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Effective actions in theories with gauge and conformal anomalies

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To my parents,
always by my side through every path

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Sommario

In questa tesi vengono presentati risultati originali derivanti dallo studio delle anomalie di gauge e dell'anomalia di traccia in teorie realistiche come un'estensione abeliana del Modello Standard (MS) o in una teoria effettiva in cui l'interazione gravitazionale è accoppiata al MS. Per questa ragione abbiamo effettuato numerosi studi perturbativi dell'azione effettiva ad un loop e, in particolare, di interazioni di gauge trilineari in presenza di simmetrie di gauge $U(1)$ anomale addizionali. L'anomalia di traccia è studiata al primo ordine perturbativo attraverso il correlatore TJJ (T indica il tensore energia-impulso e J una corrente di gauge generica). In questo secondo caso la nostra analisi è concentrata sullo studio delle azioni effettive di QED e QCD.

Mostriamo nella prima parte di questa tesi che per entrambi i tipi di anomalie l'azione effettiva è caratterizzata dalla comparsa di gradi di libertà privi di massa che sono pseudoscalari nel caso dell'anomalia chirale e scalari per l'anomalia di traccia o conforme. Nell'analisi di QED e QCD questi poli anomali possono anche essere estratti dallo studio dell'azione indotta dall'anomalia, ottenuta a sua volta come soluzione variazionale dell'equazione per l'anomalia, come mostrato in analisi precedenti. Nel caso dell'anomalia chirale, e del correlatore triangolare che ne è all'origine, è dimostrata l'equivalenza di due diverse parametrizzazioni: quella dovuta a Rosenberg e quella denominata Longitudinale/Trasversa. Uno dei risultati originali riguarda l'estensione di entrambe le parametrizzazioni a condizioni cinematiche generali. Si discute in dettaglio inoltre l'accoppiamento infrarosso dei poli sia nel caso del correlatore AVV (denotiamo con A e V rispettivamente una corrente di gauge assiale-vettoriale e una puramente vettoriale) che nel caso del correlatore TJJ, analizzando tutte le possibili regioni cinematiche in cui i poli appaiono. I risultati ottenuti recentemente da Mottola e Giannotti per il correlatore TJJ in QED nel caso conforme sono estesi al caso non conforme (sempre in QED) e sono stati riottenuti anche in QCD.

Nella seconda parte di questo lavoro di tesi sono discusse alcune caratteristiche fenomenologiche di teorie che estendono il MS con simmetrie $U(1)$ anomale e le loro nuove interazioni di gauge trilineari. Nel modello studiato la cancellazione delle anomalie è realizzata con un assione asintotico che generalizza il tradizionale assione di Peccei-Quinn (sotto forma di uno pseudoscalare di Stückelberg) e può avere una componente fisica in determinate condizioni. Esso pertanto fornisce un contesto teorico consistente per la descrizione di generiche particelle simili all'assione. Questo approccio alla cancellazione delle anomalie è alternativo al metodo di *sottrazione del polo anomalo*, introdotto in passato per riprisitinare l'invarianza di gauge di una teoria anomala. Un'analisi critica di questi due approcci è inclusa nella seconda parte della tesi.

Abstract

In this thesis some original results coming from the study of gauge and trace anomalies are presented, both analyzed in realistic theories such as an abelian extension of the Standard Model (SM) or in an effective field theory in which gravity is coupled to the SM. For this reason we perform several perturbative studies of the one loop effective action and, in particular, of the trilinear gauge interactions with the addition of extra anomalous $U(1)$ gauge symmetries. On the other hand, the trace anomaly is investigated at leading order via the TJJ correlator, where T denotes the energy momentum tensor and J a generic gauge current. In this second case our analysis is focused on the QED and QCD effective actions.

We show that in both cases the 1-particle irreducible effective action is characterized by the appearance of massless effective degrees of freedom. These are pseudoscalars in the case of the chiral anomaly and scalars for the trace/conformal anomaly and are dubbed "anomaly poles". In the QED and QCD cases these poles can also be extracted from the anomaly-induced action, which is obtained from the variational solution of the anomaly equation, as shown in previous analysis.

In the chiral case we discuss the equivalence between the Rosenberg and the Longitudinal/Transverse representations of the anomaly amplitude, showing the explicit mapping between the two in the most general external kinematical conditions. The infrared coupling of the poles is discussed both in the AVV (the correlator of Axial-Vector/Vector/Vector currents) and TJJ cases in great detail, analyzing all the possible kinematical regions where they appear. For the anomalous TJJ correlator we present its explicit form both in the conformal and in the non-conformal limits, generalizing results by Giannotti and Mottola derived in QED.

In the second part of the thesis we discuss some phenomenological features of anomalous extensions of the Standard Model and of its trilinear gauge interactions using an asymptotic axion for anomaly cancellation. This axion generalizes the traditional Peccei-Quinn axion (in the form of a Stückelberg pseudoscalar) and may develop a physical component under certain conditions, thereby providing a consistent theoretical framework for the description of axion-like particles.

This second approach to anomaly cancellation is alternative to the mechanism of *pole subtraction* for the restoration of the Ward identities of an anomalous theory. A critical investigation of these points is included.

Contents

List of publications

The published papers discussed in this thesis are (in chronological order)

- R. Armillis, C. Corianò and M. Guzzi,
  "Trilinear Anomalous Gauge Interactions from Intersecting Branes and the Neutral Currents Sector,"
  JHEP **0805**, 015 (2008) - arXiv:0711.3424 [hep-ph].

- R. Armillis, C. Corianò, L. Delle Rose and M. Guzzi,
  "Anomalous U(1) Models in Four and Five Dimensions and their Anomaly Poles,"
  JHEP **0912**, 029 (2009) - arXiv:0905.0865 [hep-ph].

- R. Armillis, C. Corianò and L. Delle Rose,
  "Anomaly Poles as Common Signatures of Chiral and Conformal Anomalies,"
  Phys. Lett. **B682**, 322-327 (2009) - arXiv:0909.4522 [hep-ph].

- R. Armillis, C. Corianò and L. Delle Rose,
  "Conformal Anomalies and the Gravitational Effective Action: The TJJ Correlator for a Dirac Fermion,"
  Phys. Rev. **D81**, 085001 (2010) - arXiv:0910.3381 [hep-ph].

- R. Armillis, C. Corianò, L. Delle Rose and L. Manni,
  "The Trace Anomaly and the Gravitational Coupling of an Anomalous U(1),"
  Int. J. Mod. Phys. **A26**, 2405-2435 (2011) - arXiv:1003.3930 [hep-ph].

- R. Armillis, C. Corianò and L. Delle Rose,
  "Trace Anomaly, Massless Scalars and the Gravitational Coupling of QCD,"
  Phys. Rev. **D82**, 064023 (2010) - arXiv:1005.4173 [hep-ph].

- R. Armillis, C. Corianò, L. Delle Rose and A. R. Fazio,
  "Comments on Anomaly Cancellations by Pole Subtractions and Ghost Instabilities with Gravity,"
  Class. Quant. Grav. **28**, 145004 (2011) - arXiv:1103.1590 [hep-ph].

Other publications

These other research papers are related to the same topics presented throughout the thesis, but are not discussed in detail

- R. Armillis, C. Corianò and M. Guzzi,
  "The Search for extra neutral currents at the LHC: QCD and anomalous gauge interactions,"
  AIP Conf. Proc. **964**, 212-217 (2007) - arXiv:0709.2111 [hep-ph].
- R. Armillis, C. Corianò, M. Guzzi and S. Morelli,
  "Axions and Anomaly-Mediated Interactions: The Green-Schwarz and Wess-Zumino Vertices at Higher Orders and g-2 of the muon,"
  JHEP **0810**, 034 (2008) - arXiv:0808.1882 [hep-ph].
- R. Armillis, C. Corianò, M. Guzzi and S. Morelli,
  "An Anomalous Extra Z Prime from Intersecting Branes with Drell-Yan and Direct Photons at the LHC,"
  Nucl. Phys. **B814**, 156-179 (2009) - arXiv:0809.3772 [hep-ph].
- F. Ambroglini, R. Armillis, P. Azzi, G. Bagliesi, A. Ballestrero, G. Balossini, A. Banfi, P. Bartalini *et al.*,
  "Proceedings of the Workshop on Monte Carlo's, Physics and Simulations at the LHC PART II," -
  arXiv:0902.0180 [hep-ph].
- F. Ambroglini, R. Armillis, P. Azzi, G. Bagliesi, A. Ballestrero, G. Balossini, A. Banfi, P. Bartalini *et al.*,
  "Proceedings of the Workshop on Monte Carlo's, Physics and Simulations at the LHC PART I," -
  arXiv:0902.0293 [hep-ph].

- R. Armillis, C. Corianò and L. Delle Rose,
  "Trilinear Gauge Interactions in Extensions of the Standard Model and Unitarity,"
  Nuovo Cim. **C32N3-4**, :261-264 (2009) - arXiv:0905.4410 [hep-ph].

- R. Armillis, C. Corianò, L. Delle Rose, M. Guzzi and A. Mariano,
  "The Effective Actions of Pseudoscalar and Scalar Particles in Theories with Gauge and Conformal Anomalies,"
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  "The Trace Anomaly and the Couplings of QED and QCD to Gravity,"
  AIP Conf. Proc. **1317**, 185-190 (2011) - arXiv:1007.2141 [hep-ph].

Introduction

The search for the identification of possible extensions of the Standard Model (SM) is a challenging research area both from the theoretical and the experimental point of view.
It is even more so with the early data distributed by the four experiments at the LHC, for which the hopes are that at least some among the many phenomenological scenarios that have been formulated in the last three decades can finally be tested.

The presence of so many wide and different possibilities render these studies very challenging. Surely, among these, the choice of simple abelian $U(1)$ extensions of the basic gauge structure $SU(3)_C \times SU(2)_L \times U(1)_Y$ of the SM is one of the simplest to take into consideration (see [1] for a review). These extensions of the Standard Model (SM) represent an economical but yet profound modification of the gauge structure of the electroweak sector, which has been tested first at Tevatron [2] and can be still tested at the LHC [3, 4, 5, 6]. They are also quite numerous. In fact $U(1)$ interactions abound in effective theories derived from string theory [5, 7, 8, 9] or from Grand Unified Theories (GUTs), with [10, 11, 12, 13, 14] or without [6, 15, 16, 17, 18] the introduction of supersymmetry. One of the common features of these models is the absence of an anomaly-free fermion spectrum, as for the SM.
In the SM case indeed all the trilinear correlators that can potentially generate gauge anomalies are set to zero by a suitable charge assignment for each SM particle under the corresponding gauge group. This mechanism for removing the anomaly is called *anomaly cancellation by charge assignment* and its realization is verified experimentally.

Nevertheless several compactifications of string theory predict the existence of *anomalous* $U(1)$ symmetries [7, 19, 20] and in these cases the mechanism of anomaly cancellation that Nature selects may not just be based on an anomaly-free spectrum, but may require a more complex pattern. This case is similar to the Green-Schwarz (GS) anomaly cancellation mechanism of string theory [21] and invokes an axion [16, 17, 22, 23, 24, 25, 26]. Interestingly enough, the same pattern appears if, for a completely different and purely dynamical reason, part of the fermion spectrum of an anomaly free theory is integrated out [27, 28], together with part of the Higgs sector [29].

The interest on the quantization of anomalous models and their proper field theoretical description has been a key topic for a long period, in an attempt to clarify under which conditions an anomalous gauge theory may be improved by the introduction of suitable interactions, so to become unitary and renormalizable [15, 16, 17, 23, 30].

One of the characteristic features of anomalous effective actions, both in the case of the chiral and of the conformal anomalies, is the presence of dynamical degrees of freedom generated by trilinear vertices. One of the open issues related to this point will be addressed in Chapter 5. There we will be discussing the difference between the cancellation of gauge anomalies obtained by the introduction of an asymptotic axion, and the same cancellation obtained by the subtraction of an anomaly pole. We will show that an anomaly pole can be described in terms of two local degrees of freedom which are kinetically mixed. A similar description emerges for the trace anomaly, with two extra scalars instead of two pseudoscalars, as in the chiral case.

It should be mentioned that the trace anomaly is part of the effective gravitational action and is not the result of any model-dependent construction. For this reason one can ask several questions regarding the true phenomenological impact of the breaking of scale invariance in the early universe.
It is also worth noticing that there are recent claims [31, 32] of the possible presence of full conformal invariance in the correlation functions of the CMB and, given the quantum origin of the CMB anisotropies, this raises significant questions in regard to the role of these scale-breaking quantum effects. The presence of new effective degrees of freedom as a signature of the trace anomaly is, for this reason, a significant feature of the quantum gravitational effective action. We will comment on these points in our conclusions.

General structure of the document

This thesis collects the results obtained throughout our investigation that try to clarify the study of the effective actions of massless pseudoscalars and scalars degrees of freedom when chiral and conformal anomalies are present in gauge theories.
Therefore the whole work has been divided into two parts, the first part covering the first four chapters and the second part the last two chapters.

The first part is focused on the identification and the fundamental properties of the anomalous trilinear correlators for chiral and conformally anomalous theories. The aim of these studies has been to show the emergence of anomaly poles for both theories in the effective actions and to describe their formulation in terms of local degrees of freedom. All the results have been obtained starting from analytical computations of the corresponding one-loop Feynman diagrams.

The second part is of phenomenological character and tries to apply, at least in part, the results of the formal studies of the first part to realistic extensions of the SM. The final chapter, which is dedicated to the study of trilinear gauge interactions in a specific extension of the SM, dubbed the MLSOM (Minimal Low Scale Orientifold Model) [15], is preceded by a more formal analysis of the relation between the different mechanisms of anomaly cancellations. In particular we will compare (Chapter 5) those invoking the subtraction of the anomaly pole for the restoration of the Ward identities in chiral gauge theories with those requiring an asymptotic Stückelberg axion. In this context, the connection between the chiral and the trace anomaly will appear to its fullest extent, since the framework in which these issues are addressed at the same time requires a supersymmetric formulation.

For this reason, we recall that in a supersymmetric theory the anomaly supermultiplet contains as its components both the trace anomaly and the chiral anomaly of a global $U(1)_R$ current, besides the gamma-trace of the supersymmetric current. In the context of supergravity the anomalous current is gauged, and the issues that we have (separately) uncovered in the first part of the thesis, for the chiral and conformal anomaly cases, will be unified. In particular we will show that the mechanism of anomaly cancellation introduced long ago by Ovrut, Cardoso [33, 34] and others as a field theory realization of the GS mechanism of string theory, performed in a supergravity context, amounts to the subtraction of specific anomaly poles in the effective action induced by the anomaly supermultiplet. This construction, as we are going to explain, induces at the level of trilinear gauge interactions some vertices whose features are unique and at variance respect to any interaction present in the SM. The current limitations of these approaches and the context in which they find justification is discussed in detail. These conclusions are reached after a careful analysis of the infrared and ultraviolet properties of the trilinear anomalous interactions, showing that the subtraction of an anomaly pole should be viewed as an ultraviolet correction and not as an exact mechanism.

Approaches to anomaly cancellation

Before entering into the detailed description of the first part of this work a comment is in order. Our attention will be mainly focused on two different approaches to anomaly cancellation: the first one involves a *polar* counterterm - as we have just mentioned - and is referred to as a generalization of the GS mechanism [21] in a four dimensional field theory, while the second one involves a Wess-Zumino term [35]. These two ways of realizing the cancellation of the anomaly are not equivalent at the level of the one-Particle-Irreducible (1PI) effective action and the issue of their completeness [24, 36], from a field theory point of view, is still open, as we are going to show with a detailed perturbative analysis.

Obviously, the investigation of the phenomenological implications of the chosen mechanism should be preceded by a complete study of those vertices which are responsible for the generation of an anomaly in perturbation theory. This motivates our studies of the anomaly vertices and of their kinematical limits, which are contained in the first 4 chapters.

The first chapter of this thesis is therefore devoted to the study of the trilinear correlator of an axial-vector and two vector currents in which the chiral anomaly appears[37, 38, 39]. The nature of these anomalous poles, in the most general kinematical case, is elucidated by performing a complete analysis of the kinematical properties of the anomaly vertex at perturbative level [40].

The computation is presented in the first chapter by the use of two independent (but equivalent) representations: the well-known Rosenberg representation [41] and the Longitudinal/Transverse (L/T) parameterization [42], used in recent studies of $g-2$ of the muon [24, 43, 44] and in the proof of non-renormalization theorems [45] of the anomaly vertex.
A dispersive analysis of this diagram had shown that this is identical to its pole counterterm only in a special situation, that is when the two external vector lines are on shell [37]. These points have been addressed in great detail in [24].

This special kinematic situation (dubbed the "collinear fermion/antifermion limit") is the only one in which the cancellation of the anomaly diagram with its counterterm is identical.
In the opposite case ("the non-collinear limit"), when the vector lines have both nonzero virtualities, the counterterm is not part of the vertex and its introduction may look rather artificial. Stated differently, the anomaly diagram appears to be pole-dominated only in certain configurations [37, 40] which affect both the infrared and the ultraviolet region of the corresponding correlator. Since these points are crucial in order to understand the origin of these singularities in perturbation theory, we will proceed from the ground up.

The first chapter then will contain the study of the two parameterizations of the anomaly diagram, the one due to Rosenberg [41] and the one that we identify as the "L/T parameterization" [42]. This second parameterization corresponds to a solution of the anomalous Ward identities used in recent studies of $g - 2$ of the muon (see also [24]).

The mapping between the two, performed in order to prove their equivalence and the isolation of the pole in both the collinear and the non-collinear limits has been analyzed in [40].

From the second chapter on we discuss the structure of the TJJ vertex, presenting its expression for QED (Quantum Electrodynamics), moving then to more complex cases.

The computation of similar diagrams, for the on-shell photon case, appears in older contributions by Berends and Gastmans [46] using dimensional regularization, in their study of the gravitational scattering of photons and by Milton using Schwinger's methods [47]. The presence of an anomaly pole in the amplitude has not been investigated nor noticed in any of these previous analysis, nor the $1/m$ expansion of the three form factors of the on-shell vertex, contained in [46], allows their identification in the S-matrix elements of the theory. Two related studies by Drummond and Hathrell, in their investigation of the gravitational contribution to the self-energy of the photon [48] and the renormalization of the trace anomaly [49] included the same on-shell vertex. Later, this same vertex has provided the ground for several elaborations concerning a possible superluminal behaviour of the photon in the presence of an external gravitational field [50]. The goal of our analysis has been to investigate the structure of this vertex, to determine its explicit off-shell expression at 1-loop order, which had not been given before, and to show that the polar contributions discovered in [51], due to the conformal anomaly, are indeed reproduced by the explicit analytical result [52].

In our approach we stress on the similarities between the case of the chiral and of the conformal anomalies, presenting the structure of the off-shell anomalous effective action for the chiral case and critically analyzing the role of the anomaly poles in this theory, building on previous investigations [6, 30, 40].

The different off-shell anomalous effective actions - in the presence of different types of external gauge currents - have been considered separately. We have investigated two vector gauge currents J_V (in the second chapter), two axial-vector gauge current J_A; the mixed case $J_V - J_A$ (in the third chapter), to conclude with the case of two non-abelian gluonic gauge currents (in the fourth chapter).
As we move in the analysis from simpler to more complex correlators, we expand substantially our technical tools. A key role in the test of our explicit perturbative results is the derivation of appropriate anomalous Ward identities which have been derived from first principles and checked on the final expressions. They allow to define consistently the anomaly vertices for a generic TJJ' correlator and are obatained by a procedure which can be easily generalized to even more complex correlators.

In the second chapter we compute in linearized gravity all the contributions to the gravitational effective action due to a virtual Dirac fermion in the presence of the trace anomaly.

The perturbative analysis deals with a trilinear correlator, called TJJ, having an energy-momentum insertion T and two vector currents J on the external lines (when axial currents are not considered, $J \equiv J_V$), all with generic virtualities. This correlator is responsible for the appearance of gauge contributions to the conformal anomaly in the effective action of gravity. The obtained results consist in the presentation of the complete anomalous off-shell effective action describing the interaction of gravity with the photons in the limit of a weak gravitational field and in the proof that this correlator exhibits an anomaly pole as well as the chiral one.
So we put in evidence that the effective action describing the interaction of gauge fields with

gravity is characterized by anomaly poles that give the same intriguing pattern of pole dominance in the UV and of decoupling in the IR (for massive or off-shell correlators), in complete analogy with the chiral case studied in the first chapter.
We conclude the chapter by noticing that *anomaly poles* are the most interesting feature of the anomalous diagrams, being them of chiral or of conformal type.

The third [53] and the fourth [54] chapter are two extensions of the second one: they both deal with the gravitational effective action in the presence of conformal anomaly, respectively showing the computation of the TJJ correlator in the case of mixed axial-vector and vector currents (third chapter) and within a non-abelian gauge theory (fourth chapter). They are both an important step of our investigation aiming at the computation of the exact effective action describing the coupling of the Standard Model to gravity via the conformal anomaly [55, 56].

The correlators that we study in the third chapter are two: the one with one vector and one axial-vector gauge currents called TJ_VJ_A and the one with two axial-vector gauge currents denoted by TJ_AJ_A.
The spectrum of the theory includes a single fermion of mass m and the investigation of the gravitational vertices has been carried out both in the massless and in the massive case [53]. This study is performed as the one previously done for the vector-like case, the difference consisting in the expansion of the trilinear correlator and in the suitable Ward identities allowing to unambiguously define it.

It turns out that the pure vector-like correlator TJJ and the corresponding chiral one (with an insertion of energy-momentum tensor and two axial-vector gauge currents) TJ_AJ_A start differing, away from the chiral limit, by contributions proportional to explicit mass breaking terms. Furthermore we conclude that the effective action obtained by coupling gravity to abelian vector/axial-vector gauge theories is characterized by effective massless degrees of freedom as well as in the pure vector case and that the anomalous poles emerge also in this case.
Understanding the physical significance of these effective actions in which a nonlocal polar counterterm can be described in terms of two auxiliary fields [30, 51, 52], one of them having a negative kinetic term, is still challenging.

The non-abelian case presented in last chapter is far more involved because the corresponding effective action is affected by the gauge choice and by ghost terms. This study confirms the general trend of the appearance of an anomaly pole which contributes to the trace part of the TJJ correlator, both in the quark and gluon sectors [54, 57]. Pole contributions, in this case, appear in each gauge-invariant subsector of the perturbative expansion.

Notwithstanding the similarities between the chiral and the conformal anomalies and the emergence of anomaly poles in both types of correlators that completely account for them, it is worth to remind that while in the case of chiral gauge theories the disappearance of the pole is necessary for ensuring the unitarization of the effective theory at high energy, in the case of conformal anomalies the corresponding poles [51, 52] play a different role. In fact gravity breaks unitarity in the UV already at Born level, and there is no compelling need to impose the cancellation of these contributions in order to preserve unitarity theory. As we have already pointed out, the local formulation of these types of theories requires two additional scalar (for conformal) or two pseudoscalar (for gauge anomalies) degrees of freedom - one of them being a ghost in both cases [30] - in order to rewrite these polar interactions in a local form.

Pole subtractions, asymptotic axions and phenomenology

We collect here few more comments concerning chapters 5 and 6. The fifth chapter presents a critical overview of the two different approaches to anomaly cancellation, the local one, based on the introduction of a Wess-Zumino term, and the non-local one, defined by a subtraction of the anomaly pole. The non-local subtraction has been proposed in the context of anomaly-free supergravities long ago. The goal of this investigation is to point out some of the issues which are still open concerning these types of effective Lagrangians. In particular we offer simple but plausible arguments to show that a mechanism of pole subtraction should be interpreted as an ultraviolet procedure which can not be extended to the far infrared. For the moment we just mention that the most successful mechanism to cancel the anomaly - beside the obvious strategy of an anomaly-free charge assignment - remains the introduction of an asymptotic axion (the local mechanism). With the term "asymptotic" we refer to a state which is part of the S matrix and is not necessarily formulated only as an intermediate effective interaction.

The mechanism of anomaly cancellation by a Wess-Zumino counterterm brings us to the final chapter of this thesis where we discuss the structure of the trilinear gauge interactions and their consistent definition in the case of anomalous abelian models. These models involve a kind of axion with different properties respect to the original axion introduced by Peccei and Quinn [58, 59] to solve the strong CP problem [60, 61]. Due to the anomaly, the shift symmetry of the axion is gauged under the anomalous $U(1)$ which extends the SM gauge group. This state is expected to play a role in the cosmology of the early Universe [26].
These extensions can be generically thought to be the result of a gauging, when the global U(1) symmetry (as for the Peccei-Quinn case) is promoted to a local one, which brings in rather tight constraints coming from the requirement of cancellation of the new gauge anomalies. As we said before, string models based on intersecting branes are one of the possible ways to generate abelian anomalous gauge interactions and axions whose interactions naturally follow into this

pattern.

The studied anomalous model contains two Higgs doublets, as in all the supersymmetric extensions of the SM, a new neutral current with its corresponding gauge boson Z' (it represents one of the phenomenological signatures of the model), together with a *physical* axion-like particle, called *axi-Higgs*. This particle can be (almost) massless, with its mass generated non-perturbatively in the QCD (Quantum Chromodynamics) vacuum as for an ordinary Peccei-Quinn axion, but can also mix with the scalars of the Higgs sector, becoming a heavy axion.

The model is also characterized by the presence of two different phases, the Stückelberg phase ([62, 63] for the original papers and [64] for a review) at high energy ($\sim TeV$) and the usual electroweak phase, called the Higgs-Stückelberg phase [16, 17]. In the first phase the additional gauge boson Z' is already massive with a mass M directly related to the Stückelberg mass scale, while the shifting axion is still a massless Nambu-Goldstone boson at this level. After the electroweak symmetry breaking the mass of the gauge boson Z' gets corrections proportional to the Higgs vacuum expectation value and, more interestingly, one linear combination of the shifting axion and a CP-odd component of the Higgs sector becomes physical: this is the so-called *axi-Higgs*. Its presence is the main distinctive feature of anomalous $U(1)$ models with this kind of anomaly cancellation mechanism.

The sixth chapter contains the details relative to the construction of the effective action at one-loop for this kind of models and an in-depth analysis of the trilinear gauge interactions appearing in this context [4, 18]. The study is carried out by means of generalized Ward identities that allow to define unambigously the necessary counterterms in each of the two phases. Our conclusions are contained in Chapter 7.

Chapter 1

The emergence of anomaly poles in the chiral anomaly

1.1 Introduction and Summary

The first chapter presents the study of the relationship between anomalies and massless degrees of freedom. The case of the axial anomaly in QED is well known, but the general behaviour of the triangle amplitude in generic kinematic conditions (i.e. when the photons are off the mass shell), its infrared aspect and above all the appearance of a massless pseudoscalar pole has not been studied in detail until recently [51]. We present a complete analysis of the 3-point function connecting three gauge currents, one of them being axial-vector and the other two of vector nature, denoted in the following as an AVV correlator. It is well-known that this diagram is the source of axial anomaly in a four dimensional gauge field theory ad that the correlator involving three axial currents, the AAA correlator can be decomposed as a sum of AVV ones.

An anomaly is in general the violation of a symmetry, valid at the quantum level, by means of quantum corrections. In the presence of a gauge anomaly, the gauge invariance of the classical Lagrangian is destroyed at the quantum level and the theory ceases to be a consistent quantum field theory.

The Dirac equation for a massive fermion ψ of mass m in QED reads as

$$- i\gamma^{\mu}(\partial_{\mu} - ieA_{\mu})\psi + m\psi = 0 \tag{1.1}$$

and implies that the vector current $J^{\mu} = \bar{\psi}\gamma^{\mu}\psi$ is conserved, so

$$\partial\mu\, J^{\mu} = 0. \tag{1.2}$$

Another current, called axial current, can be defined as $J_5^{\mu} = \bar{\psi}\gamma^{\mu}\psi$ (and $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$) and obeys at the classical level

$$\partial_\mu J_5^\mu = 2im\bar{\psi}\gamma^5\psi. \tag{1.3}$$

It can be seen that in the limit of vanishing fermion mass $m \leftarrow 0$, the classical Lagrangian exhibits a chiral $U(1)$ global symmetry under $\psi \rightarrow e^{i\alpha\gamma^5}\psi$, in addition to the $U(1)$ local gauge invariance. The current J_5^μ is the Noether current corresponding to this chiral symmetry. It turns out that both symmetries cannot be maintained simultaneously at the quantum level, so by enforcing $U(1)$ gauge invariance in Eq. (1.2), the full quantum theory results affected by a finite axial current anomaly

$$\partial_\mu \langle J_5^\mu \rangle_A \Big|_{m=0} = \frac{e^2}{16\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,F_{\mu\nu}\,F_{\rho\sigma} = \frac{e^2}{2\pi^2}\vec{E}\cdot\vec{B}, \tag{1.4}$$

with the gauge field strength $F_{\mu\nu}$ being $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\vec{E}$ and $\vec{B}$ the electric and magnetic fields respectively.

One of the subtle features of the axial anomaly is the presence of massless poles in the corresponding AVV correlator, which show up in special kinematical regions and in the chiral limit, and whose interpretation is at times rather puzzling. In fact, on several occasions the correct interpretation of these singularities have been debated at length [38, 65]. Our interest in the topic, which is one of our reasons and motivations for this analysis, has been the result of a recent work in which we have suggested the subtraction of the anomaly pole in theories involving anomalous $U(1)$'s to ensure anomaly cancellation, by defining a new gauge invariant vertex [24]. The re-defined vertex is non-local, while its Ward identity is expressed in terms of local interactions and can be interpreted diagrammatically by introducing a massless pseudoscalar - an axion field - coupled to gauge fields via Wess-Zumino terms. This coupling is induced by the anomaly and the subtraction of the anomaly pole is expected to represent the only consistent way by which a completion of an anomalous theory is supposed to work in the UV region.

However, as known from several previous studies of this vertex, the presence of a longitudinal pole in an anomaly diagram has always been established *only for special kinematical configurations* and this raises a serious concern regarding the meaning of the subtraction, introduced to restore the Ward identity at high energy, a subtraction which should be naturally performed by the UV completion of the anomalous theory. The main objective of this analysis is to show that the effective action of an anomalous gauge theory is affected by singularities which are not necessarily detected using a dispersive analysis in the infrared (IR) [66] (see also [51] for a recent study), and as such are IR decoupled. These additional poles, which account for the anomaly, can be extracted by a complete computation of the effective action and have a direct ultraviolet UV significance. For this reason, assessing the UV significance of an anomaly pole, whose identification, in the past, has always been linked to the infrared (IR) using a spectral approach, certainly helps in establishing a natural link between an anomalous theory and its completion, which should guarantee the cancellation of these contributions.

To show the existence of these singularities under the most general kinematical conditions we proceed with a complete and comparative study of the anomaly diagram in two different parameterizations which are both essential in order to understand the nature of the longitudinal subtraction. In fact, only a complete and off-shell computation of the effective action for an anomalous theory allows the identification of these terms which escape detection with the usual spectral analysis. The nature of these additional singularities of the effective action which, in some cases, are not evident due to the presence of Schouten relations, is resolved by studying a special class of amplitudes in which the presence of a pole dominance can be immediately linked to a non unitary behaviour of the theory. Having clarified these points, we proceed by discussing the structure of the anomalous effective action of a typical anomalous theory, represented by expansions in the fermion mass (m). This can be viewed as the generalization to the anomalous case of the usual Euler-Heisenberg effective action, which now contains additional (anomalous) trilinear interactions that are absent in the QED case, due to C-invariance.

1.2 Anomaly poles and general kinematics: the Rosenberg case

One of the intriguing features of the anomaly diagrams is that the poles are part of the anomaly amplitude only under some special kinematical conditions. For instance, the $\pi \to \gamma\gamma$ (pion pole) amplitude interpolates between the axial vector current (J_A) and two vector currents (J_V) and saturates the anomaly contribution (if we neglect the pion mass) given by the $\langle J_A J_V J_V \rangle$ perturbative correlator. This saturation is at the basis of 't Hooft's matching conditions, according to which the anomaly of the fermions should be reproduced by a composite particle (a pseudoscalar) in a confining theory (see also the discussion in [51]). In general, the pole appears by solving the anomalous Ward identity for the corresponding amplitude, $\Delta^{\lambda\mu\nu}(k_1, k_2)$ (we use momenta as in Fig. 1.1 with $k = k_1 + k_2$)

$$k_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) = a_n \epsilon^{\mu\nu\alpha\beta}\, k_{1\alpha}\, k_{2\beta} \tag{1.5}$$

rather trivially, using the longitudinal tensor structure

$$\Delta^{\lambda\mu\nu} \equiv w_L = a_n \,\frac{k^\lambda}{k^2}\, \epsilon^{\mu\nu\alpha\beta}\, k_{1\alpha}\, k_{2\beta}, \tag{1.6}$$

where $a_n = -i/2\pi^2$ denotes the anomaly. The presence of this tensor structure with a $1/k^2$ behaviour is the signature of the anomaly. This result holds for an AVV graph, but can be

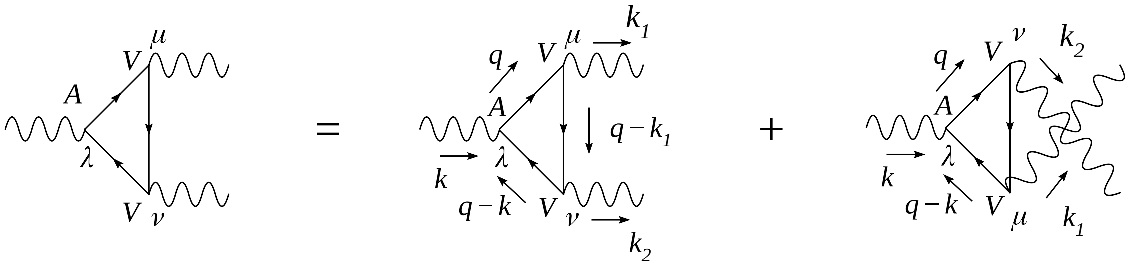


Figure 1.1: Triangle diagram with an axial-vector current (λ) and two vector currents (μ, ν). The momentum parameterization for the direct and the exchange contribution is written here in an explicit form for future reference.

trivially generalized to more general anomaly graphs, such as AAA graphs, by adding poles in the invariants of the remaining lines, i.e. $1/k_1^2$ and $1/k_2^2$, by imposing an equal distribution of the anomaly on the three axial-vector legs of the graph.

Obviously, in the chiral limit, the triangle amplitude and the pole amplitude coincide only if the two photons are on-shell. In fact, as shown by Dolgov and Zakharov [37], the pole dominance requires a special kinematics. For this reason, the pole has a nonvanishing residue only for massless photons. This, in fact, sets a limit on the validity of the matching, since the perturbative correlator and the pole amplitude are not supposed to coincide for any virtuality of the photons.

1.2.1 UV completions and decoupled poles in the IR

Being the anomaly closely related to the presence of a pole in the correlation function, the subtraction of the anomaly pole from the perturbative amplitude is sufficient to restore the Ward identities of the theory. For this to occur one has to show that the correlator has always an anomaly pole, which is not obvious. The main goal of this study is to show that the correlator responsible for the chiral gauge anomaly is always (i.e. under any kinematical conditions) characterized by the presence of a pole, and to provide an interpretation of this.

We recall that anomaly poles have been identified via an analysis in the IR which shows that the anomalous correlator has indeed a pole characterized by a nonvanishing residue. In fact, the IR coupling of the pole present in the correlator is, for a standard IR pole, rather obvious since the limit

$$\lim_{k^2 \to 0} k^2 \, \Delta^{\lambda\mu\nu} = k^\lambda \, a_n \, \epsilon^{\mu\nu\alpha\beta} \, k_{1\alpha} \, k_{2\beta} \tag{1.7}$$

allows to attribute to the anomaly amplitude a non-vanishing residue. Our main conclusion is that anomaly poles should not be searched for only by the usual dispersive analysis, which is effective only for standard IR poles, but require a complete off-shell evaluation of the anomalous effective action. We show that these additional poles are decoupled in the IR, but they nevertheless control the UV behaviour of the theory. This last point is proved by looking at a special class of amplitudes which are pole dominated in the UV and which allow to detect the non unitary behaviour of an anomalous theory rather closely.

