$$

which allows as a solution a constant value of the misalignment angle $\theta = \theta_i$. The axion energy density is given by

$$\rho = \frac{1}{2}\dot{\chi}^2 + \frac{1}{2}m_\chi^2\chi^2, \tag{6.4.5}$$

which after a harmonic averaging, due to the periodic motion, gives

$$\langle\rho\rangle = m_\chi^2\langle\chi^2\rangle. \tag{6.4.6}$$

By differentiating Eq. (6.4.5) and using the equation of motion in (6.4.4), followed by the averaging Eq. (6.4.6) one obtains the relation

$$\langle\dot{\rho}\rangle = \langle\rho\rangle \left(-3H + \frac{\dot{m}}{m}\right), \tag{6.4.7}$$

with a mass which is time-dependent through its temperature $T(t)$, while $H(t) = \dot{R}(t)/R(t)$ is the Hubble parameter. One easily finds that the solution of this equation is of the form

$$\langle\rho\rangle = \frac{m_\chi(T)}{R^3(t)} \tag{6.4.8}$$

which shows the decay of the energy density with an increasing space volume, valid even for a T-dependent mass. The condition for the oscillations of χ to take place is that the the universe has to be old at least as the the period of oscillation. Then the axion field starts oscillating and appears as dark matter, otherwise θ is misaligned but frozen. This is the physical content of the condition

$$m_\chi(T_i) = 3H(T_i), \tag{6.4.9}$$

which allows to identify the initial temperature of the coherent oscillation of the axion field χ, T_i, by equating $m_\chi(T)$ to the Hubble rate, taken as a function of temperature.
In the radiation era, the thermodynamics of all the components of the primordial state is entirely determined by the temperature T, being the system at equilibrium. This is because the contents of the early universe were in approximate thermal equilibrium, being the interaction rates of the constituents were large compared to the interaction rates H.
Pressure and entropy are then just given as a function of the temperature

$$\begin{aligned} \rho = 3p &= \frac{\pi^2}{30} g_{*,T} T^4 \\ s &= \frac{2\pi^2}{45} g_{*,S,T} T^3. \end{aligned} \tag{6.4.10}$$

Combined with the Friedmann equation they allow to relate the Hubble parameter and the energy density

$$H = \sqrt{\frac{8}{3}\pi G_N \rho}, \tag{6.4.11}$$

with $G_N = 1/M_P^2$ being the Newton constant and M_P the Planck mass. The number density of axions n_χ decreases as $1/R^3$ with the expansion, as does the entropy density $s \equiv S/R^3$, where S indicates the comoving entropy density, which remains constant in time, leaving the ratio $Y_a \equiv n_\chi/s$ conserved. An important variable is the abundance of χ at the temperatire of oscillations T_i, which is defined as

$$Y_\chi(T_i) = \frac{n_\chi}{s}\bigg|_{T_i}. \tag{6.4.12}$$

. At the beginning of the oscillations the total energy density is just the potential one

$$\rho_\chi = n_\chi(T_i)m_\chi(T_i) = 1/2 m_\chi^2(T_i)\chi_i^2, \tag{6.4.13}$$

giving for the initial abundance at $T = T_i$

$$Y_\chi(T_i) = \frac{1}{2}\frac{m_\chi(T_i)\chi_i^2}{s} = \frac{45 m_\chi(T_i)\chi_i^2}{4\pi^2 g_{*,S,T} T_i^3} \tag{6.4.14}$$

where we have used the expression of the entropy given by Eq. (6.4.10). At this point, by inserting the expression of ρ given in Eq. (6.4.10) into the expression of the Hubble rate as a function of density given by Eq. (6.4.11), the condition for oscillation Eq. (6.4.9) allows to express the axion mass at $T = T_i$ in terms of the effective massless degrees of freedom evaluated at the same temperature

$$m_\chi(T_i) = \sqrt{\frac{4}{5}\pi^3 g_{*,T_i}}\frac{T_i^2}{M_P}. \tag{6.4.15}$$

This gives for Eq. (6.4.14) the expression

$$Y_\chi(T_i) = \frac{45\sigma_\chi^2\theta_i^2}{2\sqrt{5\pi g_{*,T_i}}T_i M_P}, \tag{6.4.16}$$

where we have expressed χ in terms of the angle of misalignment θ_i at the temperature when oscillations start. Notice that we are assuming that $\theta_i = \langle\theta\rangle$ is the zero mode of the initial misalignment angle after an averaging.
$g_{*,T} = 110.75$ is the number of massless degrees of freedom of the model at the electroweak scale. Using the conservation of the abundance $Y_{a0} = Y_a(T_i)$, the expression of the contribution to the relic density is given by

$$\Omega_\chi^{mis} = \frac{n_\chi}{s}\bigg|_{T_i} m_\chi \frac{s_0}{\rho_c}. \tag{6.4.17}$$

To evaluate (6.4.17) we need the values of the critical energy density (ρ_c) and the entropy density today, which are estimated as

$$\rho_c = 5.2\cdot 10^{-6}\text{GeV/cm}^3 \qquad s_0 = 2970\ \text{cm}^{-3}, \tag{6.4.18}$$

with $\theta \simeq 1$. Given these values, the relic density as a function of $\tan\beta = v_u/v_d$, the ratio of the two Higgs vevs, is given in Fig. 6.4. In this plot we have varied the oscillation mass and plotted the relic

densities as a function of this variable. The variation of v_u has been constrained to give the values of the masses of the electroweak gauge bosons, via an appropriate choice of $\tan\beta$.
For instance, if we assume a temperature of oscillation of $T_i = 100$ GeV, an upper bound for the axi-Higgs mass, which allows the oscillations to take place, is $m_\chi(T_i) \approx 10^{-5}$eV, with $g_{*,T} \approx 100$.
In order to specify σ_χ we have assumed a value of 1 TeV for the Stueckelberg mass M_S, with a gauge coupling of the anomalous B_μ, $g_B \approx 1$, and we have taken (q_u, q_d) of order unity, obtaining $\sigma_\chi \simeq 10^2$ GeV. As we lower the oscillation temperature (and hence the mass), the corresponding curves for Ω_χ are down-shifted.
The plot shows that the values of these relic densities at current time are basically vanishing and these small results are to be attributed to the value of σ_χ, which is bound to vary around the electroweak scale. We remind that in the PQ case σ_χ is replaced by the large scale f_a at the QCD phase transition, which determines an enhancement of Ω_χ respect to the current case.
As already mentioned, nonperturbative instanton effects at the electroweak scale are expected to vastly suppress the mass of the axi-Higgs, as derived in (6.5.16), in the form

$$m_\chi^2 \sim \Lambda_{ew}^4/v^2, \qquad \text{with} \qquad \Lambda_{ew}^4 \sim \text{Exp}(-2\pi/\alpha_w(v))v^4 \tag{6.4.19}$$

$\alpha_W(v)$ being the weak charge at the scale v - which is indeed a rather small value since $\text{Exp}(-2\pi/\alpha_w(v)) \sim e^{-198}$. We will come back to this point in the next section, when discussing the possibility of raising M_S from the TeV range up to the GUT or Planck scales.
For this reason χ remains essentially a physical but frozen degree of freedom which may undergo a significant (second) misalignment only at the QCD phase transition. The possibility of sequential misalignments has been taken into account both in non supersymmetric [41] and in supersymmetric models [42]. It is the presence of a coupling of the axion to the gluons, via the color/ $U(1)_B$ mixed anomaly, that χ behaves, in this case, similar to a PQ axion. The misalignment is controlled by the periodic potential generated at the QCD phase transition, being the first misalignment at the electroweak scale irrelevant. In the absence of such mixed anomaly, χ could be classified as a quintessence axion, contributing to the dark energy content of the universe.

We show in Fig. 6.5 results of a numerical study of $\Omega_{mis}h^2$ as a function of M_S, expressed in units of 10^9 GeV. We show as a darkened area the bound coming from WMAP data [64], given as the average value plus an error band, while the monotonic curve denotes the values of $\Omega_{mis}h^2$ as a function of M_S. It is clear that the relic density of χ can contribute significantly to the dark matter content only if the Stueckelberg scale is rather large ($\sim 10^7$ GeV) and negligible otherwise.

In the next section we are going to address another scenario, where we will assume that the Stueckelberg scale is around the Planck scale and the breaking of the symmetry which allows to generate a periodic potential for the b fiels is taken at the GUT scale. This particular choice for the location of the two scales, which is well motivated in a string/brane theory context, opens up the possibility of having an ultra-light axion in the spectrum. The De Broglie wavelength of this hypothetical particle would be around 10 kpc, which is what is required to solve the issues

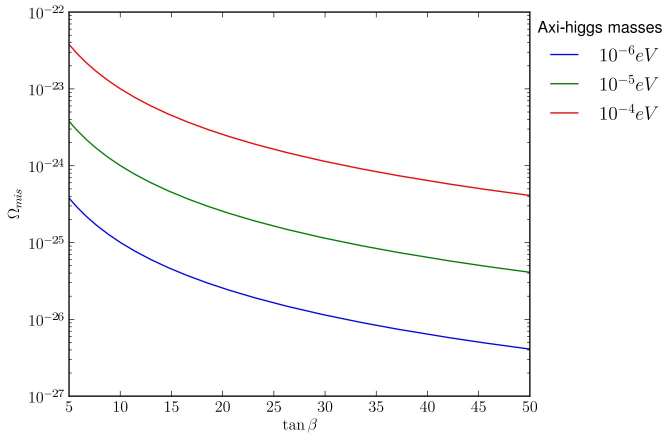

Figure 6.4 Relic density of the axi-Higgs as a function of $\tan\beta$ for several values of the mass of the axi-Higgs.

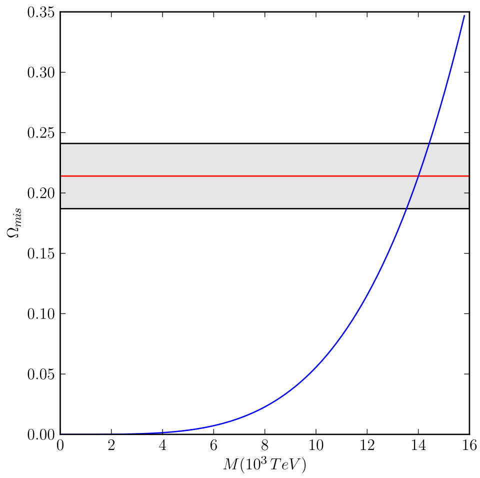

Figure 6.5 Relic density of the axi-Higgs as a function of M. The grey bar represents the measured value of $\Omega_{DM}h^2 = 0.1123 \pm 0.0035$

in the modelling of the matter distribution at the sub-galactic scale, that we have discussed in the introduction.

6.5 Stueckelberg models at the Planck/GUT scale and fuzzy dark matter

By raising the Stueckelberg mass near the Planck scale, the Stueckelberg construction acquires a fundamental meaning since it can be directly related to the cancellation of a gauge anomaly generated at the same scale [37]. As mentioned above, anomalous $U(1)$ symmetries are quite generally present in theories of intersecting branes. However, the very same structure emerges also in the low energy limit of heterotic string constructions. At the same time, as shown in [43], even in the presence of multiple anomalous abelian symmetries, only a single axion is necessary to cancel all anomalies, giving a special status to the Stueckelberg field. These considerations define a new context in which to harbour such models. In this context, it is natural to try to identify a consistent formulation within an ordinary gauge theory, by assuming that the axion emerges at the Planck scale M_P, but it acquires a mass at a scale below, which in our case is assumed to be the GUT scale. In this section therefore we are going to consider an extension of the setup discussed in previous sections, under the assumption that their dynamics is now controlled by two scales.

We will consider an E_6 based model, derived from E_8, which appeared in the heterotic string construction of [65] with an $E(8) \times E(8)$ symmetry. After a compactification of six spatial dimensions on a Calabi-Yau manifold [66] the symmetry is reduced to an $E(6)$ GUT gauge theory. Other string theory compactifications predict different GUT gauge structures, such as $SU(5)$ and $SO(10)$. The E_6, however, allows to realize a scenario where two components of dark matter are present, as we are going to elaborate. Fermions are assigned to the **27** representation of $E(6)$, which is anomaly-free. Notice that in $E(6)$ a PQ symmetry is naturally present, as shown in [40], which allows to have an ordinary PQ axion, while at the same time it is a realistic GUT symmetry which can break to the SM. This is the gauge structure to which one may append an anomalous $U(1)_X$ symmetry.
We consider a gauge symmetry of the form $E_6 \times U(1)_X$, where the gauge boson B^μ is in the Stueckelberg phase. B_α is the gauge field of $U(1)_X$ and $B_{\alpha\beta} \equiv \partial_\alpha B_\beta - \partial_\beta B_\alpha$ the corresponding field strength, while g_B its gauge coupling. As already mentioned, the $U(1)_X$ carries an anomalous coupling to the fermion spectrum.

The one-particle irreducible (1PI) effective Lagrangian of the theory at 1-loop level takes the form

$$\mathcal{L} = \mathcal{L}_{E_6} + \mathcal{L}_{St} + \mathcal{L}_{anom} + \mathcal{L}_{WZ}, \tag{6.5.1}$$

in terms of the gauge contribution of E_6 ($\mathcal{L}_{E_6}$), the Stueckelberg term $\mathcal{L}_{St}$, the anomalous 3-point functions $\mathcal{L}_{anom}$, generated by the anomalous fermion couplings to the $U(1)_X$ gauge boson, and the Wess-Zumino counterterm (WZ) $\mathcal{L}_{WZ}$. The Stueckelberg interaction to the E_6 gauge Lagrangian

$$\mathcal{L}_{E_6} = -\frac{1}{4} F^{(E_6)\,\mu\nu} F^{(E_6)}_{\mu\nu}, \tag{6.5.2}$$

which enables us to write the Stueckelberg part of the lagrangian as

$$\mathcal{L}_{Stueck} = -\frac{1}{4} B_{\alpha\beta} B^{\alpha\beta} - \frac{1}{2}(M B_\alpha - \partial_\alpha b(x))^2. \tag{6.5.3}$$

In this final form, M is the mass of the Stueckelberg gauge boson associated with $U(1)_X$ which we can be taken of the order of the Planck scale, guaranteeing the decoupling of the axion around M_{GUT}, due to the gravitational suppression of the WZ counterterms. The WZ contribution is the combination of two terms

$$\mathcal{L}_{WZ} = c_1 \frac{b}{M} F^{(E_6)\,\mu\nu} F^{(E_6)\,\rho\sigma} \epsilon_{\mu\nu\rho\sigma} + c_2 \frac{b}{M} B_{\mu\nu} B_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \tag{6.5.4}$$

needed for the cancellation of the $U(1)_X E_6 E_6$ and $U(1)_X^3$ anomalies, for appropriate values of the numerical constants c_1 and c_2, fixed by the charge assignments of the model. The three chiral familes will be assigned under $E(6) \times U(1)_X$ respectively to

$$\mathbf{27}_{X_1} \quad \mathbf{27}_{X_2} \quad \mathbf{27}_{X_3}, \tag{6.5.5}$$

in which the charges X_i ($i = 1,2,3$) are free at the moment, while the cancellation of the $U(1)^3_X$ and $E_6 \times U(1)^2_X$ anomalies implies that

$$\sum_{i=1}^{3} X_i^3 = 0, \qquad \sum_{i=1}^{3} X_i = 0. \tag{6.5.6}$$

These need to be violated in order to compensate with a Wess-Zumino term for the restoration of the gauge symmetry of the action.

