Spectrum of the $O(g^4)$ Scale-Invariant Lipatov Kernel

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(Received 28 November 1994)

An infrared scale-invariant approximation to the $O(g^4)$ Lipatov kernel has been determined by t-channel unitarity. The forward kernel responsible for parton evolution is evaluated and its eigenvalue spectrum determined. It can be written as a sum of two terms. The first term is proportional to the square of the $O(g^2)$ kernel. The second term is a new kinematic form whose spectrum shares many properties of the leading-order kernel. The full kernel gives a reduction ($\sim 68\alpha_s^2/\pi^2$) in the power growth of parton distributions at small x.

PACS numbers: 12.40.Nn, 11.10.Hi

The Balitsky-Fadon-Kuraev-Lipatov (BFKL) Pomeron [1] or, more simply, the Lipatov Pomeron, has recently attracted growing attention, both from the theoretical and the experimental sides. The BFKL equation resums leading logarithms in $1/x$. When applied in the forward direction, at large Q^2, it becomes an evolution equation for parton distributions. The Lipatov Pomeron solution of the equation predicts that a growth of the form

$$F_2(x, Q^2) \sim x^{1-\alpha_0} \sim x^{-1/2}, \tag{1}$$

where $\alpha_0 - 1$ is the leading eigenvalue of the forward $O(g^2)$ Lipatov kernel, should be observed in the small-x behavior of structure functions. The BFKL Pomeron is important in hard diffractive processes in general, for example deep-inelastic diffraction [2], and, perhaps, in rapidity-gap jet production [3]. BFKL resummation is also anticipated to play a key role in all semihard QCD processes [4], where there is a direct coupling of the hard scattering process to the Pomeron. It is one of the major results of the experimental program at the DESY ep collider HERA that a growth similar to that of (1) is observed [5].

From both a theoretical and an experimental viewpoint, it is vital to understand how the BFKL equation, and (1) in particular, is affected by next-to-leading logarithm contributions. In recent papers [6,7] a scale-invariant (in transverse momentum) infrared approximation to the $O(g^4)$, or next-to-leading order, kernel has been determined by Reggeon diagram and t-channel unitarity techniques. In this Letter we summarize some newly derived properties of this kernel, concentrating on the forward direction relevant for the evolution of parton distributions. We find that there are two components. The first has the structure of the $O(g^2)$ kernel but with additional logarithms of all the transverse momenta involved. It can be obtained by squaring the $O(g^2)$ kernel. The second component is a new kinematic form, which appears for the first time at $O(g^4)$, and has a number of interesting properties. It is separately finite and has no singularities generating infrared divergences after integration. A new eigenvalue spectrum is produced with characteristics which are suggestive of underlying holomorphic factorization and conformal symmetry properties [8].

We show that the full scale-invariant kernel gives a substantial reduction in α_0. We are unable, as yet, to give a complete result for how (1) is modified by our results. In particular, we must determine how scale invariance is broken by the off-shell renormalization scale so that, presumably, $g^2/4\pi \rightarrow \alpha_s(Q^2)$. Fadin and Lipatov have already calculated [9] the full Reggeon trajectory function (that gives the disconnected piece of the kernel) in the next-to-leading log approximation—including renormalization effects. The diagram structure we have anticipated is what is found, but there are additional scale-breaking logarithm factors involving internal transverse momenta. As outlined in [7], we hope to determine these effects in the remainder of the kernel by an extension of the Ward identity plus infrared finiteness analysis that gives the scale-invariant kernel.

It will be convenient to introduce a diagrammatic notation for transverse momentum integrals. A vertex with n incoming and m outgoing lines represents

$$(2\pi^3)\delta^2\Big(\sum k_i - \sum k_i'\Big)\Big(\sum k_i\Big), \tag{2}$$

and an n-line intermediate state represents

$$\frac{1}{(2\pi)^{3n}} \int d^2k_1 \cdots d^2k_n / k_1^2 \cdots k_n^2 . \tag{3}$$

We define all kernels (and parts of kernels) to include a momentum-conserving delta function, i.e., we write

$$K_i = (2\pi)^3 \delta^2(k_1 + k_2 - k_3 - k_4)\tilde{K}_i . \tag{4}$$

The contribution of (t-channel) four-particle nonsense states to the $O(g^4)$ kernel is given in [6] as a sum of transverse momentum diagrams of the form of Figs. 1(a)–1(d), i.e.,

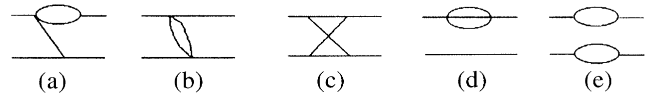

FIG. 1. (a)–(c) Connected diagrams for the $O(g^4)$ kernel; (d), (e) disconnected diagrams.