For this to happen one needs a separation of the anomaly amplitude into longitudinal and transverse components. Our results are based on direct computations, using the two parameterizations of the anomaly amplitude mentioned above. We work under the most general kinematic conditions, generalizing the L/T parameterization given in [42] away from the chiral limit and showing its exact equivalence to that of Rosenberg [41].

We start our discussion by addressing the issue of the extraction of an anomaly pole from the Rosenberg form of the anomaly diagram [41]. We review the identification of the independent structures of the AVV diagram in this formulation and then move to the L/T decomposition, illustrating the connection between the two.

1.2.2 Connecting two parameterizations

In his classic paper [41] Rosenberg provided an expression for the three-point correlator in terms of a sum of six invariant amplitudes multiplied by different tensorial structures, denoted by $A_1, \dots A_6$. These are given as parametric integrals and are easily computable only in few cases, for example when the external momenta are on-shell (massless) or with symmetric off-shell configurations of the two vector lines ($k_1^2 = k_2^2$). We will re-analyze the derivation of the amplitude, emphasizing the features of the vertex in the most general case, by focusing our attention on the special kinematical limits in which the pole appears. The AVV amplitude with off-shell external lines shown in Fig.1.1 is therefore written according to [41] in the form

$$\Delta_0^{\lambda\mu\nu} = \frac{i^3}{(2\pi)^4} \int d^4q \frac{Tr\left[\gamma^\lambda \gamma^5 (\not{q} - \not{k}) \gamma^\nu (\not{q} - \not{k}_1) \gamma^\mu \not{q}\right]}{q^2\, (q-k)^2\, (q-k_1)^2} + \text{exch.} \tag{1.8}$$

with

$$\begin{aligned} \Delta_0^{\lambda\mu\nu} &= A_1(k_1,k_2)\varepsilon[k_1,\mu,\nu,\lambda] + A_2(k_1,k_2)\varepsilon[k_2,\mu,\nu,\lambda] + A_3(k_1,k_2)\varepsilon[k_1,k_2,\mu,\lambda]{k_1}^{\nu} \\ &+ A_4(k_1,k_2)\varepsilon[k_1,k_2,\mu,\lambda]k_2^{\nu} + A_5(k_1,k_2)\varepsilon[k_1,k_2,\nu,\lambda]k_1^{\mu} + A_6(k_1,k_2)\varepsilon[k_1,k_2,\nu,\lambda]k_2^{\mu}. \end{aligned} \tag{1.9}$$

The four invariant amplitudes A_i for $i \geq 3$ are finite and given by explicit parametric integrals [41]

$$A_3(k_1,k_2) = -A_6(k_2,k_1) = -16\pi^2 I_{11}(k_1,k_2), \tag{1.10}$$

$$A_4(k_1,k_2) = -A_5(k_2,k_1) = 16\pi^2 \left[I_{20}(k_1,k_2) - I_{10}(k_1,k_2)\right], \tag{1.11}$$

where the general massive I_{st} integral is defined by

$$I_{st}(k_1,k_2) = \int_0^1 dw \int_0^{1-w} dz w^s z^t \left[z(1-z)k_1^2 + w(1-w)k_2^2 + 2wz(k_1 k_2) - m^2 \right]^{-1}, \quad (1.12)$$

whose explicit form will be worked out below. Both A_1 and A_2 are instead represented by formally divergent integrals, which can be rendered finite only by imposing the Ward identities on the two vector lines, giving

$$A_1(k_1,k_2) = k_1 \cdot k_2 \, A_3(k_1,k_2) + k_2^2 \, A_4(k_1,k_2), \quad (1.13)$$

$$A_2(k_1,k_2) = k_1^2 \, A_5(k_1,k_2) + k_1 \cdot k_2 \, A_6(k_1,k_2), \quad (1.14)$$

which allow to re-express the formally divergent amplitudes in terms of the convergent ones. The Bose symmetry on the two vector vertices with indices μ and ν is fulfilled thanks to the relations

$$A_5(k_1,k_2) = -A_4(k_2,k_1) \quad (1.15)$$

$$A_6(k_1,k_2) = -A_3(k_2,k_1). \quad (1.16)$$

1.2.3 Explicit expressions in the massless case

To extract the explicit form of the parametric integrals given by Rosenberg, we proceed with a direct computation of the invariant amplitudes of the parameterization using dimensional reduction. We perform the traces in 4 dimensions and the loop tensor integrals in D dimensions, using the common techniques of tensor reduction. We use dimensional regularization with minimal subtraction and find, as expected, the cancellation of the dependence of the result on the renormalization scale. Therefore, the parametric integral I_{11} and the combinations $I_{20} - I_{10}$ are trivially identified at the end of the computation. The result is expressed in terms of elementary functions, except for the function $\Phi(x,y)$ [67], which is related to one of the two master integrals of the decomposition, the scalar massless triangle. We obtain for generic virtualities of the external lines

$$
\begin{aligned}
A_1(s,s_1,s_2) &= -\frac{i}{4\pi^2}+\frac{i}{8\pi^2\sigma}\Big\{\Phi(s_1,s_2)\frac{s_1 s_2\,(s_2-s_1)}{s}+s_1\,(s_2-s_{12})\log\left[\frac{s_1}{s}\right] \\
&\qquad -s_2\,(s_1-s_{12})\log\left[\frac{s_2}{s}\right]\Big\}, \qquad (1.17)\\
A_3(s,s_1,s_2) &= \frac{i}{8\pi^2 s\sigma^2}\Big\{-s_1 s_2\left[4s_{12}^2+3\,(s_1+s_2)\,s_{12}+2s_1s_2\right]\Phi(s_1,s_2) \\
&\qquad -2ss_{12}\sigma-ss_1\left[2s_1s_2+s_{12}\,(3s_2+s_{12})\right]\log\left[\frac{s_1}{s}\right] \\
&\qquad -ss_2\left[s_{12}^2+s_1\,(2s_2+3s_{12})\right]\log\left[\frac{s_2}{s}\right]\Big\}, \qquad (1.18)\\
A_4(s,s_1,s_2) &= \frac{i}{8\pi^2 s\sigma^2}\Big\{s_1\left[4s_{12}^3+2\,(s_1+2s_2)\,s_{12}^2+2s_1s_2s_{12}+s_1\,(s_1-s_2)\,s_2\right]\Phi(s_1,s_2) \\
&\qquad +2ss_1\sigma+s\,(s_1+s_{12})\left(2s_{12}^2+s_1s_2\right)\log\left[\frac{s_2}{s}\right] \\
&\qquad +ss_1\left[4s_{12}^2-s_1\,(s_2-3s_{12})\right]\log\left[\frac{s_1}{s}\right]\Big\}, \qquad (1.19)
\end{aligned}
$$

where $s=k^2$, $s_1=k_1^2$, $s_2=k_2^2$, $s_{12}=k_1\cdot k_2$ with $\sigma=s_{12}^2-s_1s_2$ and the function $\Phi(x,y)$ is defined as [67]

$$
\Phi(x,y) = \frac{1}{\lambda}\left\{2[Li_2(-\rho x)+Li_2(-\rho y)]+\ln\frac{y}{x}\ln\frac{1+\rho y}{1+\rho x}+\ln(\rho x)\ln(\rho y)+\frac{\pi^2}{3}\right\}, \qquad (1.20)
$$

with

$$
\lambda(x,y)=\sqrt{\Delta}, \qquad \Delta=(1-x-y)^2-4xy, \qquad (1.21)
$$

$$
\rho(x,y)=2(1-x-y+\lambda)^{-1}, \qquad x=\frac{s_1}{s}, \qquad y=\frac{s_2}{s}. \qquad (1.22)
$$

$\Phi(x,y)$ can be traced back to the one-loop three-point massless scalar integral $C_0(s,s_1,s_2)$, as mentioned above, involved in the reduction of the tensor integrals with three denominators in Eq. (1.8) as

$$
C_0(s,s_1,s_2)=\frac{i\pi^2}{s}\Phi(x,y). \qquad (1.23)
$$

Each term in the function $\Phi(x,y)$ and also the arguments of the logarithmic functions appearing in the form factors A_i $(i=1,\dots,6)$ are real if one of these two sets of different conditions is simultaneously satisfied. In the spacelike region we may have

- $s,s_1,s_2<0$ and $s<-(\sqrt{-s_1}+\sqrt{-s_2})^2$

or in the physical region with positive kinematical invariants

- $s,s_1,s_2>0$ and $s>(\sqrt{s_1}+\sqrt{s_2})^2$.

$\varepsilon[k_1,\lambda,\mu,\nu]$	$\varepsilon[k_1,k_2,\mu,\lambda]\,k_1^\nu$	$\varepsilon[k_1,k_2,\nu,\lambda]\,k_1^\mu$	$\varepsilon[k_1,k_2,\mu,\nu]\,k_1^\lambda$
$\varepsilon[k_2,\lambda,\mu,\nu]$	$\varepsilon[k_1,k_2,\mu,\lambda]\,k_2^\nu$	$\varepsilon[k_1,k_2,\nu,\lambda]\,k_2^\mu$	$\varepsilon[k_1,k_2,\mu,\nu]\,k_2^\lambda$

Table 1.1: The eight pseudotensors in which a general amplitude $\Delta^{l\mu\nu}(k_1,k_2)$ can be expanded.

All the other regions would require some specific analytic continuations by giving to all the invariants a small imaginary part η ($\eta>0$) according to the $i\eta$ prescription with $s_i\to s_i+i\eta$.

When discussing the presence of spurious poles for $s\to 0$ we need to work with amplitudes which are well-defined around $s=0$; for this reason the analytic regularizations have been always performed before taking the $s\to 0$ limit. There is another important observation that is in order at this point. One may worry if the absence of the pole in s can be attributed to the redundancy of the Rosenberg representation [41], but, as we are going to show next, this is not the case.

1.2.4 Four amplitude decomposition in Rosenberg

In order to derive a set of a minimal number of independent invariant amplitudes we proceed from scratch. The identification of the invariant tensor structures characterizing the amplitude can be done exhaustively, by starting with the construction of all the possible tensors of rank three built out of the ε-tensor and the external momenta. We follow here an approach similar to [51] with some minor changes.
The eight tensorial structures listed in Tab.1.1 are the ones needed in the expansion of a generic triangle correlator with three indices $\{\lambda,\mu,\nu\}$ and external momenta $\{k_1,k_2\}$. Out of these 8 structures, only the six in the first three columns appear in the Rosenberg formulation and can be reduced to 4 with little effort by requiring conservation of the vector currents. If we impose the vector Ward identity on the two vector lines of the diagram and fix the divergent coefficients A_1 and A_2 in terms of the remaining amplitudes, then the form factors A_i reduce to the four ones $A_3,\dots,A_6$ and the tensor structures in front of them get automatically organized in terms of four linear combinations indicated with η_i. These four tensor amplitudes η_i are selected from a set of six quantities defined in Tab.1.2, which shows all the possible tensors entering into the expansion of a generic three-currents correlator *after* imposing the conservation of the vector current.

Coming back to our specific case, we obtain for the generic anomalous AVV vertex satisfying

η_1	$\varepsilon[k_1, k_2, \mu, \nu]\, k_1^\lambda$
η_2	$\varepsilon[k_1, k_2, \mu, \nu]\, k_2^\lambda$
η_3	$k_1 \cdot k_2 \varepsilon[k_1, \lambda, \mu, \nu] + k_1^\nu \varepsilon[k_1, k_2, \mu, \lambda]$
η_4	$k_2 \cdot k_2 \varepsilon[k_1, \lambda, \mu, \nu] + k_2^\nu \varepsilon[k_1, k_2, \mu, \lambda]$
η_5	$k_1 \cdot k_1 \varepsilon[k_2, \lambda, \mu, \nu] + k_1^\mu \varepsilon[k_1, k_2, \nu, \lambda]$
η_6	$k_1 \cdot k_2 \varepsilon[k_2, \lambda, \mu, \nu] + k_2^\mu \varepsilon[k_1, k_2, \nu, \lambda]$

Table 1.2: The six pseudotensors needed in the expansion of an amplitude $\Delta^{l\mu\nu}(k_1, k_2)$ satisfying the vector current conservation.

the vector Ward identities the parameterization

$$\begin{aligned}\Delta^{\lambda\mu\nu}_{WI} &= A_3(k_1 \cdot k_2 \varepsilon[k_1, \lambda, \mu, \nu] + k_1^\nu \varepsilon[k_1, k_2, \mu, \lambda]) + A_4(k_2 \cdot k_2 \varepsilon[k_1, \lambda, \mu, \nu] + k_2^\nu \varepsilon[k_1, k_2, \mu, \lambda]) \\ &\quad + A_5(k_1 \cdot k_1 \varepsilon[k_2, \lambda, \mu, \nu] + k_1^\mu \varepsilon[k_1, k_2, \nu, \lambda]) + A_6(k_1 \cdot k_2 \varepsilon[k_2, \lambda, \mu, \nu] + k_2^\mu \varepsilon[k_1, k_2, \nu, \lambda]) \\ &= A_3\, \eta_3^{\lambda\mu\nu}(k_1, k_2) + A_4\, \eta_4^{\lambda\mu\nu}(k_1, k_2) + A_5\, \eta_5^{\lambda\mu\nu}(k_1, k_2) + A_6\, \eta_6^{\lambda\mu\nu}(k_1, k_2).\end{aligned} \tag{1.24}$$

This is obtained after plugging Eqs. (1.13,1.14) into Eq. (1.9), where $\eta_i^{\lambda\mu\nu}(k_1, k_2)$ can be read from Tab.1.2. The remaining two homogeneous pseudotensors of degree 3 in k_1, k_2, denoted by $\eta_1^{\lambda\mu\nu}$ and $\eta_2^{\lambda\mu\nu}$

$$\eta_1^{\lambda\mu\nu}(k_1, k_2) = k_1^\lambda\, \varepsilon[k_1, k_2, \mu, \nu], \qquad \eta_2^{\lambda\mu\nu}(k_1, k_2) = k_2^\lambda\, \varepsilon[k_1, k_2, \mu, \nu], \tag{1.25}$$

are not present in the Rosenberg parameterization, although they appear in the L/T decomposition, as we show below. The reduction of these two tensors to the four ones already used as a basis can be achieved by the use of two Schouten relations

$$k_1^\lambda\, \varepsilon[k_1, k_2, \mu, \nu] = k_1^\mu\, \varepsilon[k_1, k_2, \lambda, \nu] - k_1^\nu \varepsilon[k_1, k_2, \lambda, \mu] - k_1^2 \varepsilon[k_2, \lambda, \mu, \nu] + k_1 \cdot k_2 \varepsilon[k_1, \lambda, \mu, \nu], \tag{1.26}$$

$$k_2^\lambda\, \varepsilon[k_1, k_2, \mu, \nu] = k_2^\mu\, \varepsilon[k_1, k_2, \lambda, \nu] - k_2^\nu\, \varepsilon[k_1, k_2, \lambda, \mu] - k_1 \cdot k_2 \varepsilon[k_2, \lambda, \mu, \nu] + k_2^2 \varepsilon[k_1, \lambda, \mu, \nu], \tag{1.27}$$

or equivalently,

$$\eta_1^{\lambda\mu\nu}(k_1, k_2) = \eta_3^{\lambda\mu\nu}(k_1, k_2) - \eta_5^{\lambda\mu\nu}(k_1, k_2), \tag{1.28}$$

$$\eta_2^{\lambda\mu\nu}(k_1, k_2) = \eta_4^{\lambda\mu\nu}(k_1, k_2) - \eta_6^{\lambda\mu\nu}(k_1, k_2). \tag{1.29}$$

The set of the 4 amplitudes that we have chosen in the parameterization shown in Eq. (1.24) are linearly independent and functionally independent respect to the Schouten transformations. The claim that one can make is that any tensor structure which is not of the form given in the 4-basis above can be re-expressed as a combination of these 4 structures using appropriate Schouten relations. The decomposition of the AVV diagram with respect to this basis is therefore unique. At this point it is trivial to realize that, starting from the explicit expressions of the invariant amplitudes A_i that we have given above, the absence of a residue at $s = 0$ continues to hold (for general off-shell kinematics). The important point to observe is that there is no kinematical singularity in this limit in each of the 4 independent tensor structures. The conclusion is that, in general, an AVV diagram has no massless poles. The use of a set of non-redundant amplitudes clears the ground of any doubt concerning this result. In fact, the poles appear only under special kinematical configurations, as we are going to discuss next.

1.3 The massive off-shell case for the Rosenberg parameterization

Before performing the relevant kinematical limits on the amplitude, we move one step forward and generalize the results presented in the previous section to the massive case, by writing the expression of the invariant amplitudes given by Rosenberg (and the corresponding parametric integrals) in an explicit form.

The computation is performed as in the massless case, using dimensional reduction. The modifications are minimal and mostly due to the new scalar integrals B_0 and C_0, corresponding to the massive (scalar) self-energy and triangle diagram respectively. The three-point amplitude with equal massive internal lines is given by

$$\Delta^{l\mu\nu} = \frac{i^3}{(2\pi)^4} \int d^4q \frac{Tr\left[\gamma^\lambda\gamma^5(\not{q} - \not{k} + m)\gamma^\nu(\not{q} - \not{k}_1 + m)\gamma^\mu(\not{q} + m)\right]}{(q^2 - m^2)\left((q-k)^2 - m^2\right)\left((q-k_1)^2 - m^2\right)} + \text{exch.}, \tag{1.30}$$

with $k = k_1 + k_2$, and can be again cast into the form

$$\begin{aligned}
\Delta^{\lambda\mu\nu} &= A_1(k_1, k_2, m^2)\,\varepsilon[k_1, \mu, \nu, \lambda] + A_2(k_1, k_2, m^2)\,\varepsilon[k_2, \mu, \nu, \lambda] \\
&+ A_3(k_1, k_2, m^2)\,\varepsilon[k_1, k_2, \mu, \lambda]\,{k_1}^\nu + A_4(k_1, k_2, m^2)\,\varepsilon[k_1, k_2, \mu, \lambda]\,k_2^\nu \\
&+ A_5(k_1, k_2, m^2)\,\varepsilon[k_1, k_2, \nu, \lambda]\,k_1^\mu + A_6(k_1, k_2, m^2)\,\varepsilon[k_1, k_2, \nu, \lambda]\,k_2^\mu,
\end{aligned} \tag{1.31}$$

where the tensorial structures are the same as before and the massive form factors $A_i(k_1, k_2, m^2)$ show an explicit dependence on the internal mass. They have been computed by using the tensor reduction technique to express the tensorial one-loop integrals in terms of the scalar ones. We obtain

$$
\begin{aligned}
A_1(k_1,k_2,m^2) &= -\frac{i}{4\pi^2}+\frac{1}{8\pi^4\sigma}\left\{s_1\left(s_2-s_{12}\right)D_1\left(s_1,s,m^2\right)-s_2\left(s_1-s_{12}\right)D_2\left(s_2,s,m^2\right)\right.\\
&+ \left.\left[s_1 s_2\left(s_2-s_1\right)-4\sigma m^2\right]C_0\left(s_1,s_2,s,m^2\right)\right\}, \qquad (1.32)\\
A_3(k_1,k_2,m^2) &= -\frac{i}{4\pi^2\sigma}s_{12}+\frac{1}{8\pi^4\sigma^2}\left\{-s_1\left[2s_1 s_2+s_{12}\left(3s_2+s_{12}\right)\right]D_1\left(s_1,s,m^2\right)\right.\\
&- s_2\left[2s_1 s_2+s_{12}\left(3s_1+s_{12}\right)\right]D_2\left(s_2,s,m^2\right)\\
&- \left.\left[4s_{12}\sigma m^2+s_1 s_2\left(4s_{12}^2+3\left(s_1+s_2\right)s_{12}+2s_1 s_2\right)\right]C_0\left(s_1,s_2,s,m^2\right)\right\}, \qquad (1.33)
\end{aligned}
$$

$$
\begin{aligned}
A_5(k_1,k_2,m^2) &= -\frac{i}{4\pi^2\sigma}s_2+\frac{1}{8\pi^4\sigma^2}\left\{-\left(s_2+s_{12}\right)\left(2s_{12}^2+s_1 s_2\right)D_1\left(s_1,s,m^2\right)\right.\\
&- s_2\left[s_{12}\left(3s_2+4s_{12}\right)-s_1 s_2\right]D_2\left(s_2,s,m^2\right)\\
&- \left[4s_2\sigma m^2+s_2\left(-s_2 s_1^2+\left(s_2^2+2s_{12}s_2+4s_{12}^2\right)s_1\right.\right.\\
&\left.\left.\left.+2s_{12}^2\left(s_2+2s_{12}\right)\right)\right]C_0\left(s_1,s_2,s,m^2\right)\right\}, \qquad (1.34)
\end{aligned}
$$

with $s=k^2$, $s_1=k_1^2$, $s_2=k_2^2$, $\sigma=s_{12}^2-s_1 s_2$. It is possible to check that the Bose symmetry relative to the two vector vertices

$$
\begin{aligned}
A_2(k_1,k_2,m^2) &= -A_1(k_2,k_1,m^2), \qquad (1.35)\\
A_6(k_1,k_2,m^2) &= -A_3(k_2,k_1,m^2), \qquad (1.36)\\
A_4(k_1,k_2,m^2) &= -A_5(k_2,k_1,m^2) \qquad (1.37)
\end{aligned}
$$

is respected. As mentioned above, the difference between the massless and the massive decomposition of the triangle amplitude lies in the particular set of scalar integrals involved in the tensor reduction. Here we define D_1 and D_2 as a combination of two-point scalar massive integrals (B_0) of different internal momenta

$$
D_i(s,s_i,m^2) = B_0(k^2,m^2)-B_0(k_i^2,m^2)=i\pi^2\left[a_i\log\frac{a_i+1}{a_i-1}-a_3\log\frac{a_3+1}{a_3-1}\right] \qquad i=1,2 \qquad (1.38)
$$

in which the dependence on the regularization scheme disappears in the difference of the two scalar self-energies involved in (1.38). The expression of C_0 can be given explicitly in various forms [68], for instance as

$$
C_0(s,s_1,s_2,m^2) = -i\pi^2\frac{1}{2\sqrt{\sigma}}\sum_{i=1}^{3}\left[Li_2\frac{b_i-1}{a_i+b_i}-Li_2\frac{-b_i-1}{a_i-b_i}+Li_2\frac{-b_i+1}{a_i-b_i}-Li_2\frac{b_i+1}{a_i+b_i}\right] \qquad (1.39)
$$

with

$$a_i = \sqrt{1 - \frac{4m^2}{s_i}}, \qquad b_i = \frac{-s_i + s_j + s_k}{2\sigma}, \tag{1.40}$$

where $s_3 = s$ and in the last equation $i = 1, 2, 3$ and $j, k \neq i$. Other expressions, suitable for numerical implementations, are given in [69]. The region in which all these functions have real arguments and do not need any analytic continuations are those discussed in section 1.2.3, for the massless case. In general, the prescription for $i\eta$ in the presence of a mass in the internal loop - in the fermion propagator - is taken as $m \to m - i\eta$. We have checked numerically the agreement between the expressions presented above and those given in parametric form.

1.4 The vertex in the Longitudinal/Transverse (L/T) formulation and comparisons

The second parameterization of the three-point correlator function that we are going to discuss is the one presented in [42]. One of the features of this parameterization is the presence of a longitudinal contribution for generic virtualities of the external momenta and not just in the specific configuration under which it appears in Rosenberg's formulation. Of course, the true presence of the pole in the IR has to be checked by taking the corresponding limit, since the Schouten relations allow the extraction of a pole in the IR region at the cost of extra singularities in the parameterization. For this reason we start by recalling the structure of the L/T parameterization, which separates the longitudinal from the transverse components of the anomaly vertex, which is given by

$$W^{\lambda\mu\nu} = \frac{1}{8\pi^2}\left[W^{L\,\lambda\mu\nu} - W^{T\,\lambda\mu\nu}\right], \tag{1.41}$$

where the longitudinal component

$$W^{L\,\lambda\mu\nu} = w_L\, k^\lambda \varepsilon[\mu, \nu, k_1, k_2] \tag{1.42}$$

(with $w_L = -4i/s$) describes the anomaly pole, while the transverse contributions take the form

$$\begin{aligned} W^T{}_{\lambda\mu\nu}(k_1, k_2) \;=\;& w_T^{(+)}\left(k^2, k_1^2, k_2^2\right) t^{(+)}_{\lambda\mu\nu}(k_1, k_2) + w_T^{(-)}\left(k^2, k_1^2, k_2^2\right) t^{(-)}_{\lambda\mu\nu}(k_1, k_2) \\ &+ \widetilde{w}_T^{(-)}\left(k^2, k_1^2, k_2^2\right) \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1, k_2), \end{aligned} \tag{1.43}$$

with the transverse tensors given by

$$\begin{aligned} t^{(+)}_{\lambda\mu\nu}(k_1,k_2) &= k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] - k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] \\ &\quad - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,(k_1-k_2)] + \frac{k_1^2+k_2^2-k^2}{k^2}\,k_\lambda\,\varepsilon[\mu,\nu,k_1,k_2]\,, \\ t^{(-)}_{\lambda\mu\nu}(k_1,k_2) &= \left[(k_1-k_2)_\lambda - \frac{k_1^2-k_2^2}{k^2}\,k_\lambda\right]\varepsilon[\mu,\nu,k_1,k_2] \\ \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1,k_2) &= k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] + k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,k]. \end{aligned} \tag{1.44}$$

The form factors $w_T(s,s_1,s_2)$ are all defined in the following Eqs. (1.54-1.56).

Notice that in this representation the presence of massless poles is explicit for any kinematical configuration and not just in the massless collinear limit, where the diagram takes the Dolgov-Zakharov form. A second observation concerns the presence of other pole-like singularities in the transverse invariant amplitude and tensor structures. It is then obvious that one has to wonder whether the pole present in w_L is balanced, away from the collinear region, by other contributions which are also singular. Indeed, as we are going to show, this is the case. In fact, due to the Schouten relations, we are always allowed to introduce new polar amplitudes and balance them with additional contributions on the remaining tensor structures. In fact we are going to show that the presence of such pole away from the collinear region becomes significant in the UV - at least in the perturbative approach - but not in the IR, since it decouples if one computes the residue correctly in this representation.

1.4.1 Generalization of the L/T parameterization and the anomaly pole

We can generalize the L/T formulation presented above to the case of a triangle amplitude with a massive fermion of mass m, by simply exploiting the connection between this and the Rosenberg representation [41]. We use the Schouten relations to show the equivalence between the tensor structures of both representations. This requires some care since the decomposition into L and T amplitudes requires a nonzero k, otherwise it is invalid.

At nonzero momentum, by equating the coefficients of the four invariant tensors, we obtain a linear system of four equations whose solutions return the complete matching between the two parameterizations in the form

$$A_3(k_1,k_2) = \frac{1}{8\pi^2}\left[w_L - \tilde{w}_T^{(-)} - \frac{k^2}{(k_1+k_2)^2}w_T^{(+)} - 2\,\frac{k_1\cdot k_2 - k_2^2}{k^2}w_T^{(-)}\right], \tag{1.45}$$

$$A_4(k_1,k_2) = \frac{1}{8\pi^2}\left[w_L + 2\,\frac{k_1\cdot k_2}{k^2}w_T^{(+)} - 2\,\frac{k_1\cdot k_2 + k_2^2}{k^2}w_T^{(-)}\right], \tag{1.46}$$

$$A_5(k_1,k_2) = -A_4(k_2,k_1), \qquad A_6(k_1,k_2) = -A_3(k_2,k_1), \tag{1.47}$$

and viceversa

$$w_L(k^2,\, k_1^2,\, k_2^2) \;=\; \frac{8\pi^2}{k^2}\,[A_1 - A_2]\,, \tag{1.48}$$

(we omit, for simplicity, the momentum dependence) or, after the imposition of the Ward identities in Eqs. (1.13,1.14),

$$\begin{aligned}
w_L(k^2,\, k_1^2,\, k_2^2) &= \frac{8\pi^2}{k^2}\left[(A_3 - A_6)k_1\cdot k_2 + A_4\, k_2^2 - A_5\, k_1^2\right], && (1.49)\\
w_T^{(+)}(k^2,\, k_1^2,\, k_2^2) &= -4\pi^2\,(A_3 - A_4 + A_5 - A_6)\,, && (1.50)\\
w_T^{(-)}(k^2,\, k_1^2,\, k_2^2) &= 4\pi^2\,(A_4 + A_5)\,, && (1.51)\\
\tilde{w}_T^{(-)}(k^2,\, k_1^2,\, k_2^2) &= -4\pi^2\,(A_3 + A_4 + A_5 + A_6)\,, && (1.52)
\end{aligned}$$

where $A_i \equiv A_i(k_1, k_2)$. This same mapping holds also in the massive fermion case if $A_i \equiv A_i(k_1, k_2, m)$ and leads us to the same decomposition. In this case the L/T parameterization can be obtained starting from the massive A_i coefficients shown in Eq. (1.32-1.34) and exploiting the mapping in Eqs. (1.49-1.52) between the two parameterizations. We obtain

$$\begin{aligned}
w_L(s_1, s_2, s) &= -\frac{4i}{s} && (1.53)\\
w_T^{(+)}(s_1, s_2, s) &= i\frac{s}{\sigma} + \frac{i}{2\sigma^2}\Big[(s_{12} + s_2)(3s_1^2 + s_1(6s_{12} + s_2) + 2s_{12}^2)\log\frac{s_1}{s} \\
&+ (s_{12} + s_1)(3s_2^2 + s_2(6s_{12} + s_1) + 2s_{12}^2)\log\frac{s_2}{s} \\
&+ s(2s_{12}(s_1 + s_2) + s_1 s_2(s_1 + s_2 + 6s_{12}))\Phi(s_1, s_2)\Big] && (1.54)\\
w_T^{(-)}(s_1, s_2, s) &= i\frac{s_1 - s_2}{\sigma} + \frac{i}{2\sigma^2}\Big[-(2(s_2 + s_{12})s_{12}^2 - s_1 s_{12}(3s_1 + 4s_{12}) \\
&+ s_1 s_2(s_1 + s_2 + s_{12}))\log\frac{s_1}{s} + (2(s_1 + s_{12})s_{12}^2 - s_2 s_{12}(3s_2 + 4s_{12}) \\
&+ s_1 s_2(s_1 + s_2 + s_{12}))\log\frac{s_2}{s} + s(s_1 - s_2)(s_1 s_2 + 2s_{12}^2)\Phi(s_1, s_2)\Big] && (1.55)\\
\tilde{w}_T^{(-)}(s_1, s_2, s) &= -w_T^{(-)}(s_1, s_2, s) && (1.56)
\end{aligned}$$

in the massless case, which is in complete agreement with the explicit expression given by [45], while in the massive case the same mapping gives

$$\begin{aligned}
w_L(s, s_1, s_2, m^2) &= -\frac{4i}{s} - \frac{8m^2}{\pi^2 s}C_0(s, s_1, s_2, m^2) && (1.57)\\
w_T^{(+)}(s, s_1, s_2, m^2) &= i\frac{s}{\sigma} + \frac{1}{2\pi^2\sigma^2}\Big[(s_{12} + s_2)(3s_1^2 + s_1(6s_{12} + s_2) + 2s_{12}^2)D_1(s, s_1, m^2) \\
&+ (s_{12} + s_1)(3s_2^2 + s_2(6s_{12} + s_1) + 2s_{12}^2)D_2(s, s_2, m^2) \\
&+ (4m^2 s\sigma + s(2s_{12}(s_1 + s_2) + s_1 s_2(s_1 + s_2 + 6s_{12})))C_0(s, s_1, s_2, m^2)\Big]
\end{aligned}$$

$$(1.58)$$

$$\begin{aligned}
w_T^{(-)}(s,s_1,s_2,m^2) &= i\frac{s_1-s_2}{\sigma}+\frac{1}{2\pi^2\sigma^2}\left[-(2(s_2+s_{12})s_{12}^2-s_1s_{12}(3s_1+4s_{12})\right.\\
&+ s_1s_2(s_1+s_2+s_{12}))D_1(s,s_1,m^2)+(2(s_1+s_{12})s_{12}^2-s_2s_{12}(3s_2+4s_{12})\\
&+ s_1s_2(s_1+s_2+s_{12}))D_2(s,s_2,m^2)\\
&+ \left.(4m^2\sigma(s_1-s_2)+s(s_1-s_2)(s_1s_2+2s_{12}^2))C_0(s,s_1,s_2,m^2)\right] \quad (1.59)\\
\tilde{w}_T^{(-)}(s,s_1,s_2,m^2) &= -w_T^{(-)}(s,s_1,s_2,m^2), \quad (1.60)
\end{aligned}$$

with $s_i=k_i^2$ $(i=1,2,3,\ k_3=k)$, $s_{12}=k_1\cdot k_2$, $\sigma=s_{12}^2-s_1s_2$. The functions D_i and C_0, defined in Eq. (1.38) and (1.39), are respectively a combination of two scalar bubbles and the scalar one-loop triangle. The Bose symmetry on the vector vertices is fulfilled in both representations by taking into account the way in which the A_i and the $w_L, w_T, \ldots$ transform under the exchange of k_1, k_2 and μ, ν. For the L/T invariant amplitudes we have

$$\begin{aligned}
w_T^{(+)}(k^2,\,k_1^2,\,k_2^2) &= w_T^{(+)}(k^2,\,k_1^2,\,k_2^2), \quad (1.61)\\
w_T^{(-)}(k^2,\,k_1^2,\,k_2^2) &= -w_T^{(-)}(k^2,\,k_1^2,\,k_2^2), \quad (1.62)\\
\tilde{w}_T^{(-)}(k^2,\,k_1^2,\,k_2^2) &= -\tilde{w}_T^{(-)}(k^2,\,k_1^2,\,k_2^2). \quad (1.63)
\end{aligned}$$

It is then obvious that there is complete equivalence between the two parameterizations, although there are some puzzling features that need to be investigated more closely. As we have already mentioned, the L/T parameterization appears to have a pole at $s=(k_1+k_2)^2=0$, which contributes to the anomaly. In fact, the non-vanishing Ward identity on the axial-vector line is due to the invariant amplitude w_L and to its corresponding tensor structure. Then, one obvious question to ask is if this pole is compatible with the pole structure of the Rosenberg representation [41]. The answer is affirmative as far as the computation of the residue is performed on the entire amplitude and not just on the invariant amplitudes alone. In fact, the L/T decomposition introduces kinematical singularities both in the longitudinal and in the transverse components as a price for the appearance of a longitudinal pole. This can be shown explicitly. In fact, a direct evaluation of the limit (for off shell photons) gives

$$\lim_{s\to 0} s\, w_L(k_1^2,k_2^2,k^2)(k_1+k_2)_\lambda\varepsilon[\mu,\nu,k_1,k_2] = -4i(k_1+k_2)_\lambda\varepsilon[\mu,\nu,k_1,k_2], \quad (1.64)$$

$$\lim_{s\to 0} s\, w_T^{(+)}(k_1^2,k_2^2,k^2)\, t^{(+)}_{\mu\nu\lambda}(k_1,k_2) = -\frac{2i(s_1+s_2)\log[\frac{s_1}{s_2}]}{s_1-s_2}(k_1+k_2)_\lambda\varepsilon[\mu,\nu,k_1,k_2], \quad (1.65)$$

$$\lim_{s\to 0} s\, w_T^{(-)}(k_1^2,k_2^2,k^2)\, t^{(-)}_{\mu\nu\lambda}(k_1,k_2) = \left[-4i+\frac{2i(s_1+s_2)\log(\frac{s_1}{s_2})}{s_1-s_2}\right](k_1+k_2)_\lambda\varepsilon[\mu,\nu,k_1,k_2], \quad (1.66)$$

$$\lim_{s\to 0} s\, \tilde{w}_T^{(-)}(k_1^2,k_2^2,k^2)\, \tilde{t}^{(-)}_{\mu\nu\lambda}(k_1,k_2) = 0 \quad (1.67)$$

for the several singular terms present at $s=0$. These results have been obtained after performing the analytic continuation around $s=0$ of the explicit expressions for w_L and w_T given above.