Concerning the scalar sector, this contains two $\mathbf{351}_{X_i}$ $(i = 1,2)$ irreducible representations, where the $U(1)_X$ charges X_i need to be determined. The **351** is the *antisymmetric* part of the Kronecker product $\mathbf{27} \otimes \mathbf{27}$ where **27** is the defining representation of $E(6)$. The $\mathbf{351}_X$ can be conveniently described by the 2-form $A_{\mu\nu} = -A_{\nu\mu}$ with $\mu, \nu = 1$ to 27. The most general renormalizable potential in $\mathcal{L}_{E_6}$ is expressed in terms of $A^{(1)}_{\mu\nu}$ and $A^{(2)}_{\mu\nu}$ of $U(1)_X$ of charges x_1 and x_2 respectively. If we denote the $\mathbf{27}_{X_i}$ of Eq.(6.5.5) by Ψ_μ with $\mu = 1$ to 27 then the full Lagrangian including the potential V, has an invariance under the global symmetry

$$A^{(1)}_{\mu\nu} \to e^{i\theta} A^{(1)}_{\mu\nu} \quad A^{(2)}_{\mu\nu} \to e^{i\theta} A^{(2)}_{\mu\nu} \quad \Psi_\mu \to e^{-(\frac{1}{2}i\theta)} \Psi_\mu. \tag{6.5.7}$$

This is identifiable as a Peccei-Quinn symmetry which is broken at the GUT scale when $E(6)$ is broken to $SU(5)$ [40]. This axionic symmetry can be held responsible for solving the strong CP problem. We couple $A^{(1)}_{\mu\nu}$ to the fermion families $(\mathbf{27})_{X_i}$ $i = 1,2,3$. We choose in Eq. (6.5.5), *e.g.* $X_1 = X_2 = X_3 = +1$, with the X-charge of $A^{(1)}$ fixed to $X = -2$. The second scalar representation $A^{(2)}$ is decoupled from the fermions, with an $X-$charge for $A^{(2)}$ which is arbitrary and taken for simplicity to be $X = +2$. The potential is expressed in terms of three $E_6 \times U(1)_X$ invariant components,

$$V = V_1 + V_2 + V_p, \tag{6.5.8}$$

where

$$V_1 = F(A^{(1)}, A^{(1)}) \qquad V_2 = F(A^{(2)}, A^{(2)}), \tag{6.5.9}$$

with V_1 and V_2 denoting the contributions of $(\mathbf{351})_{-2}$ and $(\mathbf{351})_{+2}$, expressed in terms of the function

[40]

$$
\begin{aligned}
F(A^{(i)}, A^{(j)}) \;\; = \;\; & M^2_{GUT} A^{(i)}_{\mu\nu} \bar{A^{(j)}}^{\mu\nu} + h_1\, (A^{(i)}_{\mu\nu} \bar{A^{(j)}}^{\mu\nu})^2 + h_2\, A^{(i)}_{\mu\nu} \bar{A}^{\nu\sigma} A^{(i)}_{\sigma\tau} \bar{A}^{\tau\mu} \\
& + h_3\, d^{\mu\nu\lambda} d_{\xi\eta\lambda} A^{(i)}_{\mu\sigma} A^{(i)}_{\nu\tau} \bar{A^{(j)}}^{\xi\sigma} \bar{A^{(j)}}^{\eta\tau} \\
& + h_4\, d^{\mu\nu\alpha} d^{\sigma\tau\beta} d_{\xi\eta\alpha} d_{\lambda\rho\beta} A^{(i)}_{\mu\sigma} A^{(i)}_{\nu\tau} \bar{A^{(j)}}^{\xi\lambda} \bar{A^{(j)}}^{\eta\rho} \\
& + h_5\, d^{\mu\nu\alpha} d^{\sigma\beta\gamma} d_{\xi\eta\beta} d_{\lambda\alpha\gamma} A^{(i)}_{\mu\sigma} A^{(i)}_{\nu\tau} \bar{A^{(j)}}^{\xi\lambda} \bar{A^{(j)}}^{\eta\tau} \\
& + h_6\, d^{\mu\nu\alpha} d^{\sigma\tau\beta} d_{\alpha\beta\gamma} d^{\gamma\zeta\xi} d_{\xi\eta\zeta} d_{\lambda\rho\chi} A^{(i)}_{\mu\sigma} \bar{A^{(j)}}^{\xi\lambda} A^{(i)}_{\nu\tau} \bar{A^{(j)}}^{\eta\rho}\,, \qquad (6.5.10)
\end{aligned}
$$

in which $d_{\alpha\beta\gamma}$ with $\alpha, \beta, \gamma = 1$ to 27 is the $E(6)$ invariant tensor.
As for the two Higgs doublet model discussed in the previous sections, also in this case we are allowed to introduce a periodic potential on the basis of the underlying gauge symmetry, of the form

$$
\begin{aligned}
V_p \;\; = \;\; & M^2_{GUT} A^{(1)}_{\mu\nu} \bar{A^{(2)}}^{\mu\nu} e^{-i4\frac{b}{M_S}} + e^{-i8\frac{b}{M_S}} \Big[(h_1\, (A^{(1)}_{\mu\nu} \bar{A^{(2)}}^{\mu\nu})^2 + h_2\, A^{(1)}_{\mu\nu} \bar{A^{(2)}}^{\nu\sigma} A^{(1)}_{\sigma\tau} \bar{A^{(2)}}^{\tau\mu} \\
& + h_3\, d^{\mu\nu\lambda} d_{\xi\eta\lambda} A^{(1)}_{\mu\sigma} A^{(1)}_{\nu\tau} \bar{A^{(2)}}^{\xi\sigma} \bar{A^{(2)}}^{\eta\tau} \\
& + h_4\, d^{\mu\nu\alpha} d^{\sigma\tau\beta} d_{\xi\eta\alpha} d_{\lambda\rho\beta} A^{(1)}_{\mu\sigma} A^{(1)}_{\nu\tau} \bar{A^{(2)}}^{\xi\lambda} \bar{A^{(2)}}^{\eta\rho} \\
& + h_5\, d^{\mu\nu\alpha} d^{\sigma\beta\gamma} d_{\xi\eta\beta} d_{\lambda\alpha\gamma} A^{(1)}_{\mu\sigma} A^{(1)}_{\nu\tau} \bar{A^{(2)}}^{\xi\lambda} \bar{A^{(2)}}^{\eta\tau} \\
& + h_6\, d^{\mu\nu\alpha} d^{\sigma\tau\beta} d_{\alpha\beta\gamma} d^{\gamma\zeta\xi} d_{\xi\eta\zeta} d_{\lambda\rho\chi} A^{(1)}_{\mu\sigma} \bar{A^{(2)}}^{\xi\lambda} A^{(1)}_{\nu\tau} \bar{A^{(2)}}^{\eta\rho} \Big] + h.c. \qquad (6.5.11)
\end{aligned}
$$

and which becomes periodic at the GUT scale after symmetry breaking, similarly to the case considered in [41, 42]. This potential is expected to be of nonperturbative origin and generated at the scale of the GUT phase transition. Also in this case the size of the contributions in V_p, generated by instanton effects at the GUT scale, are expected to be exponentially suppressed. However, the size of the suppression is related to the value of the gauge coupling at the corresponding scale.

6.5.1 The periodic potential

The breaking of the $E_6 \times U(1)_X$ symmetry at M_{GUT} can follow different routes such as $E(6) \supset SU(3)_C \times SU(3)_L \times SU(3)_H$ where

$$
\begin{aligned}
(\mathbf{351}) \;\; = \;\; & (1,3^*,3) + (1,3^*,6^*) + (1,6,3) + (3,3,1) + (3,6^*,1) + (3,3,8) + \\
& (3^*,1,3^*) + (3^*,1,6) + (3^*,8,3^*) + (6^*,3,1) + (6,1,3^*) + (8,3^*,3) \qquad (6.5.12)
\end{aligned}
$$

of which the colour singlets are only the 45 states for each of the two $(\mathbf{351})_{X_i}$

$$
(1,3^*,3)_{X_i} \qquad (1,3^*,6^*)_{X_i} \qquad (1,6,3)_{X_I} \quad i = 1,2. \qquad (6.5.13)
$$

One easily realizes that there are exactly nine colour-singlet $SU(2)_L$-doublets in the $(\mathbf{351}^{'})_{-2}$ and 9 in the $(\mathbf{351}^{'})_{+2}$, that we may denote as $H_j^{(1)}$, $H_j^{(2)}$, with $j = 1, 2 \ldots 9$, which appear in the periodic potential in the form

$$V_p \sim \sum_{j=1}^{12} \lambda_0 M_{\rm GUT}^2 (H_j^{(1)\dagger} H_j^{(2)} e^{-4ig_B \frac{b}{M_S}}) + \sum_{j,k=1}^{12} \Big[\lambda_1 (H_j^{(1)\dagger} H_j^{(2)} e^{-i4g_B \frac{b}{M_S}})^2 + \lambda_2 (H_i^{(1)\dagger} H_i)(H_i^{(1)\dagger} H_j^{(2)} e^{-i4g_B \frac{b}{M_S}}) + \lambda_3 (H_k^{(2)\dagger} H_k^{(2)})(H_j^{(1)\dagger} H_k^{(2)} e^{-i4g_B \frac{b}{M_S}}) \Big] + \text{h.c.}, \tag{6.5.14}$$

where we are neglecting all the other terms generated from the decomposition (6.5.12) which will not contribute to the breaking. The assumption that such a potential is instanton generated at the GUT scale, with parameters λ_i's induces a specifc value of the instanton suppression which is drastically different from the case of a Stueckelberg scale located at TeV/multi TeV range.

For simplicity we will consider only a typical term in the expression above, involving two neutral components, generically denoted as $H^{(1)\,0}$ and $H^{(2)\,0}$, all the remaining contributions being similar. In this simplified case the axi-Higgs χ is generated by the mixing of the CP odd components of two neutral Higgses. The analysis follows rather closely the approach discussed before, in the simplest two-Higgs doublet model, which defines the template for such constructions.
Therefore, generalizing this procedure, the structure of V_p after the breaking of the $E_6 \times U(1)_X$ symmetry can be summarised in the form

$$V_p \sim v_1 v_2 \left(\lambda_2 v_2^2 + \lambda_3 v_1^2 + \overline{\lambda_0} M_{GUT}^2\right) \cos\left(\frac{\chi}{\sigma_\chi}\right) + \lambda_1 v_1^2 v_2^2 \cos\left(2\frac{\chi}{\sigma_\chi}\right), \tag{6.5.15}$$

with a mass for the physical axion χ given by

$$m_\chi^2 \sim \frac{2 v_1 v_2}{\sigma_\chi^2} \left(\bar{\lambda}_0 v_1^2 + \lambda_2 v_2^2 + \lambda_3 v_1^2 + 4\lambda_1 v_1 v_2\right) \approx \lambda v^2 \tag{6.5.16}$$

with $v_1 \sim v_2 \sim v \sim M_{GUT}$. Assuming that M_S, the Stueckelberg mass, is of the order of $M_{\rm Planck}$ and that the breaking of the $E_6 \times U(1)_X$ symmetry takes place at the GUT scale $M_{GUT} \sim 10^{15}$ GeV, (e.g. $v_1 \sim v_2 \sim M_{GUT}$) then

$$\sigma_\chi \sim M_{\rm GUT} + \mathcal{O}(M_{\rm GUT}^2/M_{\rm Planck}^2), \qquad m_\chi^2 \sim \lambda_0 M_{\rm GUT}^2, \tag{6.5.17}$$

where all the λ_i's in V_p are of the same order. The potential V_p being generated by the instanton sector, the size of the numerical coefficients appearing in its expression are constrained to specific values. One obtains $\lambda_0 \sim e^{-2\pi/\alpha(M_{\rm GUT})}$, with the value of the coupling $4\pi g_B^2 = \alpha_{\rm GUT}$ fixed at the GUT scale. If we assume that $1/33 \leq \alpha_{GUT} \leq 1/32$, then $e^{-201} \sim 10^{-91} \leq \lambda_0 \leq e^{-205} \sim 10^{-88}$, and the mass of the axion χ takes the approximate value

$$10^{-22}\,\text{eV} < m_\chi < 10^{-20}\,\text{eV}, \tag{6.5.18}$$

which contains the allowed mass range for an ultralight axion, as discussed in recent analysis of the astrophysical constraints on this type of dark matter [36].

6.5.2 Detecting ultralight axions

One of the interesting issues on which future research has to concentrate concerns the possibility of suggesting new ways for detecting such specific class of particles. Several proposals for the detection of generic ultralight bosons [67, 68, 69] in the astrophysical context have been recently presented. For instance, it has been observed that light boson fields around spinning black holes can trigger superradiant instabilities, which can be strong enough to imprint gravitational wave detection. This could be used to set constraints on their masses and couplings. Other proposals [70] have suggested to use the precise astronomical ephemeris as a way to detect such a light dark matter, as celestial solar system bodies feel the dark matter wind which acts as a resistant force opposing their motions. The bodies feel the dark matter wind because our solar system moves with respect to the rest frame of the dark matter halo, so that the scattering off the dark matter acts as a resistant force opposing their motions.
It is at the moment an open issue, from our perspective, how to distinguish between the various proposals that have been put forward in the recent literature. The models that we have presented are, however, very specific, since they are accompanied by a well defined gauge structure and are, as such, susceptible of in depth analysis. We should also mention that another specific property of such models is their interplay with the flavour sector, especially the neutrino sector, together with their impact on leptogenesis and $SO(10)$ grand unification. This would allow to establish a possible link between the neutrino mass spectrum and the axion mass and would be an intermediate step to cover prior to a discussion of the general astrophysical suggestions for their detections mentioned above. An in-depth analysis of some of these issues is underway.

6.6 Conclusions

The invisible axion owes its origin to a global $U(1)_{PQ}$ (Peccei-Quinn, PQ) symmetry which is spontaneously broken in the early universe and explicitly broken to a discrete Z_N symmetry by instanton effects at the QCD phase transition [71]. The breaking occurs at a temperature T_{PQ} below which the symmetry is nonlinearly realized. Two distinctive features of an axion solution - as derived from the original Peccei-Quinn (PQ) proposal [23] and its extensions [72, 73, 56, 55]- such as a) the appearance of a single scale f_a ($f_a \sim 10^{10} - 10^{12}$ GeV) which controls both their mass and their coupling to the gauge fields, via an $a(x)F\tilde{F}$ operator, where $a(x)$ is the axion field and b) their non-thermal decoupling at the hadron phase transition, attributed to a mechanism of vacuum misalignment. The latter causes axions to be a component of cold rather than hot dark matter, even for small values of their mass, currently expected to be in the μeV-meV range.

The gauging of an abelian anomalous symmetry brings in a generalization of the PQ scenario. As extensively discussed in [44, 45, 46, 47, 48, 49] it enlarges the parameter space for the corresponding axion. This construction allows to bypass the mass/coupling relation for ordinary PQ axions, which has been often softened in various analyses of "axion-like particles" [74].

Original analyses of Stueckelberg models, motivated within the theory of intersecting branes, where anomalous $U(1)$'s are present, have resulted in the identification of a special pseudoscalar field, the Stueckelberg field b. Its mixing with the CP-odd scalar sector allows to extract one gauge invariant component, called the axi-Higgs χ, whose mass and couplings to the gauge fields are model dependent. If string theory via its numerous possible geometric (and otherwise) compactifications [36] provides a natural arena where axion type of fields are ubiquitously present, then the possibility that an ultralight axion of this type is a component of dark matter is quite feasible. As we have discussed, its ultralight nature is a natural consequence of the implementation of the construction reflecting the low energy structure of the heterotic string theory by involving two scales, the Planck and the GUT scale. Given the mass of such axion, it is obvious that its search has to be inferred indirectly by astrophysical observations.

In short, we have seen that Stueckelberg models with an axion provide a new perspective on an old problem and allow to open up new directions in the search for the constituents of dark matter of our universe.