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$$(g^2N)^{-2}K_{2,2}^{(4n)}(k_1,k_2,k_3,k_4)_c = K_1 + K_2 + K_3 + K_4, \tag{5}$$

with

$$\tilde{K}_1 = -\frac{2}{3}\sum_{1<->2}(2\pi)^3 k_1^2 J_2(k_1^2)k_2^2[k_3^2\delta^2(k_2 - k_4) + k_4^2\delta^2(k_2 - k_3)], \tag{6}$$

$$\tilde{K}_2 = -\sum_{1<->2}\left(\frac{k_1^2 J_1(k_1^2)k_2^2k_3^2 + k_1^2J_1(k_1^2)k_2^2k_4^2 + k_1^2k_3^2J_1(k_3^2)k_4^2 + k_1^2k_3^2k_4^2J_1(k_4^2)}{(k_1 - k_3)^2}\right), \tag{7}$$

$$\tilde{K}_3 = \sum_{1<->2} J_1[(k_1 - k_3)^2](k_2^2k_3^2 + k_1^2k_4^2), \tag{8}$$

$$\tilde{K}_4 = \sum_{1<->2} k_1^2k_2^2k_3^2k_4^2 I(k_1,k_2,k_3,k_4), \tag{9}$$

where

$$J_1(k^2) = \frac{1}{(2\pi)^3}\int d^2q\,\frac{1}{q^2(k-q)^2}, \qquad J_2(k^2) = \frac{1}{(2\pi)^3}\int d^2q\,\frac{1}{(k-q)^2}J_1(q^2), \tag{10}$$

$$I(k_1,k_2,k_3,k_4) = \frac{1}{(2\pi)^3}\int d^2p\,\frac{1}{p^2(p+k_1)^2(p+k_4)^2(p+k_1-k_3)^2}. \tag{11}$$

Diagrams of the form of Fig. 1(e) were not included in [6], essentially because they cannot be associated with higher-order Reggeization. However, for $K_{2,2}^{(4n)}$ to be properly regulated after integration, disconnected diagrams of the form of both Figs. 1(d) and 1(e) must be included. This leads us to add a further contribution, $[K_{2,2}^{(2)}]^2$, to the $O(g^4)$ kernel, from iteration of the two-particle nonsense state, as in Fig. 2. That diagrams of the form of Fig. 1(e) in $K_{2,2}^{(4n)}$ and $(K_{2,2}^{(2)})^2$ must cancel then determines that the full $O(g^4)$ kernel is given by

$$K_{2,2}^{(4)} = \frac{1}{2^3}K_{2,2}^{(4n)} - (K_{2,2}^{(2)})^2. \tag{12}$$

This is the kernel that we wish to evaluate in the "forward" direction $k_1 = -k_2 = k, k_3 = -k_4 = k'$. Our result for $K_{2,2}^{(4)}(k,-k,k',-k')$ is a much simpler expression than the full result given by (12).

In writing down (12) we have determined the overall sign by the requirement that the contribution of the four-particle state should be positive. The overall magnitude has been determined by noting that the diagrams of the form of Fig. 1(e) contain only elements that appear in the leading-order kernel and their contribution in $K_{2,2}^{(4n)}$ is unambiguous. This implies that these diagrams should occur in $K_{2,2}^{(4n)}$ [and therefore $(K_{2,2}^{(2)})^2$] with an absolute magnitude that is equal to that obtained by simple-minded iteration of the leading-order kernel. The color structure is also determined by the role of the leading-order kernel in this construction. (In a forthcoming paper [10] we will actually determine all of the coefficients of $K_{2,2}^{(4n)}$, including the color factors, directly from t-channel unitarity.)