Combining these partial contributions we obtain the total result for the residue of the entire amplitude

$$\lim_{s\to 0} s\, W_{\mu\nu\lambda} = 0, \tag{1.68}$$

which proves its vanishing at $s=0$ for off-shell photon lines. This result, in agreement with what we had anticipated, shows that in the IR also the L/T parameterization has no pole. This is expected, being the L/T and the Rosenberg parameterizations [41] equivalent descriptions of the same diagram (modulo some Schouten relations), hence it is obvious that the decoupling of the anomaly pole for off-shell external momenta has to take place in both parameterizations. Performing cautiously the limits, we can similarly proof that the pole reappears in correspondence of specific configurations of the external lines (on-shell photons), as we are going to show next. An equivalent analysis, of course, can be performed by analyzing the various cuts of the amplitudes in the L/T parameterization using a dispersive approach and looking for discontinuities proportional to $\delta(k^2)$ in the spectral density of the diagram.

1.5 Special kinematical limits in the massless case

We summarize in this section all the results concerning some specific kinematical conditions in the infrared and chiral limits of the anomaly amplitude, taken directly on the amplitude given in the previous sections.

The first analysis carried out involves the massless A_i written in Eq. (1.17, 1.19) for which we take three limits. We use the notation $A_i(s,s_1,s_2)$ to denote each invariant amplitude in the Rosenberg form for massless internal fermions. We distinguish the following cases

a) $s_1 = 0 \qquad s_2 \neq 0 \qquad s \neq 0 \qquad m = 0$

b) $s_1 = 0 \qquad s_2 = 0 \qquad s \neq 0 \qquad m = 0$

c) $s_1 = M^2 \qquad s_2 = M^2 \qquad s \neq 0 \qquad m = 0.$

While cases a) and b) will be treated here, case c) will be left to the appendix A.1, together with the same three kinematical configurations for a massive fermion. In case a) we find

$$A_1(s,0,s_2) = \frac{i}{4\pi^2}\left[\frac{s_2}{s-s_2}\log\frac{s_2}{s} - 1\right], \tag{1.69}$$

$$A_2(s,0,s_2) = \frac{i}{4\pi^2}\left[\frac{s_2}{s-s_2}\log\frac{s_2}{s} + 1\right], \tag{1.70}$$

$$A_3(s,0,s_2) = -A_6(0,s_2,s,0) = -\frac{i}{2\pi^2(s-s_2)}\left[\frac{s_2}{s-s_2}\log\frac{s_2}{s} + 1\right], \tag{1.71}$$

$$A_4(s,0,s_2) = \frac{i}{2\pi^2(s-s_2)}\log\frac{s_2}{s} \tag{1.72}$$

and a divergent $A_5(s, 0, s_2)$ which does not contribute to the physical value of the amplitude. Indeed $\Delta^{\lambda\mu\nu}$, in a physical amplitude, is contracted with the polarization vector relative to the on-shell photon with momentum k_1, giving $\epsilon_\mu(k_1)k_1^\mu = 0$, so that the contribution coming from A_5 disappears.

Notice that this amplitude satisfies the Ward identities in Eqs. (1.13,1.14) and can be written as

$$\Delta^{\lambda\mu\nu}(s, 0, s_2) = A_3(s, 0, s_2)\, \eta_3^{\lambda\mu\nu}(k_1, k_2) + A_4(s, 0, s_2)\, \eta_4^{\lambda\mu\nu}(k_1, k_2) + A_6(s, 0, s_2)\, \eta_6^{\lambda\mu\nu}(k_1, k_2), \tag{1.73}$$

with the tensors $\eta_i(k_1, k_2)$ written in Tab.1.2. Notice that the poles are located at the various thresholds of the amplitude, describing the production of a photon of invariant mass s_2, having set the first photon on-shell, and that all the residues are vanishing

$$\lim_{s\to 0} s\, A_3(s, 0, s_2) = \lim_{s\to 0} s\, A_4(s, 0, s_2) = \lim_{s\to 0} s\, A_6(s, 0, s_2) = 0, \tag{1.74}$$

including the one of the whole amplitude

$$\lim_{s\to 0} s\, \Delta^{\lambda\mu\nu}(s, 0, s_2) = 0. \tag{1.75}$$

In the L/T parameterization we find

$$w_L(s, 0, s_2) = -\frac{4i}{s}, \tag{1.76}$$

$$w_T^{(+)}(s, 0, s_2) = \frac{2i}{s - s_2}\left[\frac{s + s_2}{s - s_2}\log\frac{s_2}{s} + 2\right], \tag{1.77}$$

$$w_T^{(-)}(s, 0, s_2) = -\tilde{w}_T^{(-)}(s, 0, s_2) = \frac{2i}{s - s_2}\log\frac{s_2}{s} \tag{1.78}$$

which also show the presence of the same threshold singularity, but, in addition, also of an anomaly pole in w_L which is absent in Rosenberg's parameterization. As we have commented above, the pole is spurious, since the tensor structures are also singular in the same ($s \to 0$) limit, and there is a trivial cancellation of this contribution. Indeed we find

$$\lim_{s\to 0} s\, w_L(s, 0, s_2)\, k_\lambda \varepsilon[\mu, \nu, k_1, k_2] = -4i\, k_\lambda\, \varepsilon[\mu, \nu, k_1, k_2], \tag{1.79}$$

$$\lim_{s\to 0} s\, \left[w_T^{(+)}(s, 0, s_2)\, t^{(+)}_{\lambda\mu\nu}(k_1, k_2) + w_T^{(-)}(k_1^2, k_2^2, k^2)\, t^{(-)}_{\lambda\mu\nu}(k_1, k_2)\right] = -4i\, k_\lambda\, \varepsilon[\mu, \nu, k_1, k_2], \tag{1.80}$$

$$\lim_{s\to 0} s\, \tilde{w}_T^{(-)}(s, 0, s_2)\, \tilde{t}^{(-)}_{\lambda\mu\nu}(k_1, k_2) = 0 \tag{1.81}$$

which gives

$$\lim_{s\to 0} s\, W_{\lambda\mu\nu}(s, 0, s_2) = \frac{1}{8\pi^2}\lim_{s\to 0} s\, \left[W^{L\,\lambda\mu\nu} - W^{T\,\lambda\mu\nu}\right] = 0 \tag{1.82}$$

in agreement with Eq. (1.68).
Therefore, in this case, with only one leg on-shell, the kinematics does not allow a polar structure for the entire amplitude; in the Rosenberg parameterization this result can be derived in a straightforward way since each amplitude has a vanishing residue and the tensor structures are regular in the IR (i.e. $s \to 0$) limit. On the contrary, in this limit the L/T formulation involves both the longitudinal and the transverse components, as the tensorial structures multiplying the coefficients $w(s, 0, s_2)$ are not independent as $s \to 0$. Obviously the final result, obtained with the correct limiting procedure, is the same in both cases.

Let's take in exam another kinematical configuration, more specific than the previous one, i.e. the case in which the two photons are both on-shell and massless or

b) $s_1 = s_2 = 0 \qquad s \neq 0 \qquad m = 0.$

In this case it is well known that the AVV vertex exhibits a polar structure, as Dolgov and Zakharov showed in [37], therefore we expect to recover this amplitude in the $s \to 0$ limit. The computed form factors are extremely simple. We obtain

$$A_1(s,0,0) = -A_2(s,0,0) = -\frac{i}{4\pi^2}, \tag{1.83}$$

$$A_3(s,0,0) = -A_6(s,0,0) = -\frac{i}{2\pi^2 s} \tag{1.84}$$

which clearly exhibit the Bose symmetry for the two vector vertices, since $s_1 = s_2$. Notice that A_4, A_5 are physically nonessential, as before; indeed they are multiplied, respectively, by k_2^ν and k_1^μ in the total amplitude $\Delta^{\lambda\mu\nu}(k_1, k_2)$, and vanish after their contraction with the physical polarization vectors of the photons.

The amplitude $\Delta^{\lambda\mu\nu}(k_1, k_2)$ satisfies the Ward identities written in Eq. 1.13, since $s_{12} \to s/2$ when both photons are on-shell

$$A_1(s,0,0) = \frac{s}{2} A_3(s,0,0), \qquad A_2(s,0,0) = \frac{s}{2} A_6(s,0,0). \tag{1.85}$$

In this case the entire correlator is obtained from only two form factors A_i (A_3 and A_6), giving

$$\begin{aligned} \Delta^{\lambda\mu\nu}(s,0,0) &= A_3(s,0,0)\,\eta_3^{\lambda\mu\nu}(k_1,k_2) + A_6(s,0,0)\,\eta_6^{\lambda\mu\nu}(k_1,k_2) \\ &= \frac{i}{2\pi^2 s}\left[k_2^\mu \varepsilon[k_1,k_2,\nu,\lambda] - k_1^\nu \varepsilon[k_1,k_2,\mu,\lambda]\right] - \frac{i}{4\pi^2}\varepsilon[(k_1-k_2),\lambda,\mu,\nu]. \end{aligned} \tag{1.86}$$

This expression can be reduced to its polar Dolgov-Zakharov form after using the Schouten identities in Eqs. (1.26,1.27)

$$\Delta^{\lambda\mu\nu}(s,0,0) = -\frac{i}{2\pi^2}\frac{k^\lambda}{s}\,\varepsilon[k_1,k_2,\mu,\nu] \tag{1.87}$$

as $s_1 = s_2 = 0$.
In the L/T parameterization we expect a similar polar result, after summing over the contributions coming both from the longitudinal and transverse tensors. In this case, the only two non-vanishing coefficients are w_L and $w_T^{(+)}$

$$w_L(s,0,0) = w_T^{(+)}(s,0,0) = -\frac{4i}{s}, \tag{1.88}$$

$$w_T^{(-)}(s,0,0) = \tilde{w}_T^{(-)}(s,0,0) = 0 \tag{1.89}$$

and the residues must be computed combining them with the corresponding tensor structures. It is worth noticing that $t^{(+)}_{\lambda\mu\nu}(k_1,k_2) = 0$ for $s_1 = s_2 = 0$. This can be immediately checked starting from its definition given in Eq. (1.43) and with the aid of the two Schouten identities shown in Eqs. (1.26,1.27), which in this case become

$$k_1^\lambda\,\varepsilon[k_1,k_2,\mu,\nu] = -k_1^\nu\varepsilon[k_1,k_2,\lambda,\mu] + \frac{s}{2}\varepsilon[k_1,\lambda,\mu,\nu], \tag{1.90}$$

$$k_2^\lambda\,\varepsilon[k_1,k_2,\mu,\nu] = k_2^\mu\,\varepsilon[k_1,k_2,\lambda,\nu] - \frac{s}{2}\varepsilon[k_2,\lambda,\mu,\nu], \tag{1.91}$$

so that the unique contribution to the residue for $s \to 0$ comes from the longitudinal part

$$\begin{aligned}\lim_{s\to 0} s\,W_{\mu\nu\lambda}(s,0,0) &= \frac{1}{8\pi^2}\lim_{s\to 0} s\;W^{L\,\lambda\mu\nu}\\ &= \frac{1}{8\pi^2}\lim_{s\to 0} s\,w_L(s,0,0)\,k_\lambda\varepsilon[\mu,\nu,k_1,k_2]\\ &= -\frac{i}{2\pi^2}k^\lambda\,\varepsilon[k_1,k_2,\mu,\nu]. \end{aligned}\tag{1.92}$$

We conclude that the pole is indeed present in the L/T amplitude if the conditions $s_1 = s_2 = 0$ with $s \neq 0$ are simultaneously satisfied

$$\Delta^{\lambda\mu\nu}(s,0,0) = W_{\mu\nu\lambda}(s,0,0) = -\frac{i}{2\pi^2}\frac{k^\lambda}{s}\,\varepsilon[k_1,k_2,\mu,\nu]. \tag{1.93}$$

Another interesting case is represented by a symmetric kinematical configurations in which the external particles are massive gauge bosons of mass M. This will turn useful in the next sections, when we will discuss the behaviour of a BIM amplitude with massive external lines at high energy, showing, also in this case, its pole dominance. There are some conclusions that we can draw from this study which are important for the analysis of the next sections. Notice that in all the cases that we have discussed it is possible to isolate a $1/s$ contribution in w_L for any kinematical configurations other than the massless ($s \to 0$) one, where the L/T formulation requires a limiting procedure. This is clearly suggestive of the fact that a longitudinal component is intrinsically part of the vertex and not just of its collinear and chiral limit. This contributions is paralleled, in the Rosenberg amplitude(s) by a constant behaviour of A_1 and A_2

($A_1 = i/(4\pi^2) + ...$). Massive external gauge lines or mass corrections due to the fermion mass in the loop do not shift this $1/s$ pole.

As we have mentioned, under the general configurations contemplated in these last cases, these poles are not coupled in the IR, although this does not necessarily exclude a possible role played by these contributions in the IR region. However, the complete absence of a scale in their definition makes them suitable also of a completely different interpretation, as longitudinal contributions that survive in the asymptotic $s \to \infty$ limit of these amplitudes. In fact, we are going to show that any UV completion of these theories has necessarily to deal with the cancellation of these terms.

1.6 Effective actions and the gauge anomaly

In this section we are going to discuss the formulation of the effective action in the presence of anomaly poles, generalizing the Euler-Heisenberg (EH) result to an anomalous theory. We will focus our attention exclusively on the trilinear gauge terms, coming from the anomalous structure, which are new compared to the EH formulation.

The simplest example that we can consider is a theory describing a single anomalous gauge boson B with a Lagrangian

$$\mathcal{L}_B = \overline{\psi}\,(i\,\partial\!\!\!/ + e\,B\!\!\!/\gamma_5)\,\psi - \frac{1}{4}F_B^2. \tag{1.94}$$

The effective action of the model suffers from a trilinear gauge interaction which is anomalous (BBB). In this case the anomalous vertex is obtained by a simple symmetrization of (1.9) which generates a Δ_{AAA} vertex

$$\Delta_{AAA} = \frac{1}{3}\left(\Delta_{AVV} + \Delta_{VAV} + \Delta_{VVA}\right). \tag{1.95}$$

The anomalous gauge variation ($\delta B_\mu = \partial_\mu \theta_B$)

$$\delta\Gamma_B = \frac{i\,e^3\,a_n}{24}\int d^4x\,\theta_B(x)\,F_B \wedge F_B \tag{1.96}$$

can be reproduced by the nonlocal action

$$\Gamma_{pole} = \frac{e^3}{48\,\pi^2}\,\langle \partial B(x)\Box^{-1}(x-y)F_B(y) \wedge F_B(y)\rangle, \tag{1.97}$$

which is the variational solution of (1.96). To derive a $1/m$ expansion of the effective action, we perform an expansion of the Rosenberg form factors, obtaining

$$A_1(s,0,0,m^2) = -A_2(s,0,0,m^2) = \frac{i}{48\pi^2}\frac{s}{m^2} + \frac{i}{360\pi^2}\frac{s^2}{m^4} + O\left(\frac{1}{m^6}\right), \quad (1.98)$$

$$A_3(s,0,0,m^2) = -A_6(s,0,0,m^2) = \frac{i}{24\pi^2}\frac{1}{m^2} + \frac{i}{180\pi^2}\frac{s}{m^4} + O\left(\frac{1}{m^6}\right), \quad (1.99)$$

$$A_4(s,0,0,m^2) = -A_5(s,0,0,m^2) = \frac{i}{12\pi^2}\frac{1}{m^2} + \frac{i}{120\pi^2}\frac{s}{m^4} + O\left(\frac{1}{m^6}\right), \quad (1.100)$$

where $s \equiv k^2$. We will also use the notation s_1 and s_2 to denote the virtuality of the two external photons ($s_1 \equiv k_1^2, s_2 \equiv k_2^2$). Due to the chiral gauge anomaly, the effective action is gauge-variant. For our choice of momenta (incoming k on the axial-vector of index λ and outgoing k_1 and k_2 on the two vector currents of indices μ and ν) we obtain

$$T^{\lambda\mu\nu}_{AVV}(x,y,z) = \int \frac{d^4k\, d^4k_1\, d^4k_2}{(2\pi)^8}\, \delta^4(k-k_1-k_2)\, e^{ik\cdot z - ik_1\cdot x - ik_2\cdot y}\, \Delta^{\lambda\mu\nu}_{AVV}(k,k_1,k_2) \quad (1.101)$$

with the contribution of the anomalous vertex being given by

$$\Gamma^{(3)} = -\frac{i}{6}\int d^4x\, d^4y\, d^4z\, T^{\lambda\mu\nu}(x,y,z) B_\lambda(z)\, B_\mu(x)\, B_\nu(y), \quad (1.102)$$

where $T^{\lambda\mu\nu}(x,y,z)$ is the symmetrized correlator given by

$$T^{\lambda\mu\nu}(x,y,z) = \frac{1}{3}\left[T^{\lambda\mu\nu}_{AVV}(x,y,z) + T^{\lambda\mu\nu}_{VAV}(x,y,z) + T^{\lambda\mu\nu}_{VVA}(x,y,z)\right]. \quad (1.103)$$

The explicit form of the new anomalous contributions (the symbols $\langle\ \rangle$ denote spacetime integration) can be obtained by plugging in the expression of the various form factors expanded in $1/m$ written in Eqs. (1.98-1.100). We obtain

$$\begin{aligned}
\Gamma^{(3)} = & -\frac{i}{6}\Bigg[\frac{1}{48\pi^2 m^2}\epsilon^{\alpha\mu\nu\lambda}\left(\langle\Box B_\lambda \partial_\alpha B_\mu B_\nu\rangle - \langle\Box B_\lambda B_\mu \partial_\alpha B_\nu\rangle\right) \\
& -\frac{1}{360\pi^2 m^4}\epsilon^{\alpha\mu\nu\lambda}\left(\langle\Box^2 B_\lambda \partial_\alpha B_\mu B_\nu\rangle - \langle\Box^2 B_\lambda B_\mu \partial_\alpha B_\nu\rangle\right) \\
& +\frac{1}{24\pi^2 m^2}\left(\epsilon^{\alpha\beta\mu\lambda}\langle\partial_\alpha\partial_\nu B_\mu B_\lambda \partial_\beta B^\nu\rangle - \epsilon^{\alpha\beta\nu\lambda}\langle\partial_\alpha B_\mu B_\lambda \partial_\beta\partial^\mu B_\nu\rangle\right) \\
& -\frac{1}{180\pi^2 m^4}\left(\epsilon^{\alpha\beta\mu\lambda}\langle\partial_\alpha\partial_\nu B_\mu \Box B_\lambda \partial_\beta B^\nu\rangle - \epsilon^{\alpha\beta\nu\lambda}\langle\partial_\alpha B_\mu \Box B_\lambda \partial_\beta\partial^\mu B_\nu\rangle\right) \\
& +\frac{1}{12\pi^2 m^2}\left(\epsilon^{\alpha\beta\mu\lambda}\langle\partial_\alpha B_\mu \partial_\beta\partial_\nu B^\nu B_\lambda\rangle - \epsilon^{\alpha\beta\nu\lambda}\langle\partial_\alpha\partial_\mu B^\mu B_\lambda \partial_\beta B_\nu\rangle\right) \\
& -\frac{1}{120\pi^2 m^4}\left(\epsilon^{\alpha\beta\mu\lambda}\langle\partial_\alpha B_\mu \partial_\beta\partial_\nu \Box B_\lambda\rangle - \epsilon^{\alpha\beta\nu\lambda}\langle\partial_\alpha\partial_\mu B^\mu \Box B_\lambda \partial_\beta B_\nu\rangle\right)\Bigg]. \quad (1.104)
\end{aligned}$$

Naturally, the p/m expansion hides the nonlocal contributions which are present in the effective action. These can be identified from the off-shell expression of the anomaly vertex, which in the L/T parameterization takes a close form only in momentum space. For this reason we rewrite this parameterization as a pole ($w_L = -4i/s$) plus mass corrections in the equivalent form

$$W^{L\,\lambda\mu\nu} = (w_L - \mathcal{F}(k,k_1,k_2,m))\; k^\lambda \varepsilon[\mu,\nu,k_1,k_2] \tag{1.105}$$

$$\mathcal{F}(m,s,s_1,s_2) = \frac{8m^2}{\pi^2 s} C_0(s,s_1,s_2,m^2), \tag{1.106}$$

where C_0 has been given in Eq. (1.39). Obviously, the anomaly is completely given by w_L. The complete action is instead given by

$$\Gamma^{(3)} = \Gamma^{(3)}_{pole} + \tilde{\Gamma}^{(3)} \tag{1.107}$$

with the pole part given by

$$\Gamma^{(3)}_{pole} = -\frac{1}{8\pi^2}\int d^4x\, d^4y\, \partial\cdot B(x) \Box^{-1}_{x,y} F(y)\wedge F(y) \tag{1.108}$$

and the rest ($\tilde{\Gamma}^{(3)}$) given by a complicated nonlocal expression which contributes homogeneously to the Ward identify of the anomaly graph

$$\begin{aligned}\tilde{\Gamma}^{(3)} \;=\; & -\frac{e^3}{48\pi^2}\int d^4x\, d^4y\, d^4z\, \partial\cdot B(z) F_B(x)\wedge F_B(y) \\ & \qquad \int \frac{d^4k_1\, d^4k_2}{(2\pi)^8}\, e^{-ik_1\cdot(x-z)-ik_2\cdot(y-z)} \mathcal{F}(k,k_1,k_2,m) \\ & -\frac{e^3}{48\pi^2}\int d^4x\, d^4y\, d^4z B_\lambda(z) B_\mu(x) B_\nu(y) \\ & \qquad \int \frac{d^4k_1\, d^4k_2}{(2\pi)^8}\, e^{-ik_1\cdot(x-z)-ik_2\cdot(y-z)} W_T^{\lambda\mu\nu}(k,k_1,k_2,m),\end{aligned} \tag{1.109}$$

where $k = k_1 + k_2$. A second form of the effective action is obtained by expanding around $m = 0$, i.e. for a small mass. A simple, but very instructive case, is the one with two on-shell photons ($s_1 = s_2 = 0$) and a nonzero fermion mass. We obtain, for instance, in the AVV case the following expressions for the form factors after the series expansion around $m = 0$

$$w_L = -\frac{4\,i}{s} - \frac{4\,i\,m^2}{s^2}\log\left(-\frac{s}{m^2}\right) + O(m^3), \tag{1.110}$$

$$w_T^{(+)}(s,0,0,m^2) = \frac{12\,i}{s} - \frac{4\,i}{s}\log\left(-\frac{s}{m^2}\right) + \frac{4\,i\,m^2}{s^2}\left[2 + \log\left(\frac{s^2}{m^4}\right) - \log^2\left(-\frac{s}{m^2}\right)\right] + O(m^3). \tag{1.111}$$

It is clear that this second expansion allows to isolate the pole term from the mass corrections, and is probably a more faithful description of the anomalous content of the theory, identified by the anomaly pole.

1.7 Conclusions

The presence of anomaly poles in the perturbative expansion of the effective action, appears to be an essential property of anomalous theories, even in the most general kinematical configurations of the anomalous correlators. We have shown in this chapter that only a complete computation of the effective action allows to identify such contributions, which affect the UV behaviour of a correlator even if they are decoupled in the IR. The goal of this investigation has been to show that more general anomaly poles are present in the perturbative description of the anomaly. Previously, the appearance of these terms was considered a pure IR phenomenon, while their isolation in the L/T parameterization was probably considered an artificial result due to the presence of Schouten relations in the anomaly graph. We have also shown how the Schouten relations can "dissolve" a pole, by allowing its rewriting in terms of additional form factors which are not of polar form.

In this chapter we have performed a complete and very detailed analysis of all the relevant regions of the anomaly graph, identifying all the relevant sources of singularities in the correlator and generalized the L/T parameterization to the massive case. This result has been used to derive an effective action which generalizes the Euler-Heisenberg result to anomalous theories. In the next chapter we are going to investigate the significance of anomaly poles in the case of conformal anomaly, showing the perfect (and striking) analogy with the patterns of anomaly poles discussed in this chapter.

Chapter 2

Conformal Anomalies and the Gravitational Effective Action: The TJJ Correlator for a Chiral Fermion

2.1 Introduction

From now on we begin investigating the trilinear correlators involving an insertion of energy-momentum tensor T. In this chapter we focus on the correlator responsible for the appearance of the trace anomaly at leading order, and denoted by TJJ, where J are vector gauge currents. In the previous chapter we showed how the 1-particle irreducible effective action is characterized by the presence of massless effective degrees of freedom of pseudoscalar type when dealing with chiral anomalies. Our aim here is to discuss in detail the case of the conformal anomaly, starting from a detailed perturbative analysis of the TJJ correlator.

Investigations of conformal anomalies in gravity (see [70] for an historical overview and references) [71] and in gauge theories [72, 73, 74] as well as in string theory, have been of remarkable significance along the years. In cosmology, for instance, [75] (see also [76] for an overview) the study of the gravitational trace anomaly has been performed in an attempt to solve the problem of the "graceful exit" (see for instance [77, 78, 79, 80]). In other analysis it has been pointed out that the conformal anomaly may prevent the future singularity occurrence in various dark energy models [81, 82].

In the past the analysis of the formal structure of the effective action for gravity in four dimensions, obtained by integration of the trace anomaly [83, 84], has received a special attention, showing that the variational solution of the anomaly equation, which is non-local, can be made local by the introduction of extra scalar fields. The gauge contributions to these anomalies are identified at 1-loop level from a set of diagrams - involving fermion loops with two external gauge lines and one graviton line - and are characterized, as shown recently by Giannotti and Mottola in [51], by the presence of anomaly poles. Anomaly poles are familiar from the study of the chiral anomaly in gauge theories and describe the non-local structure of the effective action. In the case of global anomalies, as in QCD chiral dynamics, they signal the presence of a non-perturbative phase of the fundamental theory, with composite degrees of freedom (pions) which offer an equivalent description of the fundamental Lagrangian, matching the anomaly, in agreement with 't Hooft's principle. Previous studies of the role of the conformal anomaly in cosmology concerning the production of massless gauge particles and the identification of the infrared anomaly pole are those of Dolgov [37, 85], while a discussion of the infrared pole from a dispersive derivation is contained in [86].

In the first chapter and in [40] we have shown that anomaly poles are typical of the perturbative description of the chiral anomaly not just in some special kinematical conditions, for instance in the collinear region, where the coupling of the anomalous gauge current to two (on-shell) vector currents (for the AVV diagram) involves a pseudoscalar intermediate state (with a collinear and massless fermion-antifermion pair) but under any kinematical conditions. They are the most direct - and probably also the most significant - manifestation of the anomaly in the perturbative diagrammatic expansion of the effective action. On a more speculative side, the interpretation of the pole in terms of composite degrees of freedom could probably have direct physical implications, including the condensation of the composite fields, very much like Bose Einstein (BE) condensation of the pion field, under the action of gravity. Interestingly, in a recent paper, Sikivie and Yang have pointed out that Peccei-Quinn axions ([58, 59]) may form BE condensates [87]. With these motivations in mind, in this chapter, which parallels a previous investigation of the chiral gauge anomaly [40], we study the perturbative structure of the off-shell effective action showing the appearance of similar singularities under general kinematic conditions. Our investigation is a first step towards the computation of the exact effective action describing the coupling of the Standard Model to gravity via the conformal anomaly, that we hope to discuss in the future.

In our study we follow closely the work of [51]. There the authors have presented a complete off-shell classification of the invariant amplitudes of the relevant correlator responsible for the conformal anomaly, which involves the energy momentum tensor (T) and two vector currents (J), TJJ, and have thoroughly investigated it in the QED case, drawing on the analogy with the case of the chiral anomaly. The analysis of [51] is based on the use of dispersion relations, which are sufficient to identify the anomaly poles of the amplitude from the spectral density of this correlator, but not to characterize completely the off-shell effective action of the theory and the remaining non-conformal contributions, which will be discussed in this paper. The poles that we extract from the complete effective action include both the usual poles derived from the spectral analysis of the diagrams, which are coupled in the infrared (IR) and other extra poles which account for the anomaly but are decoupled in the same limit. These extra poles appear under general kinematic configurations and are typical of the off-shell as well as of the on-shell effective action, both for massive and massless fermions.

We also show, in agreement with those analysis, that the pole terms which contribute to the conformal anomaly are indeed only obtained in the on-shell limit of the external gauge lines, and identify all the mass corrections to the correlator in the general case. This analysis is obtained by working out all the relevant kinematical limits of the perturbative corrections. We present the complete anomalous off-shell effective action describing the interaction of gravity with the photons, written in a form in which we separate the non-local contribution due to the anomaly pole from the rest of the action (those which are conformally invariant in the massless fermion limit). Away from the conformal limit of the theory we present a $1/m$ expansion of the effective action as in the Euler-Heisenberg approach. This expansion, naturally, does not convey the presence of non-localities in the effective action due to the appearance of massless poles.

The computation of similar diagrams, for the on-shell photon case, appears in older contributions by Berends and Gastmans [46] using dimensional regularization, in their study of the gravitational scattering of photons and by Milton using Schwinger's methods [47]. The presence of an anomaly pole in the amplitude has not been investigated nor noticed in these previous analysis, since they do not appear explicitly in their results, nor the $1/m$ expansion of the three form factors of the on-shell vertex, contained in [46], allows their identification in the S-matrix elements of the theory. Two related analysis by Drummond and Hathrell in their investigation of the gravitational contribution to the self-energy of the photon [48] and the renormalization of the trace anomaly [49] included the same on-shell vertex. Later, this same vertex has provided the ground for several elaborations concerning a possible superluminal behaviour of the photon in the presence of an external gravitational field [50].

2.2 The conformal anomaly and gravity

In this section we briefly summarize some basic and well known aspects of the trace anomaly in quantum gravity and, in particular, the identification of the non-local action whose variation generates a given trace anomaly.

We recall that the gravitational trace anomaly in 4 spacetime dimensions generated by quantum effects in a classical gravitational and electromagnetic background is given by the expression

$$T^\mu_\mu = -\frac{1}{8}\left[2b\,C^2 + 2b'\left(E - \frac{2}{3}\Box R\right) + 2c\,F^2\right] \tag{2.1}$$

where b, b' and c are parameters that for a single fermion in the theory result $b = 1/320\,\pi^2$, $b' = -11/5760\,\pi^2$, and $c = -e^2/24\,\pi^2$; furthermore C^2 denotes the Weyl tensor squared and E is the Euler density given by

$$C^2 = C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3} \tag{2.2}$$

$$E = {}^*\!R_{\lambda\mu\nu\rho}\,{}^*\!R^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{2.3}$$

The effective action is identified by solving the following variational equation by inspection

$$-\frac{2}{\sqrt{g}}g_{\mu\nu}\frac{\delta\Gamma}{\delta g_{\mu\nu}} = T^\mu_\mu. \tag{2.4}$$

Its solution is well known and is given by the non-local expression

$$S_{anom}[g,A] = \tag{2.5}$$
$$\frac{1}{8}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\left(E - \frac{2}{3}\Box R\right)_x G_4(x,x')\left[2b\,C^2 + b'\left(E - \frac{2}{3}\Box R\right) + 2c\,F_{\mu\nu}F^{\mu\nu}\right]_{x'}.$$

Notice that we are omitting $\sqrt{g}R^2$ terms which are not necessary at one loop level. The notation $G_4(x,x')$ denotes the Green's function of the differential operator defined by

$$\Delta_4 \equiv \nabla_\mu\left(\nabla^\mu\nabla^\nu + 2R^{\mu\nu} - \frac{2}{3}Rg^{\mu\nu}\right)\nabla_\nu = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}(\nabla^\mu R)\nabla_\mu - \frac{2}{3}R\Box \tag{2.6}$$

and requires some boundary conditions to be specified. This operator is conformally covariant, in fact under a rescaling of the metric one can show that

$$g_{\mu\nu} = e^\sigma\bar{g}_{\mu\nu} \to \Delta_4 = e^{-2\sigma}\bar{\Delta}_4. \tag{2.7}$$

Notice that the general solution of (2.4) involves, in principle, also a conformally invariant part that is not identified by this method. As in ref. [51], we concentrate on the contribution proportional to F^2 and perform an expansion of this term for a weak gravitational field and drop from this action all the terms which are at least quadratic in the deviation of the metric from flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\,h_{\mu\nu} \qquad \kappa^2 = 16\,\pi\,G, \tag{2.8}$$

with G the gravitational constant. The non-local action reduces to

$$S_{anom}[g,A] = -\frac{c}{6}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\,R^{(1)}_x\,\Box^{-1}_{x,x'}\,[F_{\alpha\beta}F^{\alpha\beta}]_{x'}\,, \tag{2.9}$$

valid for a weak gravitational field. In this case

$$R^{(1)}_x \equiv \partial^x_\mu\, \partial^x_\nu\, h^{\mu\nu} - \Box\, h, \qquad h = \eta_{\mu\nu}\, h^{\mu\nu}. \tag{2.10}$$

The presence of the Green's function of the $\Box$ operator in Eq. (2.9) is the clear indication that the solution of the anomaly equation is characterized by an anomaly pole. In the next sections we are going to perform a direct diagrammatic computation of this action and reobtain from it the pole contribution identified in the dispersive analysis of [51] and the conformal invariant extra terms which are not present in Eq. (2.9). We start with an analysis of the correlator following an approach which is close to that followed in ref. [51]. The crucial point of the derivation presented in that work is the imposition of the Ward identity for the TJJ correlator (see Eq. (2.42) below) which allows to eliminate all the Schwinger (gradients) terms which otherwise plague any derivation based on the canonical formalism and are generated by the equal-time commutator of the energy momentum tensor with the vector currents. In reality, this approach can be bypassed by just imposing at a diagrammatic level the validity of an operatorial relation for the trace anomaly, evaluated at a nonzero momentum transfer, together with the conservation of the vector currents on the other two vector vertices of the correlator.