Chapter 7

Conformal Unification in a Quiver Theory and Gravitational Waves

Introduction

The detection of a stochastic background of gravitational waves can reveal details about first-order phase transitions (FOPTs) at a time of $10^{-13}s$ of the early universe. We specifically discuss quiver-type GUTs which avoid both proton decay and a desert hypothesis. A quiver based on $SU(3)^{12}$ which breaks at a $E = 4000$ GeV to trinification $SU(3)^3$ has a much larger ($g_* = 1,272$) number of effective massless degrees of freedom than the Standard Model. Assuming a FOPT for this model, we investigate the strain sensitivity of typical of this model for a wide range of FOPT parameters.
Since the discovery of gravitational waves from the merger of two black holes, each with mass $M_{BH} \sim 30\ M_{\odot}$, announced as event GW150914 in 2016[75] by the LIGO-Virgo Collaboration, it has become clear that this provides a new and invaluable window into the early universe. Many subsequent similar observations have occurred and of special interest is one where two neutron stars merger [76] where the event was shortly thereafter observed electromagnetically, thereby giving birth to multi-messenger astronomy.
The conventional way of seeking new physics at the highest possible energies is by particle colliders, with the highest energy of any active collider is at the LHC (Large Hadron Collider) with center of mass (com) energy 14 TeV. Possible colliders with center of mass energies up to 100 TeV are under discussion. In the early universe, such energy / temperature existed at cosmic times with $t < 10^{-16}$ s. To study higher energies or shorter cosmic times a method may be provided by gravitational wave detectors(GWDs) which can, in principle, be sensitive to signals generated from all cosmic times back to the Planck time $t \sim 10^{-44}$ s, which could lay bare 14 more orders of magnitude in energies up to the Planck, $M_{Planck} \sim 10^{19}$ GeV [77].
In the present article, we more conservatively study energies up to a few TeV which may overlap with accessible collider energies, yet where the detection of GWs could give additional information about

the type of phase transitions which occurred in the early universe. The discovery in [75] has already precipitated a number of papers (see, for instance, [78, 79, 80, 81, 82, 83, 84, 85, 86]) which discuss this possibility. This process takes off from the analysis of binary mergers, which can shed light on the quark-hadron phase transition [87], to far larger scales.
Although such experiments could eventually investigate phase transitions up to the GUT scale *e.g.* 10^{16} GeV, the earliest such linkage is likely to come at a much lower energy.
The advent of the AdS/CFT correspondence [88] between a maximally supersymmetric $\mathcal{N} = 4$ gauge theory with gauge group $SU(N)$, in a limit where $N \to \infty$, and a Type IIB superstring theory compactified on a manifold $AdS_5 \times S^5$, has introduced a hitherto unexpected connection between the two interactions. This has provided a number of insights into solution of problems in a broad range of theoretical physics.
To make a connection to particle phenomenology, it was then proposed that a generalisation of [88], which broke supersymmetry completely from $\mathcal{N} = 4$ to $\mathcal{N} = 0$ with finite N, should be considered. This was attained by using a generalised manifold $AdS_5 \times S^5/Z_p$, an orbifold, leading to a gauge group $SU(N)^p$ and matter fields most conveniently characterised as bifundamental and adjoint representations in a quiver diagram; hence the name quiver theory [89, 90, 91, 92].
One especially interesting example was discussed over a decade later [93, 94]. It uses the values $p = 12$ and $N = 3$ and gives rise to a theory which unifies at an unusually low energy scale $E \simeq 4$ TeV. Proton decay is absent due to the quiver construction. The goal of this work is to introduce the analysis of such models in a preliminary way, trying to uncover their possible impact on future GW research.
Given the significant interest in the detection of stochastic GWs, relics of the early universe, it is forseeable that such alternative scenarios to ordinary GUTs may draw the attention of new experimental proposals in the near future by LIGO [95], ET [96][97], MAGIS [98], AEDGE [99] and LISA [100] [101].

7.1 The Quiver Model

We use a different strategy for unification of electroweak theory with QCD than in GUTs based on $SU(5)$ or $SO(10)$. The choice of quiver is motivated by bottom-up considerations. The desert with logarithmic running of couplings is abandoned. Instead, the standard $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group is embedded in a semi-simple gauge group such as $SU(3)^p$ as suggested by gauge theories arising from compactification of the IIB superstring on an orbifold $AdS_5 \times S^5/\Gamma$ where Γ is the abelian finite group Z_p. In such non-supersymmetric quiver gauge theories the unification of couplings occurs abruptly at $\mu = M$ through the diagonal embeddings of 321 in $SU(3)^p$. The key prediction of such unification shifts from proton decay to additional particle content, in the present model at ~ 4 TeV. We use the RG β-functions from [102]. Taking the values at the Z-pole $\alpha_Y(M_Z) = 0.0101$, $\alpha_2(M_Z) = 0.0338$ and $\alpha_3(M_Z) = 0.118$, they are taken to run between M_Z and M according to the SM equations

$$\begin{aligned} \alpha_Y(M) &= (0.01014)^{-1} - (41/12\pi)\ln(M/M_Z) \\ &= 98.619 - 1.0876y \end{aligned} \tag{7.1.1}$$

$$\begin{aligned} \alpha^{-1}(M) &= 0.0338)^{-1} + (19/12\pi)\ln(M/M_Z) \\ &= 29.586 + 0.504y \end{aligned} \tag{7.1.2}$$

$$\begin{aligned} \alpha^{-1}(M) &= (0.118)^{-1} + (7/2\pi)\ln(M/M_Z) \\ &= 8.474 + 1.114y \end{aligned} \tag{7.1.3}$$

where $y = \log(M/M_Z)$.
The scale at which

$$\sin^2\theta(M) = \alpha_Y(M)/(\alpha_2(M) + \alpha_Y(M)) \tag{7.1.4}$$

satisfies $\sin^2\theta(M) = 1/4$ is at a value $M \simeq 4$ TeV.
We now focus on the ratio

$$R(M) \equiv \alpha_3(M)/\alpha_2(M). \tag{7.1.5}$$

We find that

$$R(M_Z) \simeq 3.5, \qquad R(M_3) = 3, \qquad R(M_{5/2}) = 5/2, \qquad R(M_2) = 2 \tag{7.1.6}$$

occur at the scales

$$M_3 \simeq 400\,\text{GeV}, \qquad M_{5/2} \simeq 4\,\text{TeV} \qquad \text{and} \qquad M_2 = 140\,\text{TeV}. \tag{7.1.7}$$

The proximity of $M_{5/2}$ and M, accurate to a few percent, suggests strong-electroweak unification at ~ 4 TeV. There remains the question of embedding such unification in an $SU(3)^p$ of the quiver type discussed in the Introduction.
Since the required embeddings of $SU(2)_L \times U(1)_Y$ into an $SU(3)$ necessitates $3\alpha_Y = \alpha_H$, the ratios of couplings at $\simeq 4$ TeV is

$$\alpha_{3C} : \alpha_{3W} : \alpha_{3H} :: 5 : 2 : 2 \tag{7.1.8}$$

and thus it is natural to examine $p = 12$ with diagonal embeddings of Colour (C), Weak (W) and Hypercharge (H) in $SU(3)^2, SU(3)^5, SU(3)^5$, respectively.

To accomplish this we specify the embedding of $\Gamma = Z_{12}$ in the global $SU(4)$ R-parity of the $\mathcal{N} = 4$ supersymmetry of the underlying theory.
Defining $\alpha = \exp(2\pi i/12)$, this specification can be made by

$$\mathbf{4} \equiv (\alpha^{A_1}, \alpha^{A_2}, \alpha^{A_3}, \alpha^{A_4}) \;\text{ with }\; \sum A_\mu = 0 \pmod{12} \tag{7.1.9}$$

and all $A_\mu \neq 0$ so that all four supersymmetries are broken from $\mathcal{N} = 4$ to $\mathcal{N} = 0$.
Having specified A_μ we calculate the content of complex scalars by investigating in $SU(4)$ the

$$\mathbf{6} \equiv (\alpha^{a_1}, \alpha^{a_2}, \alpha^{a_3}, \alpha^{-a_3}, \alpha^{-a_2}, \alpha^{-a_1}) \tag{7.1.10}$$

with

$$a_1 = A_1 + A_2, \quad a_2 = A_2 + A_3, \quad a_3 = A_3 + A_1 \pmod{12} \tag{7.1.11}$$

where all quantities are defined (mod 12). Finally we identify the nodes as C, W or H on the dodecahedral quiver such that the complex scalars

$$\sum_{i=1}^{3} \sum_{\alpha=1}^{12} (N_\alpha, \bar{N}_{\alpha+a_i}) \tag{7.1.12}$$

are adequate to allow the required symmetry breaking to the $SU(3)^3$ diagonal subgroup, and the chiral fermions given by

$$\sum_{\mu=1}^{4} \sum_{\alpha=1}^{12} \left(N_\alpha, \bar{N}_{\alpha+A_\mu}\right) \tag{7.1.13}$$

will be able to include the thee generations of fermions. These constraints are nontrivial but a solution was provided in [93].
The unique solution is to adopt $A_\mu \equiv (1, 2, 3, 6)$ and for the quiver nodes take the ordering:

$$- C - W - H - C - W^4 - H^4 - \tag{7.1.14}$$

with the two ends of Eq.(7.1.14) identified to form a dodecahedral quiver.
With this choice the scalars are provided by $A_I = (3, 4, 5)$ and are sufficient to break all the diagonal subgroups to

$$SU(3)_C \times SU(3)_W \times SU(3)_H \tag{7.1.15}$$

and the choice of quiver nodes in Eq. (7.1.14) generates precisely three quark lepton families which transform under Eq.(7.3.15) as

$$3\left[(3, \bar{3}, 1) + (1, 3, \bar{3}) + (\bar{3}, 1, 3)\right] \tag{7.1.16}$$

The ordering of the quiver nodes in Eq.(7.1.14) merits further explication.
The point is that breaking to a diagonal subgroup $SU(3)$ from $SU(3)^r$ is possible if and only if all the r nodes are connected by bifundamental scalars and no node is isolated. By trial and error, the reader can become convinced that Eq.(7.1.14) is the unique choice which satisfies this highly restrictive constraint.
Once the number of C, W and H nodes has been chosen in order that the three couplings accurately

unify, there is generally no quiver diagram which will allow the required symmetry breaking.
We have found only very few successful examples, one of which is studied assiduously in this article.
The choice of gauge group and matter fields is far less arbitrary than it may seem at first sight. The choice is unique, or nearly unique.
Anomaly freedom of the superstring guarantees that the only possible combination of chiral fermions is as in Eq. (7.1.16). This fact makes it easier to confirm the occurrence of three familes in the complicated quiver diagram because one needs to check only one of the three representations, for example the colour triplets which all originate from C nodes.
Further breaking to the SM group gives the correct light chiral states. The couplings run up to $E = M$ and then become frozen for at least a finite energy range provided conformal invariance sets in as expected by analogy with the supersymmetric case in [88].
At $M \sim 4$TeV, there are many new particles predicted by this scenario: gauge bosons, fermions and scalars. These are necessary to satisfy the conformal constraints discussed in [89].
This quiver model is interesting because it ameliorates the hierarchy problem in $SU(5)$ and $SO(10)$ GUTs between the weak / Higgs mass scale and the GUT scale. It predicts correctly the value of $\sin^2\theta(M_Z)$, of $\alpha_3(M_Z)$ and the appearance of exactly three families.
One final advantage is that the unification of the three SM couplings at $M \sim 4$ TeV is very precise, more accurate even than in SusyGUTs. This was shown, together with the robustness of the model, in [93].
We believe grand unification at 4 TeV has no disadvantage relative to unification at a trillion times higher scale, and has the advantage of avoiding the dubious desert hypothesis.
To clarify the quiver theory construction, we explain in more detail the case of the Z_{12} orbifold by exhibiting the relevant quiver diagrams. In this case the quiver diagram is a dodecahedron, like a clockface, with nodes labeled as indicated in Eq. (7.1.14).
Certain shortcuts make use of the symmetries of the quiver diagram and obviate including every possible link which will make the diagram very dense with links and more difficult to understand.
Let us begin with the chiral fermions which are denoted by oriented arrows between two nodes.
The quarks can be counted by examining the $C \rightarrow W$ links and subtracting the $W \rightarrow C$ links, noting that anomaly freedom dictates that the chiral fermions will necessarily appear only in the specific combination of Eq. (7.1.16) and so no other C links need to be checked. The relevant quiver diagram is shown in Fig. 1.

We see that there are five families and two antifamilies, resulting in precisely three light chiral families as required. The family-antifamily pairs are not chiral, but vector-like, and can therefore acquire Dirac masses.
Next we exhibit in Fig. 2 and Fig. 3 two further Z_{12} quiver diagrams which illustrate the scalar sector. Complex scalars are denoted by unoriented dashed lines. We must ensure and check that there are sufficient scalars whose VEVs can spontaneously break the $SU(3)^2_C$ down to $SU(3)_C$, the $SU(3)^5_W$

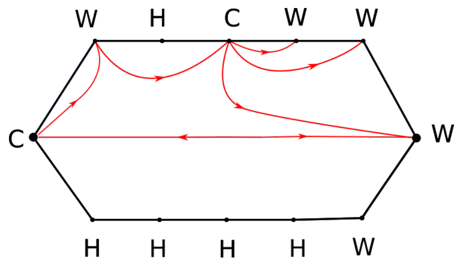

Figure 7.1 Quiver diagram showing quark states.

down to $SU(3)_W$ and finally $SU(3)_H^5$ down to $SU(3)_H$.

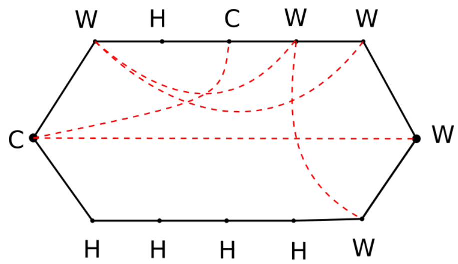

Figure 7.2 Scalar states which break $SU(3)_W^5$

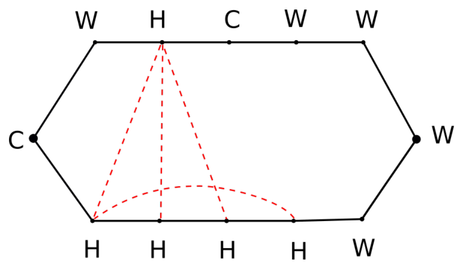

Figure 7.3 scalar states that break $SU(3)_H^5$

To break any $SU(3)^n$ down to the diagonal subgroup, a necessary and sufficient condition is that the bifundamental scalars link all n of the original gauge groups together. The $SU(3)$ gauge groups cannot be disconnected into subgroups nor can the bifundamental scalars separate into disconnected parts. In Fig. 2 the breakings of $SU(3)_C^2$ down to $SU(3)_C$ and of $SU(3)_W^5$ down to $SU(3)_W$ are shown to satisfy all these tight constraints so that the required spontaneous symmetry breaking is possible.
In Fig. 3 it is shown that the symmetry breaking $SU(3)_H^5$ down to $SU(3)_H$ also satisfies the same unforgiving connectivity requirements.
We note that this symmetry breaking is very nontrivial and is what underlies the correct identification of the nodes in Eq.(7.1.14), which is unique in allowing the required outcomes for both chiral fermions and complex scalars.
In fact, for 4 TeV grand unification without a desert, the $SU(3)^{12}$ construction appears to be unique

when one insists that one arrives at three chiral families under trinification, and hence under the Standard Model, as well as ensuring that correct symmetry breaking is permitted.
The GUT gauge group $SU(3)^{12}$ has dimension 96 which is bigger than the dimensions 24 and 45 of $SU(5)$ and $SO(10)$ respectively. This can be regarded as the price to pay to avoid the desert. The wealth of additional state at 4TeV also changes the nature of the phase transition $SU(3)^{12} \to SU(3)^3$ which can generate the gravitational waves studied in the next section.