We should note that since we are specifically considering only the infrared structure of the $O(g^4)$ color zero interaction of two Reggeized gluons, the even-signature color octet trajectory found as a bound state of two Reggeized gluons in [2] does not enter our discussion. If we were to simultaneously consider the leading-order coupling of two Reggeized gluons to four Reggeized gluons, which is also $O(g^4)$ and is discussed in [6], this would not be the case. As we noted in [6], solving the BFKL equation in the $O(g^4)$ approximation for the kernel should consistently involve also the 2-4 coupling (and simultaneously a possible coupling of two Reggeized gluons to two even-signature octet Reggeons), but this would lead to a complete Reggeon field theory and not a self-contained integral equation.

The major technical problem in determining $K_{2,2}^{(4)}(k,-k,k',-k')$ is the evaluation of the box graph, i.e., $I(k,-k,k',-k')$ defined in (11). As we will show in detail in [11], if we regularize I with a mass term m^2 in each propagator, it can be evaluated as a sum of logarithms associated with each of the possible "two-particle" thresholds in the external momenta. As $m^2 \to 0$, we obtain

$$2\pi^2 I[k,k'] = \frac{A_{12}}{k'^2}\ln[k'^2/m^2] + \frac{A_{23}}{k^2}\ln[k^2/m^2] + \frac{A_{34}}{k'^2}\ln[k'^2/m^2] + \frac{A_{13}}{(k+k')^2}\ln[(k+k')^2/m^2]$$
$$+ \frac{A_{14}}{k^2}\ln[k^2/m^2] + \frac{A_{24}}{(k-k')^2}\ln[(k-k')^2/m^2], \tag{13}$$

where

$$A_{12} = -A_{14} = -A_{23} = A_{34} = \frac{k^2 - k'^2}{k^2(k+k')^2(k-k')^2}, \qquad A_{13} = A_{24} = \frac{1}{k^2k'^2}, \tag{14}$$

and so, as $m^2 \to 0$,

$$\tilde{K}_4 \to \frac{-k^2k'^2}{2\pi^2}\left(\frac{2(k'^2 - k^2)}{(k+k')^2(k-k')^2}\ln\left[\frac{k'^2}{k^2}\right] + \frac{1}{(k-k')^2}\ln\left[\frac{(k-k')^2}{m^2}\right] + \frac{1}{(k+k')^2}\ln\left[\frac{(k+k')^2}{m^2}\right]\right). \tag{15}$$

$\tilde{K}_3$ simply gives a contribution of the same form as the last two terms in (15), i.e., as $m^2 \to 0$

$$\tilde{K}_3 \to \frac{k^2k'^2}{2\pi^2}\left(\frac{1}{(k-k')^2}\ln\left[\frac{(k-k')^2}{m^2}\right] + \frac{1}{(k+k')^2}\ln\left[\frac{(k+k')^2}{m^2}\right]\right). \tag{16}$$

Similarly $\tilde{K}_2$ gives

$$\tilde{K}_2 \to \frac{-k^2k'^2}{2\pi^2}\left\{\frac{1}{(k-k')^2}\left(\ln\left[\frac{k^2}{m^2}\right] + \ln\left[\frac{k'^2}{m^2}\right]\right) + \frac{1}{(k+k')^2}\left(\ln\left[\frac{k^2}{m^2}\right] + \ln\left[\frac{k'^2}{m^2}\right]\right)\right\}. \tag{17}$$

The infrared finiteness of $\tilde{K}_c^{(4n)} = \tilde{K}_2 + \tilde{K}_3 + \tilde{K}_4$ is now apparent and we can write

$$\begin{aligned}2\pi^2\tilde{K}_c^{(4n)} &= \left(\frac{k^2k'^2}{(k-k')^2}\ln\left[\frac{(k-k')^4}{k^2k'^2}\right] + \frac{k^2k'^2}{(k+k')^2}\ln\left[\frac{(k+k')^4}{k^2k'^2}\right]\right) - \left(\frac{2k^2k'^2(k^2-k'^2)}{(k-k')^2(k+k')^2}\ln\left[\frac{k^2}{k'^2}\right]\right)\\ &= (\mathcal{K}_1) - (\mathcal{K}_2).\end{aligned} \tag{18}$$

Note that only $\mathcal{K}_1$ gives infrared divergences (at $k' = \pm k$) when integrated over k'. These divergences are canceled by the disconnected part of the kernel which we implicitly include in $\mathcal{K}_1$ for the rest of our discussion.