2.3 The construction of the full amplitude $\Gamma^{\mu\nu\alpha\beta}(p,q)$

We consider the standard QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\,\bar{\psi}\gamma^\mu(\partial_\mu - i\,e\,A_\mu)\psi - m\bar{\psi}\psi, \tag{2.11}$$

with the energy momentum tensor split into the free fermionic part T_f, the interacting fermion-photon part T_{fp} and the photon contribution T_{ph} which are given by

$$T^{\mu\nu}_f = -i\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi + g^{\mu\nu}(i\bar{\psi}\gamma^\lambda\overleftrightarrow{\partial}_\lambda\psi - m\bar{\psi}\psi), \tag{2.12}$$

$$T^{\mu\nu}_{fp} = -\,eJ^{(\mu}A^{\nu)} + eg^{\mu\nu}J^\lambda A_\lambda\,, \tag{2.13}$$

and

$$T^{\mu\nu}_{ph} = F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho}, \tag{2.14}$$

where the current is defined as

$$J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)\,. \tag{2.15}$$

In the coupling to gravity of the total energy momentum tensor

$$T^{\mu\nu} \equiv T^{\mu\nu}_f + T^{\mu\nu}_{fp} + T^{\mu\nu}_{ph} \tag{2.16}$$

we keep terms linear in the gravitational field, of the form $h_{\mu\nu}T^{\mu\nu}$, and we have introduced some standard notation for the symmetrization of the tensor indices and left-right derivatives $H^{(\mu\nu)} \equiv (H^{\mu\nu} + H^{\nu\mu})/2$ and $\overleftrightarrow{\partial}_\mu \equiv (\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)/2$. It is also convenient to introduce a partial energy momentum tensor T_p, corresponding to the sum of the Dirac and interaction terms

$$T_p^{\mu\nu} \equiv T_f^{\mu\nu} + T_{fp}^{\mu\nu} \tag{2.17}$$

which satisfies the inhomogeneous equation

$$\partial_\nu T_p^{\mu\nu} = -\partial_\nu T_{ph}^{\mu\nu}. \tag{2.18}$$

Using the equations of motion for the e.m. field $\partial_\nu F^{\mu\nu} = J^\mu$, the inhomogeneous equation becomes

$$\partial_\nu T_p^{\mu\nu} = F^{\mu\lambda} J_\lambda. \tag{2.19}$$

There are two ways to identify the contributions of $T^{\mu\nu}$ and $T_p^{\mu\nu}$ in the perturbative expansion of the effective action. In the formalism of the background fields, the T_pJJ correlator can be extracted from the defining functional integral

$$\begin{aligned}\langle T_p^{\mu\nu}(z)\rangle_A &\equiv \int D\psi D\bar{\psi}\; T_p^{\mu\nu}(z)\; e^{i\int d^4x\mathcal{L} + \int J\cdot A(x) d^4x} \\ &= \langle T_p^{\mu\nu}\, e^{i\int d^4x\, J\cdot A(x)}\rangle \end{aligned} \tag{2.20}$$

expanded through second order in the external field A. The relevant terms in this expansion are explicitly given by

$$\langle T_p^{\mu\nu}(z)\rangle_A = \frac{1}{2!}\langle T_f^{\mu\nu}(z)(J\cdot A)(J\cdot A)\rangle + \langle T_{fp}^{\mu\nu}(J\cdot A)\rangle + \ldots, \tag{2.21}$$

with $(J\cdot A) \equiv \int d^4x J\cdot A(x)$. The corresponding diagrams are extracted via two functional derivatives respect to the background field A_μ and are given by

$$\Gamma^{\mu\nu\alpha\beta}(z;x,y) \equiv \left.\frac{\delta^2 \langle T_p^{\mu\nu}(z)\rangle_A}{\delta A_\alpha(x)\delta A_\beta(y)}\right|_{A=0} = V^{\mu\nu\alpha\beta} + W^{\mu\nu\alpha\beta} \tag{2.22}$$

$$V^{\mu\nu\alpha\beta} = (i\,e)^2 \left\langle T_f^{\mu\nu}(z) J^\alpha(x) J^\beta(y)\right\rangle_{A=0} \tag{2.23}$$

$$\begin{aligned} W^{\mu\nu\alpha\beta} &= \left.\frac{\delta^2 \left\langle T_{fp}^{\mu\nu}(z)\,(J\cdot A)\right\rangle}{\delta A_\alpha(x)\,\delta A_\beta(y)}\right|_{A=0} \\ &= \delta^4(x-z)\, g^{\alpha(\mu}\Pi^{\nu)\beta}(z,y) + \delta^4(y-z)\, g^{\beta(\mu}\Pi^{\nu)\alpha}(z,x) \\ &\qquad - g^{\mu\nu}\left[\delta^4(x-z) - \delta^4(y-z)\right]\Pi^{\alpha\beta}(x,y) \end{aligned} \tag{2.24}$$

These two contributions are of $O(e^2)$. Alternatively, one can directly compute the matrix element

$$\mathcal{M}^{\mu\nu} = \langle 0|T_p^{\mu\nu}(z)\int d^4w d^4w' J\cdot A(w) J\cdot A(w')|\gamma\gamma\rangle, \tag{2.25}$$

which generates the diagrams (b) and (c) shown in Fig.2.1, respectively called the "triangle" and the "t-p-bubble" ("t-" stays for tensor), together with the two ones obtained for the exchange of p with q and α with β.

The conformal anomaly appears in the perturbative expansion of T_p and involves these four diagrams. The electromagnetic contribution is responsible for other two diagrams whose invariant amplitudes are well-defined and will be used to fix the ill-defined amplitudes present in the tensor expansion of $T_p^{\mu\nu}$ by using a Ward identity.

The lowest order contribution is obtained, in the background field formalism, from Maxwell's e.m. tensor, and is given by

$$S^{\mu\nu\alpha\beta} = \frac{\delta^2 \left\langle T_{ph}^{\mu\nu}(z)\right\rangle}{\delta A_\alpha(x)\delta A_\beta(y)}\bigg|_{A=0}. \tag{2.26}$$

Equivalently, it can be obtained from the matrix element

$$\langle 0|T_{ph}^{\mu\nu}|\gamma\gamma\rangle \tag{2.27}$$

which allows to identify the vertex in momentum space. Using (2.26) we easily obtain

$$\begin{aligned} S_{\mu\nu\alpha\beta}(z,x,y) &= 2g_{\alpha\beta}\partial_{(\mu}\delta_{xz}\partial_{\nu)}\delta_{yz} - 2g_{\beta(\mu}\partial_{\nu)}\delta_{xz}\partial_\alpha\delta_{yz} - 2g_{\alpha(\nu}\partial_{\mu)}\delta_{yz}\partial_\beta\delta_{xz} \\ &+ g_{\alpha\mu}g_{\beta\nu}\partial_\lambda\delta_{yz}\partial^\lambda\delta_{xz} + g_{\alpha\nu}g_{\beta\mu}\partial_\lambda\delta_{yz}\partial^\lambda\delta_{xz} + g_{\mu\nu}\partial_\beta\delta_{xz}\partial_\alpha\delta_{yz} - \partial_\rho\delta_{yz}\partial_\rho\delta_{xz}g_{\alpha\beta}g_{\mu\nu} \end{aligned} \tag{2.28}$$

where $\partial_\mu\delta_{xz} \equiv \partial/\partial x^\mu \delta(x-z)$ and so on. In momentum space this lowest order vertex is given by

$$\begin{aligned} S^{\mu\nu\alpha\beta} &= \left(p^\mu q^\nu + p^\nu q^\mu\right)g^{\alpha\beta} + p\cdot q\left(g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu}\right) - g^{\mu\nu}\left(p\cdot q\, g^{\alpha\beta} - q^\alpha p^\beta\right) \\ &- \left(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu\right)q^\alpha - \left(g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu\right)p^\beta. \end{aligned} \tag{2.29}$$

The corresponding vertices which appear respectively in the triangle diagram and on the t-bubble at $O(e^2)$ are given by

$$V'^{\mu\nu}(k_1,k_2) = \frac{1}{4}\left[\gamma^\mu(k_1+k_2)^\nu + \gamma^\nu(k_1+k_2)^\mu\right] - \frac{1}{2}g^{\mu\nu}\left[\gamma^\lambda(k_1+k_2)_\lambda - 2m\right], \tag{2.30}$$

$$W'^{\mu\nu\alpha} = -\frac{1}{2}(\gamma^\mu g^{\nu\alpha} + \gamma^\nu g^{\mu\alpha}) + g^{\mu\nu}\gamma^\alpha, \tag{2.31}$$

where $k_1(k_2)$ is outcoming (incoming). Using the two vertices $V'^{\mu\nu}(k_1,k_2)$ and $W'^{\mu\nu\alpha}$ we obtain

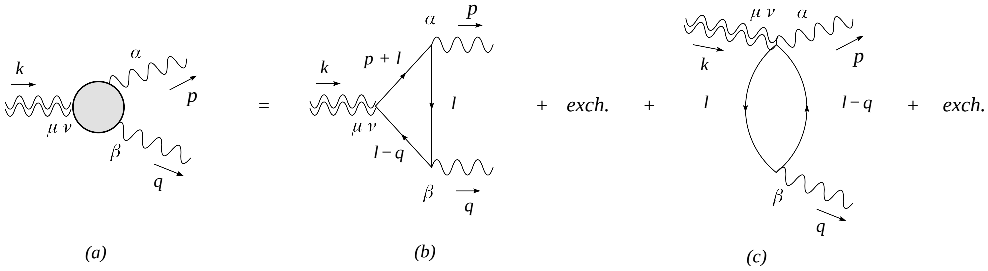


Figure 2.1: The complete one-loop vertex (a) given by the sum of the 1PI contributions called $V^{\mu\nu\alpha\beta}(p,q)$ (b) and $W^{\mu\nu\alpha\beta}(p,q)$ (c).

for the diagrams (b) and (c) of Fig.2.1

$$V^{\mu\nu\alpha\beta}(p,q) = -(-ie)^2 i^3 \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{tr}\left\{V'^{\mu\nu}(l+p, l-q)(\not l - \not q + m)\gamma^\beta\,(\not l + m)\,\gamma^\alpha(\not l + \not p + m)\right\}}{[l^2 - m^2]\,[(l-q)^2 - m^2]\,[(l+p)^2 - m^2]}\,, \tag{2.32}$$

and

$$W^{\mu\nu\alpha\beta}(p,q) \quad = \quad -(ie^2)i^2 \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{tr}\left\{W'^{\mu\nu\alpha}\,(\not l + m)\gamma^\beta(\not l - \not q + m)\right\}}{[l^2 - m^2][(l-q)^2 - m^2]}, \tag{2.33}$$

so that the one-loop amplitude in Fig. 2.1 results

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = V^{\mu\nu\alpha\beta}(p,q) + V^{\mu\nu\beta\alpha}(q,p) + W^{\mu\nu\alpha\beta}(p,q) + W^{\mu\nu\beta\alpha}(q,p). \tag{2.34}$$

The bare Ward identity which allows to define the divergent amplitudes that contribute to the anomaly in Γ in terms of the remaining finite ones is obtained by re-expressing the classical equation

$$\partial_\nu T^{\mu\nu}_{ph} = -F^{\mu\nu} J_\nu \tag{2.35}$$

as an equation of generating functionals in the background electromagnetic field

$$\partial_\nu \langle T^{\mu\nu}_{ph} \rangle_A = -F^{\mu\nu} \langle J_\nu \rangle_A, \tag{2.36}$$

which can be expanded perturbatively as

$$\partial_\nu \langle T^{\mu\nu}_{ph} \rangle_A = -F^{\mu\nu} \langle J_\nu \int d^4 w (ie) J \cdot A(w) \rangle_+ \dots . \tag{2.37}$$

Notice that we have omitted the first term in the Dyson's series of $\langle J_\nu \rangle_A$, shown on the r.h.s of (2.37) since $\langle J_\nu \rangle = 0$. The bare Ward identity then takes the form

$$\partial_\nu \Gamma^{\mu\nu\alpha\beta} = \frac{\delta^2 \left(F^{\mu\lambda}(z)\,\langle J_\lambda(z)\rangle_A\right)}{\delta A_\alpha(x)\delta A_\beta(y)}\Bigg|_{A=0} \tag{2.38}$$

$p^\mu p^\nu p^\alpha p^\beta$ $q^\mu q^\nu q^\alpha q^\beta$	$p^\mu p^\nu p^\alpha q^\beta$ $p^\mu p^\nu q^\alpha p^\beta$ $p^\mu q^\nu p^\alpha p^\beta$ $q^\mu p^\nu p^\alpha p^\beta$	$p^\mu p^\nu q^\alpha q^\beta$ $p^\mu q^\nu p^\alpha q^\beta$ $q^\mu p^\nu p^\alpha q^\beta$	$p^\mu q^\nu q^\alpha p^\beta$ $q^\mu p^\nu q^\alpha p^\beta$ $q^\mu q^\nu p^\alpha p^\beta$	$p^\mu q^\nu q^\alpha q^\beta$ $q^\mu p^\nu q^\alpha q^\beta$ $q^\mu q^\nu p^\alpha q^\beta$ $q^\mu q^\nu q^\alpha p^\beta$	$g^{\mu\nu} g^{\alpha\beta}$ $g^{\alpha\mu} g^{\beta\nu}$ $g^{\alpha\nu} g^{\beta\mu}$
$p^\mu p^\nu g^{\alpha\beta}$ $p^\mu q^\nu g^{\alpha\beta}$ $q^\mu p^\nu g^{\alpha\beta}$ $q^\mu q^\nu g^{\alpha\beta}$	$p^\beta p^\nu g^{\alpha\mu}$ $p^\beta q^\nu g^{\alpha\mu}$ $q^\beta p^\nu g^{\alpha\mu}$ $q^\beta q^\nu g^{\alpha\mu}$	$p^\beta p^\mu g^{\alpha\nu}$ $p^\beta q^\mu g^{\alpha\nu}$ $q^\beta p^\mu g^{\alpha\nu}$ $q^\beta q^\mu g^{\alpha\nu}$	$p^\alpha p^\nu g^{\beta\mu}$ $p^\alpha q^\nu g^{\beta\mu}$ $q^\alpha p^\nu g^{\beta\mu}$ $q^\alpha q^\nu g^{\beta\mu}$	$p^\mu p^\alpha g^{\beta\nu}$ $p^\mu q^\alpha g^{\beta\nu}$ $q^\mu p^\alpha g^{\beta\nu}$ $q^\mu q^\alpha g^{\beta\nu}$	$p^\alpha p^\beta g^{\mu\nu}$ $p^\alpha q^\beta g^{\mu\nu}$ $q^\alpha p^\beta g^{\mu\nu}$ $q^\alpha q^\beta g^{\mu\nu}$

Table 2.1: The 43 tensor monomials built up from the metric tensor and the two independent momenta p and q into which a general fourth rank tensor can be expanded.

which takes contribution only from the first term on the r.h.s of Eq. (2.37). This relation can be written in momentum space. For this we use the definition of the vacuum polarization

$$\Pi^{\alpha\beta}(x,y) \equiv -ie^2 \langle J_\alpha(x) J_\beta(y)\rangle, \tag{2.39}$$

or

$$\begin{aligned}\Pi^{\alpha\beta}(p) &= -i^2\,(-ie)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{tr}\left\{\gamma^\alpha\,(\not{l}+m)\gamma^\beta(\not{l}+\not{p}+m)\right\}}{[l^2-m^2]\,[(l+p)^2-m^2]} \\ &= (p^2 g^{\alpha\beta} - p^\alpha p^\beta)\,\Pi(p^2,m^2)\end{aligned} \tag{2.40}$$

with

$$\Pi(p^2,m^2) = \frac{e^2}{36\,\pi^2\,p^2}\left[6\,\mathcal{A}_0(m^2) + p^2 - 6\,m^2 - \,3\,\mathcal{B}_0(p^2,m^2)\left(2m^2+p^2\right)\right], \tag{2.41}$$

which obviously satisfies the Ward identity $p_\alpha\,\Pi^{\alpha\beta}(p) = 0$. The expressions of the $\mathcal{A}_0$ and $\mathcal{B}_0$ contributions are given in Appendix A.2.

Using these definitions, the unrenormalized Ward identity which allows to completely characterize the form of the correlator in momentum space becomes

$$\begin{aligned}k_\nu\,\Gamma^{\mu\nu\alpha\beta}(p,q) &= \left(q^\mu p^\alpha p^\beta - q^\mu g^{\alpha\beta} p^2 + g^{\mu\beta} q^\alpha p^2 - g^{\mu\beta} p^\alpha p\cdot q\right)\Pi(p^2) \\ &+ \left(p^\mu q^\alpha q^\beta - p^\mu g^{\alpha\beta} q^2 + g^{\mu\alpha} p^\beta q^2 - g^{\mu\alpha} q^\beta p\cdot q\right)\Pi(q^2)\,.\end{aligned} \tag{2.42}$$

2.3.1 Tensor expansion and invariant amplitudes of Γ

The full one-loop amplitude Γ can be expanded on the basis provided by the 43 monomial tensors listed in Tab.2.1

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{43} A_i(k^2,p^2,q^2)\, l_i^{\mu\nu\alpha\beta}(p,q). \tag{2.43}$$

Since the amplitude $\Gamma^{\mu\nu\alpha\beta}(p,q)$ has total mass dimension equal to 2 it is obvious that not all the coefficients A_i are convergent. They can be divided into 3 groups:

a) $A_1 \leq A_i \leq A_{16}$ - multiplied by a product of four momenta, they have mass dimension -2 and therefore are UV finite;

b) $A_{17} \leq A_i \leq A_{19}$ - these have mass dimension 2 since the four Lorentz indices of the amplitude are carried by two metric tensors

c) $A_{20} \leq A_i \leq A_{43}$ - they appear next to a metric tensor and two momenta, have mass dimension 0 and are divergent.

The way in which the 43 invariant amplitudes will be managed in order to reduce them to the 13 named $F_i(k^2,p^2,q^2)$ is the subject of this section. The reduction is accomplished in 4 different steps and has as a guiding principle the elimination of the divergent amplitudes A_i in terms of the convergent ones after imposing some conditions on the whole amplitude. We require

a) the symmetry in the two indices μ and ν of the symmetric energy-momentum tensor $T^{\mu\nu}$;

b) the conservation of the two vector currents on p^α and q^β;

c) the Ward identity on the vertex with the incoming momentum k defined above in Eq. (2.3.1).

Condition a) becomes

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \Gamma^{\nu\mu\alpha\beta}(p,q), \tag{2.44}$$

giving a linear system of 43 equations; 15 of them being identically satisfied because the tensorial structures are already symmetric in the exchange of μ and ν, while the remaining 14 conditions are

$$\begin{array}{lllll} A_5 = A_6, & A_8 = A_9, & A_{10} = A_{11}, & A_{13} = A_{14}, & A_{18} = A_{19}, \\ A_{21} = A_{22}, & A_{24} = A_{28}, & A_{25} = A_{29}, & A_{26} = A_{30}, & A_{27} = A_{31}, \\ A_{32} = A_{36}, & A_{34} = A_{37}, & A_{33} = A_{38}, & A_{35} = A_{39}, & \end{array} \tag{2.45}$$

where all A_i are thought as functions of the invariants k^2, p^2, q^2. After substituting (2.45) into $\Gamma^{\mu\nu\alpha\beta}(p,q)$ the 43 invariant tensors of the decomposition are multiplied by only 29 invariant amplitudes. Condition b), which is vector current conservation on the two vertices with indices α and β, allows to re-express some divergent A_i in terms of other finite ones

$$p_\alpha \, \Gamma^{\mu\nu\alpha\beta}(p,q) = q_\beta \, \Gamma^{\mu\nu\alpha\beta}(p,q) = 0. \tag{2.46}$$

This constraint generates two sets of 14 independent tensor structures each, so that in order to fulfill (2.46) each coefficient is separately set to vanish. The first Ward identity leads to a linear system composed of 10 equations

$$p_\alpha \, \Gamma^{\mu\nu\alpha\beta}(p,q) = 0 \Rightarrow \begin{cases} A_{19} + A_{36}\, p\cdot p + A_{37}\, p\cdot q = 0, \\ A_{38}\, p\cdot p + A_{39}\, p\cdot q = 0, \\ A_{17} + A_{40}\, p\cdot p + A_{42}\, p\cdot q = 0, \\ A_{41}\, p\cdot p + A_{43}\, p\cdot q = 0, \\ A_{20} + 2A_{28} + A_1\, p\cdot p + A_4\, p\cdot q = 0, \\ 2\,A_{30} + A_3\, p\cdot p + A_7\, p\cdot q = 0, \\ A_{22} + A_{29} + A_6\, p\cdot p + A_{11}\, p\cdot q = 0, \\ A_{31} + A_9\, p\cdot p + A_{14}\, p\cdot q = 0, \\ A_{23} + A_{12}\, p\cdot p + A_{16}\, p\cdot q = 0, \\ A_{15}\, p\cdot p + A_2\, p\cdot q = 0; \end{cases} \tag{2.47}$$

we choose to solve it for the set $\{A_{15}, A_{17}, A_{19}, A_{23}, A_{28}, A_{29}, A_{30}, A_{31}, A_{39}, A_{43}\}$ in which only the first one is convergent and the others are UV divergent. The set would not include all the divergent A_i since in the last equations appear two convergent coefficients, A_{15} and A_2.
Following our choice the result is

$$A_{15} = -A_2\,\frac{p\cdot q}{p\cdot p}, \qquad A_{17} = -A_{40}\, p\cdot p - A_{42}\, p\cdot q, \tag{2.48}$$
$$A_{19} = -A_{36}\, p\cdot p - A_{37}\, p\cdot q, \qquad A_{23} = -A_{12}\, p\cdot p - A_{16}\, p\cdot q, \tag{2.49}$$
$$A_{28} = \frac{1}{2}\Big[- A_{20} - A_1\, p\cdot p - A_4\, p\cdot q\Big], \qquad A_{29} = -A_{22} - A_6\, p\cdot p - A_{11}\, p\cdot q, \tag{2.50}$$
$$A_{30} = -\frac{1}{2}\Big[A_3 p\cdot p + A_7\, p\cdot q\Big], \qquad A_{31} = -A_9\, p\cdot p - A_{14}\, p\cdot q, \tag{2.51}$$
$$A_{39} = -A_{38}\frac{p\cdot p}{p\cdot q}, \qquad A_{43} = -A_{41}\,\frac{p\cdot p}{p\cdot q}. \tag{2.52}$$

In an analogous way we go on with the second Ward identity (WI) after replacing the solution of the previous system in the original amplitude. The new one is indicated by $\Gamma_b^{\mu\nu\alpha\beta}(p,q)$, where the subscript b is there to indicate that we have applied condition b) on Γ. The constraint gives

$$q_\beta\,\Gamma_b^{\mu\nu\alpha\beta}(p,q)=0\Rightarrow\begin{cases}A_{40}\,p\cdot q+A_{41}\,q\cdot q\ =0,\\ A_1\,p\cdot q+A_3\,q\cdot q\ =0,\\ A_{20}+A_4\,p\cdot q+A_7\,q\cdot q\ =0,\\ A_{36}+A_6\,p\cdot q+A_9\,q\cdot q\ =0,\\ A_{22}+A_{37}+A_{11}\,p\cdot q+A_{14}\,q\cdot q\ =0,\\ 2A_{38}+A_{12}\,p\cdot q-A_2\frac{p\cdot q\,q\cdot q}{p\cdot p}=0.\end{cases}\tag{2.53}$$

We solve these equations determining the amplitudes in the set $\{A_1, A_{20}, A_{22}, A_{36}, A_{38}, A_{40}\}$ in terms of the remaining ones, obtaining

$$A_{38}=-\frac{A_{12}\,p\cdot p\;p\cdot q\;-A_2\,p\cdot q\;q\cdot q}{2\,p\cdot p},\qquad A_{40}=-\frac{A_{41}\,q\cdot q}{p\cdot q},\tag{2.54}$$

$$A_1=-\frac{A_3\,q\cdot q}{p\cdot q},\qquad A_{20}=-A_4\,p\cdot q\;-A_7\,q\cdot q\,,\tag{2.55}$$

$$A_{22}=-A_{37}-A_{11}\,p\cdot q\;-A_{14}\,q\cdot q\,,\qquad A_{36}=-A_6\,p\cdot q\;-A_9\,q\cdot q.\tag{2.56}$$

The manipulations above have allowed a reduction of the number of invariant amplitudes from the initial 43 to 13 using the $\{\mu,\nu\}$ symmetry (14 equations), the first WI on p_α (10 equations) and the second WI on q_β (6 equations).
The surviving invariant amplitudes in which the amplitude $\Gamma_c^{\mu\nu\alpha\beta}(p,q)$ can be expanded using the form factors are $\{A_2, A_3, A_4, A_6, A_7, A_9, A_{11}, A_{12}, A_{14}, A_{16}, A_{37}, A_{41}, A_{42}\}$. This set still contains 3 divergent amplitudes, (A_{37}, A_{41}, A_{42}). The amplitude $\Gamma_c^{\mu\nu\alpha\beta}(p,q)$ is indeed ill-defined until we impose on it condition c), that is Eq. (2.42). This condition gives

$$\text{Eq.(2.42)}\Rightarrow\begin{cases}-A_3\left[1+\frac{p\cdot p}{2\,p\cdot q}\right]\ +A_6+\frac{1}{2}\,A_7\ -A_9\ -\ \frac{A_{41}}{p\cdot q}=0,\\ A_{37}+A_{42}+A_4\,[p\cdot p\ +p\cdot q]\ +\ A_{11}\,p\cdot q\ +\frac{1}{2}\ A_7\ q\cdot q+\\ \qquad\qquad +A_{11}\ q\cdot q\ +\frac{1}{2}\,A_3\ \frac{p\cdot p\ q\cdot q}{p\cdot q}=0,\\ \frac{1}{2}A_2\frac{p\cdot q\,q\cdot q}{p\cdot p}-A_{41}\frac{p\cdot p+q\cdot q}{p\cdot q}-\frac{1}{2}A_3\,p\cdot p+A_7(\,p\cdot p+\frac{1}{2}\,p\cdot q)+A_6\,p\cdot q\\ \qquad +A_{12}(\frac{1}{2}\,p\cdot q+q\cdot q)+A_{14}(\,p\cdot q+2\,q\cdot q)+2A_{37}-\Pi(p^2)-\Pi(q^2)=0\end{cases}$$

From this condition we obtain

$$A_{37} = -\frac{A_2}{4}\frac{p\cdot q\, q\cdot q}{p\cdot p} + \frac{1}{4}A_3\, p\cdot p - \frac{1}{4}A_7\,(2\,p\cdot p + p\cdot q) + \frac{1}{2}A_{41}\left(\frac{p\cdot p + q\cdot q}{p\cdot q}\right)$$
$$-\frac{1}{2}A_6\, p\cdot q - \frac{1}{4}A_{12}\,(p\cdot q + 2\,q\cdot q) - \frac{1}{2}A_{14}\,(p\cdot q + 2q\cdot q) + \frac{1}{2}\left[\Pi(p^2) + \Pi(q^2)\right] \tag{2.57}$$

$$A_{41} = -\frac{A_3}{2}\,p\cdot p - \,(A_3 - A_6 - A_7 + A_9)\; p\cdot q \tag{2.58}$$

$$A_{42} = \frac{A_3}{2}\,p\cdot p\left(\frac{p\cdot p}{p\cdot q} + 1 - \frac{q\cdot q}{p\cdot q}\right) + \frac{1}{2}A_7\,(p\cdot p + p\cdot q - q\cdot q) - A_4\;(p\cdot p + p\cdot q)$$
$$-(A_6 - A_9)\,p\cdot p + (A_{14} - A_{11})(q\cdot q + p\cdot q). \tag{2.59}$$

After these steps we end up with an expression for Γ written in terms of only 10 invariant amplitudes, that are $\mathcal{X} \equiv \{A_2, A_3, A_4, A_6, A_7, A_9, A_{11}, A_{12}, A_{14}, A_{16}\}$, significantly reduced respect to the original 43. Further reductions are possible (down to 8 independent invariant amplitudes), however, since these reductions just add to the complexity of the related tensor structures, it is convenient to select an appropriate set of reducible (but finite) components characterized by a simpler tensor structure and present the result in that form. The 13 amplitudes introduced in the final decomposition are, in this respect, a good choice since the corresponding tensor structures are rather simple. These tensors are combinations of the 43 monomials listed in Tab.2.1.

The set $\mathcal{X}$ is very useful for the actual computation of the tensor integrals and for the study of their reduction to scalar form. To compare with the previous study of Giannotti and Mottola [51] we have mapped the computation of the components of the set $\mathcal{X}$ into their structures F_i ($i = 1, 2, .., 13$). Also in this case, the truly independent amplitudes are 8. One can extract, out of the 13 reducible amplitudes, a consistent subset of 8 invariant amplitudes. The remaining amplitudes in the 13 tensor structures are, in principle, obtainable from this subset.

2.3.2 Reorganization of the amplitude

Before obtaining the mapping between the amplitudes in $\mathcal{X}$ and the structures F_i, we briefly describe the tensor decomposition introduced in [51] which defines these 13 structures. We define the rank-2 tensors

$$u^{\alpha\beta}(p,q) \equiv (p\cdot q)\, g^{\alpha\beta} - q^\alpha\, p^\beta\,, \tag{2.60}$$

$$w^{\alpha\beta}(p,q) \equiv p^2\, q^2\, g^{\alpha\beta} + (p\cdot q)\, p^\alpha\, q^\beta - q^2\, p^\alpha\, p^\beta - p^2\, q^\alpha\, q^\beta\,, \tag{2.61}$$

which are Bose symmetric,

$$u^{\alpha\beta}(p,q) = u^{\beta\alpha}(q,p)\,, \tag{2.62}$$

$$w^{\alpha\beta}(p,q) = w^{\beta\alpha}(q,p)\,, \tag{2.63}$$

and conserve vector current,

$$p_\alpha\, u^{\alpha\beta}(p,q) = q_\beta\, u^{\alpha\beta}(p,q) = 0\,, \tag{2.64}$$

$$p_\alpha\, w^{\alpha\beta}(p,q) = q_\beta\, w^{\alpha\beta}(p,q) = 0\,. \tag{2.65}$$

These two tensors are used to build the set of 13 tensors catalogued in Table 2.2. They are linearly independent for generic k^2, p^2, q^2 different from zero. Five of the 13 tensors are Bose symmetric, namely,

$$t_i^{\mu\nu\alpha\beta}(p,q) = t_i^{\mu\nu\beta\alpha}(q,p)\,, \qquad i = 1,2,7,8,13\,, \tag{2.66}$$

while the remaining eight tensors form four pairs which are overall related by Bose symmetry

$$t_3^{\mu\nu\alpha\beta}(p,q) = t_5^{\mu\nu\beta\alpha}(q,p)\,, \tag{2.67}$$

$$t_4^{\mu\nu\alpha\beta}(p,q) = t_6^{\mu\nu\beta\alpha}(q,p)\,, \tag{2.68}$$

$$t_9^{\mu\nu\alpha\beta}(p,q) = t_{10}^{\mu\nu\beta\alpha}(q,p)\,, \tag{2.69}$$

$$t_{11}^{\mu\nu\alpha\beta}(p,q) = t_{12}^{\mu\nu\beta\alpha}(q,p)\,. \tag{2.70}$$

The amplitude in (2.34) can be expanded in this basis composed as

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{13} F_i(s; s_1, s_2, m^2)\; t_i^{\mu\nu\alpha\beta}(p,q)\,, \tag{2.71}$$

where the invariant amplitudes F_i are functions of the kinematical invariants $s = k^2 = (p+q)^2$, $s_1 = p^2$, $s_2 = q^2$ and of the internal mass m. In [51] the authors use the Feynman parameterization and momentum shifts in order to identify the expressions of these amplitudes in terms of parametric integrals, which was the approach followed also by Rosenberg in his original identification of the 6 invariant amplitudes of the AVV anomaly diagram. If we choose to reorganize all the monomials into the simpler set of 13 tensor groups shown in Tab.2.2, then we need to map the A_i in χ and the F_i's. The mapping is given by

$$\begin{aligned} F_1 &= \frac{1}{3\,k^2}\Big[A_4(4\,p\cdot q + 3\,p\cdot p) + 2A_{11}(\,p\cdot q\, +\, 2\,q\cdot q) + 2A_6\,p\cdot p \\ &\quad + 2A_7\,q\cdot q\, - 2A_{14}\,q\cdot q\, - A_{16}\,q\cdot q + 2A_3\,\frac{p\cdot p\; q\cdot q}{p\cdot q}\Big], \end{aligned} \tag{2.72}$$

$$F_2 = \frac{1}{3\,k^2}\left[-2A_3\left(\frac{p\cdot p}{p\cdot q} + 2\right) + 4A_6\, + A_7\, - 2A_9\, - A_{12}\right], \tag{2.73}$$

i	$t_i^{\mu\nu\alpha\beta}(p,q)$
1	$\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) u^{\alpha\beta}(p.q)$
2	$\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) w^{\alpha\beta}(p.q)$
3	$\left(p^2 g^{\mu\nu} - 4p^\mu p^\nu\right) u^{\alpha\beta}(p.q)$
4	$\left(p^2 g^{\mu\nu} - 4p^\mu p^\nu\right) w^{\alpha\beta}(p.q)$
5	$\left(q^2 g^{\mu\nu} - 4q^\mu q^\nu\right) u^{\alpha\beta}(p.q)$
6	$\left(q^2 g^{\mu\nu} - 4q^\mu q^\nu\right) w^{\alpha\beta}(p.q)$
7	$\left[p\cdot q\, g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)\right] u^{\alpha\beta}(p.q)$
8	$\left[p\cdot q\, g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)\right] w^{\alpha\beta}(p.q)$
9	$\left(p\cdot q\, p^\alpha - p^2 q^\alpha\right)\left[p^\beta\left(q^\mu p^\nu + p^\mu q^\nu\right) - p\cdot q\left(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu\right)\right]$
10	$\left(p\cdot q\, q^\beta - q^2 p^\beta\right)\left[q^\alpha\left(q^\mu p^\nu + p^\mu q^\nu\right) - p\cdot q\left(g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu\right)\right]$
11	$\left(p\cdot q\, p^\alpha - p^2 q^\alpha\right)\left[2\,q^\beta q^\mu q^\nu - q^2(g^{\beta\nu}q^\mu + g^{\beta\mu}q^\nu)\right]$
12	$\left(p\cdot q\, q^\beta - q^2 p^\beta\right)\left[2\,p^\alpha p^\mu p^\nu - p^2(g^{\alpha\nu}p^\mu + g^{\alpha\mu}p^\nu)\right]$
13	$\left(p^\mu q^\nu + p^\nu q^\mu\right)g^{\alpha\beta} + p\cdot q\left(g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu}\right) - g^{\mu\nu}u^{\alpha\beta}$ $-\left(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu\right)q^\alpha - \left(g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu\right)p^\beta$

Table 2.2: Basis of 13 fourth rank tensors satisfying the vector current conservation on the external lines with momenta p and q.

$$\begin{aligned}
F_3 &= \frac{1}{12\,k^2}\left[A_4(2\,p\cdot q + 3\,q\cdot q) - 2A_{11}(p\cdot q + 2\,q\cdot q) - 2A_6\,p\cdot p\right.\\
&\quad\left. -2A_7\,q\cdot q + 2A_{14}\,q\cdot q + A_{16}\,q\cdot q - 2A_3\,\frac{p\cdot p\; q\cdot q}{p\cdot q}\right] \qquad (2.74)\\
F_4 &= \frac{A_7}{4\,p\cdot p} + \frac{1}{12\,k^2}\left[2A_3\left(\frac{p\cdot p}{p\cdot q} + 2\right) - 4A_6 - A_7 + 2A_9 + A_{12}\right] \qquad (2.75)\\
F_5 &= \frac{A_{16}}{4} + \frac{1}{12\;k^2}\left[-2\,A_6\,p\cdot p - 2\,A_3\,\frac{q\cdot q\,p\cdot p}{p\cdot q} + A_4\,(-3\,p\cdot p - 4\,p\cdot q)\right.\\
&\quad\left. -2\,A_{11}\,(p\cdot q + 2q\cdot q) - 2\,A_7\,q\cdot q + 2\,A_{14}\,q\cdot q + A_{16}\,q\cdot q\right], \qquad (2.76)\\
F_6 &= \frac{A_{12}}{4\,q\cdot q} + \frac{1}{12\,k^2}\left[-4\,A_6 - A_7 + 2\,A_9 + A_{12} + 2\,A_3\left(\frac{p\cdot p}{p\cdot q} + 2\right)\right], \qquad (2.77)\\
F_7 &= \frac{A_{11}}{2} + \frac{1}{p\cdot q^2}\left(A_9\,q\cdot q\,p\cdot p + \frac{A_6}{2}\,p\cdot p\,p\cdot q + \frac{A_{14}}{2}\,q\cdot q\,p\cdot q\right)\\
&\quad + \frac{1}{6\,k^2}\left[A_4(-4\,p\cdot q - 3\,p\cdot p) - 2\,A_{11}(p\cdot q + 2\,q\cdot q) - 2\,A_6\,p\cdot p - 2\,A_7\,q\cdot q\right.\\
&\quad\left. + 2\,A_{14}\,q\cdot q + A_{16}\,q\cdot q - 2\,A_3\frac{p\cdot p\;q\cdot q}{p\cdot q}\right], \qquad (2.78)
\end{aligned}$$

$$F_8 = \frac{1}{6\,k^2}\Big[2\,A_3\left(\frac{p\cdot p}{p\cdot q}+2\right)-3\,\frac{A_9}{p\cdot q}\,(\,p\cdot p+q\cdot q)-4\,A_6\,-\,A_7\,-\,4\,A_9+\,A_{12}\Big] \quad (2.79)$$

$$F_9 = \frac{A_6}{p\cdot q}\,+\,A_9\frac{q\cdot q}{p\cdot q^{\,2}}, \quad (2.80)$$

$$F_{10} = A_9\,\frac{p\cdot p}{p\cdot q^{\,2}}+\frac{A_{14}}{p\cdot q}, \quad (2.81)$$

$$F_{11} = \frac{A_{12}}{2\,q\cdot q}-\frac{A_2}{2\,p\cdot p}, \quad (2.82)$$

$$F_{12} = \frac{A_3}{2\,p\cdot q}+\frac{A_7}{2\,p\cdot p}, \quad (2.83)$$

$$\begin{aligned} F_{13} = {} & \frac{1}{2}A_6\,(p\cdot p\,+p\cdot q\,-q\cdot q\,)+\frac{1}{4}A_7\,(p\cdot p\,+p\cdot q\,-q\cdot q\,)+\frac{A_2\,p\cdot q\,q\cdot q}{4\,p\cdot p} \\ & +A_{14}\left(\frac{p\cdot q}{2}+q\cdot q\right)+\frac{1}{4}A_{12}\,(p\cdot q\,+2\,q\cdot q\,) \\ & +\frac{A_3}{4\,p\cdot q}\left(p\cdot p^{\,2}+(p\cdot q\,+q\cdot q\,)\,p\cdot p\,+2p\cdot q\,q\cdot q\right) \\ & +\frac{1}{2}A_9\left[q\cdot q\,+p\cdot p\,\left(\frac{2q\cdot q}{p\cdot q}+1\right)\right]-\frac{1}{2}\,[\Pi(p)+\Pi(q)]. \end{aligned} \quad (2.84)$$

We have shown how to obtain the 13 F_i' s, starting from our derivation of the one-loop full amplitude $\Gamma^{\mu\nu\alpha\beta}(p,q)$ leading to the ten invariant amplitudes of the set $\mathcal{X}$. Since we know the analytical expression of the A_i involved, we can go one step further and give all the F_i' s in their analytical form in the most general kinematical configuration.