7.2 Gravitational Waves

In the presence of a cosmological FOPT a new phase begins to nucleate as the universe cools down, with the region inside the bubbles containing the new phase. The latent free energy released detonates the transition on the bubble wall. A scalar field acquires a vev in a state of true vacuum in the interior of the bubble, and of false vacuum in the exterior, causing the expansion and the collision of the bubble walls. This takes place at a specific nucleation temperature T_n, which can be determined only by a combination of numerical simulations and an accurate study of the scalar potential, with the inclusion of thermal effects. The transition stops when the bubbles occupy all the volume.
Clearly, one expects a driving potential derived from the scalar sector which, in the simplest examples, is characterised by at least two separate scales. These scales identify the two local minima, separated by a maximum of considerable height, in order to guarantee a state of false vacuum of the system at the beginning of the nucleation phase [16, 17][103]. The presence of a false vacuum with a sufficient amount of supercooling is a natural requirement for having a strongly FOPT. In turn, these qualitative conditions point towards the possibility of having a significant emission of GWs.
As we have already mentioned, in the present quiver model, there is only one cosmological phase transition at a scale above the electroweak scale, at an energy/temperature of 4 TeV. As we are going to show, the quiver model allows a certain conformal scalar potential which breaks to trinification $SU(3)^3$, and that can be identified in its symmetry structure starting from its symmetry content. Therefore, this breaking is expected to induce vacuum transitions, from a local metastable minimum which is sufficiently trapped, to the true vacuum. We are going to describe it below. An additional element which afftects the transition are thermal effects. As usual, they can be taken into account by the inclusion of a Coleman-Weinberg term [104] $\mathcal{V}_{Eff}^{CW}$ at one-loop level and a finite temperature contribution [105] $\mathcal{V}_{Eff}^{T}$ to arrive at an effective potential

$$\mathcal{V}_{Eff} = \mathcal{V}_{Eff}^{Tree} + \mathcal{V}_{Eff}^{CW} + \mathcal{V}_{Eff}^{T}. \tag{7.2.1}$$

This defines the most general class of effective potentials discussed [106, 107, 108].
In our case, if we denote by Φ (without labels) a generic scalar field taken from any of the groups of scalar fields in Eq.(7.3.13), Eq.(7.3.16) and Eq.(7.3.18), the Coleman-Weinberg [104] term in the potential $\mathcal{V}_{Eff}$ is then

$$\mathcal{V}^{CW}_{Eff} = \frac{\lambda}{4!}\Phi^4 + \frac{\lambda^2\Phi^4}{256\phi^2}\left(\ln\frac{\Phi^2}{M^2} - \frac{25}{5}\right) \tag{7.2.2}$$

The Dolan-Jackiw-Weinberg finite temperature correction in terms of our generic scalar field Φ can be written [105, 109] as

$$\mathcal{V}^{T}_{Eff} = \frac{\pi^2 T^4}{90} + \frac{M^2T^2}{24} - \frac{1}{12\pi}M^3T - \frac{1}{64\pi^2}M^4\ln M^2T^2 + \frac{c}{64\pi^2}M^4 + O(M^6/T^2). \tag{7.2.3}$$

from which the effective potential is given by Eq. (7.2.1). We will comment on the the structure of the potential at zero temperature below.

7.3 Specific features of the quiver theory

Even in the absence of additional information about the way in which this transition takes place, due to the complexity of the scalar sector of the model, and within the assumption of a strongly FOPT, the goal of our analysis is to investigate the dependence of the GW emission on two peculiar parameters of the model, which are its large number of massless degrees of freedom and the relatively low transition temperature $T \sim 4$ TeV.

We shall need g_*, the equivalent number of massless degrees degrees of freedom for the quiver theory, defined by

$$g_* = n_B + \frac{7}{8}n_F \tag{7.3.1}$$

where n_B, n_F is the number for bosons, fermions respectively. It is easier to count g_* before spontaneous symmetry breaking, although of course the result is the same.

In the standard model with three families we have

$$\begin{aligned} n_B(\text{spin}=1) &= 12\times 2 = 24 \\ n_B(\text{spin}=0) &= 4 \\ n_F(\text{spin}=1/2) &= 3\times 15\times 2 = 90 \end{aligned} \tag{7.3.2}$$

so that in this case

$$g_* = 28 + \frac{7}{8}(90) = 106.75 \tag{7.3.3}$$

which will also be g_* for the quiver theory at energies $E < 4$ TeV.

In our present $SU(3)^{12}$ quiver theory we recall from the previous section that the scalars are in the bifundamental representations

$$\sum_{i=1}^{3}\sum_{\alpha=1}^{12}(3_\alpha, \bar{3}_{\alpha+a_i}) \tag{7.3.4}$$

with $a_1 = 3, 4, 5$, and the chiral fermions are in bifundamentals

$$\sum_{\mu=1}^{4}\sum_{\alpha=1}^{12}\left(3_\alpha, \bar{3}_{\alpha+A_\mu}\right) \tag{7.3.5}$$

with $A_\mu = 1, 2, 3, 6$.

The equivalent massless degrees of freedom are

$$\begin{aligned} n_B(\text{spin} = 1) &= 96 \times 2 = 192 \\ n_B(\text{spin} = 0) &= 12 \times 9 \times 3 = 324 \\ n_F(\text{spin} = 1/2) &= 12 \times 18 \times 4 = 864 \end{aligned} \tag{7.3.6}$$

so that for the full quiver theory

$$g_* = 516 + \frac{7}{8}(864) = 1,272 \tag{7.3.7}$$

which is the number of effective massless degrees of freedom for $E \geq 4$ GeV. In our present $SU(3)^{12}$ quiver theory we recall from the previous section that the scalars are in the bifundamental representations

$$\sum_{i=1}^{3}\sum_{\alpha=1}^{12}(3_\alpha, \bar{3}_{\alpha\pm a_i}) \tag{7.3.8}$$

with $a_1 = 3, 4, 5$, and the chiral fermions are in bifundamentals

$$\sum_{\mu=1}^{4}\sum_{\alpha=1}^{12}\left(3_\alpha, \bar{3}_{\alpha+A_\mu}\right) \tag{7.3.9}$$

with $A_\mu = 1, 2, 3, 6$.
The equivalent massless degrees of freedom are

$$\begin{aligned} n_B(\text{spin} = 1) &= 96 \times 2 = 192 \\ n_B(\text{spin} = 0) &= 12 \times 9 \times 3 = 324 \\ n_F(\text{spin} = 1/2) &= 12 \times 18 \times 4 = 864 \end{aligned} \tag{7.3.10}$$

so that for the full quiver theory

$$g_* = 516 + \frac{7}{8}(864) = 1,272 \tag{7.3.11}$$

which is the number of effective massless degrees of freedom for $E \geq 4$ GeV.
We pause for few remarks.
The nature of the phase transition depends on the effective potential of the theory. Eq.(7.3.8) exhibits the scalars present in the quiver and the twelve nodes of the quiver are identified in Eq.(7.1.14). The dodecahedral quiver has nodes which we label clockwise by 1 to 12 by Color(C), Weak (W) and Hypercharge (H) as follows:

$$(1)\,C \to (2)\,W \to (3)\,H \to (4)\,C \to (5-8)\,W \to (9-12)\,H \tag{7.3.12}$$

We are initially concerned with the breaking $SU(3)^{12} \to SU(3)^3$ at scale $E = 4$ TeV. This can be studied separately for C, W and H in Eq.(7.3.12).
Let us define lower-case Greek indices $\alpha_i, \beta_i, \gamma_i, \delta_i \ldots = 1, 2, 3$ for the SU(3) group of the i^{th} node and discriminate between subscripts which represent defining representations and superscripts which denote anti-defining representations.
From Eq. (7.3.12) the SM color gauge group arises from the diagonal subgroup of the SU(3)'s at nodes 1 and 4 respectively, and this symmetry breaking is achieved by VEVs of the complex scalar bifundamentals:

$$\Phi_{\alpha_1}^{\beta_4} \text{ and } \Phi_{\alpha_4}^{\beta_1} \tag{7.3.13}$$

In the effective potential at tree level there are quadratic and quartic terms involving the 1 to 4 bifundamentals as follows

$$\mathcal{V}_{Eff}^{(C,Tree)} = \mathcal{C}_2^{(14)} \left(\Phi_{\alpha_1}^{\beta_4} \Phi_{\alpha_1}^{\beta_4}\right) + \mathcal{C}_4^{(14)} \left(\Phi_{\alpha_1}^{\beta_4} \Phi_{\alpha_1}^{\beta_4}\right)^2 + \mathcal{C}_4^{(14)'} \left(\Phi_{\alpha_1}^{\beta_4} \Phi_{\beta_4}^{\gamma_1} \Phi_{\gamma_1}^{\delta_4} \Phi_{\delta_4}^{\alpha_1}\right) \tag{7.3.14}$$

To break to the trinification group

$$SU(3)_C \times SU(3)_W \times SU(3)_H \tag{7.3.15}$$

a similar combination of bifundamental scalars conspire to arrive at diagonal subgroups for both the five $SU(3)_W$ nodes and the five $SU(3)_H$ nodes respectively. Another intermediate symmetry-breaking stage is where $SU(3)_W$ in Eq.(7.3.15) breaks to the $SU(2)_L$ of the SM, also $SU(3)_W \times SU(3)_H$ breaks to the $U(1)_Y$ of the SM but for our analysis of gravitational radiation we shall focus only on a FOPT where the quiver group $SU(3)^{12}$ breaks at $E = 4$ TeV to the trinification group in Eq.(7.3.15).
For W we use scalars connecting nodes 2-5-6-7-8 and the relevant scalar bifundamentals in Eq.(7.3.8)

are

$$\Phi_{\alpha_2}^{\beta_5} \;,\; \Phi_{\alpha_2}^{\beta_6} \;,\; \Phi_{\alpha_2}^{\beta_7} \text{ and } \Phi_{\alpha_5}^{\beta_8} \tag{7.3.16}$$

The corresponding quadratic and quartic terms in the tree-level effective potential composed of the scalars in Eq.(7.3.16) are

$$\begin{aligned}
\mathcal{V}_{Eff}^{(W,Tree)} &= \mathcal{C}_2^{(25678)} \left(\Phi_{\alpha_2}^{\beta_5}\Phi_{\beta_5}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_6}\Phi_{\beta_6}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_7}\Phi_{\beta_7}^{\alpha_2} + \Phi_{\alpha_5}^{\beta_8}\Phi_{\beta_8}^{\alpha_5} \right) \\
&+ \mathcal{C}_4^{(25678)} \left(\Phi_{\alpha_2}^{\beta_5}\Phi_{\beta_5}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_6}\Phi_{\beta_6}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_7}\Phi_{\beta_7}^{\alpha_2} + \Phi_{\alpha_5}^{\beta_8}\Phi_{\beta_8}^{\alpha_5} \right)^2 \\
&+ \mathcal{C}_4^{(25678)'} \left(\Phi_{\alpha_2}^{\beta_5}\Phi_{\beta_5}^{\gamma_2}\Phi_{\gamma_2}^{\delta_5}\Phi_{\delta_5}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_6}\Phi_{\beta_6}^{\gamma_2}\Phi_{\gamma_2}^{\delta_6}\Phi_{\delta_6}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_7}\Phi_{\beta_7}^{\gamma_2}\Phi_{\gamma_2}^{\delta_7}\Phi_{\delta_7}^{\alpha_2} + \Phi_{\alpha_5}^{\beta_8}\Phi_{\beta_8}^{\gamma_5}\Phi_{\gamma_5}^{\delta_8}\Phi_{\delta_8}^{\alpha_5} \right)
\end{aligned} \tag{7.3.17}$$

For H we use scalars connecting nodes 3-9-10-11-12 and the relevant scalar bifundamentals in Eq.(7.3.8) are.

$$\Phi_{\alpha_3}^{\beta_{10}} \;,\; \Phi_{\alpha_3}^{\beta_{11}} \;,\; \Phi_{\alpha_3}^{\beta_{12}} \text{ and } \Phi_{\alpha_9}^{\beta_{12}} \tag{7.3.18}$$

The corresponding quadratic and quartic terms in the tree-level effective potential composed of the scalars in Eq.(7.3.18) are

$$\begin{aligned}
\mathcal{V}_{Eff}^{(H,Tree)} &= \mathcal{C}_2^{(39101112)} \left(\Phi_{\alpha_3}^{\beta_{10}}\Phi_{\beta_{10}}^{\alpha_3} + \Phi_{\alpha_3}^{\beta_{11}}\Phi_{\beta_{11}}^{\alpha_3} + \Phi_{\alpha_3}^{\beta_{12}}\Phi_{\beta_{12}}^{\alpha_3} + \Phi_{\alpha_9}^{\beta_{12}}\Phi_{\beta_{12}}^{\alpha_9} \right) \\
&+ \mathcal{C}_4^{(39101112)} \left(\Phi_{\alpha_3}^{\beta_{10}}\Phi_{\beta_{10}}^{\alpha_3} + \Phi_{\alpha_3}^{\beta_{11}}\Phi_{\beta_{11}}^{\alpha_3} + \Phi_{\alpha_3}^{\beta_{12}}\Phi_{\beta_{12}}^{\alpha_3} + \Phi_{\alpha_9}^{\beta_{12}}\Phi_{\beta_{12}}^{\alpha_9} \right)^2 \\
&+ \mathcal{C}_4^{(39101112)'} \left(\Phi_{\alpha_3}^{\beta_{10}}\Phi_{\beta_{10}}^{\gamma_3}\Phi_{\gamma_3}^{\delta_{10}}\Phi_{\delta_{10}}^{\alpha_3} + \Phi_{\alpha_3}^{\beta_{11}}\Phi_{\beta_{11}}^{\gamma_3}\Phi_{\gamma_3}^{\delta_{11}}\Phi_{\delta_{11}}^{\alpha_3} \right. \\
&\qquad \left. + \Phi_{\alpha_3}^{\beta_{12}}\Phi_{\beta_{12}}^{\gamma_3}\Phi_{\gamma_3}^{\delta_{12}}\Phi_{\delta_{12}}^{\alpha_3} + \Phi_{\alpha_9}^{\beta_{12}}\Phi_{\beta_{12}}^{\gamma_9}\Phi_{\gamma_9}^{\delta_{12}}\Phi_{\delta_{12}}^{\alpha_9} \right)
\end{aligned} \tag{7.3.19}$$

Because the quiver theory above 4 TeV is conformal we must impose $C_2 = 0$ in all the quadratic terms. Next, before adding the three $\mathcal{V}_{Eff}^{Tree}$ expressions, let us examine the symmetries of the dodecahedral quiver which imply that

$$\begin{aligned}
C_4^{(25678)} &= C_4^{(39101112)} \equiv D_4 \\
C_4^{(25678)'} &= C_4^{(39101112)'} \equiv D_4'
\end{aligned} \tag{7.3.20}$$

whereupon, suppressing superscripts, the most general tree-level effective potential is

$$\mathcal{V}_{Eff}^{Tree} = \mathcal{C}_4 \left(\Phi_{\alpha_1}^{\beta_4}\Phi_{\alpha_1}^{\beta_4}\right)^2 + \mathcal{C}_4' \left(\Phi_{\alpha_1}^{\beta_4}\Phi_{\beta_4}^{\gamma_1}\Phi_{\gamma_1}^{\delta_4}\Phi_{\delta_4}^{\alpha_1}\right) + \mathcal{D}_4 \left(\Phi_{\alpha_2}^{\beta_5}\Phi_{\beta_5}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_6}\Phi_{\beta_6}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_7}\Phi_{\beta_7}^{\alpha_2} + \Phi_{\alpha_5}^{\beta_8}\Phi_{\beta_8}^{\alpha_5}\right)^2$$
$$+\mathcal{D}_4' \left(\Phi_{\alpha_2}^{\beta_5}\Phi_{\beta_5}^{\gamma_2}\Phi_{\gamma_2}^{\delta_5}\Phi_{\delta_5}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_6}\Phi_{\beta_6}^{\gamma_2}\Phi_{\gamma_2}^{\delta_6}\Phi_{\delta_6}^{\alpha_2} + \Phi_{\alpha_2}^{\beta_7}\Phi_{\beta_7}^{\gamma_2}\Phi_{\gamma_2}^{\delta_7}\Phi_{\delta_7}^{\alpha_2} + \Phi_{\alpha_5}^{\beta_8}\Phi_{\beta_8}^{\gamma_5}\Phi_{\gamma_5}^{\delta_8}\Phi_{\delta_8}^{\alpha_5}\right). \tag{7.3.21}$$

7.4 Production of GW and the parameters choice

As mentioned in the previous sections, the two main features of our quiver model are the large number of massless degrees of freedom present at the phase transition and the relatively low temperature at which the unification of the gauge couplings is reached. Given the complexity of the scalar sector of the model, it is beyond the scope of the current analysis to provide further details about the way the conformal symmetry is broken, with the generation of appropriate scales in the potential which would allow vacuum and thermal transitions of significant strength. As already mentioend, we will simply assume that this is possible, leaving a discussion of this issue to future work.
We recall that the energy density of the gravitational wave is measured (today) by the variables

$$h_0^2\Omega_{GW}(f) \equiv \left(\frac{h_0^2}{\rho_c}\frac{d\,\rho_{GW}}{d\,\log f}\right)_0 \tag{7.4.1}$$

expressed in frequency (f) octaves, in which we are going to separate the various contributions.
Thus we may write, for the final contribution to the energy density, as a fraction of the critical density:

$$\Omega_{GW}(f) = \Omega_{GW}^{Coll}(f) + \Omega_{GW}^{SW}(f) + \Omega_{GW}^{Turb}(f) \tag{7.4.2}$$

where the terms on the right hand side denote contributions sourced by bubble collisions, sound waves, and plasma turbulence, respectively.