Apart from the logarithmic factors, $\mathcal{K}_1$ has the same structure as the forward (connected) $O(g^2)$ kernel. Indeed, if we evaluate all of the diagrams generated by Fig. 2 that survive in the forward direction, it is straightforward to show that

$$\mathcal{K}_1 = (2\pi)^2\widetilde{(K_{2,2}^{(2)})^2}, \tag{19}$$

implying from (12) and (18) that

$$\tilde{K}_{2,2}^{(4)} = -\frac{1}{2^4\pi^2}(3\mathcal{K}_1 + \mathcal{K}_2). \tag{20}$$

The appearance of $\mathcal{K}_2$ is a particularly interesting feature, since its symmetry properties (the antisymmetry of $\ln[k^2/k'^2]$ compensates for that of $(k^2 - k'^2)$) determine that it can only appear at the first logarithmic level.

We now move on to the eigenvalues of $K_{2,2}^{(4)}$. Our integration measure will be $\int d^Dk'/(k'^2)^2$, and so we use as eigenfunctions

$$\phi_{\mu,n}(k') = (k'^2)^{\mu}e^{in\theta}, \qquad \mu = \frac{1}{2} + i\nu, \qquad n = 0, \pm1, \pm2, \ldots, \tag{21}$$

which are complete and orthogonal (in $D = 2$).

The eigenvalues of $(K_{2,2}^{(2)})^2$ are trivially given by the square of the $O(g^2)$ eigenvalues, and so the essential problem is to determine the eigenvalues of $\mathcal{K}_2$. We first define $\mathcal{K}_2$ in D dimensions as

$$\mathcal{K}_2 = 2\eta\,\frac{k^2k'^2(k^2 - k'^2)}{(k+k')^2(k-k')^2}[(k^2)^{D/2-1} - (k'^2)^{D/2-1}], \tag{22}$$

where

$$\eta = \frac{\Gamma[2 - D/2]\Gamma[D/2 - 1]^2}{\Gamma[D-2]} \underset{D\to2}{\to} \frac{2}{D-2}. \tag{23}$$

We then look for eigenvalues $\lambda(\mu, n)$ such that

$$\int\frac{d^Dk'}{(k'^2)^2}\mathcal{K}_2(k,k')\phi_{\mu,n}(k') = \lambda(\mu,n)\phi_{\mu,n}(k). \tag{24}$$

If we define $\cos\chi = k\cdot\hat{x}$ and $\cos\theta = k'\cdot\hat{x}$, where $\hat{x}$ is an arbitrarily chosen unit vector, the only nontrivial angular integral is

$$I_\chi[n] = \int_0^{2\pi} d\theta\,\frac{e^{in\theta}}{1 - z(k,k')\sin^2(\theta - \chi)} = 2\pi e^{in\chi}\left(\frac{k^2 - k'^2}{k^2 + k'^2}\right)\left[\left(\frac{k'}{k}\right)^n\Theta[k - k'] - \left(\frac{k}{k'}\right)^n\Theta[k' - k]\right], \tag{25}$$

$$z[k,k'] = -\frac{4k^2k'^2}{(k^2 - k'^2)^2},$$

if n is an even integer (≥ 0). $I_\chi[n]$ vanishes if n is an odd integer and $I_\chi[-n] = I_\chi[|n|]$.

$I_\chi[n]$ is symmetric under the exchange of k and k', and also is invariant under $k \to 1/k$, $k' \to 1/k'$. This last invariance is sufficient to show that

$$\lambda(\mu, n) = \lambda(1 - \mu, n). \tag{26}$$

Using (25) we obtain from (24) that, as $D \to 2$,

$$\begin{aligned}\lambda(\mu,n) \to 2\eta\,\frac{\pi^{D/2}}{\Gamma[D/2]}[&\beta(|n|/2 + D/2 + \mu - 1) - \beta(|n|/2 - D/2 - \mu + 2)\\ &- \beta(|n|/2 + D + \mu - 2) + \beta(|n|/2 - D - \mu + 3)],\end{aligned} \tag{27}$$