2.4 Trace condition in the non-conformal case

Similarly to the chiral case, we can fix the correlator by requiring the validity of a trace condition on the amplitude, besides the two Ward identities on the conserved vector currents and the Bose symmetry in their indices. This approach is alternative to the imposition of the Ward identity (2.42) but nevertheless equivalent to it. At a diagrammatic level we obtain

$$g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}(p,q)=\Lambda^{\alpha\beta}(p,q)-\frac{e^2}{6\pi^2}\,u^{\alpha\beta}(p,q). \quad (2.85)$$

We comment below on this equation, in relation to the scales present in the perturbative expansion of the correlator, which are, besides the fermion mass m, the energy at which we probe the correlator (s) and the subtraction point after renormalization (μ or M). We have also defined

$$\Lambda^{\alpha\beta}(p,q) = -m\,(ie)^2\int d^4x\,d^4y\,e^{ip\cdot x+iq\cdot y}\left\langle\bar{\psi}\psi J^\alpha(x)J^\beta(y)\right\rangle \quad (2.86)$$

A direct computation gives

$$\Lambda^{\alpha\beta}(p,q) = G_1(s,s_1,s_2,m^2)\, u^{\alpha\beta}(p,q) + G_2(s,s_1,s_2,m^2)\, w^{\alpha\beta}(p,q), \tag{2.87}$$

where

$$3\, s\, F_1(s,s_1,s_2,m^2) = G_1(s,s_1,s_2,m^2) - \frac{e^2}{6\pi^2} \tag{2.88}$$

$$3\, s\, F_2(s,s_1,s_2,m^2) = G_2(s,s_1,s_2,m^2) \tag{2.89}$$

and

$$\begin{aligned} G_1(s,s_1,s_2,m^2) &= \frac{e^2\gamma m^2}{\pi^2\sigma} + \frac{e^2\,\mathcal{D}_2(s,s_2,m^2)\, s_2 m^2}{\pi^2\sigma^2}\left[s^2 + 4s_1 s - 2s_2 s - 5s_1^2 + s_2^2 + 4s_1 s_2\right] \\ &- \frac{e^2\,\mathcal{D}_1(s,s_1,m^2)\, s_1 m^2}{\pi^2\sigma^2}\left[-\left(s-s_1\right)^2 + 5s_2^2 - 4\left(s+s_1\right)s_2\right] \\ &- e^2\,\mathcal{C}_0(s,s_1,s_2,m^2)\left\{\frac{m^2\gamma}{2\pi^2\sigma^2}\left[\left(s-s_1\right)^3 - s_2^3 + \left(3s+s_1\right)s_2^2\right.\right. \\ &\left.\left. + \left(-3s^2 - 10s_1 s + s_1^2\right)s_2\right] - \frac{2m^4\gamma}{\pi^2\sigma}\right\}, \end{aligned} \tag{2.90}$$

$$\begin{aligned} G_2(s,s_1,s_2,m^2) &= -\frac{2e^2m^2}{\pi^2\sigma} - \frac{2e^2\,\mathcal{D}_2(s,s_2,m^2)m^2}{\pi^2\sigma^2}\left[\left(s-s_1\right)^2 - 2s_2^2 + \left(s+s_1\right)s_2\right] \\ &- \frac{2\,e^2\mathcal{D}_1(s,s_1,m^2)\, m^2}{\pi^2\sigma^2}\left[s^2 + \left(s_1 - 2s_2\right)s - 2s_1^2 + s_2^2 + s_1 s_2\right] \\ &- e^2\mathcal{C}_0(s,s_1,s_2,m^2)\left[\frac{4m^4}{\pi^2\sigma} + \frac{m^2}{\pi^2\sigma^2}\left[s^3 - \left(s_1+s_2\right)s^2 - \left(s_1^2 - 6s_2 s_1 + s_2^2\right)s\right.\right. \\ &\left.\left. + \left(s_1 - s_2\right)^2\left(s_1+s_2\right)\right]\right], \end{aligned} \tag{2.91}$$

where $\gamma \equiv s - s_1 - s_2$, $\sigma \equiv s^2 - 2(s_1+s_2)\,s + (s_1-s_2)^2$ and the scalar integrals $\mathcal{D}_1(s,s_1,m^2)$, $\mathcal{D}_2(s,s_1,m^2)$, $\mathcal{C}_0(s,s_1,s_2,m^2)$ for generic virtualities and masses are defined in Appendix A.2. We have checked that the final expressions of the form factors in the most general case, obtained either by imposing this condition on the energy momentum tensor or the Ward identity in the form given by Eq. (2.36) exactly coincide. In Appendix A.3 we discuss this relation in the simpler case of a massless fermion in the loop.

2.5 The off-shell massive $\langle TJJ \rangle$ correlator

To obtain the explicit expression of the parametric integrals which describe the form factors, we follow an approach similar to that of [40], for the case of the chiral gauge anomaly. These have been obtained by re-computing the anomaly diagrams by dimensional reduction together with the tensor-to-scalar decomposition of the Feynman amplitudes. For instance, in [40] we have given the explicit expressions of the parametric integrals of Rosenberg using this trick. The correctness of the result can be checked numerically by comparing the parametric forms to the explicit computation. In this case the procedure is identical, though the computations are very involved. By comparing the two approaches we extract, indirectly, an explicit expression of the parametric forms of these integrals, introduced in [51]. We have checked that indeed there is perfect numerical agreement between our computation and the parametric result, as discussed in Appendix A.4.

We introduce in this section the main results of our computation which will be used in the next sections for further analysis. The complete expressions of the form factors F_i $(i = 1, \ldots, 13)$ in the massive and then in the massless case are contained in Appendix A.5 and A.6 respectively, whereas the master integrals are collected in Appendix A.2. In both cases the virtualities of the external lines are generic and denoted by s_1, s_2. After presenting the complete expressions, we discuss several kinematical limits of the result, in particular the on-shell limit for the two vector lines ($s_1 \to 0$, $s_2 \to 0$) in order to better understand the structure of the whole correlator. The appearance of generalized anomaly poles in the correlator and their IR decoupling under the most general conditions will be discussed thoroughly.

Notice that F_{13} contains two vacuum polarization diagrams with the two photon momenta which are divergent and we are bond to define a suitable renormalization of the 2-point function which will affect the running of the coupling. In the next section we will address the explicit relation between renormalization schemes and running of the coupling in the context of the renormalization of the correlator.

2.5.1 Anomaly poles and their UV/IR significance

There are close similarities between the effective action in the case of the chiral gauge anomaly and the conformal case, due to the presence of massless poles. In [40] we have analyzed the fact that in the chiral case the anomaly is entirely generated by the longitudinal component w_L, which is indeed isolated for *any* configuration of the photon momenta. This is somehow unexpected since the dispersive analysis shows that the pole in w_L is coupled only under a specific kinematic condition, and is usually interpreted as an infrared effect. Nevertheless there is a complete equivalence between the representation of the anomaly diagram in the Rosenberg representation [41] - where the pole is not extracted as an independent component - and the L/T representation in which the pole is isolated under any kinematical configuration (and even in the massive case). This is apparent from the broken anomalous Ward identities satisfied by the AVV diagram where the mass corrections and the anomaly term can be separately identified [40].

To illustrate the emergence of a similar behaviour in the case of the conformal anomaly, it is sufficient to notice in the expression of F_1 given in Eq. (A.94) the presence of the isolated contribution ($F_{1\,pole} \equiv -e^2/(18\pi^2 s)$) which survives in the massless limit but is present also in the massive case. This component, indeed, is responsible for the trace anomaly also in the massive case, even though there appear extra corrections with mass-dependent terms. Obviously also in this case, which is generic from the kinematical point of view, one can clearly show that the pole does not couple in the infrared if we compute the residue of the entire amplitude. The anomaly pole, in fact, appears in the spectral function only in a special kinematic configuration when the fermion-antifermion pair of the anomaly diagram is collinear. However both in the case of the AVV diagram and in the conformal case, as evident from the expression of F_1, it reappears as an extra contribution and is responsible for the trace anomaly. It is rather easy to show the pole dominance of the anomaly away from the conformal point (massive case) at high energy, since the non anomalous terms present in F_1 and F_2 are subleading at large s. We are entitled to separate the pole contribution, which describes the non-local contribution to the trace anomaly, from the rest, and rewrite the F_1 form factor and effective action, respectively, as

$$F_1 = F_{1\,pole} + \tilde{F}_1 \tag{2.92}$$

and

$$\mathcal{S} = \mathcal{S}_{pole} + \tilde{\mathcal{S}}. \tag{2.93}$$

The reminder ($\tilde{\mathcal{S}}$) includes all the remaining contributions coming from the several form factors of the expansion, while the pole part gives

$$\mathcal{S}_{pole} = -\frac{e^2}{36\pi^2} \int d^4x d^4y \left(\Box h(x) - \partial_\mu \partial_\nu h^{\mu\nu}(x)\right) \Box^{-1}_{x\,y} F_{\alpha\beta}(x) F^{\alpha\beta}(y). \tag{2.94}$$

As we have just mentioned, it is not difficult to show that the anomaly pole in F_1, in the general kinematical case (e.g. for off-shell photons and a massive fermion in the loop) decouples in the infrared (i.e. its residue vanishes) while it remains coupled in the massless on-shell limit. In other configurations (for any of the two photons off-shell) is also decoupled. This behaviour is in perfect analogy with the chiral case [40].

2.5.2 Massive and massless contributions to anomalous Ward identities and the trace anomaly

Anomalous effects are associated with massless fermions, and for this reason, when we analyze the contribution to the anomaly for a massive correlator, we need to justify the distinction between massless and massive contributions. The latter contribute to the anomalous Ward identity, in our approach, via terms of $O(m^2/s^2)$, where $s = k^2$ is the virtuality of the graviton vertex. At nonzero momentum transfer ($k \neq 0$) the second term on the right-hand side of Eq. (2.85) is interpreted as an anomalous contribution, proportional to an asymptotic β function (β_{as}) of the theory, coming from the residue of the anomaly pole which appears in the form factor F_1. While the appearance of the asymptotic β function of the theory (which coincides with the β function of the $\overline{MS}$ scheme) is expected at large s, where all the remaining scales of the theory (s_1, s_2, m) can be dropped, corrections to the asymptotic description in the ultraviolet (UV) are expected. At the same time, in the far infrared (IR) region, below the fermion mass, the anomalous contribution should approach zero in a certain fashion, which will be specified below.

A complete quantitative understanding of this point for a general kinematics (e.g. for $s \neq 0$) remains, in a way, an open issue, but much more can be said for the simpler case of zero momentum transfer, where a consistent pattern of separation between massless and massive contributions to the correlator emerge in the UV region. In this case the virtuality of the two photons and the fermion mass m (plus a renormalization scale μ or M) are the scales which appear in the renormalized perturbative expansion. Related analysis have been presented in [88] and [51] and our conclusions do not differ from these previous investigations. We summarize the main points.

Respect to the case of the chiral anomaly, the trace anomaly is connected with the regularization procedure involved in the computation of the diagrams. In our analysis we have used dimensional regularization (DR) and we have imposed conservation of the vector currents, the symmetry requirements on the correlator and the conservation of the energy momentum tensor. As we move from 4 to d spacetime dimensions (before that we renormalize the theory), the anomaly pole term appears quite naturally in the expression of the correlator. This is not surprising, since QED in $d \neq 4$ dimensions is not even classically conformal invariant and the trace of the energy momentum tensor in the classical theory involves both a F^2 term ($\sim (d-4)F^2$) beside, for a massive correlator, a $\bar{\psi}\psi$ contribution. Let's summarize the basic features concerning the renormalization property of the correlator as they emerge from our direct computation.

1) The anomalous Ward identity obtained by tracing the correlator ($\Gamma^{\mu\nu\alpha\beta}$) with $g_{\mu\nu}$ involves only the F_1 and F_2 form factors in the massive case; in the massless case the scale breaking appears uniquely due to F_1 via the term $e^3/(12\pi^2)u^{\alpha\beta}(p,q)$, as pointed out before. The finiteness of the two form factors involved in the trace of the correlator is indeed evident. 2) The residue of the pole term ($e^3/(12\pi^2)$) in F_1 is affected by the renormalization of the entire correlator (the form factor F_{13} is the only one requiring renormalization) only by the re-definition of the bare coupling (e^2) in terms of the renormalized coupling (e_R^2) through the renormalization factor Z_3.

At this point, the interpretation of the residue at the pole as a contribution proportional to the β function of the theory is, in a way, ambiguous [89], since the β function is related to a given renormalization scheme. We stress once more that Eq. (2.85) does not involve a renormalization scheme - which at this point has not yet been defined - but just a regularization. We have regulated the infinities of the theory but we have not specified a subtraction of the infinities. For this reason, the substitution

$$(e^3/(12\pi^2)) \to 2\beta_{as}(e)/e \tag{2.95}$$

which attributes the mass-independent term in F_1 to a specific β function, the asymptotic one (β_{as}), as we are going to elaborate below, requires some clarification.

To fully appreciate this point, it is convenient to go back to the unrenormalized Ward identity (2.42) and differentiate it with respect to the momentum q and then set $p = -q$ ($k = 0$) by going to zero momentum transfer. One obtains the derivative Ward identity

$$g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}(p,-p) = 2p^2 \frac{d\Pi}{dp^2}(p^2)(p^2 g^{\alpha\beta} - p^\alpha p^\beta). \tag{2.96}$$

The appearance of the derivative of the scalar self-energy of the photon on the right-had side of the previous equation is particularly illuminating, since it allows to relate this expression to a particular β function of the theory, which is not the asymptotic β_{as} considered in Eq. (2.95). This β function is useful for describing the IR running of the coupling.

To illustrate this point we start from the expression of the scalar amplitude appearing in the photon self-energy in DR

$$\Pi(p^2,m) = \frac{e^2}{2\pi^2}\left(\frac{1}{6\epsilon} - \frac{\gamma}{6} - \int_0^1 dx x(1-x)\log\frac{m^2 - p^2 x(1-x)}{4\pi\mu^2}\right) \tag{2.97}$$

whose renormalization at zero momentum gives

$$\Pi_R(p^2,m) = \Pi(p^2,m) - \Pi(0,m) = -\frac{e^2}{2\pi^2}\int_0^1 x(1-x)\log\frac{m^2 - p^2 x(1-x)}{m^2}. \tag{2.98}$$

Using this expression, we can easily compute

$$2p^2\frac{d\Pi}{dp^2} = 2p^2\frac{d\Pi_R}{dp^2} = -\frac{e^2}{6\pi^2} + \frac{e^2\, m^2}{\pi^2}\int_0^1 dx\,\frac{x(1-x)}{m^2 - p^2 x(1-x)}. \tag{2.99}$$

Notice that this result does not depend on the renormalization scheme due to the presence of the derivative respect to p^2. Notice also that the β function of the theory evaluated in the zero momentum subtraction scheme is exactly given by the right-hand side of the previous expression

$$2p^2\frac{d\Pi_R}{dp^2} = -\frac{\beta(e^2,p^2)}{e^2}, \tag{2.100}$$

(where $\beta(e^2, p^2) = 2e\beta(e, p^2)$), but this result does not hold, generically, in any scheme. The identification of anomalous (massless) effects in the theory, as exemplified by these simpler Ward identity, should then be obtained by extracting the appropriate β function of the theory, whose running should be driven by the effective massless degrees of freedoms (fermions, in our case) at the relevant observation scale (p^2).

Clearly, in the case of Eq. (2.100) all the mass contributions have been absorbed into the very definition of the β function. Notice that if $p^2 \ll m^2$ this β function, after a rearrangement gives

$$-\frac{\beta(e^2, p^2)}{e^2} = \frac{e^2}{\pi^2} \int_0^1 dx \frac{p^2 x^2 (1-x)^2}{m^2 - p^2 x(1-x)} \tag{2.101}$$

and therefore it vanishes as $\beta \sim O(p^2/m^2)$ for $p^2 \to 0$. Equivalently, by taking the $m \to \infty$ limit we recover the expected decoupling of the fermion (due to a vanishing β) since we are probing the correlator at a scale (p^2) which is not sufficient to resolve the contribution of the fermion loop. On the contrary, as $p^2 \to \infty$, with m fixed, the running of the β function is the usual asymptotic one $\beta_{as}(e^2) \sim e^4/(6\pi^2)$ modified by corrections $O(m^2/p^2)$. The UV limit is characterized by the same running typical of the massless case, as expected.

Notice that the right-hand side of Eq. (2.96), as we have already remarked, does not depend on the renormalization scheme, while the β function does and Eq. (2.100) should be understood as a definition. For this reason, $\beta(e^2, p^2)$ correctly describes the IR running of the coupling as $p^2 \ll m^2$, and in this case it is obvious that massless anomalous effects of scale breaking are not present in this specific limit.

In the case of regularization scheme different from zero momentum subtraction, there are some differences which should be taken into consideration. For instance, in a mass-dependent scheme one subtracts the value of the graph at a Euclidean momentum point $p^2 = -M^2$, redefining the scalar self-energy as

$$\Pi^R(p^2, m, M) = \Pi(p^2, m) - \Pi(p^2 = -M^2, m) = \frac{e^3}{2\pi^2}\left[\int_0^1 dx\ x(1-x)\ \log \frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)}\right] \tag{2.102}$$

which gives, respect to the previous ($M = 0$) scheme, a β function now of the form

$$\begin{aligned}\beta(e) &= -\frac{e}{2} M \frac{d}{dM} \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \log \frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)} \\ &= \frac{e^3}{2\pi^2} \int_0^1 dx x(1-x) \frac{M^2 x(1-x)}{m^2 + M^2 x(1-x)}.\end{aligned} \tag{2.103}$$

For large values of M, this β function describes the usual UV running since

$$\beta(e) \sim \frac{e^3}{2\pi^2} \int_0^1 dx x(1-x) = \beta(e)_{as} = \frac{e^3}{12\pi^2}. \tag{2.104}$$

In this second scheme, the (regularization independent) right-hand side of Eq. (2.96) can be interpreted as due to an anomalous contribution coming from the pole plus some explicit mass corrections, as obvious from the first and second term of (2.99). We conclude with some considerations on a third (mass-independent) scheme.

In the $\overline{MS}$ scheme, the renormalization of the photon self-energy is performed via the subtraction

$$\Pi_R(p^2, m, \mu) = \Pi(p^2, m, \mu) - \frac{e^2}{12\pi^2}\left(\frac{1}{\epsilon} + \gamma - \log 4\pi\right) \tag{2.105}$$

which gives directly an asymptotic β function since

$$\begin{aligned}\beta(e) &= \frac{e}{2}\mu\frac{d}{d\mu}\Pi_R(p^2, m, \mu) \\ &= \frac{e^3}{2\pi^2}\int_0^1 dx x(1-x) = \frac{e^3}{12\pi^2}.\end{aligned} \tag{2.106}$$

It is clear, from these considerations, that a judicious definition of the β function allows a correct interpretation of the right-hand side of (2.96) and (2.99). In the $\overline{MS}$ scheme, the breaking of scale invariance can be attributed to a UV running of the coupling (for $p^2 \gg m^2$) plus mass corrections which are suppressed as $O(m^2/p^2)$. Notice that in this case the renormalization scale (μ^2) should be $O(p^2)$, since we should not allow large logarithms to be present in the perturbative expansion. In this sense, the extrapolation of the $\overline{MS}$ result to $p^2 \sim \mu^2 \ll m^2$ should be forbidden by the same criterion, since large logs of the relevant scales ($\log(m/\mu)$) would otherwise be generated. In the far infrared region $p^2 \ll m^2$ the use of the same β function is indeed not appropriate, since the same scheme does not correctly describe the decoupling of the anomaly, which instead should occur, since there is no massless fermion in the theory.

To conclude this discussion we just mention that the $\overline{MS}$ scheme can be used, obviously, both to describe the far IR and the far UV regions of the theory, with the condition that we are bound to choose a vanishing β function at $p^2 \ll m^2$ and an asymptotic one for $p^2 \gg m^2$ and assuring continuity of the gauge coupling across the fermion mass scale though the β-function is discontinuous. This is the standard procedure followed in the $\overline{MS}$ scheme as, for instance, in QCD factorization, improved with the inclusion of threshold effects at the crossing scales (see for instance [90, 91]) where the number of massless flavours change.

2.5.3 The off-shell massless $\langle TJJ \rangle$ correlator

Clearly, as we perform the massless limit on the amplitude, the residue of the same anomaly pole - identified above in the contribution $F_{1\,pole}$ - is still present, but will now be decoupled in the infrared.

In the massless case the scalar functions F_i depend only on the kinematic invariants s, s_1, s_2 but we still retain the last entry of these functions and set it equal to 0 for clarity, using the notation $F_i \equiv F_i(s; s_1, s_2, 0)$. These new functions are computed starting from the massive ones and letting $m \to 0$ and $\mathcal{A}_0(m^2) \to 0$, i.e. eliminating all the massless tadpoles generated in the zero fermion mass limit.

The off-shell massless invariant amplitudes $F_i(s; s_1, s_2, 0)$ are here given in terms of a new set of master integrals listed in Appendix A.2. We give here only the simplest invariant amplitudes, leaving the remaining ones to the appendix A.6. The anomaly pole is clearly present in F_1, which is given by

$$\underline{\underline{\mathbf{F_1(s;\, s_1,\, s_2,\, 0)}}} \quad = \quad -\frac{e^2}{18\pi^2 s}, \tag{2.107}$$

while

$$\underline{\underline{\mathbf{F_2(s;\, s_1,\, s_2,\, 0)}}} \quad = \quad 0. \tag{2.108}$$

The complete $\langle TJJ \rangle$ correlator is very complicated in this case as the long expressions of the form factors show, but a deeper analysis of its poles by computing the residue in $s = 0$ can be useful to draw some conclusions. The single pieces of $\Gamma^{\mu\nu\alpha\beta}(s;\, s_1,\, s_2,\, 0)$ indeed contribute as

$$\begin{aligned}
\lim_{s\to 0}\, sF_1(s;\, s_1,\, s_2,\, 0)\, t_1^{\mu\nu\alpha\beta} &= -\frac{e^2}{18\,\pi^2}\, t_1^{\mu\nu\alpha\beta}\big|_{s=0}, & (2.109)\\
\lim_{s\to 0}\, sF_3(s;\, s_1,\, s_2,\, 0)\, t_3^{\mu\nu\alpha\beta} &= \frac{e^2}{72\,\pi^2}\, t_3^{\mu\nu\alpha\beta}\big|_{s=0}, & (2.110)\\
\lim_{s\to 0}\, sF_5(s;\, s_1,\, s_2,\, 0)\, t_5^{\mu\nu\alpha\beta} &= \frac{e^2}{72\,\pi^2}\, t_5^{\mu\nu\alpha\beta}\big|_{s=0}, & (2.111)\\
\lim_{s\to 0}\, sF_7(s;\, s_1,\, s_2,\, 0)\, t_7^{\mu\nu\alpha\beta} &= \frac{e^2}{36\,\pi^2}\, t_7^{\mu\nu\alpha\beta}\big|_{s=0}, & (2.112)
\end{aligned}$$

while F_2 is absent in the massless case. The residues of the $F_i(s;\, s_1,\, s_2,\, 0)$ not included in the equation above are all vanishing. Combining the results given above one can easily check that the entire correlator is completely free from anomaly poles as

$$\lim_{s\to 0}\, s\,\Gamma^{\mu\nu\alpha\beta}(s;\, s_1,\, s_2,\, 0) = 0 \tag{2.113}$$

in this rather general configuration. A similar result holds for the correlator responsible for the chiral anomaly and shows the decoupling of polar contributions in the infrared.

2.5.4 The on-shell massive $\langle TJJ \rangle$ correlator

A particular case of the $\langle TJJ \rangle$ correlator is represented by its on-shell version with a massive fermion in the loop. If we contract $u^{\alpha\beta}(p, q)$ and $w^{\alpha\beta}(p, q)$ with the polarization tensors $\epsilon_\alpha(p)$ and

$\epsilon_\beta(q)$ requiring $\epsilon_\alpha(p)\,p^\alpha = 0$, $\epsilon_\beta(p)\,p^\beta = 0$, the first tensor remains unchanged while $w^{\alpha\beta}(p,q)$ becomes $\widetilde{w}^{\alpha\beta}(p,q) = s_1\,s_2\,g^{\alpha\beta}$. This will be carefully taken into account when computing the $s_1 \to 0$, $s_2 \to 0$ limit of the product of the invariant amplitudes F_i with their corresponding tensors $t_i^{\mu\nu\alpha\beta}$ $(i = 1,\ldots,13)$.
The invariant amplitudes reported below describe $F_i(s;0,0,m^2)$ whose tensors $t_i^{\mu\nu\alpha\beta}$ are also finite and non-vanishing. They are

$$
\begin{aligned}
\underline{\underline{\mathbf{F_1(s;\,0,\,0,\,m^2)}}} &= -\frac{e^2}{18\pi^2 s} + \frac{e^2 m^2}{3\pi^2 s^2} - \frac{e^2\,m^2}{3\pi^2 s}\mathcal{C}_0(s,0,0,m^2)\left[\frac{1}{2} - \frac{2m^2}{s}\right],\\
\underline{\underline{\mathbf{F_3(s;\,0,\,0,\,m^2)}}} &= -\frac{e^2}{144\pi^2 s} - \frac{e^2 m^2}{12\pi^2 s^2} - \frac{e^2\,m^2}{4\pi^2 s^2}\mathcal{D}(s,0,0,m^2)\\
&\quad - \frac{e^2\,m^2}{6\pi^2 s}\mathcal{C}_0(s,0,0,m^2)\,\left[\frac{1}{2} + \frac{m^2}{s}\right],\\
\underline{\underline{\mathbf{F_5(s;\,0,\,0,\,m^2)}}} &= \underline{\underline{\mathbf{F_3(s;\,0,\,0,\,m^2)}}},\\
\underline{\underline{\mathbf{F_7(s;\,0,\,0,\,m^2)}}} &= -4\,\underline{\underline{\mathbf{F_3(s;\,0,\,0,\,m^2)}}}\\
\underline{\underline{\mathbf{F_{13\,R}(s;\,0,\,0,\,m^2)}}} &= \frac{11e^2}{144\pi^2} + \frac{e^2 m^2}{4\pi^2 s} + e^2\mathcal{C}_0(s,0,0,m^2)\,\left[\frac{m^4}{2\pi^2 s} + \frac{m^2}{4\pi^2}\right]\\
&\quad + e^2\mathcal{D}(s,0,0,m^2)\,\left[\frac{5m^2}{12\pi^2 s} + \frac{1}{12}\right],
\end{aligned}
\tag{2.114}
$$

where the on-shell scalar integrals $\mathcal{D}(s,0,0,m^2)$ and $\mathcal{C}_0(s,0,0,m^2)$ are computed in Appendix A.2; here $F_{13\,R}$ denotes the renormalized amplitude, obtained by first removing the UV pole present in the photon self-energy by the usual renormalization of the photon wavefunction and then taking the on-shell limit. The remaining invariant amplitudes $F_i(s,0,0,m^2)$ are zero or multiply vanishing tensors in this kinematical configuration so they do not contribute to the correlator.
The limit from the massive on-shell form factors to the massless ones is clearer by looking at the series expansion of the scalar integrals around $m = 0$

$$
C_0(s,0,0,m^2) = \frac{1}{2\,s}\,\left[\log\left(-\frac{s}{m^2}\right)\right]^2 - \frac{2\,m^2}{s^2}\log\left(-\frac{s}{m^2}\right) + O(m^3) \tag{2.115}
$$

and from this we obtain for F_1'

$$
F_1'(s,0,0,m^2) = \frac{e^2\,m^2}{3\pi^2\,s^2}\left\{1 - \frac{1}{4}\left[\log\left(-\frac{s}{m^2}\right)\right]^2\right\}, \tag{2.116}
$$

where the notation F_1' denotes the first form factor after the subtraction of the pole in $1/s$.

Using the results given above, the full massive on-shell amplitude is given by

$$
\begin{aligned}
\Gamma^{\mu\nu\alpha\beta}(s;0,0,m^2) = \quad & F_1\,(s;0,0,m^2)\,\widetilde{t}_1^{\mu\nu\alpha\beta} + F_3\,(s;0,0,m^2)\,(\widetilde{t}_3^{\mu\nu\alpha\beta} + \widetilde{t}_5^{\mu\nu\alpha\beta} - 4\,\widetilde{t}_7^{\mu\nu\alpha\beta})\\
& + F_{13,\,R}\,(s;0,0,m^2)\,\widetilde{t}_{13}^{\mu\nu\alpha\beta},
\end{aligned}
\tag{2.117}
$$

so that the invariant amplitudes reduce from 13 to 3 and the three linear combinations of the tensors can be taken as a new basis

$$\begin{aligned}
\widetilde{t}_1^{\mu\nu\alpha\beta} &= \lim_{s_1,s_2\to 0} t_1^{\mu\nu\alpha\beta} = (s\, g^{\mu\nu} - k^\mu k^\nu)\, u^{\alpha\beta}(p,q) && (2.118)\\
\widetilde{t}_3^{\mu\nu\alpha\beta} + \widetilde{t}_5^{\mu\nu\alpha\beta} - 4\,\widetilde{t}_7^{\mu\nu\alpha\beta} &= \lim_{s_1,s_2\to 0} (t_3^{\mu\nu\alpha\beta} + t_5^{\mu\nu\alpha\beta} - 4\, t_7^{\mu\nu\alpha\beta}) = \\
& -2\, u^{\alpha\beta}(p,q)\,(s\, g^{\mu\nu} + 2(p^\mu\, p^\nu + q^\mu\, q^\nu) - 4\,(p^\mu\, q^\nu + q^\mu\, p^\nu)) && (2.119)\\
\widetilde{t}_{13}^{\mu\nu\alpha\beta} &= \lim_{s_1,s_2\to 0} t_{13}^{\mu\nu\alpha\beta} = \big(p^\mu q^\nu + p^\nu q^\mu\big) g^{\alpha\beta} + \frac{s}{2}\big(g^{\alpha\nu} g^{\beta\mu} + g^{\alpha\mu} g^{\beta\nu}\big)\\
& -g^{\mu\nu}\big(\frac{s}{2} g^{\alpha\beta} - q^\alpha p^\beta\big) - \big(g^{\beta\nu} p^\mu + g^{\beta\mu} p^\nu\big) q^\alpha - \big(g^{\alpha\nu} q^\mu + g^{\alpha\mu} q^\nu\big) p^\beta, && (2.120)
\end{aligned}$$

as previously done in the literature [46]. If we extract the residue of the full amplitude we realize that even though some functions $F_i(s,0,0,m^2)$ have kinematical singularities in $1/s$ this polar structure is no longer present in the complete massive correlator

$$\lim_{s\to 0} s\, \Gamma^{\mu\nu\alpha\beta} = 0 \qquad (2.121)$$

showing that in the massive case the $\langle TJJ \rangle$ correlator exhibits no poles. In a following section we will comment on the interpretation of these massless poles exploiting the analogy with a similar situation encountered in the case of the gauge anomaly.

2.6 The general effective action and its various limits

In this section we present results for the correlator in various kinematical limits. We start from its expression in the on-shell massive case and then perform its expansion in $1/m$ which will be used in a next section to extract the corresponding effective action. As a final step we show the on-shell structure of the invariant amplitudes in the conformal limit.