For the contribution $\Omega_{GM}^{Coll}(f)$ from the bubble collisions, we may write, when $\beta/H^* \gg 1$,

$$\Omega_{GW}^{Coll}(f) = \Omega_{GW}^{Coll}(f_{peak})S^{Coll}(f) \tag{7.4.3}$$

where the spectral function is given by [110]

$$S^{Coll}(f) = \frac{(a+b)f_{Peak}f^a}{bf_{Peak}^{a+b} + af^{a+b}} \tag{7.4.4}$$

where $(a, b) \simeq (3, 1.0)$. The peak amplitude is provided by [111]

$$h^2\Omega_{GW}^{Coll}(f_{Peak}) \simeq 1.7 \times 10^{-5}\kappa^2\Delta\left(\frac{\beta}{H_*}\right)^{-2}\left(\frac{\alpha}{1+\alpha}\right)^2\left(\frac{g_*}{100}\right)^{-\frac{1}{3}} \tag{7.4.5}$$

where the efficiency factor κ was first derived by Steinhardt [112] as

$$\kappa = \frac{1}{1 + A\alpha}\left(A\alpha + \frac{4}{27}\sqrt{\frac{3\alpha}{2}}\right) \tag{7.4.6}$$

with $A = 0.715$.

The peak frequency in Eq.(7.4.5) is given by

$$f_{Peak} \simeq 17\left(\frac{f_*}{\beta}\right)\left(\frac{\beta}{H_*}\right)\left(\frac{T_*}{10^8 GeV}\right)\left(\frac{g_*}{100}\right)^{\frac{1}{6}} \text{Hz} \tag{7.4.7}$$

$$\frac{f_*}{\beta} = \frac{0.62}{1.8 - 0.1 v_b + v_b^2} \tag{7.4.8}$$

while in the same equation the dependence of Δ on the velocity v_b of the bubble wall is given by [112, 113, 114, 115]

$$v_b(\alpha) = \frac{\frac{1}{\sqrt{3}} + \sqrt{\alpha^2 + \frac{2\alpha}{3}}}{1 + \alpha}, \tag{7.4.9}$$

with

$$\Delta = \frac{0.11 v_b^3}{0.42 + v_b^2}. \tag{7.4.10}$$

For the second term in Eq.(7.4.2) we may similarly write

$$\Omega_{GW}^{SW}(f) = \Omega_{GW}^{SW}(f_{peak}) S^{SW}(f) \tag{7.4.11}$$

with [116, 117, 113]

$$h^2 \Omega_{GW}^{SW}(f_{Peak}) \simeq 2.7 \times 10^{-6} \kappa_v^2 v_b \left(\frac{\beta}{H_*}\right)^{-1}\left(\frac{\alpha}{1+\alpha}\right)\left(\frac{g_*}{100}\right)^{-\frac{1}{3}}(H_* \tau_{SW}) \tag{7.4.12}$$

$$\kappa_v \simeq \frac{\alpha}{0.73 + 0.083\sqrt{\alpha} + \alpha}. \tag{7.4.13}$$

According to [110] the peak sound wave frequency is provided by

$$f_{Peak} \simeq 19 \frac{1}{v_b}\left(\frac{\beta}{H_*}\right)\left(\frac{T_*}{10^8 GeV}\right)\left(\frac{g_*}{100}\right)^{\frac{1}{6}} \text{Hz} \tag{7.4.14}$$

while, according to [106], the sound-wave spectral function in Eq.(7.4.11) is

$$S^{SW}(f) = \left(\frac{f}{f_{Peak}}\right)^3 \left(\frac{7}{4+3\left(\frac{f}{f_{Peak}}\right)^2}\right)^{\frac{7}{2}}. \tag{7.4.15}$$

The sound waves remain active for a time τ_{SW}

$$\tau_{SW} = \frac{R_*}{U_f} \tag{7.4.16}$$

in which R_* is the root mean bubble separation $R_* \simeq (8\pi)^{\frac{1}{3}} v_b/\beta$ and U_f is the root mean square of the fluid velocity [117]

$$U_f^2 \simeq \frac{3}{4}\left(\frac{\alpha}{1+\alpha}\right)\kappa_v. \tag{7.4.17}$$

We note that the last factor in Eq. (7.4.12) represents an important comparison of the sonic period to the Hubble time of cosmic expansion[118, 119, 120, 82, 84, 121].

For the third and final term in Eq.(7.4.2) representing plasma turbulence, we write similarly again:

$$\Omega_{GW}^{Turb}(f) = \Omega_{GW}^{Turb}(f_{peak}) S^{Turb}(f) \tag{7.4.18}$$

in which the factors are given by the estimates [122]

$$h^2\Omega_{GW}^{Turb}(f_{Peak}) \simeq 3.4\times 10^{-4} v_b \left(\frac{\beta}{H_*}\right)^{-1} \left(\frac{\kappa_{Turb}\alpha}{1+\alpha}\right)^{\frac{3}{2}} \left(\frac{g_*}{100}\right)^{-\frac{1}{3}} \tag{7.4.19}$$

$$f_{Peak} \simeq 37\frac{1}{v_b}\left(\frac{\beta}{H_*}\right)\left(\frac{T_*}{10^8 GeV}\right)\left(\frac{g_*}{100}\right)^{\frac{1}{6}} .\text{Hz}. \tag{7.4.20}$$

The spectral function $S_{GW}^{Turb}(f)$ in Eq.(7.4.18) is provided by [106, 123, 77]

$$S^{Turb}(f) = \frac{\left(\frac{f}{f_{Peak}}\right)^3}{\left(1+\frac{f}{f_{Peak}}\right)^{11/3}\left(1+\frac{8\pi f}{h_*}\right)} \tag{7.4.21}$$

$$h_* = 17\left(\frac{T_*}{10^8 GeV}\right)\left(\frac{g_*}{100}\right)^{\frac{1}{6}} \text{Hz}. \tag{7.4.22}$$

wherein we set $\kappa_{Turb} \simeq 0.05\kappa_v$.
In order to derive numerical predictions for the peak frequency of the quiver model and compare the results with other models, we pause for some considerations. One of the most important parameters appearing in all the equations presented above is β/H_*, which is derived from the tunneling action

around the time when the transition occurs (t_*) at the temperature T_*, using the adiabatic time-temperature relation

$$\frac{dt}{dT} = -\frac{1}{TH(T)}, \tag{7.4.23}$$

in the form [124, 111][125]

$$S(t) = S(t_*) - \beta(t - t_*) + O((t - t_*)^2) \qquad \text{with} \qquad \frac{\beta}{H_*} = T_* \frac{dS}{dT}\Big|_{T=T_*}. \tag{7.4.24}$$

$\beta = \dot{\Gamma}/\Gamma_*$ measures the time variation of the nucleation rate and $\tau \equiv 1/\beta$ characterizes the time scale of the phase transition at the transition time t_*. The parameter β/H_* is defined by the ratio between τ and the Hubble time $1/H_*$. It is one of the most important parameters controlling the energy released into GWs at the phase transition. A more in depth analysis shows that one should set a distinction between the thermal and the vacuum tunneling contributions to $\Gamma(t)$, which involves either the three dimensional S_3 or four dimensional S_4 Euclidean bouncing solutions, which will not be of our concern in this work, as well as a finer characterization of the nucleation temperature (see for instance [84]).
If we denote the nucleation rate with $\Gamma(t)$, the temperature of the transition is defined to be the temperature at which the probablility of nucleating one bubble per Hubble volume per Hubble time is one

$$\Gamma(t) \sim T^4 e^{-S(t)} \qquad \frac{\Gamma}{H^4} \sim 1 \tag{7.4.25}$$

which gives for the tunneling action the expression

$$S(T_*) \sim -4 \log \frac{T_*}{m_P}, \tag{7.4.26}$$

where M_P is the Planck mass. As an order of magnitude estimate one can set $\beta \sim H_* S_*$ [122, 126] which gives a value $\beta/H_* \sim O(10^2)$ at the electroweak scale, and is a good approximation also in our case, due to the logarithmic dependence of $S(T_*)$ on T_*.
Therefore we set

$$\frac{\beta}{H_*} \sim 100 - 300, \qquad T_* \sim 4000\,\text{GeV}, \qquad g_* = 1732 \tag{7.4.27}$$

and vary α, the strength of the PT.

7.5 Results

We show in Tables 1, 2 and 3 the values of the relevant parameters the results for the collisional, sound waves and turbolence contributions to $h_0^2\Omega$, for $\beta/H^* = 100, 200$ and 300 and parametric values of α, the strength of the transition, varying from 0.6 to 0.8. In the collisional sector, shown in Table 1, peak frequency emissions are in the range of 10^{-2} Hz, with contributions which, for a fixed β/H^*, are

Table 7.1 Numerical values of the peak frequency and GW emissions for the collisional contributions.

β/H_*	α	$f_{peak}(Hz)$	$h_0^2\Omega_{coll}$
100	0.60	2.7×10^{-2}	0.97×10^{-12}
100	0.65	2.7×10^{-2}	1.2×10^{-12}
100	0.70	2.7×10^{-2}	1.4×10^{-12}
100	0.75	2.7×10^{-2}	1.7×10^{-12}
100	0.80	2.6×10^{-2}	1.9×10^{-12}
200	0.60	5.4×10^{-2}	2.41×10^{-13}
200	0.65	5.3×10^{-2}	2.9×10^{-13}
200	0.70	5.3×10^{-2}	3.5×10^{-13}
200	0.75	5.3×10^{-2}	4.1×10^{-13}
200	0.80	5.3×10^{-2}	4.8×10^{-13}
300	0.60	8.0×10^{-2}	1.1×10^{-13}
300	0.65	8.0×10^{-2}	1.3×10^{-13}
300	0.70	8.0×10^{-2}	1.6×10^{-13}
300	0.75	8.0×10^{-2}	1.8×10^{-13}
300	0.80	8.9×10^{-2}	2.1×10^{-13}

essentially stable, as we vary α. The percentile variation of $h_0^2\Omega_{coll}$ is around 20%, for a fixed β/H^*, as α increases by about 10% stepwise ($\Delta\alpha = 0.5$). For the same, fixed value of α, as we increase β/H^* from 100 to 300, the reduction of the gravitational wave emission is about 90 %.

Table 2 summarizes the results for the GW emission due to sound waves in the plasma. In this case, the peaks of the emissions are centered at larger frequencies ($\sim 10^{-1}Hz$) compared to the collisional contributions, and show, similarly to Table 1, very small variations ($< 1\%$) as we vary α, for a given value of β/H^*.
At a fixed value of the ratio β/H^*, the GW emission increases in a slightly milder way (by $\sim 10-15\%$) for $\beta/H^* = 100$ as we raise α, while it is about 20%, as in the previous Table, for $\beta/H^* = 200, 300$. As in Table 1, the emission into sound waves, for a given α, gets suppressed by 90% in its size as we vary β/H^* from 100 to 300.

We show in Table 3 results for the GW emissions due to turbulence. The pattern, also in this case, is similar to those of the previous two cases. The peak frequencies are larger ($\sim 10^{-1}$), by factors of 10 and 100 respect to the sound waves and to the collisional contributions, respectively. The increase in the GW emission, as we vary α, is about 10%, for a given β/H^*, while the reduction in the energy of the GW gets reduced about $60-70\%$ as we increase β/H^* from 100 to 300. In all cases, the turbolence contributions are larger than those coming from the collisional and the sound waves at their respective peak frequences, with a factor approximately to 20 for the ratio between $\Omega sw \sim 20\times\Omega_{coll}$ and $\Omega_{turb} \sim 1000-2000\times\Omega_{coll}$.

We can compare our results against the discovery potential of the space detector LISA in few plots using PTPlot [108], assuming in all cases a value of $\beta/H^* = 100$.
It is clear that the maximum sensitivity for this proposed experiment is for GW amplitudes with a

Table 7.2 Numerical values for the PT parameters for the sound waves contributions.

β/H_*	α	$f_{peak}(Hz)$	$h_0^2\Omega_{sw}$
100	0.60	1.3×10^{-1}	1.9×10^{-11}
100	0.65	1.3×10^{-1}	2.2×10^{-11}
100	0.70	1.4×10^{-1}	2.5×10^{-11}
100	0.75	1.3×10^{-1}	2.8×10^{-11}
100	0.80	1.3×10^{-1}	3.1×10^{-11}
200	0.60	2.7×10^{-1}	4.7×10^{-12}
200	0.65	2.7×10^{-1}	5.5×10^{-12}
200	0.70	2.7×10^{-1}	6.2×10^{-12}
200	0.75	2.7×10^{-1}	7.0×10^{-12}
200	0.80	2.6×10^{-1}	7.8×10^{-12}
300	0.60	4.0×10^{-1}	2.1×10^{-12}
300	0.65	4.0×10^{-1}	2.4×10^{-12}
300	0.70	4.0×10^{-1}	2.8×10^{-12}
300	0.75	4.0×10^{-1}	3.1×10^{-12}
300	0.80	4.0×10^{-1}	3.5×10^{-12}

peak around a few mHz and an energy density of the GW $h^2\Omega \sim 10^{-11}$.

We show 4 plots in Figs. 7.4 and 7.5 which illustrate the difference between the quiver model and typical models characterised by a lower number of degrees of freedom ($\sim$ 150), and a transition temperature comparable with that of the electroweak scale ($\sim$ 200 GeV). In Fig. 7.4 we show results for $h_0^2\Omega$ for a typical choice of parameters $\alpha = 0.6$ and 0.2. In the first case the GW energy density follows into the sensitivity region of LISA, while in the second case the curve lapses the region of sensitivity, being tangent to it. A similar study can be performed in the quiver model, as shown in Fig. 7.5, where the plots show that the increase in temperature by few TeV's increases the frequency of such stochastic background. While the overall energy released as GWs is comparable with the one generated by a transition temperature typical of transitions around electroweak scale, the peak in frequency is shifted upward, and located around $\sim 5 \times 10^{-2}$ Hz, beyond the sensitivity of LISA.

Obviously, one can investigate the parametric dependence of $h_0^2\Omega$ in a general way, by simply varying the parameters which affect the emission of GWs. For instance, we can vary T_* from 200 GeV to 1000 GeV, as well as the number of massless degrees of freedom in ρ_{rad}, assuming that in both cases a FOPT is ensured by a sufficiently large value of α.

The first variation of parameters is shown in Fig. 7.6, where on the left we plot the GW emission for the lower case $T_* = 200$ GeV and on the right for the higher temperature case $T_* = 1000$, keeping a value of $g_* = 200$, which departs rather modestly from the simplest extensions of the Standard Model compared to the quiver case. The dependence of such models on the number of degrees of freedom g_* is rather mild, as one can realize from Fig. 7.7, where we plot $h_0^2\Omega$ in models with $T_* = 300$ GeV and $g_* = 300$ (left) and 1000 (right). In general an increase in g_* moves the value of f_{peak} slightly towards higher frequency, although it is clear that the dominant effect is related to the drastic change

Table 7.3 Numerical values for the PT parameters for the turbulence contributions.