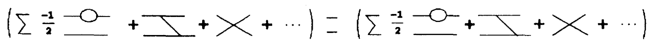

FIG. 2. Iteration of the leading-order kernel via two-particle nonsense states.

where $\beta(x)$ is the incomplete beta function. Writing $\Lambda(\nu,n) \equiv \lambda(\frac{1}{2}+i\nu,n)$ we obtain, for $D=2$,

$$\Lambda(\nu,n) = -2\pi\left[\beta'\left(\frac{|n|+1}{2}+i\nu\right) + \beta'\left(\frac{|n|+1}{2}-i\nu\right)\right]. \quad (28)$$

Since

$$\beta'(x) = \frac{1}{4}\left[\psi'\left(\frac{x+1}{2}\right) - \psi'\left(\frac{x}{2}\right)\right], \quad (29)$$

and

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2}, \quad (30)$$

$\beta'(x)$ is a real analytic function, and it follows from (28) that the eigenvalues $\Lambda(\nu,n)$ are all real.

The symmetry property (26) is reflected directly in the appearance of the two terms in $\Lambda(\nu,n)$, one depending on $(i\nu+1/2+n/2)$ and the other on $(i\nu+1/2-n/2)$. A similar decomposition of the $O(g^2)$ eigenvalues leads ultimately to the holomorphic factorization property [8] closely related to conformal symmetry. Since (26) is a consequence of inversion in k space one might also suspect that it is related to underlying conformal symmetry properties in the conjugate coordinate space. If the eigenvalues of the full $O(g^4)$ kernel are independent of q^2 [as seems likely based on analogy with the $O(g^2)$ kernel] then we do expect the coordinate space conformal symmetry and holomorphic factorization properties of the full kernel to be reflected in the forward eigenvalue spectrum.

Moving on to the modification of α_0, we note that to obtain the contribution to the eigenvalue of $\tilde{K}_{2,2}^{(4)}$ we multiply $\Lambda(\nu,n)$ by $-1/2^4\pi^2$. To compare with $\alpha_0 - 1$ we have to multiply, in addition, by $N^2 g^4/(2\pi)^3$, where $N=3$ for QCD. It follows from the above that the leading eigenvalue is $\Lambda(0,0)$, as it is for the $O(g^2)$ kernel. From (28)–(30), and writing $\alpha_s = g^2/4\pi$, we obtain the contribution to α_0 due to the $\mathcal{K}_2$ term in (20) as

$$-\frac{9\alpha_s^2}{(2\pi)^3}\Lambda(0,0) = \frac{9\alpha_s^2}{2\pi^2}\beta'(1/2) = -\frac{9\alpha_s^2}{8\pi^2}\left(\sum_{n=0}^{\infty}\frac{1}{(n+1/4)^2} - \sum_{n=0}^{\infty}\frac{1}{(n+3/4)^2}\right) \sim -16.5\,\frac{\alpha_s^2}{\pi^2}. \quad (31)$$

Combining with the $\mathcal{K}_1$ term in $K_{2,2}^{(4)}$ gives, for the full correction to α_0,

$$-\frac{3}{4}\left(\frac{12}{\pi}\ln[2]\alpha_s\right)^2 - \frac{9\alpha_s^2}{2\pi^2}\beta'(1/2) \sim -68\,\frac{\alpha_s^2}{\pi^2}, \quad (32)$$

which is a substantial negative correction.

We close by noting that the t-channel transverse-momentum integral formalism, on which our results are based, may well be valid only when a cutoff is introduced [10]. For small-x physics at relatively large transverse momentum, the cutoff dependence of our kernels will be crucial and, of course, without knowing this we cannot reliably estimate the relative magnitude of our contributions in the full physical kernel. In this sense our results give only the size of effects that are obtained by naively extending the infrared behavior of the theory to infinite transverse momentum.

We thank J. Bartels, R. K. Ellis, V. Fadin, R. Kirschner, L. Lipatov, and M. Wuesthoff for informative discussions and comments. A. R. W. acknowledges support from the U.S. Department of Energy, Division of High Energy Physics, Contract No. W-31-109-ENG-38.

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