It is possible to identify from them the structure of the effective action in its most general form. If we denote by $\mathcal{S}_i$ the contribution to the effective action due to each form factor F_i, then we can write it in the form

$$\mathcal{S}_i = \int d^4x\, d^4y\, d^4z\, \hat{t}_i^{\mu\nu\alpha\beta}(z,x,y) h_{\mu\nu}(z) A_\alpha(x) A_\beta(y) \int \frac{d^4p\, d^4q}{(2\pi)^8} e^{-ip\cdot(x-z)-iq\cdot(y-z)} F_i(k,p,q) \qquad (2.122)$$

where $k \equiv p+q$. We have introduced the operatorial version of the tensor structures $t_i^{\mu\nu\alpha\beta}$, denoted by $\hat{t}_i$ that will be characterized below. Defining

$$\hat{p}_x^\alpha \equiv i\frac{\partial}{\partial x_\alpha}, \qquad \hat{q}_y^\alpha \equiv i\frac{\partial}{\partial y_\alpha}, \qquad \hat{k}_z^\alpha \equiv -i\frac{\partial}{\partial z_\alpha} \qquad (2.123)$$

and using the identity

$$\hat{F}_i(\hat{k}_z,\hat{p}_x,\hat{q}_y)\delta^4(x-z)\delta^4(y-z) = \int \frac{d^4p\, d^4q}{(2\pi)^8} e^{-ip\cdot(x-z)-iq\cdot(y-z)} F_i(k,p,q) \tag{2.124}$$

where formally $\hat{F}_i$ is the operatorial version of F_i, we can arrange the anomalous effective action also in the form

$$\mathcal{S}_i = \int d^4x d^4y d^4z \hat{F}_i(\hat{k}_z,\hat{p}_x,\hat{q}_y)\left[\delta^4(x-z)\delta^4(y-z)\right] \hat{t}_i^{\mu\nu\alpha\beta}(z,x,y) h_{\mu\nu} A_\alpha(x) A_\beta(y). \tag{2.125}$$

For instance we get

$$\hat{t}_1^{\mu\nu\alpha\beta}(z,x,y)h_{\mu\nu}A_\alpha(x)A_\beta(y) = \frac{1}{2}\left(\Box_z h(z) - \partial^z_\mu\partial^z_\nu h^{\mu\nu}(z)\right) F_{\alpha\beta}(x)F^{\alpha\beta}(y), \tag{2.126}$$

$$\hat{t}_2^{\mu\nu\alpha\beta}(z,x,y)h_{\mu\nu}A_\alpha(x)A_\beta(y) = \left(\Box_z h(z) - \partial^z_\mu\partial^z_\nu h^{\mu\nu}(z)\right) \partial_\mu F^\mu_\lambda(x)\partial_\nu F^{\nu\lambda}(y), \tag{2.127}$$

$$\hat{t}_3^{\mu\nu\alpha\beta}(z,x,y)h_{\mu\nu}A_\alpha(x)A_\beta(y) = \frac{1}{2}h^{\mu\nu}(z)\left(\Box_x g_{\mu\nu} - 4\partial^x_\mu\partial^x_\nu\right) F_{\alpha\beta}(x)F^{\alpha\beta}(y), \tag{2.128}$$

$$\hat{t}_4^{\mu\nu\alpha\beta}(z,x,y)h_{\mu\nu}A_\alpha(x)A_\beta(y) = h^{\mu\nu}(z)\left(\Box_x g_{\mu\nu} - 4\partial^x_\mu\partial^x_\nu\right) \partial_\mu F^\mu_\lambda(x)\partial_\nu F^{\nu\lambda}(y), \tag{2.129}$$

$$\hat{t}_5^{\mu\nu\alpha\beta}(z,x,y)h_{\mu\nu}A_\alpha(x)A_\beta(y) = \frac{1}{2}h^{\mu\nu}(z)\left(\Box_y g_{\mu\nu} - 4\partial^y_\mu\partial^y_\nu\right) F_{\alpha\beta}(x)F^{\alpha\beta}(y), \tag{2.130}$$

$$\hat{t}_6^{\mu\nu\alpha\beta}(z,x,y)h_{\mu\nu}A_\alpha(x)A_\beta(y) = h^{\mu\nu}(z)\left(\Box_y g_{\mu\nu} - 4\partial^y_\mu\partial^y_\nu\right) \partial_\mu F^\mu_\lambda(x)\partial_\nu F^{\nu\lambda}(y), \tag{2.131}$$

$$\hat{t}_7^{\mu\nu\alpha\beta}(z,x,y)h_{\mu\nu}A_\alpha(x)A_\beta(y) = \frac{1}{2}h^{\mu\nu}(z)\left(\partial^{x\,\lambda}\partial^y_\lambda g_{\mu\nu} - 2(\partial^y_\mu\partial^x_\nu + \partial^y_\nu\partial^x_\mu)\right) F_{\alpha\beta}(x)F^{\alpha\beta}(y), \tag{2.132}$$

$$\hat{t}_8^{\mu\nu\alpha\beta}(z,x,y)h_{\mu\nu}A_\alpha(x)A_\beta(y) = h^{\mu\nu}(z)\left(\partial^{x\,\lambda}\partial^y_\lambda g_{\mu\nu} - 2(\partial^y_\mu\partial^x_\nu + \partial^y_\nu\partial^x_\mu)\right) \partial_\mu F^\mu_\lambda(x)\partial_\nu F^{\nu\lambda}(y) \tag{2.133}$$

and similar expressions for the remaining tensor structures. However, the most useful forms of the effective action involve an expansion in the fermions mass, as in the $1/m$ formulation (the Euler-Heisenberg form) or for small m. In this second case the non-local contributions obtained from the anomaly poles appear separated from the massive terms, showing the full-fledged implications of the anomaly. This second formulation allows a smooth massless limit, where the breaking of the conformal anomaly is entirely due to the massless fermion loops.

In the $1/m$ case, for on-shell gauge bosons, the result turns out to be particularly simple. We obtain

$$F_1(s,0,0,m^2) = \frac{7e^2}{2160\pi^2}\frac{1}{m^2} + \frac{e^2 s}{3024\,\pi^2}\frac{1}{m^4} + O\left(\frac{1}{m^6}\right), \tag{2.134}$$

$$F_3(s,0,0,m^2) = F_5(s,0,0,m^2) = \frac{e^2}{4320\,\pi^2}\frac{1}{m^2} + \frac{e^2 s}{60480\,\pi^2}\frac{1}{m^4} + O\left(\frac{1}{m^6}\right), \tag{2.135}$$

$$F_7(s,0,0,m^2) = -4\,F_3(s,0,0,m^2) \tag{2.136}$$

$$F_{13,R}(s,0,0,m^2) = \frac{11e^2 s}{1440\,\pi^2}\frac{1}{m^2} + \frac{11e^2 s^2}{20160\,\pi^2}\frac{1}{m^4} + O\left(\frac{1}{m^6}\right), \tag{2.137}$$

which can be rearranged in terms of three independent tensor structures. Going to configuration space, the linearized expression of the contribution to the gravitational effective action due to the TJJ vertex, in this case, can be easily obtained in the form

$$\begin{aligned} S_{TJJ} &= \int d^4x d^4y d^4z \, \Gamma^{\mu\nu\alpha\beta}(x,y,z) A_\alpha(x) \, A_\beta(y) \, h_{\mu\nu}(z) \\ &= \frac{7\, e^2}{4320\, \pi^2\, m^2} \int d^4x \, (\Box h - \partial^\mu \partial^\nu h_{\mu\nu}) \, F^2 \\ &- \frac{e^2}{4320\, \pi^2\, m^2} \int d^4x \left(\Box h F^2 - 8 \partial^\mu F^{\alpha\beta} \partial^\nu F_{\alpha\beta} h_{\mu\nu} + 4 (\partial^\mu \partial^\nu F_{\alpha\beta}) F^{\alpha\beta} h_{\mu\nu} \right) \\ &+ \frac{11\, e^2}{1440\, \pi^2\, m^2} \int d^4x T^{\mu\nu}_{ph} \Box h_{\mu\nu}. \end{aligned} \tag{2.138}$$

which shows three independent contributions linear in the (weak) gravitational field.

2.7 The massless (on-shell) $\langle TJJ \rangle$ correlator

The non-local structure of the effective action, as we have pointed out in the previous sections, is not apparent within an expansion in $1/m$, nor this expansion has a smooth match with the massless case.

The computation of the correlator $\Gamma^{\mu\nu\alpha\beta}(s; 0, 0, 0)$ hides some subtleties in the massless fermion limit (with on-shell external photons), as the form factors F_i and the tensorial structures t_i both contain the kinematical invariants s_1, s_2. For this reason the limit of both factors (form factor and corresponding tensor structure) $F_i \, t_i^{\mu\nu\alpha\beta}$ has to be taken carefully, starting from the expression of the massless $F_i(s; s_1, s_2, 0)$ listed in Appendix A.6 and from the tensors $t_i^{\mu\nu\alpha\beta}$ contracted with the physical polarization tensors. In this case only few form factors survive and in particular

$$F_1(s,0,0,0) = -\frac{e^2}{18\pi^2 s}, \tag{2.139}$$

$$F_3(s,0,0,0) = F_5(s,0,0,0) = -\frac{e^2}{144\pi^2\, s}, \tag{2.140}$$

$$F_7(s,0,0,0) = -4\, F_3(s,0,0,0), \tag{2.141}$$

$$F_{13,R}(s,0,0,0) = -\frac{e^2}{144\pi^2} \left[12 \log\left(-\frac{s}{\mu^2} \right) - 35 \right], \tag{2.142}$$

and hence the whole correlator with two onshell photons on the external lines is

$$\begin{aligned} \Gamma^{\mu\nu\alpha\beta}(s;0,0,0) &= F_1(s,0,0,0) \, \tilde{t}_1^{\mu\nu\alpha\beta} + F_3(s,0,0,0) \left(\tilde{t}_3^{\mu\nu\alpha\beta} + \tilde{t}_5^{\mu\nu\alpha\beta} - 4\, \tilde{t}_7^{\mu\nu\alpha\beta} \right) + F_{13,R} \, \tilde{t}_{13}^{\mu\nu\alpha\beta} \\ &= -\frac{e^2}{48\pi^2\, s} \left[\left(2\, p^\beta \, q^\alpha - s\, g^{\alpha\beta} \right) \left(2\, p^\mu \, p^\nu + 2\, q^\mu \, q^\nu - s\, g^{\mu\nu} \right) \right] + F_{13,R} \, \tilde{t}_{13}^{\mu\nu\alpha\beta}, \end{aligned} \tag{2.143}$$

where $\widetilde{t}_i^{\mu\nu\alpha\beta}$ are the tensors defined in Eqs. (2.118-2.120).

The study of the singularities in $1/s$ for this correlator requires a different analysis for F_1 and the remaining form factors, as explicitly shown in eq. 2.143, where F_1 has been kept aside from the others, even if it is proportional to F_3. Indeed F_1 is the only form factor multiplying a non zero trace tensor, $\widetilde{t}_1^{\mu\nu\alpha\beta}$, and responsible for the trace anomaly. If we take the residue of the onshell correlator for physical polarizations of the photons in the final state we see how the 4 form factors and their tensors combine in such a way that the result is different from zero as

$$\lim_{s\to 0} s\,\Gamma^{\mu\nu\alpha\beta}(s;0,0,0) = -\frac{e^2}{12\pi^2}\,p^\beta\,q^\alpha(p^\mu\,p^\nu + q^\mu\,q^\nu), \tag{2.144}$$

where clearly each singular part in $1/s$ present in F_1, F_3, F_5, F_7 added up and the logarithmic behaviour in s of F_{13} has been regulated by the factor s in front when taking the limit. The result shows that the pole, in this case, is coupled in the IR, as shown by the dispersive analysis.

2.8 Conclusions

We have presented in this chapter a computation of the TJJ correlator, responsible for the appearance of gauge contributions to the conformal anomaly in the effective action of gravity. We have used our results to present the general form of the gauge contributions to this action, in the limit of a weak gravitational field. One interesting feature of this correlator is the presence of an anomaly pole [51].

Usually anomaly poles are interpreted as affecting the infrared region of the correlator and appear only in one special kinematical configuration, which requires massless fermions in the loop and on-shell conditions for the external gauge lines. In general, however, the anomaly pole affects the UV region even if it is not coupled in the infrared. This surprising feature of the anomaly is present both in the case of the chiral anomaly [40] *and* in the conformal anomaly. Here we have extracted explicitly this behaviour by a general analysis of the correlator, extending our previous study of the chiral gauge anomaly.

As we noticed at the end of the previous chapter, anomaly poles are the most interesting feature, at perturbative level, of the anomaly, being it conformal or chiral, and are described by mixed diagrams involving either a scalar (gravitational case) [51] or a pseudoscalar (chiral case) [30, 40]. The connection between the infrared and the ultraviolet, signalled by the presence of these contributions, should not be too surprising in an anomalous context. The pole-like behaviour of an anomalous correlator is usually "captured" by a variational solution of a given anomaly equation, which implicitly assumes the presence of a pole term in the integrated functional [92]. By rediscovering the pole in perturbation theory, obviously, one can clearly conclude that variational solutions of the anomaly equations are indeed correct, although they miss homogeneous solutions to the Ward identity, that indeed must necessarily be identified by an off-shell perturbative analysis of the correlators. This is the approach followed here and in [40].

We have also seen that the identification of the massless anomaly pole allows to provide a "mixed" formulation of the effective action in which the pole is isolated from the remaining mass terms, extracted in the $\tilde{\mathcal{S}}_{pole}$ part of the anomalous action, which could be used for further studies. We have also emphasized that a typical $1/m$ expansion of the anomalous effective action fails to convey fully the presence of scaleless contributions.

Chapter 3

The Trace Anomaly and the Gravitational Coupling of an Anomalous $U(1)$

3.1 Introduction

In the previous chapter we have presented a complete computation of the off-shell graviton-photon-photon vertex for an abelian gauge theory, which is derived from the correlator of the energy-momentum tensor (T) with two vector currents (J) (the TJJ correlator) [51, 52]. Previous studies of this correlator, included those of [46, 48, 49, 85], were limited to the QED case, while, surprisingly, there has not been any previous attempt to discuss the structure of more general vertices, such the TJ_AJ_A or TJ_VJ_A correlators, carrying one insertion of the energy momentum tensor and of one or more chiral currents. They become the object of investigation of this third chapter.

These correlators appear indeed in the expression of the 1 particle irreducible (1PI) effective action which describes the interaction of gravity with the fields of a chiral theory, such as the Standard Model, and contribute, to leading order in the gauge coupling expansion, to the radiative breaking of scale invariance. In turn, this is the prominent perturbative feature of the trace anomaly, which appears to be generated by specific pole terms, as we are going to elaborate in the following of this chapter.

Correlators of this type can potentially carry mixed anomalies. Specifically, this can be a trace anomaly, due to the insertion of an energy momentum tensor, in combination with a chiral anomaly, due to the presence of axial-vector currents. This anomaly mixing, in principle, is expected to be present both in the case that we investigate - involving one or two axial-vector currents - and in higher point functions. In the latter case they may involve a larger number of axial-vector gauge currents, such as the $TJ_AJ_AJ_A$ vertex and many others, which are divergent by power-counting, as one can easily figure out, and contribute to higher perturbative orders.

As in the case of the AVV diagram (with Axial-vector/Vector/Vector currents), responsible for the chiral anomaly and discussed in the first chapter, also in the case under analysis one of the crucial points relies on the derivation of the correct Ward identities which allow to define this trilinear vertex consistently. This point requires some care, due to the formal manipulations involved in the handling of the functional integral and to the presence of mass corrections. In the massless case, instead, the computation of this correlator can be formally related to the vector case (the TJJ case) of [51, 52] by a naive manipulation of the chiral projectors in the loops. Our investigation addresses all these points in some detail, offering a general approach that can be applied to the realistic case of the Standard Model. In this respect, the study of the gravitational coupling of a chiral abelian theory (with one anomalous $U(1)$) contains all the issues that appear in of the fermion sector of the non-abelian case.

3.1.1 The anomalous effective action

As we have mentioned above, one of the key features of the trace anomaly is the appearance in the 1PI effective action of dynamical massless poles which mediate the anomalous interaction [51, 52]. The story of massless poles in anomaly-mediated interactions, obviously, is not new, and goes back to Dolgov and Zakharov [37], in their analysis of the chiral anomaly. The nonlocal "$1/\Box$" structure of the effective anomalous interaction, due to the pole term in the correlator, is, in fact, a distinctive feature of the diagrammatic expansion of these effective theories. These can be made local at the cost of introducing two pseudoscar (auxiliary) fields [30]. In the case of conformal anomalies, the identification of similar massless poles and their interpretation has been addressed recently in [51], and in [52], by direct computations. These singularities, as discussed throughout this thesis, affect both the infrared and the ultraviolet region of the anomaly diagrams, as we will illustrate in the next sections. These features, present in the QED and QCD cases, are naturally shared by an anomalous abelian theory when it gets coupled to gravity.

The possible physical implications of this behaviour of the effective action have been discussed in [93], and for this reason similar analysis in the complete Standard Model and for other correlators (such as the TTT vertex) are underway.

3.1.2 Aspects of the computation

Coming to other features of our computation, it should be remarked that a direct derivation from first principles of correlators with axial-vector/vector currents and energy momentum insertions, in general, runs into difficulties. This is due to the appearance of commutators of the energy momentum tensor with the chiral current, situation that we will try to avoid.

As in the vector-like case, we will provide explicit expressions of all the form factors appearing in the correlator, for a simple theory. We have selected an abelian model with two vector/axial-vector currents and a single massive fermion. One important point that we intend to stress is that the local (gauge) or global nature of the two currents, in the example that we provide, is not relevant for the conclusions and the goals of this analysis, being the two gauge fields to which the two currents couple just classical background fields. For this reason, our investigation is essentially the search of the correct conditions for defining anomalous correlators of the form TJ_VJ_A and TJ_AJ_A (with a single insertion of $T_{\mu\nu}$). The approach is the exact analogous of the one followed in the investigation of the AVV graph of the chiral anomaly, and in principle could be generalized to more complex correlators. Unfortunately, however, the explicit test of the Ward identities containing higher point functions becomes increasingly difficult in perturbation theory.

Another remark concerns the use of Dimensional Reduction (DRED) with a 4-dimensional γ_5 [94] in our analysis. Typically, in these types of studies, it is necessary at each step to check the consistency of the perturbative result against the constraints posed by the anomalous Ward identities. Our results, which are more complex than in a previous analysis of the TJJ vertex, indeed satisfy these conditions. It has also been checked that Dimensional Regularization (DR) and DRED give the same expression for the TJ_AJ_A vertex, while they differ in the case of the TJ_VJ_A vertex by infinite contributions. In this second case, as we are going to show, both the condition of charge conjugation invariance (C-invariance) and the Ward identity extracted from the functional integral imply that this specific vertex is required to vanish identically for any fermion mass.

3.2 The Lagrangian and the off-shell effective action

To establish notations, here we will briefly summarize our conventions. The diagrammatic contributions will be presented both in the usual V/A (vector/axial-vector) form, with Dirac spinors, and in the L/R (Left-Right) form, using chiral fermions. We will include mass effects in the fermion loops and we will keep all the external lines off their mass-shell in order to establish the most general form of the corresponding effective action.

We consider a theory with a Dirac fermion ψ and two abelian gauge bosons, namely V and A, described by the Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4} F_{V\,\mu\nu} F_V^{\mu\nu} - \frac{1}{4} F_{A\,\mu\nu} F_A^{\mu\nu} + \bar{\psi}\gamma^\mu (i\partial_\mu + g V_\mu + g\gamma^5 A_\mu)\psi - m\bar{\psi}\psi, \tag{3.1}$$

where the fermion couples to the two gauge bosons with, respectively, a vector and an axial-vector interaction. In our conventions, the axial-vector gauge boson is denoted by A, while the vector one is denoted by V. The axial current will be denoted $J_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$, and sometimes we will be using a suffix "5" to emphasize its axial-vector character. For instance Π_{55} will denote the axial-axial two-point function while $\Pi \equiv \Pi_{VV}$ will denote the corresponding two-point function of the vector case. In the derivation of the Ward identities which will be discussed below, the gauge fields will be considered as external background fields both in the V/A and in the L/R formulation. This theory couples to gravity in the weak gravitational field limit via the energy momentum tensor of (3.1).

In particular, the corresponding effective action will be formally defined as the sum of

1) the tree-level action given by (3.1)

$$\mathcal{S}_0 = \int d^4x \mathcal{L}_0 \tag{3.2}$$

and 2) the trilinear interactions TJ_AJ_V, TJ_VJ_V and TJ_AJ_A. These extra graphs appear as leading corrections to the effective action, which is defined as

$$\mathcal{S}_{anom} \equiv \langle \Gamma_{AA} hAA \rangle + \langle \Gamma_{VA} hVA \rangle + \langle \Gamma_{VV} hVA \rangle \tag{3.3}$$

with

$$\langle \Gamma_{hAA} \rangle \equiv \int d^4z\, d^4x\, d^4y\, \Gamma_{AA}^{\mu\nu\alpha\beta} h_{\mu\nu}(z)\, A_\alpha(x)\, A_\beta(y) \tag{3.4}$$

and similarly for all the other terms. The field $h_{\mu\nu}$ denotes the linearized fluctuations of the metric around a flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \qquad\qquad \kappa = \sqrt{16\pi G_N} \tag{3.5}$$

with G_N being the 4-dimensional Newton's constant.

One of the principal goals of our investigation is to provide a correct definition of $\mathcal{S}_{anom}$ by deriving the essential Ward identities of the anomalous correlators. At the same time we will show, as in a previous case study for QED, that the effective action is characterized by massless anomaly poles. The extraction of these singularities, in our case, is not based on dispersion theory as in [51] but the results are obviously equivalent to the dispersive treatment [52] in the massless case, with a generalization for massive fermions.

3.2.1 Symmetries and the energy momentum tensor

The Lagrangian in (3.1) remains invariant under the local vector gauge transformation $U(1)_V$

$$\begin{aligned} \psi &\rightarrow e^{ig\,\alpha(x)}\psi, && (3.6)\\ \bar{\psi} &\rightarrow \bar{\psi}e^{-ig\,\alpha(x)}, && (3.7)\\ V^\mu &\rightarrow V^\mu + \partial^\mu\alpha(x), && (3.8) \end{aligned}$$

which implies the conservation of the vector current $J^\mu_V \equiv J^\mu = \bar{\psi}\gamma^\mu\psi$. If the fermion mass is zero the Lagrangian is also invariant under a local axial-vector gauge transformation $U(1)_A$

$$\begin{aligned} \psi &\rightarrow e^{ig\,\beta(x)\gamma_5}\psi, && (3.9)\\ \bar{\psi} &\rightarrow \bar{\psi}e^{ig\,\beta(x)\gamma_5}, && (3.10)\\ A^\mu &\rightarrow A^\mu + \partial^\mu\beta(x), && (3.11) \end{aligned}$$

implying the conservation of the axial-vector current J_A. Obviously, this is explicitly broken by the contributions of massive fermions

$$\partial_\mu J^\mu_A = 2im\,\bar{\psi}\gamma_5\psi. \tag{3.12}$$

The energy-momentum tensor consists of four contributions: the free fermion part T_f, the fermion-boson interaction parts T_{i_V} and T_{i_A}, due to the interactions of the axial and vector gauge fields with the fermions, and the gauge term T_g which are given by

$$T^{\mu\nu}_f = -i\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi + g^{\mu\nu}(i\bar{\psi}\gamma^\lambda\overleftrightarrow{\partial}_\lambda\psi - m\bar{\psi}\psi), \tag{3.13}$$

$$T^{\mu\nu}_{i_V} = -\,gJ^{(\mu}V^{\nu)} + gg^{\mu\nu}J^\lambda V_\lambda\,, \tag{3.14}$$

$$T^{\mu\nu}_{i_A} = -\,gJ^{(\mu}_A A^{\nu)} + gg^{\mu\nu}J^\lambda_A A_\lambda\,, \tag{3.15}$$

and

$$T^{\mu\nu}_g = F^{\mu\lambda}_V F^\nu_{V\,\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\rho}_V F_{V\,\lambda\rho} + F^{\mu\lambda}_A F^\nu_{A\,\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\rho}_A F_{A\,\lambda\rho}. \tag{3.16}$$

The complete energy-momentum tensor is

$$T^{\mu\nu} = T^{\mu\nu}_f + T^{\mu\nu}_{i_V} + T^{\mu\nu}_{i_A} + T^{\mu\nu}_g, \tag{3.17}$$

which couples to gravity with a linearized term of the form $h_{\mu\nu}T^{\mu\nu}$. The Lagrangian (3.1) can be rewritten in the chiral basis decomposing the fields in terms of their left-handed and right-handed components by using the chirality projectors

$$P_L = \frac{1-\gamma_5}{2}, \qquad P_R = \frac{1+\gamma_5}{2}. \tag{3.18}$$

We define the chiral fermion fields as

$$\psi_L = P_L \psi, \qquad \psi_R = P_R \psi \tag{3.19}$$

and the left and right gauge fields, A_L and A_R, as

$$A_L^\mu = V^\mu - A^\mu, \tag{3.20}$$
$$A_R^\mu = V^\mu + A^\mu, \tag{3.21}$$

so that the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{4} F_{L\,\mu\nu}\, F_L^{\mu\nu} - \frac{1}{4} F_{R\,\mu\nu}\, F_R^{\mu\nu} + \bar{\psi}_L \gamma_\mu (i\,\partial^\mu + g A_L^\mu)\,\psi_L + \bar{\psi}_R \gamma_\mu (i\,\partial^\mu + g A_R^\mu)\,\psi_R \tag{3.22}$$

when the mass term has been set to vanish. The energy momentum is separated into the various chiral contributions

$$T_{f,L}^{\mu\nu} = -i\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\, P_L\psi + g^{\mu\nu} i\bar{\psi}\gamma^\lambda \overleftrightarrow{\partial}_\lambda\, P_L\psi, \tag{3.23}$$
$$T_{f,R}^{\mu\nu} = -i\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\, P_R\psi + g^{\mu\nu} i\bar{\psi}\gamma^\lambda \overleftrightarrow{\partial}_\lambda\, P_R\psi, \tag{3.24}$$
$$T_{i,L}^{\mu\nu} = -\,g\,(J_L^{(\mu} A_L^{\nu)} - g^{\mu\nu} J_L^\lambda A_{L\lambda})\,, \tag{3.25}$$
$$T_{i,R}^{\mu\nu} = -\,g\,(J_R^{(\mu} A_R^{\nu)} - g^{\mu\nu} J_R^\lambda A_{R\lambda})\,, \tag{3.26}$$

with

$$J_L^\mu(x) = \bar{\psi}(x)\gamma^\mu P_L \psi(x), \tag{3.27}$$
$$J_R^\mu(x) = \bar{\psi}(x)\gamma^\mu P_R \psi(x). \tag{3.28}$$

Notice that the Lagrangian in (3.22) is invariant under the chiral transformation $U(1)_L \times U(1)_R$.

3.2.2 Perturbative expansion of the axial-vector contributions

The analysis of the vector-like contributions, i.e. of the $\langle TJJ \rangle$ correlator, has been performed in great detail in [52]. For this reason we will consider, at this point, a vanishing vector contribution ($V \to 0$) in the defining Lagrangian (3.1) and we will focus our discussion at the moment on its axial part. A relation between the vector and axial contributions will be worked out in the later sections, where we will show that mixed vector-axial vector correlators vanish for any nonzero m. We will also show how to relate pure vector like to axial vector like contributions, as indicated below in Eq. 3.93.

To extract the one-loop contributions to the $\langle T J_A J_A \rangle$ correlator in the perturbative expansion and identify those due to the conformal anomaly, it is sufficient to consider only the partial energy-momentum tensor T_p given by the Dirac and the interaction term in Eqs. (3.13) and (3.15)

$$T_p^{\mu\nu} = T_f^{\mu\nu} + T_{iA}^{\mu\nu}, \tag{3.29}$$

while the gauge term in Eq. (3.16) is only responsible, to second order (g^2), of two non-amputated diagrams removed from the perturbative expansion of the effective action. We also recall that the conservation of the energy momentum tensor can be reformulated as a partial conservation equation

$$\partial_\nu T_p^{\mu\nu} = -\partial_\nu T_{Ag}^{\mu\nu}, \tag{3.30}$$

with

$$T_{Ag}^{\mu\nu} \equiv F_A^{\mu\lambda} F_{A\,\lambda}^{\nu} - \frac{1}{4} g^{\mu\nu} F_A^{\lambda\rho} F_{A\,\lambda\rho}. \tag{3.31}$$

Using diffeomorphism invariance one can derive formally a quantum relation similar to (3.30), which takes the form

$$\partial_\nu \langle T_p^{\mu\nu} \rangle_A = g\, F_A^{\mu\lambda} \, \langle J_{A\lambda} \rangle_A. \tag{3.32}$$

This relation is the analogue - for the axial case - of the relation identified in [51], which allows to extract the momentum conservation Ward identity in the case of the TJJ (for vector currents). In (3.32) the functional average of $T_p^{\mu\nu}$ is now defined as

$$\langle T_p^{\mu\nu}(z) \rangle_A \quad \equiv \quad \int D\psi D\bar{\psi} \; T_p^{\mu\nu}(z) \; e^{i\int d^4x\, \mathcal{L}_k(\psi) + ig\int d^4x\, J_A\cdot A(x)} \tag{3.33}$$

with

$$\mathcal{L}_k(\psi) \equiv \bar{\psi} i \gamma^\mu \partial_\mu \psi \tag{3.34}$$

being the kinetic fermion Lagrangian in flat spacetime, and we will denote by $\mathcal{S}_k(\psi)$ the corresponding action. Notice that equation (3.32) can be naively thought as the quantum counterpart of the non-homogeneous equation

$$\partial_\nu T_p^{\mu\nu} = g\, F_A^{\mu\lambda} \, J_{A\lambda} \tag{3.35}$$

satisfied by $T_p^{\mu\nu}$. Here the axial vector field A is taken as a background. A rigorous derivation of this relation requires the use of invariance under diffeomorphism of the generating functional of the full theory (expressed in terms of $g_{\mu\nu}$ and a A_μ) and an expansion around flat space, as can be checked.

The conservation equation (3.32) is relevant for the extraction of one of the Ward identities necessary to define the correlator. Notice that the expectation value of T_p in the background of the gauge field A is the generating functional of the correlation functions that we need. These are obtained by an expansion through second order in the external field A. The relevant terms in this expansion are explicitly given by

$$\langle T_p^{\mu\nu}(z)\rangle_A = \frac{(ig)^2}{2!}\,\langle T_f^{\mu\nu}(z)\,(J_A\cdot A)\,(J_A\cdot A)\rangle + ig\,\langle T_{iA}^{\mu\nu}(z)\,(J_A\cdot A)\rangle + \ldots\,, \tag{3.36}$$

with $(J_A\cdot A)\equiv\int d^4x\, J_A\cdot A(x)$.

The corresponding diagrams are extracted via two functional derivatives respect to the background field A and are given by

$$\Gamma_{AA}^{\mu\nu\alpha\beta}(z;x,y)\equiv\frac{\delta^2\,\langle T_p^{\mu\nu}(z)\rangle_A}{\delta A_\alpha(x)\delta A_\beta(y)}\bigg|_{A=0} = V_{55}^{\mu\nu\alpha\beta}(z;x,y)+W_{55}^{\mu\nu\alpha\beta}(z;x,y), \tag{3.37}$$

where

$$V_{55}^{\mu\nu\alpha\beta}(z;x,y) = (i\,g)^2\,\left\langle T_f^{\mu\nu}(z)J_A^\alpha(x)J_A^\beta(y)\right\rangle_{A=0}, \tag{3.38}$$

and

$$\begin{aligned} W_{55}^{\mu\nu\alpha\beta}(z;x,y) &= (i\,g)\,\frac{\delta^2\left\langle T_{iA}^{\mu\nu}(z)\,(J_A\cdot A)\right\rangle}{\delta A_\alpha(x)\,\delta A_\beta(y)}\bigg|_{A=0} \\ &= \delta^4(x-z)\,g^{\alpha(\mu}\Pi_{AA}^{\nu)\beta}(z,y)+\delta^4(y-z)\,g^{\beta(\mu}\Pi_{AA}^{\nu)\alpha}(z,x) \\ &\quad - g^{\mu\nu}\left[\delta^4(x-z)+\delta^4(y-z)\right]\Pi_{AA}^{\alpha\beta}(x,y), \end{aligned} \tag{3.39}$$

is a second term expressed in terms of the correlator of two axial currents

$$\Pi_{AA}^{\alpha\beta}(x,y) = -ig^2\langle J_A^\alpha(x)J_A^\beta(y)\rangle\bigg|_{A=0}. \tag{3.40}$$

3.3 Ward identities

The consistent definition of the $\langle TJ_AJ_A\rangle$ correlator requires the imposition of some Ward identities on it, that we are going to derive below. We start from the Ward identity to be satisfied by the axial vector current and then move to the conservation equation of the energy momentum tensor.

3.3.1 Axial vector Ward identities

The axial vector Ward identity is given by

$$\partial_\alpha^x\,\Gamma_{AA}^{\mu\nu\alpha\beta}(z;x,y) = \partial_\alpha^x\left[V_{55}^{\mu\nu\alpha\beta}(z;x,y)+W_{55}^{\mu\nu\alpha\beta}(z;x,y)\right]. \tag{3.41}$$

The two terms in the previous equation take the form

$$\begin{aligned}
\partial^x_\alpha\, V^{\mu\nu\alpha\beta}_{55}(z;x,y) &= (i\,g)^2\,\partial^x_\alpha\,\left\langle T^{\mu\nu}_f(z) J^\alpha_A(x) J^\beta_A(y)\right\rangle\,, &(3.42)\\
\partial^x_\alpha\, W^{\mu\nu\alpha\beta}_{55}(z;x,y) &= g^{\alpha(\mu}\Pi^{\nu)\beta}_{AA}(z,y)\,\partial^x_\alpha\delta^4(x-z) + 2mi\,\delta^4(y-z) g^{\beta(\mu}\Pi^{\nu)}_{AP}(z,x)\\
&- g^{\mu\nu}\Pi^{\alpha\beta}_{AA}(x,y)\partial^x_\alpha\delta^4(x-z) - 2mi\,g^{\mu\nu}[\delta^4(x-z)+\delta^4(y-z)]\Pi^\beta_{AP}(x,y), &(3.43)
\end{aligned}$$

while $\Pi^\alpha_{AP}(x,y)$ is defined by

$$\Pi^\alpha_{AP}(x,y) = -ig^2\langle J^\alpha_5(x) P(y)\rangle\Big|_{A=0}\,, \tag{3.44}$$

Here, P denotes the pseudoscalar current $P \equiv \bar\psi\gamma_5\psi$, and $\Pi^\alpha_{AP}\,, \Pi^{\alpha\beta}_{AA}$ are related by the PCAC condition

$$2i\,m\,\Pi^\beta_{AP}(x,y) = \partial^x_\alpha\,\Pi^{\alpha\beta}_{AA}(x,y). \tag{3.45}$$

The derivative of the correlator with the insertion of the free energy momentum tensor (T_f) can be calculated using functional techniques. For this purpose we consider the generating functional with the fermionic sources η and $\bar\eta$ and the classical background sources V^μ and A^μ coupled respectively to the current operators $J^\mu_V = \bar\psi\gamma^\mu\psi$ and $J^\mu_A = \bar\psi\gamma^\mu\gamma_5\psi$

$$\langle T^{\mu\nu}_f(z)\rangle_{V,A,\eta,\bar\eta} = \int D\psi D\bar\psi\; T^{\mu\nu}_f(z)\, e^{i\mathcal{S}_k(\psi)+i\int d^4x\,(g\,J_V\cdot V + g\,J_A\cdot A + \bar\psi\eta + \bar\eta\psi)} \tag{3.46}$$

and exploit the consequence of a chiral transformation on the corresponding Green's functions. The functional integral must be invariant under a reparameterization of the integration variables, giving the identity

$$\begin{aligned}
&\int D\psi D\bar\psi\; T^{\mu\nu}_f(z)\, e^{i\,\mathcal{S}_k(\psi)+i\int d^4x\,(g\,J_V\cdot V + g\,J_A\cdot A + \bar\psi\,\eta + \bar\eta\,\psi)} =\\
&\int D\psi' D\bar\psi'\; T^{\mu\nu}_f(z)'\, e^{i\,\mathcal{S}_k(\psi')+i\int d^4x\,(g\,J'_V\cdot V + g\,J'_A\cdot A + \bar\psi'\,\eta + \bar\eta\,\psi')}.
\end{aligned} \tag{3.47}$$

For a local infinitesimal chiral transformation of the fermion fields defined by

$$\psi \to \psi' = \psi + i\,g\,\epsilon(x)\,\gamma_5\,\psi, \tag{3.48}$$

$$\bar\psi \to \bar\psi' = \bar\psi + i\,g\,\epsilon(x)\,\bar\psi\,\gamma_5, \tag{3.49}$$

we can compute the variation of the action $\mathcal{S}$ and of $T^{\mu\nu}_p$ appearing on the right hand side (r.h.s.) of Eq. (3.47). The action changes as

$$\mathcal{S}_k(\psi')' = \mathcal{S}_k(\psi) + \int d^4x\,\epsilon(x)(\partial_\alpha J^\alpha_A(x) - 2i\,mP(x)), \tag{3.50}$$

whereas the vector and the axial-vector currents are obviously invariant

$$J'^{\mu}_V = J^{\mu}_V, \qquad J'^{\mu}_A = J^{\mu}_A. \tag{3.51}$$

The variation of the free energy-momentum tensor is instead given by

$$\delta T^{\mu\nu}_f(z) = \frac{1}{2}\Big[J^{\mu}_A(z)\partial^{\nu}\epsilon(z) + J^{\nu}_A(z)\partial^{\mu}\epsilon(z)\Big] - g^{\mu\nu}\Big[J^{\lambda}_A(z)\partial_{\lambda}\epsilon(z) - 2mi\epsilon(z)P(z)\Big]. \tag{3.52}$$