β/H_*	α	$f_{peak}(Hz)$	$h_0^2\Omega_{turb}$
100	0.60	1.9×10^{-1}	8.7×10^{-10}
100	0.65	1.9×10^{-1}	1.0×10^{-10}
100	0.70	1.9×10^{-1}	1.1×10^{-9}
100	0.75	1.9×10^{-1}	1.3×10^{-9}
100	0.80	1.9×10^{-1}	1.4×10^{-9}
200	0.60	3.8×10^{-1}	4.3×10^{-10}
200	0.65	3.8×10^{-1}	5.0×10^{-10}
200	0.70	3.8×10^{-1}	5.7×10^{-10}
200	0.75	3.8×10^{-1}	6.4×10^{-10}
200	0.80	3.8×10^{-1}	7.1×10^{-10}
300	0.60	5.8×10^{-1}	2.9×10^{-10}
300	0.65	5.8×10^{-1}	3.3×10^{-10}
300	0.70	5.7×10^{-1}	3.8×10^{-10}
300	0.75	5.7×10^{-1}	4.2×10^{-10}
300	0.80	5.7×10^{-1}	4.7×10^{-10}

of temperature in the transition, which plays a decisive role in the study of such models. We note that the final factor in Eq.(7.4.12) is not included in our plots which were produced using PTPlot software.

7.6 Discussion

About forty years ago, around 1980, it appeared likely that minimal $SU(5)$ grand unification theory [127] [128] would agree with experiment and proton decay would soon be observed, with a lifetime $\sim 10^{30}$ years and with the decay modes and branching ratios in agreement with the predictions of minimal $SU(5)$ GUT theory. If so, it would have been a huge leap forward by factor of at least a trillion (10^{12}) in energy scale above the electroweak scale ~ 100 GeV.

Unfortunately for this simplest GUT, the proton lifetime for the predicted dominant decay mode $p \to e^+\pi^0$ was found experimentally to be 100 times too long, now known to be 10,000 times too long, to agree with minimal SU(5). The reason that minimal $SU(5)$ theory failed was surely because of the desert hypothesis that there exists no new physics in the huge hierarchy between the weak scale and the putative GUT scale.

In the present paper, therefore, we have avoided this desert hypothesis by employing a quiver GUT which makes no assumption about new physics at scales above 4 TeV, except that the theory is expected to become conformally invariant up to much higher scales. Proton decay is absent at tree level because of the quiver inspired assignments of the quarks and leptons. If we assume that the breaking of conformal symmetry is characterised by a FOPT at a relatively small scale, in this scenario one should consider the production of gravitational waves in a frequency interval ($10^{-3} - 10^{-1}$ Hz) which is in the range of proposed recent experiments. We should also mention that direct simulations [129] may give

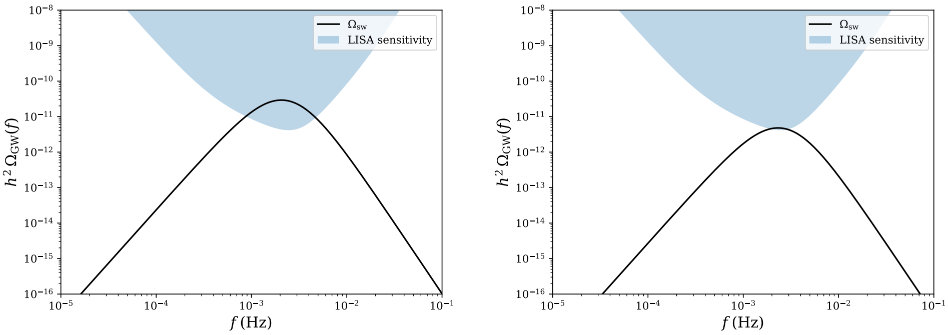

Figure 7.4 Typical GW emission in extensions of the Standard Model ($g_* = 130$) in a FOPT with (left) $\alpha = 0.6, v_b = 0.9$, $\beta/H_* = 100$; (right) $\alpha = 0.2, v_b = 0.8$. We have set $T_* = 200$ GeV.

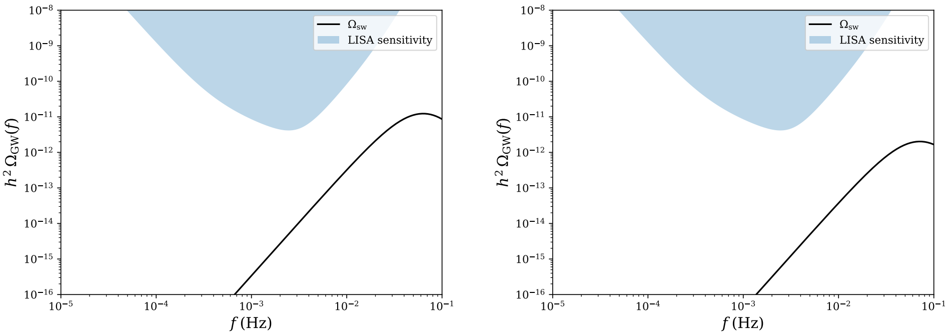

Figure 7.5 GW emission in the quiver model with $g_* = 1732$ in a FOPT with (left) $\alpha = 0.6, v_b = 0.9$, $\beta/H_* = 100$; (right) $\alpha = 0.2, v_b = 0.8$. We have set $T_* = 4000$ GeV.

the opportunity to improve systematically on previous approximations, especially for what concerns the contribution of f_{turb} to the GEW emissions.

We have suggested that a phase transition in the early universe, expected by the $SU(3)^{12}$ quiver GUT theory described in this article, could source GWs in the mHz region, but slightly too large in frequency to be detectable by the forthcoming LISA gravitational wave detector, both for a three and a seven year run of this experiment. However, the wide array of experiments proposed in the future may be able to detect or exclude models with larger transition temperatures respect to those taken into account in the past.

Such models are characterised by a rather large number of massless degrees of freedom compared to the Standard Model or other simpler models, such as the 2-Higgs doublet model, which modify minimally the Standard Model and allow a FOPT to take place rather close to the electroweak scale. In the quiver model that we have discussed, the larger transition temperature T_* and the larger number of degrees of freedom, present a new challenge for their detection both at theoretical and at experimental level.

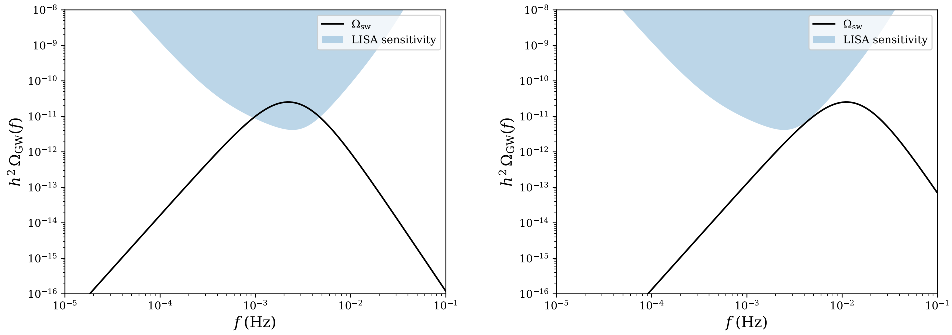

Figure 7.6 GW emission in extensions of the Standard Model with (left) $T_* = 200$ GeV, $g_* = 200$, $\alpha = 0.6$, $v_b = 0.9$, $\beta/H_* = 100$; (right) $T_* = 1000$ GeV, $g_* = 200$ $\alpha = 0.6$, $v_b = 0.9$.

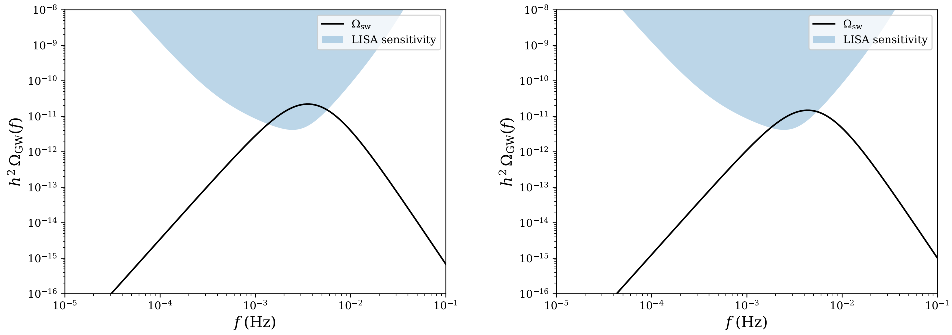

Figure 7.7 GW emission in extensions of the Standard Model with (left) $T_* = 300$ GeV, $g_* = 300$, $\alpha = 0.6$, $v_b = 0.9$, $\beta/H_* = 100$; (right) $T_* = 300$ GeV, $g_* = 1000$ $\alpha = 0.2$, $v_b = 0.8$.

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Chapter 8

An Axion-Like Particle from an $SO(10)$ Seesaw with $U(1)_X$

We investigate the decoupling of heavy right handed neutrinos in the context of an SO(10) GUT model, where a remnant anomalous symmetry is $U(1)_X$. In this model the see-saw mechanism which generates the neutrino masses is intertwined with the Stueckelberg mechanism, which leaves the CP-odd phase of a very heavy Higgs in the low energy spectrum as an axion-like particle. Such pseudoscalar is predicted to be ultralight, in the 10^{-20} eV mass range. In this scenario, the remnant anomalous X symmetry of the particles of the Standard Model is interpreted as due to the incomplete decoupling of the right handed neutrino sector. We illustrate this scenario including its realisation in the context of SO(10).

8.1 Introduction

Recently there has been considerable interest in the occurrence of axion-like particles [130, 131, 132] including the appearance in model building of anomalous $U(1)$ symmetries with a Stueckelberg field [37, 133, 42, 134, 44, 45, 45, 47, 48, 49, 50, 135, 52, 53, 133, 49]. In this paper we examine the simplest GUT example where this phenomenon is closely related to the see-saw mechanism [136] for generating the neutrino masses and may provide a link between axions and right-handed neutrinos.
At the same time our scenario establishes a possible link between leptogenesis and dark matter [137, 138] in a generalized setting, due to the prediction of an axion in the low energy spectrum. Stueckelberg axions ($b(x)$) appear in the field theory realization of the Green-Schwarz mechanism of anomaly cancellation of string theory, in the dualization of a 3-form, and correspond to pseudoscalar gauge degrees of freedom (see also the discussion in [133]). As ordinary Nambu-Goldstone modes they undergo a local shift

$$b(x) \to b(x) + M\theta(x) \tag{8.1.1}$$

under an Abelian gauge transformation and are coupled to the anomaly via a dimension-5 operators

of the form $b(x)/MF \wedge F$ where F is, generically, the field strength of the gauge fields which share a mixed anomaly with the $U(1)$ symmetry, and M is the Stueckelberg scale.
In these scenarios, pseudoscalar gauge degrees of freedom may develop physical components only after the breaking of the shift symmetry by some extra potential. This is expected to occur in the case of phase transitions in a non-abelian gauge theory, when instanton interactions naturally arise and induce a mixing between the Stueckelberg field and the Higgs sector of the theory, with the generation of a periodic potential, after spontaneous symmetry breaking.
This scenario in which the CP odd phases of the scalar sector mix and generate such a potential, has provided the basic template for the emergence of a physical CP odd state, in a way which is very close to what was conjectured to occur in the case of the electroweak or DFSZ version of the Peccei-Quinn [23] axion (see the review [39]), where the anomalous symmetry is a global rather than a local one.
Indeed, we recall that in the DFSZ case one writes down a general potential, function of three scalar fields, which is $SU(2) \times U(1)$ invariant. The simplest realization of this scenario is in the two-Higgs doublet model, where the Higgs fields H_u and H_d are assigned the global symmetry

$$H_u \to e^{i\alpha X_u} H_u, \qquad H_d \to e^{i\alpha X_d} H_d \tag{8.1.2}$$

under $U(1)_{PQ}$ and are accompanied by an additional scalar Φ, which is singlet under the Standard Model (SM) symmetry

$$\Phi \to e^{i\alpha X_\Phi} \Phi \tag{8.1.3}$$

with $X_u + X_d = -2X_\Phi$. The potential is given by a combination of terms of the form

$$V = V(|H_u|^2, |H_d|^2, |\Phi|^2, |H_u H_d^\dagger|^2, |H_u \cdot H_d|^2, H_u \cdot H_d, \Phi^2) \tag{8.1.4}$$

(with $H_u \cdot H_d \equiv H_u^\alpha H_d^\beta \epsilon_{\alpha\beta}$) which is invariant under the Standard Model gauge symmetry and is in addition invariant under the global $U(1)_{PQ}$.
As pointed out in [49] a similar effective theory can be obtained in the case of a gauge symmetry, in a scenario that leaves most of the intermediate steps in the generation of Stueckelberg-like Lagrangian unchanged. In this realization of the Stueckelberg Lagrangian, the Stueckelberg pseudoscalar emerges from the phase of the complex scalar field which is responsible for the breaking of the gauged $U(1)$ symmetry. The breaking takes place at the GUT (Grand Unified Theory) scale, which takes the role of the Stueckelberg mass for the low energy effective theory.
In our case such abelian symmetry is contained within $SO(10)$ and it is identified with $U(1)_X$. This provides the basic observation which motivates our work, which connects the decoupling of a gauge boson corresponding to an $U(1)_X$ symmetry within $SO(10)$ and of a right-handed neutrino to the appearance of an axion in the spectrum of the low energy theory. Being the construction sequential in each of the three generations, this scenario predicts three axions in the spectrum. Building on a similar analysis by two of us in [37] based on a $E_6 \times U(1)_X$, such axions are expected to be ultralight, in the 10^{-20} eV mass range.

8.1.1 Incomplete decoupling of a chiral fermion and global anomalous $U(1)_X$

We believe it is useful to scrutinise this within a transparent model where two examples of physics beyond the standard model, the non-zero neutrino masses and the Stueckelberg axion are closely related. Since we know from experiment [139] that the first extension exists in Nature, it increases our expectation that the second should be realised. We shall review the group theory of SO(10) including the available irreducible representations for the matter particles and the symmetry breaking.
SO(10) naturally provides three right-handed neutrinos which can participate in the see-saw. Because of the decoupling of these additional neutrino states at high masses, the resultant effective theory possesses an anomalous $U(1)_X$ symmetry. Since we shall discuss neutrino masses it is worth recalling the various possibilities for introducing them into the minimal SM. We shall mention four of these, one being the see-saw mechanism, and reveal why the other three are less attractive. One of them, introduced in [140], once appeared to be compelling when based only on the SuperKamiokande experiment [139] but it predicted maximal solar neutrino mixing which unfortunately was subsequently excluded by the SNO experiment [141]. This left as the most popular possibility the see-saw mechanism which we shall employ in the present model. When neutrino masses were established experimentally in 1998 there was confusion about to whom priority for the see-saw idea belonged and it was temporarily assigned to a number of theory papers published in 1979. Further scholarship revealed, however, that priority belonged to a 1977 paper by Minkowski [136].

8.2 SO(10) Grand Unification

The SO(10) model for unifying quarks and leptons was invented over forty years ago in [142, 143]. After non-zero masses for neutrinos were discovered, it became the most popular GUT superseding the otherwise more economical SU(5) GUT [127]. A recent discussion of an SO(10) GUT can be found in [144]. In the minimal Standard Model (SM), as in the minimal SU(5) GUT, the neutrinos were assumed to be massless. In the SO(10) GUT, each family in a **16** contains, in addition to the fifteen helicity states of the minimal SM, a right- handed neutrino N. This gives rise to several additional features, beyond the most obvious one that the neutrinos can acquire mass through the see-saw mechanism. An SU(5) GUT subsumes the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ but an SO(10) GUT with one additional rank includes also a $U(1)_{(X)}$. It is this gauged (X) symmetry and its breaking which will play a central role in our present discussion.
The group theory underlying the SO(10) GUT is well-known and reviewed in many papers; one reliable such reference is [145].
For the purposes of establishing notation we shall briefly discuss this with special emphasis on the role of (X) symmetry which will be treated further in subsequent subsections.