We note that this change of variables is not a gauge transformation; V and A are therefore invariant. For this reason, using also the invariance of the two currents, the interaction terms $T_{i,A}$ and $T_{i,V}$ of the energy momentum tensor remain invariant as well. It follows that the variation of $T^{\mu\nu}_p(z)$ is due only to the free contribution shown above.
If we rewrite the infinitesimal parameter $\epsilon(z)$ as $\epsilon(z) = \int d^4x\, \epsilon(x)\delta^4(z-x)$, the energy momentum variation can be recast in the following form

$$\delta T^{\mu\nu}_f(z) = \int d^4x\, \epsilon(x)\, \mathcal{H}^{\mu\nu}(x,z), \tag{3.53}$$

where this definition of $\mathcal{H}^{\mu\nu}(x,z)$

$$\begin{aligned}\mathcal{H}^{\mu\nu}(x,z) = \frac{1}{2}\, J^{\mu}_A(z)\, \partial^{\nu}_z\, \delta^4(z-x) + \frac{1}{2}\, J^{\nu}_A(z)\, \partial^{\mu}_z\, \delta^4(z-x) \\ -g^{\mu\nu}\left(J^{\lambda}_A(z)\, \partial^z_{\lambda}\, \delta^4(z-x) - 2i\, mP(z)\, \delta^4(z-x)\right)\end{aligned} \tag{3.54}$$

will turn useful in the following. Given the chiral nature of the transformation, we include also the anomalous variation of the measure

$$D\psi' D\bar{\psi}' = D\psi D\bar{\psi}\, \exp\left\{i\int d^4x\, \epsilon(x)a_n\left[\frac{1}{3}\epsilon^{\alpha\beta\mu\nu}F^A_{\alpha\beta}F^A_{\mu\nu} + \epsilon^{\alpha\beta\mu\nu}F^V_{\alpha\beta}F^V_{\mu\nu}\right]\right\} \tag{3.55}$$

where $a_n = \frac{g^2}{16\pi^2}$ is the anomaly coefficient. Expanding the r.h.s. of Eq. (3.47) to the first order in ϵ and taking into account the variation of the measure we obtain the Schwinger-Dyson equation

$$\begin{aligned} 0 &= \int d^4x\, \epsilon(x)\int D\psi D\bar{\psi}\Big\{i\, T^{\mu\nu}_f(z)\Big[\partial_{\alpha}J^{\alpha}_A(x) - 2miP(x) + ig\bar{\psi}(x)\gamma_5\eta(x) + ig\bar{\eta}(x)\gamma_5\psi(x) \\ &+ a_n\left(\frac{1}{3}F^A(x)\tilde{F}^A(x) + F^V(x)\tilde{F}^V(x)\right)\Big] + \mathcal{H}^{\mu\nu}(x,z)\Big\}e^{i\mathcal{S}_k(\psi)+i\int d^4x\,(g\,J_V\cdot V + g\,J_A\cdot A + \bar{\psi}\eta + \bar{\eta}\psi)}\end{aligned}$$

(with $F\tilde{F} \equiv \epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$). The expression takes a simplified form if we set the sources η, V and $\bar{\eta}$ to zero, and hence we obtain the anomalous Ward identity

$$i\langle T^{\mu\nu}_f(z)\partial\cdot J_A(x)\rangle_A = -2m\langle T^{\mu\nu}(z)P(x)\rangle_A - ia_n\frac{1}{3}F^A(x)\tilde{F}^A(x)\langle T^{\mu\nu}_f(z)\rangle_A - \langle\mathcal{H}^{\mu\nu}(x,z)\rangle_A. \tag{3.56}$$

From Eq. (3.56) we can extract Ward identities on correlation functions which contain one insertion of the energy-momentum tensor and several gauge currents just by functional differentiation respect to the external sources. For example, taking a derivative of (3.56) with respect to background field A^μ we obtain the constraint

$$\partial^x_\alpha \frac{\delta}{\delta A^\beta(y)} \langle T^{\mu\nu}_f(z) J^\alpha_A(x)\rangle_{V,A,\eta,\bar\eta}\bigg|_{V,A,\eta,\bar\eta=0} = \frac{\delta}{\delta A^\beta(y)} \bigg\{ 2mi\langle T^{\mu\nu}_f(z)\, P(x)\rangle_{V,A,\eta,\bar\eta} + i\,\langle \mathcal{H}^{\mu\nu}(x,z)\rangle_{V,A,\eta,\bar\eta} \bigg\}\bigg|_{V,A,\eta,\bar\eta=0}, \tag{3.57}$$

and performing explicitly the functional derivative we obtain the axial Ward identity

$$\partial^x_\alpha\, \langle T^{\mu\nu}_f(z) J^\alpha_A(x) J^\beta_A(y)\rangle = 2mi\langle T^{\mu\nu}_f(z) P(x) J^\beta_A(y)\rangle + i\langle \mathcal{H}^{\mu\nu}(x,z) J^\beta_A(y)\rangle \tag{3.58}$$

where the last term is given by

$$\begin{aligned} \langle \mathcal{H}^{\mu\nu}(x,z) J^\beta_A(y)\rangle_{V,A,\eta,\bar\eta}\bigg|_{V,A,\eta,\bar\eta=0} &= (-ig^2)^{-1}\bigg\{ \frac{1}{2}\Pi^{\mu\beta}_{AA}(z,y)\partial^\nu_z \delta^4(z-x) \\ &\quad + \frac{1}{2}\,\Pi^{\nu\beta}_{AA}(z,y)\,\partial^\mu_z\,\delta^4(z-x) \\ &\quad - g^{\mu\nu}\Big[\Pi^{\lambda\beta}_{AA}(z,y)\partial^z_\lambda \delta^4(z-x) - 2mi\,\Pi^\beta_{AP}(z,y)\delta^4(z-x)\Big]\bigg\}. \end{aligned} \tag{3.59}$$

Notice that Eq. (3.58) allows to derive indirectly the vacuum expectation value of the commutator of T_f with J_A by comparison with the canonical expression

$$\partial^x_\alpha\, \langle T^{\mu\nu}_f(z) J^\alpha_A(x) J^\beta_A(y)\rangle = 2mi\langle T^{\mu\nu}_f(z) P(x) J^\beta_A(y)\rangle + \langle \big[T^{\mu\nu}(z), J^\alpha_A(x)\big] g_{\alpha,0}\delta(x_0-z_0) J^\beta_A(y)\rangle \tag{3.60}$$

or

$$\langle \big[T^{\mu\nu}(z), J^\alpha_A(x)\big] g_{\alpha,0}\delta(x_0-z_0) J^\beta_A(y)\rangle = i\langle \mathcal{H}^{\mu\nu}(x,z) J^\beta_A(y)\rangle. \tag{3.61}$$

Proceeding with the functional differentiation one can derive further unrenormalized Ward identities for correlators of the form $T J_A J_A J_A$

$$\begin{aligned} (ig)^2\, \partial^x_\lambda\, \langle T^{\mu\nu}_f(z)\, J^\lambda_A(x)\, J^\alpha_A(y) J^\beta_A(w)\rangle &= (ig)^2\, \langle T^{\mu\nu}_f(z)\, 2\, m\, iP(x)\, J^\alpha_A(y)\, J^\beta_A(w)\rangle \\ &+ \frac{8}{3}\, a_n\, \epsilon^{\alpha\beta\rho\sigma}\, \partial_\rho\, \delta^4(x-y)\, \partial_\sigma\, \delta^4(x-w) \langle T^{\mu\nu}_f(z)\rangle \\ &+ i\,(ig)^2\, \langle \mathcal{H}^{\mu\nu}(x,z)\, J^\alpha_A(y)\, J^\beta_A(w)\rangle, \end{aligned} \tag{3.62}$$

which can be analyzed and checked in perturbation theory in a specific regularization scheme.

3.3.2 The axial Ward identity in momentum space

The Ward identity on the $\langle T J_A J_A\rangle$ vertex is extracted combining Eqs. (3.58) and (3.59) with Eqs. (3.42) and (3.43) and it is explicitly given by

$$\begin{aligned}
\partial^x_\alpha\,\Gamma^{\mu\nu\alpha\beta}_{AA}(z;x,y) &= 2\,mi\,(i\,g)^2\,\left\langle T^{\mu\nu}_f(z)P(x)J^\beta_A(y)\right\rangle + \left\{\frac{1}{2}\,\Pi^{\mu\beta}_{AA}(z,y)\,\partial^\nu_z\,\delta^4(z-x)\right.\\
&+ \frac{1}{2}\,\Pi^{\nu\beta}_{AA}(z,y)\,\partial^\mu_z\,\delta^4(z-x)\\
&- \left. g^{\mu\nu}\left[\Pi^{\lambda\beta}_{AA}(z,y)\,\partial^z_\lambda\,\delta^4(z-x) - 2m\,i\,\Pi^\beta_{AP}(z,y)\,\delta^4(z-x)\right]\right\}\\
&+ g^{\alpha(\mu}\,\Pi^{\nu)\beta}_{AA}(z,y)\,\partial^x_\alpha\,\delta^4(x-z) + 2m\,i\,\delta^4(y-z)g^{\beta(\mu}\Pi^{\nu)}_{AP}(z,x)\\
&- g^{\mu\nu}\,\Pi^{\alpha\beta}_{AA}(x,y)\partial^x_\alpha\delta^4(x-z) - 2\,m\,i\,g^{\mu\nu}\left[\delta^4(x-z)+\delta^4(y-z)\right]\Pi^\beta_{AP}(x,y).
\end{aligned} \tag{3.63}$$

By defining

$$(2\pi)^4\,\delta^4(k-p-q)\,\Gamma^{\mu\nu\alpha\beta}_{AA}(k,p,q) = \int d^4x\,d^4y\,d^4z\,e^{-i\,k\cdot z+i\,p\cdot x+i\,q\cdot y}\,\Gamma^{\mu\nu\alpha\beta}_{AA}(z;x,y) \tag{3.64}$$

and

$$(2\pi)^4\,\delta^4(k-p-q)\,\Delta^{\mu\nu\beta}_{AP}(k,p,q) = \int d^4x\,d^4y\,d^4z\,e^{-i\,k\cdot z+i\,p\cdot x+i\,q\cdot y}\,\left\langle T^{\mu\nu}_f(z)P(x)J^\beta_A(y)\right\rangle, \tag{3.65}$$

we obtain its form in momentum space

$$\begin{aligned}
-\,ip_\alpha\,\Gamma^{\mu\nu\alpha\beta}_{AA}(k,p,q) &= 2mi(i\,g)^2\,\Delta^{\mu\nu\beta}_{AP}(k,p,q) + \left\{\frac{1}{2}ip^\nu\Pi^{\mu\beta}_{AA}(q)\right.\\
&+ \left.\frac{1}{2}ip^\mu\Pi^{\nu\beta}_{AA}(q) - g^{\mu\nu}\left[ip_\lambda\Pi^{\lambda\beta}_{AA}(q) - 2m\,i\,\Pi^\beta_{AP}(q)\right]\right\}\\
&- ip_\alpha g^{\alpha(\mu}\Pi^{\nu)\beta}_{AA}(q)\; + 2m\,i\,g^{\beta(\mu}\Pi^{\nu)}_{AP}(p)\\
&+ g^{\mu\nu}ip_\alpha\Pi^{\alpha\beta}_{AA}(q) - 2m\,i\,g^{\mu\nu}\left[\Pi^\beta_{AP}(q)+\Pi^\beta_{AP}(p)\right].
\end{aligned} \tag{3.66}$$

We will be using this identity in the definition of the correlator with two axial-vector currents.

3.3.3 Ward identity for the conservation of $T_{\mu\nu}$

Moving to the conservation equation of the energy momentum tensor, the derivation of the corresponding Ward identity involves the functional relation (3.32) which is given by

$$\begin{aligned}
\frac{\partial}{\partial z^\nu}\Gamma^{\mu\nu\alpha\beta}_{AA}(z;x,y) &= -\frac{\partial}{\partial z_\mu}\,\delta^4(z-x)\,\Pi^{\alpha\beta}_{AA}(z,y) + g^{\alpha\mu}\,\frac{\partial}{\partial z^\lambda}\,\delta^4(z-x)\,\Pi^{\lambda\beta}_{AA}(z,y)\\
&-\frac{\partial}{\partial z_\mu}\,\delta^4(z-y)\,\Pi^{\alpha\beta}_{AA}(x,z) + g^{\beta\mu}\,\frac{\partial}{\partial z^\lambda}\,\delta^4(z-y)\,\Pi^{\lambda\alpha}_{AA}(z,x),
\end{aligned} \tag{3.67}$$

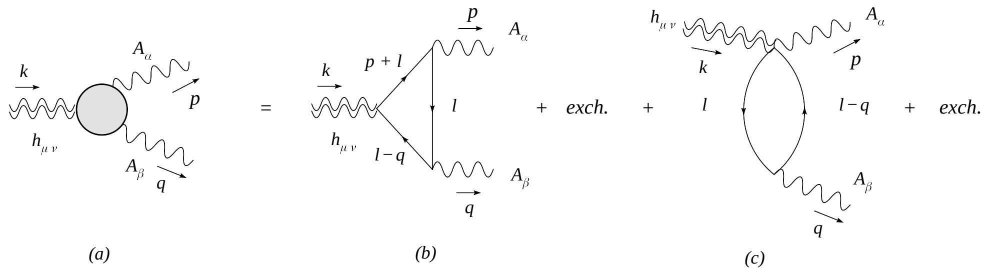


Figure 3.1: The complete one-loop vertex (a) given by the sum of the 1PI contributions called $V_{55}^{\mu\nu\alpha\beta}(p,q)$ (b) and $W_{55}^{\mu\nu\alpha\beta}(p,q)$ (c) with a graviton $h_{\mu\nu}$ in the initial state and two gauge bosons with axial-vector couplings A_α, A_β in the final state.

which can be simplified using the PCAC relation (3.45). In momentum space it gives

$$k_\nu \Gamma_{AA}^{\mu\nu\alpha\beta}(p,q) = (g^{\alpha\mu}\, k_\nu - g_\nu^\alpha\, p_\mu)\, \Pi_{AA}^{\beta\nu}(q) + (g^{\beta\mu}\, k_\nu - g_\nu^\beta\, q_\mu)\, \Pi_{AA}^{\alpha\nu}(p). \tag{3.68}$$

The complete set of defining conditions of each vertex, beside the two Ward identities derived above, is the request of a symmetry on its μ,ν indices, i.e. $\Gamma_{AA}^{\mu\nu\alpha\beta} = \Gamma_{AA}^{\nu\mu\alpha\beta}$. We will be using these conditions in order to fix the entire structure of the correlator and check the consistency of a given regularization scheme.

3.4 Diagrammatic expansion

The relevant diagrams responsible for the conformal anomaly are shown in Fig. 3.1 and take the form of Eqs. (3.38) and (3.39). They consist of an amplitude with triangular topology (see Fig. 3.1b) and of a bubble-like diagram (called a "t-bubble", see Fig. 3.1c). This has the topology of a self-energy loop inserted on each of the gauge lines and attached from one side to the T vertex. These contributions are all of $O(g^2)$. At this point, we recall that the tree-level vertex with a graviton and a Dirac fermion, namely $V'^{\mu\nu}$, and the trilinear graviton-gauge boson-fermion coupling, i.e. $W_5'^{\mu\nu\alpha}$, induced by the two contributions T_f and T_{iA} are respectively given by

$$V'^{\mu\nu}(k_1,k_2) = \frac{1}{4}\left[\gamma^\mu(k_1+k_2)^\nu + \gamma^\nu(k_1+k_2)^\mu\right] - \frac{1}{2}g^{\mu\nu}\left[\gamma^\lambda(k_1+k_2)_\lambda - 2m\right] \tag{3.69}$$

$$W_5'^{\mu\nu\alpha} = -\frac{1}{2}(\gamma^\mu\gamma_5 g^{\nu\alpha} + \gamma^\nu\gamma_5 g^{\mu\alpha}) + g^{\mu\nu}\gamma^\alpha\gamma_5 \tag{3.70}$$

where k_1 and k_2 are generic momenta, incoming and outgoing, respectively. Notice that the first contribution is vector-like, derived from (3.13) and, naturally, is the same appearing in the

previous analysis of the $\langle TJJ\rangle$ correlator in [52]. The second one, $W_5'^{\mu\nu\alpha}$, due to (3.15), differs from the analogous vertex $W'^{\mu\nu\alpha}$ appearing in the case of the $\langle TJJ\rangle$ correlator because of the presence of the γ_5 matrix.
If we denote with k the incoming momentum of the graviton and with p and q the two outgoing momenta of the A gauge bosons we obtain

$$(2\,\pi)^4\,\delta^4(k-p-q)\,V_{55}^{\mu\nu\alpha\beta}(p,q) \equiv \int d^4x\,d^4y\,d^4z\,e^{-ik\cdot z+ip\cdot x+iq\cdot y}\,\langle T_f^{\mu\nu}(z)\,J_A^{\alpha}(x)\,J_A^{\beta}(y)\rangle \tag{3.71}$$

$$(2\,\pi)^4\,\delta^4(k-p-q)\,W_{55}^{\mu\nu\alpha\beta}(p,q) \equiv \int d^4x\,d^4y\,d^4z\,e^{-ik\cdot z+ip\cdot x+iq\cdot y}\,\langle T_{iA}^{\mu\nu}(z)J_A^{\alpha}(x)J_A^{\beta}(y)\rangle. \tag{3.72}$$

Explicitly

$$V_{55}^{\mu\nu\alpha\beta}(p,q) = -ig^2\int\frac{d^4l}{(2\pi)^4}\,\frac{\mathrm{tr}\left\{V'^{\mu\nu}(l+p,l-q)(\not l-\not q+m)\gamma^\beta\gamma_5\,(\not l+m)\,\gamma^\alpha\gamma_5(\not l+\not p+m)\right\}}{[l^2-m^2]\,[(l-q)^2-m^2]\,[(l+p)^2-m^2]}\,, \tag{3.73}$$

$$W_{55}^{\mu\nu\alpha\beta}(p,q) = -ig^2\int\frac{d^4l}{(2\pi)^4}\,\frac{\mathrm{tr}\left\{W_5'^{\mu\nu\alpha}\,(\not l+m)\,\gamma^\beta\gamma_5(\not l+\not q+m)\right\}}{[l^2-m^2]\,[(l+q)^2-m^2]}, \tag{3.74}$$

so that the complete one-loop amplitude (see Fig. 3.1) is built up by symmetrizing on the external boson lines as

$$\Gamma_{AA}^{\mu\nu\alpha\beta}(p,q) = V_{55}^{\mu\nu\alpha\beta}(p,q) + V_{55}^{\mu\nu\beta\alpha}(q,p) + W_{55}^{\mu\nu\alpha\beta}(p,q) + W_{55}^{\mu\nu\beta\alpha}(q,p). \tag{3.75}$$

3.5 Tensor decomposition and naive manipulations

As we have mentioned, the correlator is completely defined by a set of Ward identities, which amount to renormalization conditions which should be imposed in such a way 1) to respect its Bose symmetry and 2) the conservation of the fundamental currents of the theory. This is the case for all the anomalous correlators, both for chiral and conformal anomalies. At the same time, one needs a good regularization scheme in order to proceed with the actual implementation of these conditions, which could be obviously violated. This may require a (final) finite renormalization of the result in order to force the result to satisfy the original Ward identities. In this respect, various regularization schemes are available for chiral vertices, from a partially [95] to a totally anticommuting γ_5. As we have already mentioned, in the computation of the correlator we have used DRED [94], with loop momenta computed in D spacetime dimensions and traces performed in 4 dimensions, and we have verified all the Ward identities formally derived in this chapter.

3.5.1 Vanishing of the TJ_VJ_A correlator

We start our analysis by studying the TJ_VJ_A correlator.

For this reason we just recall that this specific correlation function can be extracted by the generating functional

$$\begin{aligned} \langle T_p^{\mu\nu}(z)\rangle_{V,A} &\equiv \int D\psi D\bar{\psi}\; T_p^{\mu\nu}(z)\; e^{i\int d^4x(\mathcal{L}_k(\psi)+g\,J_V\cdot V(x)+g\,J_A\cdot A(x))} \\ &= \langle T_p^{\mu\nu}\, e^{i\int d^4x\, g\,(\,J_V\cdot V(x)+J_A\cdot A(x))}\rangle. \end{aligned} \tag{3.76}$$

Here we have introduced two independent sources J_V and J_A. The corresponding correlators are obtained via functional variations respect to the background fields V and A, namely

$$\Gamma_{VA}^{\mu\nu\alpha\beta}(z;x,y) \equiv \frac{\delta^2\,\langle T_p^{\mu\nu}(z)\rangle_{V,A}}{\delta V_\alpha(x)\delta A_\beta(y)}\bigg|_{V,A=0} = V_5^{\mu\nu\alpha\beta}(z;x,y) + W_5^{\mu\nu\alpha\beta}(z;x,y). \tag{3.77}$$

whose expressions in momentum space are (for the direct and the exchange contributions)

$$\begin{aligned} V_{5\,dir}^{\mu\nu\alpha\beta}(p,q) &= -(-ig)^2 i^3 \int \frac{d^4l}{(2\pi)^4}\, \frac{1}{[l^2-m^2]\,[(l-q)^2-m^2]\,[(l+p)^2-m^2]}\cdot \\ &\cdot\left[\mathrm{tr}\left\{V'^{\mu\nu}(l+p,l-q)(\not{l}-\not{q}+m)\gamma^\beta\,(\not{l}+m)\,\gamma^\alpha\gamma_5(\not{l}+\not{p}+m)\right\}\right], \end{aligned} \tag{3.78}$$

$$\begin{aligned} V_{5\,ex}^{\mu\nu\alpha\beta}(p,q) &= -(-ig)^2 i^3 \int \frac{d^4l}{(2\pi)^4}\, \frac{1}{[l^2-m^2]\,[(l+q)^2-m^2]\,[(l-p)^2-m^2]}\cdot \\ &\cdot\left[\mathrm{tr}\left\{V'^{\mu\nu}(l-p,l+q)(\not{l}-\not{p}+m)\gamma^\alpha\gamma_5\,(\not{l}+m)\,\gamma^\beta(\not{l}+\not{q}+m)\right\}\right], \end{aligned} \tag{3.79}$$

$$W_{5\,dir}^{\mu\nu\alpha\beta}(p,q) = -(-ig)^2 i^3 \int \frac{d^4l}{(2\pi)^4}\, \frac{\mathrm{tr}\left\{W'^{\mu\nu\alpha}\gamma_5(\not{l}+m)\gamma^\beta\,(\not{l}+\not{q}+m)\,\right\}}{[l^2-m^2]\,[(l+q)^2-m^2]}, \tag{3.80}$$

$$W_{5\,ex}^{\mu\nu\alpha\beta}(p,q) = -(-ig)^2 i^3 \int \frac{d^4l}{(2\pi)^4}\, \frac{\mathrm{tr}\left\{W'^{\mu\nu\beta}(\not{l}+m)\gamma^\alpha\gamma_5\,(\not{l}+\not{p}+m)\,\right\}}{[l^2-m^2]\,[(l+p)^2-m^2]}, \tag{3.81}$$

and where the vertices $V'^{\mu\nu}$ and $W'^{\mu\nu\alpha}$ are defined as

$$V'^{\mu\nu}(k_1,k_2) = \frac{1}{4}\,[\gamma^\mu(k_1+k_2)^\nu + \gamma^\nu(k_1+k_2)^\mu] - \frac{1}{2}g^{\mu\nu}[\gamma^\lambda(k_1+k_2)_\lambda - 2m], \tag{3.82}$$

$$W'^{\mu\nu\alpha} = -\frac{1}{2}(\gamma^\mu g^{\nu\alpha} + \gamma^\nu g^{\mu\alpha}) + g^{\mu\nu}\gamma^\alpha. \tag{3.83}$$

We will use the same trick used for the proof of Furry's theorem to show the vanishing of this correlator, which is formally divergent and therefore ill-defined. For this reason one needs some external Ward identities in order to resolve its structure. For the TJ_VJ_A vertex the situation is quite peculiar since one can show, using DRED and by allowing momentum shifts, that the three Ward identities are indeed homogeneous

$$k_\mu\Gamma_{VA}^{\mu\nu\alpha\beta} = p_\alpha\Gamma_{VA}^{\mu\nu\alpha\beta} = q_\beta\Gamma_{VA}^{\mu\nu\alpha\beta} = 0, \tag{3.84}$$

while the properties of symmetry of the correlator are respected. Obviously, this indicates that there is a regularization scheme in which the anomaly of the axial-vector current J_A does not appear. A closer inspection shows that this result is caused by a cancellation between the direct and the exchange contribution, since the ϵ-tensor is present in each of the two (direct and exchange) diagrams contributing to the vertex, but not in their sum. Indeed, this clearly seems to indicate that this correlator may be vanishing identically. A second argument, based on charge conjugation invariance brings to identical conclusions.

For this reason, we take the expression of the triangle diagram and insert the identity $C^{-1}C = 1$ - involving the charge conjugation matrix C between every γ matrix - together with the relations

$$C\,\gamma^\mu\,C^{-1} = -(\gamma^\mu)^T, \qquad C\,\gamma_5\,C^{-1} = \gamma_5, \tag{3.85}$$

so that the trace in Eq. (3.78) becomes

$$\begin{aligned}\mathcal{T} &= \mathrm{tr}\left\{\tilde{V}'^{\mu\nu}(l+p,l-q)^T(\not{l}-\not{q}-m)^T(\gamma^\beta)^T\,(\not{l}-m)^T\,(\gamma^\alpha)^T\gamma_5(\not{l}+\not{p}-m)^T\right\} \\ &= -\mathrm{tr}\left\{\tilde{V}'^{\mu\nu}(l+p,l-q)(\not{l}+\not{p}-m)\gamma^\alpha\gamma_5\,(\not{l}-m)\,\gamma^\beta\,(\not{l}-\not{q}-m)\right\}\end{aligned} \tag{3.86}$$

where $\tilde{V}'^{\mu\nu}$ differs from $V'^{\mu\nu}$ only for the sign of the mass term

$$\tilde{V}'^{\mu\nu}(k_1,k_2) = \frac{1}{4}\left[\gamma^\mu(k_1+k_2)^\nu + \gamma^\nu(k_1+k_2)^\mu\right] - \frac{1}{2}g^{\mu\nu}[\gamma^\lambda(k_1+k_2)_\lambda + 2m]. \tag{3.87}$$

Changing the integration variable $l \to -l$ in Eq. (3.86) we get

$$\mathcal{T} = -\mathrm{tr}\left\{V'^{\mu\nu}(l-p,l+q)(\not{l}-\not{p}+m)\gamma^\alpha\gamma_5\,(\not{l}+m)\,\gamma^\beta\,(\not{l}+\not{q}+m)\right\}, \tag{3.88}$$

while the three denominators in Eq. (3.78) change according to

$$\frac{1}{[l^2-m^2]\,[(l-q)^2-m^2]\,[(l+p)^2-m^2]} \to \frac{1}{[l^2-m^2]\,[(l+q)^2-m^2]\,[(l-p)^2-m^2]}. \tag{3.89}$$

Combining Eq. (3.88) and (3.89) it is easy to recognize that

$$V^{\mu\nu\alpha\beta}_{5\,dir}(p,q) = -V^{\mu\nu\alpha\beta}_{5\,ex}(p,q) \tag{3.90}$$

so that the sum of the two triangles vanishes.

The last point to check in order to be sure of the vanishing of the vertex concerns the contributions from the t-bubble diagrams. These have been defined in Eq. (3.80) and (3.81) and their topology is the one showed in Fig. 3.1c. These are both separately equal to zero because they consists of a combination of 2-point functions of the form $\Pi^{\alpha\beta}_{VA}(p)$ given by

$$\Pi^{\alpha\beta}_{VA}(p) = -g^2\int\frac{d^4l}{(2\pi)^4}\,\frac{\mathrm{tr}\left\{\gamma^\alpha\gamma_5\,(\not{l}+m)\gamma^\beta\,(\not{l}+\not{p}+m)\right\}}{[l^2-m^2]\,[(l+p)^2-m^2]} \tag{3.91}$$

which are also identically vanishing.

$p^\mu p^\nu p^\alpha p^\beta$ $q^\mu q^\nu q^\alpha q^\beta$	$p^\mu p^\nu p^\alpha q^\beta$ $p^\mu p^\nu q^\alpha p^\beta$ $p^\mu q^\nu p^\alpha p^\beta$ $q^\mu p^\nu p^\alpha p^\beta$	$p^\mu p^\nu q^\alpha q^\beta$ $p^\mu q^\nu p^\alpha q^\beta$ $q^\mu p^\nu p^\alpha q^\beta$	$p^\mu q^\nu q^\alpha p^\beta$ $q^\mu p^\nu q^\alpha p^\beta$ $q^\mu q^\nu p^\alpha p^\beta$	$p^\mu q^\nu q^\alpha q^\beta$ $q^\mu p^\nu q^\alpha q^\beta$ $q^\mu q^\nu p^\alpha q^\beta$ $q^\mu q^\nu q^\alpha p^\beta$	$g^{\mu\nu} g^{\alpha\beta}$ $g^{\alpha\mu} g^{\beta\nu}$ $g^{\alpha\nu} g^{\beta\mu}$
$p^\mu p^\nu g^{\alpha\beta}$ $p^\mu q^\nu g^{\alpha\beta}$ $q^\mu p^\nu g^{\alpha\beta}$ $q^\mu q^\nu g^{\alpha\beta}$	$p^\beta p^\nu g^{\alpha\mu}$ $p^\beta q^\nu g^{\alpha\mu}$ $q^\beta p^\nu g^{\alpha\mu}$ $q^\beta q^\nu g^{\alpha\mu}$	$p^\beta p^\mu g^{\alpha\nu}$ $p^\beta q^\mu g^{\alpha\nu}$ $q^\beta p^\mu g^{\alpha\nu}$ $q^\beta q^\mu g^{\alpha\nu}$	$p^\alpha p^\nu g^{\beta\mu}$ $p^\alpha q^\nu g^{\beta\mu}$ $q^\alpha p^\nu g^{\beta\mu}$ $q^\alpha q^\nu g^{\beta\mu}$	$p^\mu p^\alpha g^{\beta\nu}$ $p^\mu q^\alpha g^{\beta\nu}$ $q^\mu p^\alpha g^{\beta\nu}$ $q^\mu q^\alpha g^{\beta\nu}$	$p^\alpha p^\beta g^{\mu\nu}$ $p^\alpha q^\beta g^{\mu\nu}$ $q^\alpha p^\beta g^{\mu\nu}$ $q^\alpha q^\beta g^{\mu\nu}$

Table 3.1: The 43 tensor monomials called $l_i^{\mu\nu\alpha\beta}(p,q)$ built up from the metric tensor and the two independent momenta p and q into which a general fourth rank tensor can be expanded.

3.5.2 The computation of the $\langle TJ_AJ_A\rangle$ correlator

We now going to address the computation of the TJ_AJ_A vertex, but prior to that we briefly review the vector/vector case. As discussed in [51] and in [52] the full one-loop amplitude with the energy momentum tensor coupled to two vector currents, $\Gamma_{VV}^{\mu\nu\alpha\beta}$, can be expanded on the basis provided by the 43 monomial tensors $l_i^{\mu\nu\alpha\beta}(p,q)$ listed in Tab. 3.1

$$\Gamma_{VV}^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{43} A_i(k^2,p^2,q^2)\, l_i^{\mu\nu\alpha\beta}(p,q), \tag{3.92}$$

whose form factors $A_i(k^2,p^2,q^2)$ are not all convergent, since the amplitude has total mass dimension equal to 2. It has been shown in [52] that they can be divided into 3 groups:

a) $A_1 \le A_i \le A_{16}$ - multiplied by a product of four momenta, they have mass dimension -2 and therefore are UV finite;

b) $A_{17} \le A_i \le A_{19}$ - these have mass dimension 2 since the four Lorentz indices of the amplitude are carried by two metric tensors

c) $A_{20} \le A_i \le A_{43}$ - they appear next to a metric tensor and two momenta, have mass dimension 0 and are divergent.

In [51] the 43 invariant amplitudes $A_i(k^2,p^2,q^2)$ have been cleverly reduced to the 13 named $F_i(k^2,p^2,q^2)$. A similar result is obtained in [52] using a different intermediate basis. This reorganization of the amplitude shows conclusively that the effective action of theories with conformal anomalies is affected by anomaly poles which contain the entire signature of the anomaly [92].

i	$t_i^{\mu\nu\alpha\beta}(p,q)$
1	$\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) u^{\alpha\beta}(p.q)$
2	$\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) w^{\alpha\beta}(p.q)$
3	$\left(p^2 g^{\mu\nu} - 4p^\mu p^\nu\right) u^{\alpha\beta}(p.q)$
4	$\left(p^2 g^{\mu\nu} - 4p^\mu p^\nu\right) w^{\alpha\beta}(p.q)$
5	$\left(q^2 g^{\mu\nu} - 4q^\mu q^\nu\right) u^{\alpha\beta}(p.q)$
6	$\left(q^2 g^{\mu\nu} - 4q^\mu q^\nu\right) w^{\alpha\beta}(p.q)$
7	$\left[p\cdot q\, g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)\right] u^{\alpha\beta}(p.q)$
8	$\left[p\cdot q\, g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)\right] w^{\alpha\beta}(p.q)$
9	$\left(p\cdot q\, p^\alpha - p^2 q^\alpha\right)\left[p^\beta\left(q^\mu p^\nu + p^\mu q^\nu\right) - p\cdot q\left(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu\right)\right]$
10	$\left(p\cdot q\, q^\beta - q^2 p^\beta\right)\left[q^\alpha\left(q^\mu p^\nu + p^\mu q^\nu\right) - p\cdot q\left(g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu\right)\right]$
11	$\left(p\cdot q\, p^\alpha - p^2 q^\alpha\right)\left[2\, q^\beta q^\mu q^\nu - q^2(g^{\beta\nu}q^\mu + g^{\beta\mu}q^\nu)\right]$
12	$\left(p\cdot q\, q^\beta - q^2 p^\beta\right)\left[2\, p^\alpha p^\mu p^\nu - p^2(g^{\alpha\nu}p^\mu + g^{\alpha\mu}p^\nu)\right]$
13	$\left(p^\mu q^\nu + p^\nu q^\mu\right)g^{\alpha\beta} + p\cdot q\left(g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu}\right) - g^{\mu\nu}u^{\alpha\beta}$ $-\left(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu\right)q^\alpha - \left(g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu\right)p^\beta$

Table 3.2: The 13 fourth rank tensors $t_i^{\mu\nu\alpha\beta}(p,q)$ satisfying the vector current conservation on the external lines with momenta p and q.

As we are going to show, a similar result holds also for the $\langle TJ_AJ_A\rangle$ vertex. At the same time, we are going to demonstrate the appearance only of conformal anomalies, since the mixed anomalies cancel, and present the complete expression of this vertex.

To illustrate this point, we observe that the insertion of the non-chiral component of $T^{\mu\nu}$ (represented by $T_f^{\mu\nu}$) in the correlator V_{55}, defines one of the two subamplitudes which may potentially generate mixed anomalies. On the other hand, it is however obvious - by a glance at the structure of the correlator - that we could remove symmetrically the chiral matrix all together. Therefore, the $\langle TJ_AJ_A\rangle$ correlator can be split in two terms, the first being the correlator with two vector currents called TJ_VJ_V, while the second is an extra contribution, proportional to the fermion mass m, denoted by Ω

$$\Gamma_{AA}^{\mu\nu\alpha\beta}(p,q) = \Gamma_{VV}^{\mu\nu\alpha\beta}(p,q) + \Omega^{\mu\nu\alpha\beta}(p,q). \tag{3.93}$$

The explicit computation of the correlator with two vector currents $\Gamma_{VV}^{\mu\nu\alpha\beta}$ can be borrowed from [52], but the computation of the extra terms is very involved, due to the need to select a specific number of tensor structures in its expansion. Notice that the decomposition in Eq. (3.93) is particularly useful because shows that the vector and axial-vector cases coincide in the chiral

limit, i.e. for $\Omega^{\mu\nu\alpha\beta} = 0$.
As we have just mentioned above, the amplitude $\Gamma_{VV}^{\mu\nu\alpha\beta}$ can be expanded in the reduced basis given in Tab. 3.2

$$\Gamma_{VV}^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{13} F_i(s; s_1, s_2, m^2)\; t_i^{\mu\nu\alpha\beta}(p,q)\,, \tag{3.94}$$

where the invariant amplitudes $F_i(s; s_1, s_2, m^2)$ are functions of the kinematical invariants $s = k^2 = (p+q)^2$, $s_1 = p^2$, $s_2 = q^2$. Their explicit expressions in the general case have been given in [52]. In the simplest case, i.e. for an internal zero mass fermion ($m = 0$) and on-shell photons on the external lines ($s_1 = s_2 = 0$), the only non-vanishing $F_i(s; s_1, s_2, m^2)$ are given by

$$F_1(s,0,0,0) = -\frac{g^2}{18\pi^2 s}, \tag{3.95}$$

$$F_3(s,0,0,0) = F_5(s,0,0,0) = -\frac{g^2}{144\pi^2\, s}, \tag{3.96}$$

$$F_7(s,0,0,0) = -4\, F_3(s,0,0,0), \tag{3.97}$$

$$F_{13,R}(s,0,0,0) = -\frac{g^2}{144\pi^2}\left[12\log\left(-\frac{s}{\mu^2}\right) - 35\right], \tag{3.98}$$

(with $s < 0$) where F_{13} is affected by charge renormalization (with a scale μ). As we are going to discuss next, F_1 is the only form factor contributing to the trace anomaly in the massless case, and contains an anomaly pole. In this sense we can say that the pole *saturates* the anomaly and completely accounts for it. In [51] this $1/s$ terms is identified by a spectral analysis of the correlator, while the same structure emerges form the complete expressions of the form factors derived in [52] and presented above.