8.2.1 Breaking patterns

The gauge group $SO(10)$ has the dimension 45 of its adjoint. An adjoint of scalars can break the symmetry while preserving rank-5 to

$$SO(10) \rightarrow [SU(3)_C \times SU(2)_L \times U(1)_Y]_{SM} \times U(1)_X \tag{8.2.1}$$

with $3X = 12Y - 15(B-L)$. We shall need more scalars to give mass to the fermions. Each family is in a **16** irreducible representation. For example the first family is

$$16 \equiv (u^r, u^g, u^b, d^r, d^g, d^b; u^r, u^g, u^b, d^r, d^g, d^b; \nu_e, e^-, N, e^+)_L \tag{8.2.2}$$

where we have designated the colours as r, g, b (= red, green, blue). The Yukawa couplings which can provide fermion masses require scalar fields which are included in

$$16 \times 16 = 10_s + 120_a + 126_s \tag{8.2.3}$$

where the subscripts s, a specify symmetric, antisymmetric. The 10 is the vector representation of SO(10), although the spinor representation 16 is really the defining representation, because one can make 10 from 16, as in Eq.(8.2.3), but not *vice versa*. We first consider the decomposition of SU(5) into $SU(3)_c \times SU(2)_L \times U(1)_Y$, adopting the notation $(SU(3)_C, SU(2)_L)_Y$ with the result that

$$\begin{aligned}
\bar{5} &= (\bar{3},1)_{+2/3} + (1,2)_{-1} \\
10 &= (3,2)_{+1/3} + (\bar{3},1)_{-4/3} + (1,1)_{+2} \\
1\bar{5} &= (6,1)_{-4/3} + (3,2)_{+1/3} + (1,3)_{+2} \\
24 &= (8,1)_0 + (3,2)_{-5/3} + (\bar{3},2)_{+5/3} + (1,3)_0 + (1,1)_0 \\
45 &= (8,2)_{+1} + (\bar{6},1)_{-2/3} + (\bar{3},2)_{-7/3} + (\bar{3},1)_{-4/3} + \\
&\quad (3,3)_{-2/3} + (3,1)_{-2/3} + (1,2)_{+1} \\
5\bar{0} &= (8,2)_{+1} + (6,1)_{+8/3} + (\bar{6}.3)_{-2/3} + (\bar{3},2)_{-7/3} + \\
&\quad (3,1)_{-2/3} + (1,1)_{-4}
\end{aligned} \tag{8.2.4}$$

The states in the first two lines of Eq.(8.2.4) are the familiar ones of one SM family, without a right-handed neutrino, which is why $(10 + \bar{5})$ is used in an SU(5) GUT. The scalars in the SU(5) Yukawa couplings must be among

$$\begin{aligned}
\bar{5} \times \bar{5} &= 10_a + 15_s \\
10 \times \bar{5} &= 5 + 45 \\
10 \times 10 &= \bar{5}_s + 4\bar{5}_a + 5\bar{0}_s
\end{aligned} \tag{8.2.5}$$

and we note that the usual Higgs boson, which in this notation is the complex doublet $(1,2)_{\pm 1}$, appears uniquely in the **5** and **45** of SU(5), as can be seen from Eq.(8.2.4). Armed with these preliminaries about SU(5), it is rendered almost trivial to extend the analysis to SO(10), but the (X) symmetry means we must tread carefully. We return to Eq.(8.2.3) and adopt a new notation in the SO(10) decompositions of $(SU(5))_X$. From [145] we are able to decompose the scalar SO(10) irreducible representations into their SU(5) components:

$$\begin{aligned} 10 &= 5_2 + \bar{5}_{-2} \\ 120 &= 5_2 + \bar{5}_{-2} + 10_{-6} + \bar{10}_6 + 45_2 + \bar{45}_{-2} \\ 126 &= 1_{-10} + \bar{5}_{-2} + 10_{-6} + \bar{15}_6 + 45_2 + \bar{45}_{-2} \\ 45 &= 24_0 + 10_4 + \bar{10}_{-4} + 1_0 \end{aligned} \tag{8.2.6}$$

All of 10, 120 and 126 necessarily contain a candidate for the SM complex Higgs doublet. From Eq. (8.2.4), we can, if needed, translate the SU(5) representations in Eq.(8.2.6) into SM representations. This provides all the group theory we shall need in the present article. In the following we shall focus on the breaking of $U(1)_{(X)}$ which is intimately related to the mass of the right-handed neutrinos N in Eq.(8.2.2) and hence to the see-saw mechanism.

8.2.2 The two complex singlet scalars in the effective potential

If we introduce a scalar field Φ, singlet under SU(5) with lepton number L=+2, we can write the Majorana mass M of the right-handed neutrino N_R^i (i,j =1,2,3) of the three generations as

$$\lambda_{ij} N_R^i N_R^j \Phi. \tag{8.2.7}$$

The masses $\lambda_{ij}\langle\Phi\rangle$ may be taken to be $\sim 10^{10}$ GeV, far above the weak scale, whereupon we may integrate out the right-handed neutrino N to derive an effective field theory with interesting properties. In particular, the gauged $U(1)_{(X)}$ of the SO(10) GUT has become anomalous, because in the $(X)^3$ triangle diagram N has been removed from the internal states.

We note that the 126 of scalars in Eq.(8.2.6) contains an SU(5) singlet, charged under $(B-L)$, in addition to the SU(5) singlet in the 45 of Eq. 8.2.6, Φ. The presence of two such states in our model will be relevant in our subsequent analysis.

Let us step back to a purely bottom-up approach. Consider the original minimal standard model (MSM) with massless neutrinos. In perturbation theory, it conserves baryon number (B) and lepton number (L) so there is a global $U(1)_{(B-L}$ which, without a right-handed neutrino, is anomalous. Such a statement is obviously not connected to grand unification. Of course, this model is ruled out because neutrinos have non-zero masses so some modification is necessary to the MSM and there is a number of possibilities[146]. The most popular is the addition of right-handed neutrinos which permit the see-saw mechanism for generating neutrino masses. This is achieved most naturally in $SO(10)$ unification.

Now we carefully discuss a top-down analysis of $SO(10)$ spontaneous symmetry breaking. At the GUT scale (10^{15-16} GeV) the adjoint 45 is used to break the symmetry in a necessarily rank-preserving manner according to

$$SO(10) \to SU(5) \times U(1)_X \to SU(3) \times SU(2) \times U(1)_Y \times U(1)_X \tag{8.2.8}$$

so that the $U(1)_X$, with $3X = 12Y - 15(B - L)$, is still unbroken and its gauge boson is massless. At an intermediate scale $M_I \sim 10^{10-11}$ GeV the complex 126 is used spontaneously to break $U(1)_X$ and to give Majorana masses to the three right-handed neutrinos. This arises from a VEV of the $SU(5)$-singlet complex component in Eq.(8.4.9) which has the Mexican-hat type of potential required for the Higgs mechanism.

8.3 See-Saw Mechanism

In the MSM neutrinos are massless. The minimal standard model involves three chiral neutrino states, but it does not admit renormalizable interactions that can generate neutrino masses. Nevertheless, experimental evidence suggests that both solar and atmospheric neutrinos display flavor oscillations, and hence that neutrinos do have mass. Two very different neutrino squared-mass differences are required to fit the data:

$$6.9 \times 10^{-5} eV^2 \leq \Delta_s \leq 7.9 \times 10^{-5} eV^2 \text{ and } \Delta_a \sim (2.4 - 2.7) \times 10^{-3} eV^2, \tag{8.3.1}$$

where the neutrino masses m_i are ordered such that:

$$\Delta_s = |m_2^2 - m_1^2| \text{ and } \Delta_a = |m_3^2 - m_2^2| \simeq |m_3^2 - m_1^2| \tag{8.3.2}$$

and the subscripts s and a pertain to solar (s) and atmospheric (a) oscillations respectively. The large uncertainty in Δ_s reflects the several potential explanations of the observed solar neutrino flux: in terms of vacuum oscillations or large-angle or small-angle MSW solutions, but in every case the two independent squared-mass differences must be widely spaced with

$$r = \Delta_s / \Delta_a \sim 3 \times 10^{-2}. \tag{8.3.3}$$

In a three-family scenario, four neutrino mixing parameters suffice to describe neutrino oscillations, akin to the four Kobayashi-Maskawa parameters in the quark sector. Solar neutrinos may exhibit an energy-independent time-averaged suppression due to Δ_a, as well as energy-dependent oscillations depending on Δ_s / E. Atmospheric neutrinos may exhibit oscillations due to Δ_a, but they are almost entirely unaffected by Δ_s. It is convenient to define neutrino mixing angles as follows:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} c_2c_3 & c_2s_3 & s_2e^{-i\delta} \\ +c_1s_3+s_1s_2c_3e^{i\delta} & -c_1c_3-s_1s_2s_3e^{i\delta} & -s_1c_2 \\ +s_1s_3-c_1s_2c_3e^{i\delta} & -s_1c_3-c_1s_2s_3e^{i\delta} & +c_1c_2 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{8.3.4}$$

with s_i and c_i standing for sines and cosines of θ_i. For neutrino masses satisfying (8.3.1), the vacuum survival probability of solar neutrinos is:

$$P(\nu_e \to \nu_e)|_s \simeq 1 - \frac{\sin^2 2\theta_2}{2} - \cos^4\theta_2 \sin^2 2\theta_3 \sin^2(\Delta_s R_s/4E) \tag{8.3.5}$$

whereas the transition probabilities of atmospheric neutrinos are:

$$\begin{aligned} P(\nu_\mu \to \nu_\tau)|_a &\simeq \sin^2 2\theta_1 \cos^4\theta_2 \sin^2(\Delta_a R_a/4E) \\ P(\nu_e \to \nu_\mu)|_a &\simeq \sin^2 2\theta_2 \sin^2\theta_1 \sin^2(\Delta_a R_a/4E) \\ P(\nu_e \to \nu_\tau)|_a &\simeq \sin^2 2\theta_2 \cos^2\theta_1 \sin^2(\Delta_a R_a/4E) \end{aligned} \tag{8.3.6}$$

None of these probabilities depend on δ, the measure of CP violation. Let us turn to the origin of neutrino masses. Among the many renormalizable and gauge-invariant extensions of the standard model that can do the trick are [146] (i) The introduction of a complex triplet of mesons (T^{++}, T^+, T^0) coupled bilinearly to pairs of lepton doublets. They must also couple bilinearly to the Higgs doublet(s) so as to avoid spontaneous (X) violation and the appearance of a massless and experimentally excluded majoron. This mechanism can generate an arbitrary complex symmetric Majorana mass matrix for neutrinos. (ii) The introduction of singlet counterparts to the neutrinos with very large Majorana masses. The interplay between these mass terms and those generated by the Higgs boson, the so-called see-saw mechanism, yields an arbitrary but naturally small Majorana neutrino mass matrix. (iii) The introduction of a charged singlet meson f^+ coupled antisymmetrically to pairs of lepton doublets, and a doubly-charged singlet meson g^{++} coupled bilinearly both to pairs of lepton singlets and to pairs of f-mesons. An arbitrary Majorana neutrino mass matrix is generated in two loops. (iv) The introduction of a charged singlet meson f^+ coupled antisymmetrically to pairs of lepton doublets and (also antisymmetrically) to a pair of Higgs doublets. This simple mechanism was first proposed in [140] and results at one loop in a Majorana mass matrix in the flavor basis (e, μ, τ) of a special form:

$$\begin{pmatrix} 0 & m_{e\mu} & m_{e\tau} \\ m_{e\mu} & 0 & m_{\mu\tau} \\ m_{e\tau} & m_{\mu\tau} & 0 \end{pmatrix} \tag{8.3.7}$$

This Zee model is attractive as an simple extension of the SM. It predicts maximal solar neutrino mixing, $\theta_{12} = \frac{\pi}{4}$, a value which was strongly disfavoured by SNO data[141, 147]. Of all the models preserving only the three chiral left-handed neutrinos of the SM - models (i), (iii) and (iv) above -

model (iv) is surely the most appealing and it fails. Therefore one is led to additional neutrino states, typically two or more massive right-handed neutrinos which we denote N_I $(i = 1, 2, \ldots, p)$.
In the model we shall discuss p is necessarily $p = 3$ because each of the three quark-lepton families is in a **16** of $SO(10)$ and each contains one N state. There has been considerable interest in more minimal models with $p = 2$ as introduced in the so-called FGY model of [148]. This choice has the property of reducing the number of free parameters such that the CP-violating phase in N_i mixing matrix is simply related to the CP-violating phase, δ, in Eq.(8.3.4). This means that the measurement of δ in long-baseline neutrino oscillation experiment would shine light on the origin of matter-antimatter asymmetry arising from leptogenesis[149] where it arises from N_i decay. In general, this connection does not exist so that an optimistic logic could argue that the FGY model, sometimes called the minimal see-saw, is possibly correct.
For the present case of $p = 3$ we introduce a mass basis

$$(\nu_e.\nu_\mu.\nu_\tau, N_1, N_2, N_3) \tag{8.3.8}$$

so that there is a 6×6 mass matrix in four 3×3 blocks with the top-left block vanishing and the bottom-right being the large Majorana masses for the N_i. The two off-diagonal blocks are Dirac masses coupling the ν_{IL} to the N_{iR}.

The effective mass matrix of the light Majorana neutrinos is given by

$$M = M_D (M_R)^{-1} M_D^T \tag{8.3.9}$$

where M_D and M_R are the 3×3 mass matrices for the Dirac and right-handed Majorana neutrinos, respectively. M_D^T designates the transpose.
The see-saw strategy is immediately evident from Eq.(8.3.9). Denoting the mean values of the 3×3 blocks by m and M

$$\begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \tag{8.3.10}$$

the eigenvalues for $m \ll M$ are close to m^2/M and M. This shows how large the N_i masses are expected to be. Taking the first family, with a typical quark mass 10 MeV and electron neutrino mass $10^{-5} eV$, we find $M \sim 10^{10}$ GeV. Coincidentally, and suggestively, such a mass fits well with the mass required for successful leptogenesis[149].
This discussion exhibits the great advantage of the see-saw mechanism compared to the alternative models discussed above: the smallness of the neutrino masses relative to those of the quarks and leptons occurs naturally. That being said, the other side of the coin is that experimental observation of the very massive N_i is challenging.
The crucial observation for our present purposes is to consider the $U(1)_X$ triangle anomalies. If we

keep all the states in Eq.(8.2.2) for one family

$$16 \equiv (u^r, u^g, u^b, d^r, d^g, d^b; u^r, u^g, u^b, d^r, d^g, d^b; \nu_e, e^-, N, e^+)_L, \quad (8.3.11)$$

then we can examine this question.
The pure gauge anomaly $U(1)_X^3$ has cancelling contributions from the states in Eq.(8.3.11) as follows

$$6\left(\frac{1}{27}\right) + 6\left(-\frac{1}{27}\right) + 2(+1) + 2(-1) = 0 \quad (8.3.12)$$

For the gravitational triangle anomaly which has only one $U(1)_X$ vertex the respective cancelling contributions are

$$6\left(\frac{1}{3}\right) + 6\left(-\frac{1}{3}\right) + 2(+1) + 2(-1) = 0. \quad (8.3.13)$$

When we decouple the N state in Eq.(8.3.11) by taking it to very high mass, the right hand sides of Eq.(8.3.12)and Eq.(8.3.13) both change from zero to -1, the anomalies do not cancel, and therefore there exists in the effective theory an anomalous $U(1)$ symmetry of the sort considered in different contexts in *e.g.* [150, 151, 152, 153].

8.4 Anomalous $U(1)_X$

Let us introduce the matter fields in our model. The fermions are in three **16**'s, Ψ_i $(i = 1, 2, 3)$. Each **16** contains a right-handed neutrino N_R^i with $(X) = +1$.
$SO(10)$ contains the usual $SU(5)$ subgroup [127] which plays a rôle in containing the minimal standard model (MSM) as if without neutrino mass. To provide mass to N_R without breaking $SU(5)$ we introduce a complex scalar Φ in the **126** of $SO(10)$ which under $SU(5)$ contains

$$126 \subset 1 + 5 + \bar{10} + 15 + \bar{45} + 50 \quad (8.4.1)$$

and the N_R^i acquire mass as in Eq. 8.2.7 when the $SU(5)$-singlet component of Φ in Eq.(8.4.9) gains an intermediate mass scale VEV

$$< \Phi >= M_I \quad (8.4.2)$$

where for the see-saw mechanism the intermediate mass scale M_I is typically $\sim 10^{10}$ GeV.
To break the symmetry $SU(5)$ to that of the standard model we introduce more scalars in the representations of $SO(10)$ which are the adjoint A in a **45**, the vector V in a **10** and finally a spinor $B(16)$.
The adjoint **45** decomposes under $SU(5)$ as

$$45 \supset 1 + 10 + \bar{10} + 24 \quad (8.4.3)$$

so that the **24** can provide the rank-preserving $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$.
We recall that in $SO(10)$, 45 decomposes as in Eq. 8.2.6 within which the 24 can provide the rank-

preserving $SU(5) \to SU(3) \times SU(2) \times U(1)_Y$ symmetry breaking. The fermion masses arise from the Yukawa couplings

$$\mathcal{L}_{Yukawa} = \Psi \left(Y_V V + Y_\Phi \Phi \right) \Psi \tag{8.4.4}$$

which may be understood to contain the coupling of Eq.(8.2.7).
We adopt the convention that Latin indices $a, b, c, \ldots$ run from 1 to 10 and Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to 16. The vector field V is V_a and the adjoint A is $A_{ab} = -A_{ba}$ so that all the V and A couplings up to quartic in the Higgs potential can be written, bearing in mind that

$$\begin{aligned} 10 \times 10 &\supset 1 + 45 + 50 \\ 10 \times 45 &\supset 10 + 120 + 320 \\ 45 \times 45 &\supset 1 + 45 + 54 + 210 + 770 + 945. \end{aligned} \tag{8.4.5}$$

in the form

$$\mathcal{V}(V, A) = V_a V_a + (V_a V_a)^2 + A_{ab} A_{ab} + (A_{ab} A_{ab})^2 + (V_a V_a)(A_{bc} A_{bc}) + \ldots \tag{8.4.6}$$

among other terms.
To deal with the **126** it is essential to introduce the Γ matrices

$$\Gamma^a_{\alpha\beta} \tag{8.4.7}$$

which are ten 16×16 matrices which roughly generalise the four 4×4 Dirac matrices γ^μ pertinent to $O(4)$, and likewise satisfy a Clifford algebra. The Φ field of the **126** is a symmetric scalar field satisfying the trace condition

$$\Gamma^a_{ij} \Phi_{ji} = Tr(\Gamma^a \Phi) = 0 \tag{8.4.8}$$

Now, in addition to Eq.(8.4.5), we shall need

$$\begin{aligned} 126 \times 10 &\supset 210 + 1050. \\ 126 \times 45 &\supset 120 + 126 + 1728 + 3696. \\ 126 \times 126 &\supset 54_S + 945_A + 1050_S + 2772_S + 4125_S + 6930_A. \end{aligned} \tag{8.4.9}$$

to write the Higgs potential terms involving Φ such as

$$\begin{aligned} \mathcal{V}(\Phi) &= \Phi_{ij}\Phi_{ij} + (\Phi_{ij}\Phi_{ij})^2 + \Phi_{ij}\Phi_{jk}\Phi_{kl}\Phi_{li} \\ &\quad + \Gamma^a_{ij}\Phi_{jk}\Phi_{kl}\Phi_{lm}\Phi_{mn}\Gamma^a_{ni} + \ldots \end{aligned} \tag{8.4.10}$$

among other terms including mixed $\Phi - A$ terms possible under $SO(10)$ symmetry, as can be seen from Eqs.(8.4.5) and (8.4.9). We take note of the cubic scalar coupling $16.16.\overline{126}$ which may be written

$$B_\alpha B_\beta \Phi^*_{\alpha\beta} \tag{8.4.11}$$

and which we shall use in the next section.

8.5 Stueckelberg Axion

In order to illustrate how the mixing of the CP-odd phases takes place in the breaking of $SO(10) \to SU(5) \times U(1)$ we consider specific terms in the potential, describing the conditions which need to be satisfied in order to generate a periodic potential function of a single gauge invariant field. The latter takes the role of a physical axion and will be denoted by χ.
The periodic potential is generated at the scale at which $SU(5) \times U(1)$ is broken necessarily at $> 10^{15}$ GeV to avoid too-fast proton decay. At this GUT scale, instanton effects are present. In order to understand why this happens, we consider an $SO(10)$ invariant term in the original theory such as

$$16 \times 16 \times \overline{126} \tag{8.5.1}$$

which is built out of the spinorial (16) of $SO(10)$ and the complex conjugate of the 126. The $SO(10)$ singlet is obtained from

$$16 \times 16 = 10_s + 120_a + 126_s \tag{8.5.2}$$

by combining the 126_s taken from the symmetric part of the product $(16 \times 16)_s = 126_s + 10_s$ with the $\overline{126}$. We can specialize (8.2.3) by indicating the X content of the decomposition using

$$16 = 1_{-5} + \bar{5}_{+3} + 10_{-1} \tag{8.5.3}$$

from which gives for their antisymmetric product

$$\begin{aligned} 120_a &= (16 \times 16)_a \\ &= \bar{5}_{-2} + 10_{-6} + (5 + 45)_{+2} + \overline{10}_{+6} + \overline{45}_{-2} \end{aligned} \tag{8.5.4}$$

while the symmetric component can be specialized in the form

$$(16 \times 16)_s = 126_s + 10_s \tag{8.5.5}$$

$$= (1_{-10} + \bar{5}_{-2} + 10_{-6} + \overline{15}_{+6} + 45_{+2} + \overline{50}_{-2}) + (5_{+2} + \bar{5}_{-2}) \tag{8.5.6}$$

where the two contributions in brackets refer respectively to the 126_s and to the 10_s of $SO(10)$.
A periodic potential can be extracted from the decomposition above starting from the $126_s \times \overline{126}$, $SO(10)$ singlet, by combining the 1_{-10} in Eq. (8.5.6) with the 1_{+10} in the $\overline{126}$, the latter obtained by conjugation of (8.4.9) - with the inclusion of its complete $SU(5) \times U(1)_X$ content -

$$\overline{126} = 1_{+10} + 5_{+2} + \overline{10}_{+6} + 15_{-6} + \overline{45}_{-2} + 50_{+2}. \tag{8.5.7}$$

A term in this form in the potential allows to induce a mixing of the CP-odd phases of the two SU(5) singlet representations in such a way that one linear combination of these will correspond to a physical axion while the second one will be part of the Nambu-Goldstone mode generated by the breaking of $U(1)_X$.
We will be denoting with σ and ϕ the two fields corresponding to the 1_{-10} and 1_{10} respectively, denoting their vevs with v_σ and v_ϕ respectively. We will assume that v_ϕ will be large in such a way to provide a mass term for the right-handed neutrino, as specified in (8.2.7) using the Majorana operator $N_R N_R \phi$.

In order to characterise the structure of the Stueckelberg Lagrangian at classical level we focus our attention on the extra (periodic) potential related to σ and ϕ

$$V_p = \lambda M_I^2 \sigma\phi + \text{ h.c.} \tag{8.5.8}$$

Since there must be an $SU(5)$ singlet it is important to realise that the other parts of Eq.(8.5.7) do not contribute. The coupling λ is instanton generated at the scale M_{GUT}, a fact which provides a drastic suppression in V_p. We parameterize both fields around their vevs as

$$\begin{aligned} \sigma &= \frac{v_\sigma + \sigma_1 + i\sigma_2}{\sqrt{2}} \\ &= \frac{v_\sigma + \rho_\sigma}{\sqrt{2}} e^{iF_\sigma(x)/(g_B v_\sigma)} \\ \phi &= \frac{v_\phi + \rho_\phi}{\sqrt{2}} e^{ib(x)/v_\phi} \end{aligned} \tag{8.5.9}$$

and v_ϕ is at the GUT scale $M_{GUT} \sim 10^{15}$ GeV. The parameterization of V_p in a broken phase is made possible by the remaining - non periodic - general scalar potential which will assume a typical mexican-hat shape as for an ordinary $U(1)$ symmetry. Both σ and ϕ are charged under $U(1)_{(X)}$ and therefore their vevs break the gauged (X) which as we have discussed survives as an anomalous $U(1)$ in the effective theory at low energies. We denote with g_B the gauge coupling of the $U(1)_X$ gauge boson (B_μ), while $\pm q_B$ will denote the corresponding X charges of the scalars. Their normalization, equal to ± 10 in the normalization of [145], is indeed arbitrary. The role of the Stueckelberg field is taken by $b(x)$ in the polar parameterization of ϕ, which is normalized to 1 in mass dimension, while F_σ is massless.

The two covariant derivatives of the scalars take the form

$$\begin{aligned} D_\mu \sigma &= \left(\partial_\mu + i q_B g_B B_\mu\right)\sigma \\ D_\mu \phi &= \left(\partial_\mu + i q_B g_B B_\mu\right)\phi \end{aligned} \tag{8.5.10}$$

with the typical Stueckelberg kinetic term generated from the decoupling of the radial fluctuations of the ϕ field

$$|D_\mu \phi|^2 = \frac{1}{2}\partial_\mu \rho_\phi \partial^\mu \rho_\phi + \frac{1}{2}(\partial_\mu b - M B_\mu)^2 \tag{8.5.11}$$

with $M = q_B g_B v_\phi \sim M_I$ takes the role of the Stueckelberg scale. In general it is natural to assume that both v_ϕ and v_σ are of the same order, and the mass of B_μ, the X gauge boson, will be given as a mean of both vevs

$$M_B = \sqrt{(q_B g_B v_\sigma)^2 + M^2} \tag{8.5.12}$$

The quadratic action, neglecting the contribution of the radial excitations of σ and ϕ, can be easily written down for such $\sigma - \phi$ combination

$$\begin{aligned} \mathcal{L}_q &= \frac{1}{2}\left(\partial_\mu \sigma_2\right)^2 + \frac{1}{2}\left(\partial_\mu b\right)^2 + \frac{1}{2}M_B^2 B_\mu B^\mu \\ &\quad + B_\mu \partial^\mu \left(M_1 b + v_\sigma g_B q_B \sigma_2\right), \end{aligned} \tag{8.5.13}$$

from which, after diagonalization of the mass terms we obtain

$$\begin{aligned} \mathcal{L}_q &= \frac{1}{2}\left(\partial_\mu \chi_B\right)^2 + \frac{1}{2}\left(\partial_\mu G_B\right)^2 + \frac{1}{2}\left(\partial_\mu h_1\right)^2 + \frac{1}{2}M_B^2 B_\mu B^\mu - \frac{1}{2}m_1^2 h_1^2 \\ &\quad + M_B B^\mu \partial_\mu G_B. \end{aligned} \tag{8.5.14}$$

where we are neglecting all the other terms generated from the decomposition which will not contribute to the breaking. We can identify the linear combinations

$$\begin{aligned} \chi_B &= \frac{1}{M_B}\left(-M\,\sigma_2 + q_B g_B v_\sigma\, b\right), \\ G_B &= \frac{1}{M_B}\left(q_B g_B v_\sigma\, \sigma_2 + M\, b\right), \end{aligned} \tag{8.5.15}$$

corresponding to the physical axion χ_B, and to a massless Nambu-Goldstone mode G_B. The rotation matrix that allows the change of variables $(\sigma_2, b) \to (\chi, G_B)$ is given by

$$U = \begin{pmatrix} -\cos\theta_B & \sin\theta_B \\ \sin\theta_B & \cos\theta_B \end{pmatrix} \tag{8.5.16}$$

with

$$\theta_B = \arcsin(q_B g_B v_\sigma / M_B). \tag{8.5.17}$$

The potential, as shown in similar analysis [133], is periodic in χ / f_χ where $f_\chi \sim M_I$ takes the role of the axion decay constant. As already stressed before, the origin of this potential is nonperturbative and linked to the presence of instantons at the SO(10) GUT phase transition. For such reason, the size of the constants λ in such potential are exponentially suppressed with $\lambda_i \sim e^{-2\pi/\alpha_{GUT}}$, with the value of the coupling α_{GUT} fixed at the scale M_{GUT} when the SO(10) instantons are exact. The value of α_{GUT} here is in the range $1/33 \leq \alpha_I \leq 1/32$, giving $10^{-91} \leq \lambda_{ij} \leq 10^{-88}$, determining an axion mass given by $m_\chi^2 \sim \lambda M_I^2$ in the range

$$10^{-22}\mathrm{e}V < m_\chi < 10^{-20}\mathrm{e}V \tag{8.5.18}$$

corresponding to an ultralight axion, which has been invoked for the resolution of several astrophysical constraints[36].

8.6 Conclusions

We have investigated the possibility that the decoupling of a right-handed neutrino in the context of an $SO(10)$ GUT can be accompanied by an axion-like particle. Such a particle shares many of the properties already considered for a similar model discussed by two of us in the context of an $E_6 \times U(1)_X$ unification, interpreted as low-energy GUT theory derived from string theory [37].

While, in the previous construction, the Stueckelberg Lagrangian was generated by the dualisation of a 3-form and required an anomalous $U(1)$ gauge symmetry, in this construction we have simply considered the possibility that the $U(1)_X$ symmetry of the Standard Model has an interesting implication.

Starting from an $SO(10)$ symmetry, broken to an $SU(5) \times U(1)_X$ GUT symmetry, the decoupling of a right-handed neutrino leaves at low energy an action which is Stueckelberg like, with a global anomaly which couples to a CP-odd phase, $\chi(x)$. We have invoked the generation of a periodic potential in the $SU(5) \times U(1)_X$ effective theory in order to extract such gauge invariant degree of freedom in the pseudoscalar sector which couples to a global anomaly. Such Stueckelberg-like pseudoscalars are expected to be ultralight, around 10^{-20} eV and to decouple at the scale corresponding to the mass of the right-handed neutrino. An earlier paper which relates the lightness of the axion to neutrino mass is [154].

We have illustrated, by analysing the representation content of the scalar sector of the $SO(10)$ and $SU(5) \times U(1)_X$ theories how this could be achieved.
We believe that we have merely identified the general tracts of this mechanism to which we hope to return in the near future in a more extensive analysis.

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Conclusions

In this thesis we have presented in a unified way the determinant role played by axion in the context of the evolution of the universe in its primordial phases. After an introduction to the formalisms necessary to carry on the discussion and having given a context in which to place the discussion we have presented an extension of the original PQ symmetry which enlarges the parameter space for axion and overcomes the mass/coupling constant relation. We have shown that Stueckelberg's models provide a new insight into the ancient axion problem and open up exciting new avenues on the search for dark matter. The previous observations were subsequently reinforced and given a more solid mathematical structure. The starting point was to consider the $SO(10)$ group subsequently broken into $SU(5) \times U(1)_X$ .In particular, the possibility that the decoupling of the right-handed neutrino, in the context of an algebraic structure of SO(10), leads to the appearance of an axion-like particle was discussed. It is seen that this particle possesses some features shared with other models interpreted as low-energy GUT theory, derived from string theory. We have seen how symmetry breaks from SO(10) leading to neutrino decoupling and leaving a Stueckelberg action equipped with global anomaly. Moreover, the association of this action with a periodic potential helps us to extract the gauge invariant degree of freedom in the pseudo-scalar sector. We finally show how this Stueckelberg-like sector turns out to be ultra-light at the scale corresponding to the mass of the right-handed neutrino. Finally in this thesis has been presented another work, always contextualized in the research regarding the development of the primordial phases of the universe. Since the first detection of gravitational waves there has been a huge interest in the study of this new type of cosmological messenger; gravitational waves can be considered as tools to probe the existence of new physics. In fact the gravitational wave generated by cosmological phase transitions of the first order possesses a spectrum dependent on the model that describes the phase transition. It is therefore possible to predict what are the implications of the GUT models on the phase transition of the early Universe and therefore on the spectrum and detection of gravitational waves. Therefore starting from the measurements it is possible to confirm possible unification models. It is in progress the design phase, by ESA, of LISA, a space mission that will engage the first space-based laser interferometer, whose launch is scheduled for 2034. The activity carried out in one of our papers was therefore focused on the study of a Quiver-type GUT model. The parameters characterizing the model have been calculated and it has been shown that the gravitational wave emitted by the phase transition described by this model is compatible with the sensitivity of the LISA experiment.

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