Coming instead to the new contribution $\Omega^{\mu\nu\alpha\beta}$ appearing in Eq. (3.93), this can be written as

$$\Omega^{\mu\nu\alpha\beta}(p,q) = \Omega_V^{\mu\nu\alpha\beta}(p,q) + \Omega_V^{\mu\nu\beta\alpha}(q,p) + \Omega_W^{\mu\nu\alpha\beta}(p,q) + \Omega_W^{\mu\nu\beta\alpha}(q,p), \tag{3.99}$$

where the amplitudes $\Omega_V^{\mu\nu\alpha\beta}$ and $\Omega_W^{\mu\nu\alpha\beta}$ are given by

$$\Omega_V^{\mu\nu\alpha\beta}(p,q) = -2m(-ig^2)\int \frac{d^4l}{(2\pi)^4}\,\frac{\mathrm{tr}\left\{V'^{\mu\nu}(l+p,l-q)(\not l - \not q + m)\gamma^\beta\gamma^\alpha(\not l + \not p + m)\right\}}{[l^2-m^2]\,[(l-q)^2-m^2]\,[(l+p)^2-m^2]}\,, \tag{3.100}$$

$$\Omega_W^{\mu\nu\alpha\beta}(p,q) = -2m(-ig^2)\int \frac{d^4l}{(2\pi)^4}\,\frac{\mathrm{tr}\left\{W'^{\mu\nu\alpha}\,\gamma^\beta(\not l + \not q + m)\right\}}{[l^2-m^2]\,[(l+q)^2-m^2]}, \tag{3.101}$$

with the $V'^{\mu\nu}$ and $W'^{\mu\nu\alpha}$ defined in eqs (3.82) and (3.83). The remaining two terms in Eq. (3.99) are simply the Bose symmetric amplitudes obtained exchanging the indices α and β and the momenta p and q of (3.100) and (3.101). The extra term $\Omega^{\mu\nu\alpha\beta}$ can be expanded on the basis provided by the 43 monomial tensors $l_i^{\mu\nu\alpha\beta}(p,q)$ listed in Tab. 3.1

$$\Omega^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{43} E_i(k^2,p^2,q^2,m^2)\, l_i^{\mu\nu\alpha\beta}(p,q), \tag{3.102}$$

where the form factors $E_i(k^2,p^2,q^2,m^2)$ are some functions of the kinematical variables and of the mass of the fermion in the loop. This needs to be identified by a direct inspection. The explicit computation shows that not all the 43 invariant amplitudes $E_i(k^2,p^2,q^2,m^2)$ are really present in this expansion and therefore the surviving ones can be appropriately combined in a lower number of composite tensor structures. This result can be organized in a more compact form after introducing a new tensor basis whose elements $f_i^{\mu\nu\alpha\beta}(p,q)$ $(i = 1,\ldots,9)$ are listed in Tab.3.3. We obtain

$$\Omega^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{9} R_i(s,s_1,s_2,m^2)\, f_i^{\mu\nu\alpha\beta}(p,q), \tag{3.103}$$

where the invariant amplitudes $R_i(s,s_1,s_2,m^2)$ depend on the kinematical variables $s = k^2 = (p+q)^2$, $s_1 = p^2$, $s_2 = q^2$ besides the fermion mass m.
Three of the nine tensors are Bose symmetric, namely,

$$f_i^{\mu\nu\alpha\beta}(p,q) = f_i^{\mu\nu\beta\alpha}(q,p)\,, \qquad i = 1,6,9\,, \tag{3.104}$$

while the remaining ones form three pairs related by Bose symmetry

$$f_2^{\mu\nu\alpha\beta}(p,q) = f_3^{\mu\nu\beta\alpha}(q,p)\,, \tag{3.105}$$
$$f_4^{\mu\nu\alpha\beta}(p,q) = f_5^{\mu\nu\beta\alpha}(q,p)\,, \tag{3.106}$$
$$f_7^{\mu\nu\alpha\beta}(p,q) = f_8^{\mu\nu\beta\alpha}(q,p)\,. \tag{3.107}$$

This basis is particularly useful because only the first three of the nine tensors have a non-zero trace

$$g_{\mu\nu} f_1^{\mu\nu\alpha\beta}(p,q) = 3k^2 g^{\alpha\beta}\,, \tag{3.108}$$
$$g_{\mu\nu} f_2^{\mu\nu\alpha\beta}(p,q) = g_{\mu\nu} f_3^{\mu\nu\alpha\beta}(p,q) = 2(p^\alpha q^\beta - p^\beta q^\alpha)\,, \tag{3.109}$$

while the remaining six tensors are traceless

$$g_{\mu\nu} f_i^{\mu\nu\alpha\beta}(p,q) = 0\,, \qquad i = 4,5,6,7,8,9\,. \tag{3.110}$$

At this point, the goal is to express the amplitude $\Omega^{\mu\nu\alpha\beta}(p,q)$ in an analytical form. We start from the evaluation of the integrals in Eqs. (3.100) and (3.101), obtaining the form factors E_i. At

i	$f_i^{\mu\nu\alpha\beta}(p,q)$
1	$\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) g^{\alpha\beta}$
2	$p^\nu q^\beta g^{\alpha\mu} + p^\mu q^\beta g^{\alpha\nu} - p^\nu q^\alpha g^{\beta\mu} - p^\mu q^\alpha g^{\beta\nu}$
3	$p^\alpha q^\nu g^{\beta\mu} + p^\alpha q^\mu g^{\beta\nu} - p^\beta q^\nu g^{\alpha\mu} - p^\beta q^\mu g^{\alpha\nu}$
4	$p^\nu p^\beta g^{\alpha\mu} + p^\mu p^\beta g^{\alpha\nu} - p^\nu p^\alpha g^{\beta\mu} - p^\mu p^\alpha g^{\beta\nu}$
5	$q^\alpha q^\nu g^{\beta\mu} + q^\alpha q^\mu g^{\beta\nu} - q^\beta q^\nu g^{\alpha\mu} - q^\beta q^\mu g^{\alpha\nu}$
6	$(p^\mu q^\nu + q^\mu p^\nu) g^{\alpha\beta} + p\cdot q\,(g^{\alpha\nu} g^{\beta\mu} + g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu})$
7	$\left(p^2 g^{\mu\nu} - 4 p^\mu p^\nu\right) g^{\alpha\beta}$
8	$\left(q^2 g^{\mu\nu} - 4 q^\mu q^\nu\right) g^{\alpha\beta}$
9	$\left(p\cdot q g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)\right) g^{\alpha\beta}$

Table 3.3: Basis of 9 fourth rank tensors called $f_i^{\mu\nu\alpha\beta}(p,q)$.

a second stage we map them into the new parameterization defined in eq. (3.103), determining in this way the coefficients R_i. The relations between the two sets $\{E_i\}_{i=1,\dots,43}$ and $\{R_i\}_{i=1,\dots,9}$, for the most general external momenta are

$$
\begin{aligned}
R_1 &= \frac{1}{3k^2}\left(E_{20}\,p^2 + 2E_{21}\,p\cdot q + E_{23}\,q^2 + 4E_{17} + 2E_{18}\right), && (3.111)\\
R_2 &= E_{26}, && (3.112)\\
R_3 &= E_{33}, && (3.113)\\
R_4 &= E_{26}, && (3.114)\\
R_5 &= E_{33}, && (3.115)\\
R_6 &= \frac{E_{18}}{p\cdot q}, && (3.116)\\
R_7 &= -\frac{1}{12k^2}\left(E_{20}\,p^2 + 2E_{21}\,p\cdot q + E_{23}\,q^2 + 4E_{17} + 2E_{18}\right) - \frac{E_{20}}{4}, && (3.117)\\
R_8 &= -\frac{1}{12k^2}\left(E_{20}\,p^2 + 2E_{21}\,p\cdot q + E_{23}\,q^2 + 4E_{17} + 2E_{18}\right) - \frac{E_{23}}{4}, && (3.118)\\
R_9 &= -\frac{1}{6k^2}\left(E_{20}\,p^2 + 2E_{21}\,p\cdot q + E_{23}\,q^2 + 4E_{17} + 2E_{18}\right) + \frac{E_{18}}{2p\cdot q} - \frac{E_{21}}{2}, && (3.119)
\end{aligned}
$$

where all the dependence on the kinematical invariants k^2, p^2, q^2 and m^2 appearing in the sets R_i and E_i has been omitted. The explicit expressions in DRED of the form factors R_i have been collected in Appendix A.9 and represent an important step in the computation of the $\langle TJ_AJ_A\rangle$ correlator. These form factors are affected by the usual ultraviolet singularities, which in a renormalizable theory would be removed by standard renormalization counterterms. In our case they turn out to be proportional to 2-point functions.

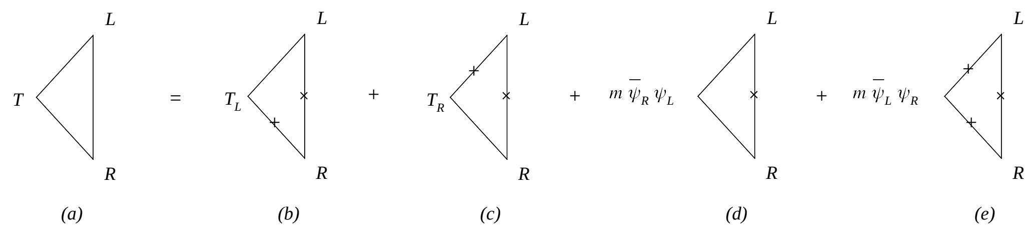


Figure 3.2: Chiral decomposition of the correlator.

Except for these possible counterterms, the main techniques and methods used in this analysis remain invariant and are of an easy application also in the case of the Standard Model. Notice, in particular, that the main equation (3.93) implies that the non-renormalizable contributions are proportional to mass corrections contributing to Ω, and the non-renormalizable terms indeed involve correlators of two axial-vector currents, as just mentioned above. The renormalization of the first contribution Γ_{VV} is canonical, and is attributed to the form factor F_{13} of Eq. (3.98), which is induced by a renormalization of 2-point functions of vector currents.

Before coming to the analysis of the other vertices, in closing this section we just remark that our analysis in the V/A basis can be rewritten completely in terms of chiral L/R currents, since the following relations hold for nonzero m

$$\langle TJ_VJ_V\rangle = \langle TJ_LJ_L\rangle + \langle TJ_RJ_R\rangle + \langle TJ_LJ_R\rangle + \langle TJ_RJ_L\rangle, \tag{3.120}$$

$$\langle TJ_AJ_A\rangle = \langle TJ_LJ_L\rangle + \langle TJ_RJ_R\rangle - \langle TJ_LJ_R\rangle - \langle TJ_RJ_L\rangle, \tag{3.121}$$

$$\langle TJ_AJ_A\rangle = \langle TJ_VJ_V\rangle - 2\left(\langle TJ_LJ_R\rangle + \langle TJ_RJ_L\rangle\right). \tag{3.122}$$

$$\langle TJ_LJ_L\rangle = \langle TJ_RJ_R\rangle = \frac{1}{4}\left(\langle TJJ\rangle + \langle TJ_AJ_A\rangle\right), \tag{3.123}$$

while

$$\langle T_LJ_LJ_L\rangle = \langle T_RJ_RJ_R\rangle = \frac{1}{2}\langle TJJ\rangle \tag{3.124}$$

is valid for a vanishing fermion mass m. The formulation in terms of L/R currents is the most convenient for the study of vertices containing trace anomalies, in the case of realistic theories such as the Standard Model.

3.6 Trace anomaly of the $\langle TJ_AJ_A\rangle$ correlator

We now move to analyze the trace of the $\langle TJ_AJ_A\rangle$ correlator. We consider generic virtualities of the external lines and a massive fermion.

In the absence of anomalies, the naive trace of the $\Gamma^{\mu\nu\alpha\beta}_{AA}$ amplitude is simply obtained by replacing the partial energy-momentum tensor $T^{\mu\nu}_p$ in the $\langle TJ_AJ_A\rangle$ correlator with its classical trace $T^{\mu}_{p\,\mu} = -m\bar{\psi}\psi$ and it is given by

$$\begin{aligned}\Lambda^{\alpha\beta}_{AA}(p,q) &= -m\,(ig)^2\int d^4x\,d^4y\,e^{ip\cdot x+iq\cdot y}\langle\bar{\psi}\psi J^{\alpha}_A(x)J^{\beta}_A(y)\rangle\\ &= -m\,g^2\int\frac{d^4l}{(2\pi)^4}\,tr\left\{\frac{i}{\not l-\not q-m}\gamma^{\beta}\gamma_5\frac{i}{\not l-m}\gamma^{\alpha}\gamma_5\frac{i}{\not l+\not p-m}\right\}+\text{exch.}\end{aligned} \tag{3.125}$$

As in Eq. (3.93) we can split the $\Lambda^{\alpha\beta}_{AA}$ into two terms: the first, $\Lambda^{\alpha\beta}_{VV}$, being the classical trace obtained from the $\langle TJ_VJ_V\rangle$ correlator, whereas the second, $\Lambda^{\alpha\beta}_{\Omega}$, takes into account the axial contribution to the amplitude as

$$\Lambda^{\alpha\beta}_{AA}(p,q) = \Lambda^{\alpha\beta}_{VV}(p,q)+\Lambda^{\alpha\beta}_{\Omega}(p,q). \tag{3.126}$$

The $\Lambda^{\alpha\beta}_{VV}$ amplitude refers to the $\langle TJ_VJ_V\rangle$ correlator. It can be written in the form

$$\Lambda^{\alpha\beta}_{VV}(p,q) = G_1(s,s_1,s_2,m^2)\,u^{\alpha\beta}(p,q)+G_2(s,s_1,s_2,m^2)\,w^{\alpha\beta}(p,q), \tag{3.127}$$

where the rank-2 tensors are defined by

$$u^{\alpha\beta}(p,q) \equiv (p\cdot q)\,g^{\alpha\beta}-q^{\alpha}\,p^{\beta}\,, \tag{3.128}$$

$$w^{\alpha\beta}(p,q) \equiv p^2\,q^2\,g^{\alpha\beta}+(p\cdot q)\,p^{\alpha}\,q^{\beta}-q^2\,p^{\alpha}\,p^{\beta}-p^2\,q^{\alpha}\,q^{\beta}, \tag{3.129}$$

with coefficients $G_i(s,s_1,s_2,m^2)$ which are left to an Appendix (Appendix A.10).
The second term $\Lambda^{\alpha\beta}_{\Omega}$ in Eq. (3.126) can be decomposed into two tensorial structures as

$$\Lambda^{\alpha\beta}_{\Omega}(p,q) = H_1(s,s_1,s_2,m^2)g^{\alpha\beta}+H_2(s,s_1,s_2,m^2)(p^{\alpha}q^{\beta}-q^{\alpha}p^{\beta}) \tag{3.130}$$

where the functions H_i are related to the invariant amplitudes R_i listed in Appendix A.9 by the relations

$$3sR_1(s,s_1,s_2,m^2) = H_1(s,s_1,s_2,m^2)-\frac{g^2m^2}{\pi^2}\,, \tag{3.131}$$

$$2R_2(s,s_1,s_2,m^2)+2R_3(s,s_1,s_2,m^2) = H_2(s,s_1,s_2,m^2)\,. \tag{3.132}$$

The analytical expressions of the off-shell $H_i(s,s_1,s_2,m^2)$ form factors are given by

$$\begin{aligned} H_1(s,s_1,s_2,m^2) &= \frac{g^2\,m^2}{2\,\pi^2}\Big[\mathcal{D}_1(s,s_1,m^2)+\mathcal{D}_2(s,s_2,m^2)-2\,\mathcal{B}_0(s^2,m^2) \\ &\quad +(s-4m^2)\,\mathcal{C}_0(s,s_1,s_2,m^2)\Big], \end{aligned} \tag{3.133}$$

$$\begin{aligned} H_2(s,s_1,s_2,m^2) &= \frac{g^2\,m^2}{\pi^2\,\sigma}\Big[(s+s_1-s_2)\,\mathcal{D}_1(s,s_1,m^2)+(s-s_1+s_2)\,\mathcal{D}_2(s,s_2,m^2) \\ &\quad +s\,(s-s_1-s_2)\,\mathcal{C}_0(s,s_1,s_2,m^2)\Big], \end{aligned} \tag{3.134}$$

where $\sigma \equiv s^2-2(s_1+s_2)\,s+(s_1-s_2)^2$ and the scalar integrals $\mathcal{B}_0(s^2,m^2)$, $\mathcal{D}_1(s,s_1,m^2)$, $\mathcal{D}_2(s,s_1,m^2)$, $\mathcal{C}_0(s,s_1,s_2,m^2)$ for generic virtualities and masses are defined in Appendix A.2.

Tracing the $\Gamma^{\mu\nu\alpha\beta}_{AA}$ correlator we obtain the relation

$$g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}_{AA}(p,q)=\Lambda^{\alpha\beta}_{AA}(p,q)-\frac{g^2}{6\pi^2}u^{\alpha\beta}(p,q)-\frac{g^2m^2}{\pi^2}g^{\alpha\beta}, \tag{3.135}$$

where the first term on the right-hand-sice is the trace anomaly appearing already in the $\langle TJ_VJ_V\rangle$ correlator. The second term, proportional to m^2, comes from the axial extra term $\Omega^{\mu\nu\alpha\beta}$ and denotes an additional explicit breaking related to the fermion mass. In particular, the anomaly $-\frac{g^2}{6\pi^2}u^{\alpha\beta}$ is carried by the form factor F_1, whose expression is given in [52], whereas the mass correction $-g^2m^2/\pi^2g^{\alpha\beta}$ is induced by R_1. This additional contribution is gauge variant and its origin can be traced back to the breaking of the $U(1)_A$ gauge symmetry due to the fermion mass term.

In the conformal limit the anomalous trace equation (3.135) takes a simpler form because, as we have already discussed in the previous sections, the $\langle TJ_AJ_A\rangle$ correlator reduces to the $\langle TJ_VJ_V\rangle$ and we obtain

$$g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}_{AA}(p,q)\Big|_{m=0}=g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}_{VV}(p,q)\Big|_{m=0}=-\frac{g^2}{6\pi^2}u^{\alpha\beta}(p,q). \tag{3.136}$$

We give in Appendix A.9 the general expression of the form factors R_i $(i=1,\ldots,9)$, which, combined with the results of the 13 form factors F_j, characterize completely the contributions to the effective action of a vector/axial-vector abelian theory mediated by the conformal anomaly.

Concerning the connection between the anomalous contribution and the β function of the theory, also in this case remain valid our previous conclusions, given in [51, 52]. Specifically, we just recall, at this point, that in the (mass independent) regularization scheme $\overline{MS}$ scheme, the e^2 term in the trace is directly related to the β function in this scheme since $\beta(g)=g^3/(12\pi^2)$. In particular, the form factor F_{13} is affected by renormalization via the electric charge [51] [52]. We close this section with few remarks concerning the structure of the effective action for

Figure 3.3: Polar form of the correlator for external on-shell lines: (a) the contribution to the spectral density from the collinear on-shell region of the anomaly loop; (b) the pole as virtual exchange in Γ^{anom}.

these types of theories, which can be identified from the variational integration of the anomaly equation [83]. This approach is, in a way, complementary to the strategy that we follow, based on a direct computation. As shown in [51] there is perfect agreement between the operatorial structure of variational solution, which also exhibits a $1/\Box$ effective interaction, and the anomaly pole found in our analysis. In the variational solution of [83], the $1/s$ massless exchange appears after a linearization of the same solution around the flat spacetime limit, as pointed out in [51]. In fact, one obtains in the weak gravitational field limit

$$S_{anom}[g, A] = -\frac{c}{6} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'}\, R^{(1)}_x \,\Box^{-1}_{x,x'} \,[F_{\alpha\beta} F^{\alpha\beta}]_{x'}\,, \tag{3.137}$$

$(c = -g^2/(24\pi^2))$. In this case

$$R^{(1)}_x \equiv \partial^x_\mu \,\partial^x_\nu \, h^{\mu\nu} - \Box\, h, \qquad h = \eta_{\mu\nu}\, h^{\mu\nu} \tag{3.138}$$

is the linearized scalar curvature. As in the case of the TJJ correlator [51] the anomalous contribution to the trace is all contained in the (conformal) anomaly pole (Fig. 3.3 b)

$$\Gamma^{\mu\nu\alpha\beta}_{anom}(p, q) = \int d^4x \int d^4y \, e^{ip\cdot x + iq\cdot y} \,\frac{\delta^2 T^{\mu\nu}_{anom}(0)}{\delta A_\alpha(x) \delta A_\beta(y)} = \frac{g^2}{18\pi^2} \frac{1}{k^2} \left(g^{\mu\nu} k^2 - k^\mu k^\nu\right) u^{\alpha\beta}(p, q)\,, \tag{3.139}$$

where [51]

$$T^{\mu\nu}_{anom}(z) = \frac{c}{3} \left(g^{\mu\nu}\Box - \partial^\mu \partial^\nu\right)_z \int d^4x' \,\Box^{-1}_{z,x'} \left[F_{\alpha\beta} F^{\alpha\beta}\right]_{x'}. \tag{3.140}$$

This effective action is trivially obtained from the tensor structure $F_1 t_1^{\mu\nu\alpha\beta}$, present in the expansion of $\Gamma^{\mu\nu\alpha\beta}$ and accounts for the full trace of the correlator in the massless fermion limit, as shown in Eq. (3.136).

3.6.1 Infrared couplings of the anomaly poles and UV behaviour

Before coming to conclusions, we pause here in order to comment on these results and on their meaning on a wider perspective.

We recall that a similar analysis in the QED case [51, 52] also manifests such pole singularities, which appear to be rather generic in anomaly amplitudes. They can be attributed, diagrammatically, to specific configurations of the loop momenta, as illustrated in Fig. (3.3). The diagram in this figure describes a massive external line decaying into two massless intermediate fermions, in turn decaying into two on-shell axial (or vector) lines (the equivalence between the axial and the vector case in the massless limit is the content of Eq. 3.93 ($\Omega \to 0$)).

The pole is detected by a computation of the spectral density ($\rho(s)$), which turns out to be proportional to a delta-function ($\rho(s) \sim \delta(s)$). $\rho(s)$ can be found just by evaluating the s-channel cut of the anomalous graph using Cutkovsky rules. This approach, as discussed before [51, 52], allows to identify the anomaly poles which are of infrared origin ($s \sim 0$). Other contributions, also characterized by form factors of the form $1/s$, as we have shown, appear in the anomalous amplitude when one performs an *off-shell* computation of the anomalous correlator. These contributions describe the UV behaviour of an anomalous amplitude ($s \to \infty$) and as such they are usually referred to as "ultraviolet poles", although the name is slightly misleading, being only generated after an asymptotic expansion of the massive correlator. In fact, the residue of the correlator as $s \to 0$ is indeed vanishing in the massive fermion case [52], showing that no pole is coupled in this limit. Apart from this important detail, it is however correct to retain their appearance in a perturbative computation - even in the UV region - as a manifestation of the same phenomenon of the trace anomaly. In the case of the chiral anomaly the situation is identical.

These computations [52] show that the asymptotic expansion - at large energy - of the regulated graphs responsible for the trace anomaly can be accompanied by corrections which are suppressed as m^2/s^2 (as $s \gg m^2$) in the high energy limit, where m is the mass of the fermion in the virtual loop. This organization of the effective action in the UV region allows to recover the ordinary radiative breaking of scale invariance at high energy, being mass corrections negligible in this regime. The use of a mass-independent regularization scheme, such as DRED or DR, is perfectly well taylored in this case, since the separation between pole term and mass corrections involves an asymptotic expansion (at high energy). In particular the β function computed in such schemes consistently accounts for the UV running of the coupling [52].

We have described this point at length in the case of the gauge anomaly in [40], to which we refer for more details. This implies that the anomaly is saturated by a pole in very different kinematical regions, in agreement with previous analysis performed in chiral theories [40, 42].

These conclusions show that the description of the effective action in terms of two auxiliary fields - which are introduced in order to recover the local form of the Lagrangian - is significant both in massless theories [51, 96] (for instance on null surfaces, i.e. $s = 0$), but also in the high energy domain, for large values of s. We refer to [51, 96] for a discussion of the auxiliary field formulation. Similar arguments have been presented in [30, 40, 24] for the axion pole in the chiral coupling of anomalous $U(1)$'s (in the AVV vertex), proving that these auxiliary degrees of freedom are the most significant signature of chiral and conformal anomalies.

3.7 Conclusions

We have presented in this chapter an off-shell computation of the correlator of the energy momentum tensor and two vector/axial-vector currents in a chiral theory with an anomalous fermion spectrum, useful for the study of the coupling of anomalous $U(1)$'s to gravity. These interactions are mediated by the trace anomaly. Starting directly from the functional integral, we have derived the Ward identities for the corresponding vertices. These apply, in general, to any correlator of similar type. All the computations have been performed using DRED, and we have shown the cancellation of mixed chiral/conformal anomalies for these types of vertices.

Our computation can be viewed as the generalization of the classical analysis of the AVV diagram to these new vertices and as the extension of the studies contained in the first and second chapter. We have allowed explicit mass breaking terms to investigate the most general form of the Ward identities for these correlators, that are of crucial importance for the more general analysis in the Standard Model case.

Obviously, the inclusion of this study into a theory with spontaneous symmetry breaking and Yukawa couplings, such as the Standard Model, would allow to relate the explicit chiral symmetry breaking terms (mass terms) to the extra interactions of the theory, in particular to the Higgs sector.

We have also shown that, similarly to the case of a vector-like theory studied in the second chapter, also in the case of a mixed vector/axial-vector theory, the effective action obtained by coupling gravity to the gauge currents is characterized by effective massless degrees of freedom. An extension of these analyses to the QCD case and then to the coupling of gravity to non-abelian gauge currents will be presented in next chapter.

Chapter 4

Trace Anomaly, Massless Scalars and the Gravitational Coupling of QCD

4.1 Introduction

The study of the effective action describing the coupling of a gauge theory to gravity via the trace anomaly [97] is an important aspect of quantum field theory, which is not deprived also of direct phenomenological implications. This coupling is mediated by the correlator involving the energy momentum tensor together with two vector currents (or TJJ vertex), which describes the interaction of a graviton with two photons or two gluons in QED and QCD, respectively. At the same time, the vertex has been at the center of an interesting case study of the renormalization properties of composite operators in Yang Mills theories [98], in the context of an explicit check of the violation of the Joglekar-Lee theorem [99] on the vanishing of S-matrix elements of BRST exact operators. In this second case it was computed on-shell, but only at zero momentum transfer. In this chapter we are going to extend this computation and investigate the presence of massless singularities in its expression. These contribute to the trace anomaly and play a leading role in fixing the structure of the effective action that couples QCD to gravity. The analysis of [98], which predates our study, unfortunately does not resolve the issue about the presence or the absence of the anomaly pole in the anomalous effective action of QCD because of the restricted kinematics involved in that analysis of the TJJ vertex, and for this reason we have to proceed with a complete re-computation.

As we have already stressed, anomaly poles characterize quite universally (gravitational and chiral) anomalous effective actions, in the sense that account for their anomalies. They have been identified and discussed in the abelian case both by a dispersive analysis [51] and by a direct explicit computation of the related anomalous Feynman amplitudes quite recently

[52, 92]. Extensive analysis in the case of chiral gauge theory for anomalous $U(1)$ models have shown the close parallel between solutions of the Ward identities, the coupling of the poles in the ultraviolet and in the infrared region and the gravity case [24, 40].

It is therefore important to check whether similar contributions appear also in non-abelian gauge theories coupled to gravity. We recall that the same pole structure is found in the variational solution of the expression of the trace anomaly, where one tries to identify an action whose energy momentum tensor reproduces the trace anomaly. This action, found by Riegert long ago [83], is nonlocal and involves the Green's function of a quartic (conformally covariant) operator. The action describes the structure of the singularities of anomalous correlators with any number of insertions of the energy momentum tensor and two photons (T^nJJ), which are expected to correspond both to single and to higher order poles, for a sufficiently high n. For obvious reasons, explicit checks of this effective action using perturbation theory - as the number of external graviton lines grows - becomes increasingly difficult to handle. The TJJ correlator is the first (leading) contribution to this infinite sum of correlators in which the anomalous gravitational effective action is expanded.

Given the presence of a quartic operator in Riegert's nonlocal action, the proof that this action contains a single pole to lowest order (in the TJJ vertex), once expanded around flat space, has been given in [51] by Giannotti and Mottola, and provides the basis for the discussion of the anomalous effective action in terms of massless auxiliary fields contained in their work. The auxiliary fields are introduced in order to rewrite the action in a local form. We show by an explicit computation of the lowest order vertex that Riegert's action is indeed consistent in the non-abelian case as well, since its pole structure is recovered in perturbation theory, similarly to the abelian case. Therefore, one can reasonably conjecture the presence of anomaly poles in each gauge invariant subsets of the diagrammatic expansion, as the computation for the non-abelian TJJ shows (here for the case of the single pole). In particular, this is in agreement with the result of [51], where it is shown that, after expanding around flat spacetime, the quartic operator in Riegert's action becomes a simple $1/\square$ nonlocal interaction (for the TJJ contribution), i.e. a pole term. We remark that the identification of a pole term in this and in others similar correlators, as we are going to emphasize in the following sections (at least in the case of QED and for the sector of QCD mediated by quark loops), requires an extrapolation to the massless fermion limit, and for this reason its interpretation as a long-range dynamical effect in the gravitational effective action requires some caution. In QCD, however, there is an extra sector that contributes to the same correlator, entirely due to virtual loops of gluons in the anomaly graphs, which remains unaffected by the massless fermion limit. The appearance of such a singularity in the effective action, however, does not necessarily imply that its contribution survives in the physical S-matrix. We will also establish the appearance of other singularities in the trace-free form factors which, obviously, are not part of Riegert's action.

We will comment at the end of this chapter on the possible implications of these results and on some recent proposals to link this type of behaviour [100, 101] to cosmology and to the dark energy problem. We also remark that, in general, the coefficient in front of the trace anomaly, for a given theory, can be computed in terms of its massless fields content, and as such it is well known. However, the structure of the effective action and the characterization of its fundamental form factors at nonzero momentum transfer and its complete analytical structure is a novel result. In this respect, the classification of all the relevant tensor structures which appear in the computation of this correlator is rather involved and has been performed in the completely off-shell case. We remark that the complexity of the final expression, in the off-shell case, prevents us from presenting its form. For this reason we will give only the on-shell version of the complete vertex, which is expressed, as we have mentioned, only in terms of three fundamental form factors.

Concerning the phenomenological relevance of this vertex, we just mention that it plays an essential role in the study of NLO corrections to processes involving a graviton exchange. In fact, in theories with extra dimension, where a low-gravity scale and the presence of Kaluza-Klein excitations may enhance the rates for processes mediated by gluons and gravitons, the vertex appears in the hard scattering of the corresponding factorization formula [102] and has been computed in dimensional regularization. However, to our knowledge, in all cases, there has been no separate discussion of the general structure of the vertex (i.e. as an amplitude) nor of its Ward identities, which, in principle, would require a more careful investigation because of the trace anomaly. Anomalous amplitudes, in fact, are defined by the fundamental Ward identities imposed on them, that we are going to derive from general principles. We cover this gap and show, that both dimensional regularization and dimensional reduction reproduce the correct Ward identity satisfied by this vertex, showing at the same time that the use of these regularizations is indeed appropriate. Results for this vertex will be given only in the on-shell case, since in this case the result can be expressed in terms of just three form factors. We have computed also the off-shell effective action, but its expression is rather lengthy and will not be discussed here, since it is gauge dependent and of less significance compared to the on-shell result. Most of our work is concerned with a technical derivation of the leading contribution to the anomalous effective action of QCD coupled to gravity. We have summarized in our conclusions a brief discussion of the relevance of this study in the ongoing attempt to link the trace anomaly and QCD to a possible alternative solution of the problem of dark energy, using this effective action as an intermediate step [100, 101].

4.2 Anomalous effective actions and their variational solutions

In this section we briefly review the topic of the variational solutions of anomalous effective actions, and on the local formulations of these using auxiliary fields.

One well known result of quantum gravity is that the effective action of the trace anomaly is given by a nonlocal form when expressed in terms of the spacetime metric $g_{\mu\nu}$. This was obtained [83] from a variational solution of the equation for the trace anomaly [97]

$$T^\mu_\mu = b\,F + b'\left(E - \frac{2}{3}\Box\,R\right) + b''\,\Box\,R + c\,F^{a\,\mu\nu}F^a_{\mu\nu}, \tag{4.1}$$

(see also [103, 104] for an analysis of the gravitational sector) which takes in $D = 4$ spacetime dimensions the form

$$S_{anom}[g, A] = \tag{4.2}$$
$$\frac{1}{8}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\left(E - \frac{2}{3}\Box R\right)_x \Delta_4^{-1}(x,x')\left[2b\,F + b'\left(E - \frac{2}{3}\Box R\right) + 2\,c\,F_{\mu\nu}F^{\mu\nu}\right]_{x'}.$$

Here, the parameters b and b' are the coefficients of the Weyl tensor squared,

$$F = C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3} \tag{4.3}$$

and the Euler density

$$E = {}^*\!R_{\lambda\mu\nu\rho}\,{}^*\!R^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{4.4}$$

respectively of the trace anomaly in a general background curved spacetime. Notice that the last term in (4.2) is the contribution generated in the presence of a background gauge field, with coefficient c. For a Dirac fermion in a classical gravitational ($g_{\mu\nu}$) and abelian (A_α) background, the values of the coefficients are $b = 1/(320\,\pi^2)$, and $b' = -11/(5760\,\pi^2)$, and $c = -e^2/(24\,\pi^2)$, with e being the electric charge of the fermion. One crucial feature of this solution is its origin, which is purely variational. Obtained by Riegert long ago, the action was derived by solving the variational equation satisfied by the trace of the energy momentum tensor. $\Delta_4^{-1}(x,x')$ denotes the Green's function inverse of the conformally covariant differential operator of fourth order, defined by

$$\Delta_4 \equiv \nabla_\mu\left(\nabla^\mu\nabla^\nu + 2R^{\mu\nu} - \frac{2}{3}Rg^{\mu\nu}\right)\nabla_\nu = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}(\nabla^\mu R)\nabla_\mu - \frac{2}{3}R\Box\,. \tag{4.5}$$

Given a solution of a variational equation, it is mandatory to check whether the solution is indeed justified by a perturbative computation. One specific feature of these solutions is the presence of anomaly poles. In the previous chapters we have elaborated on the significance of

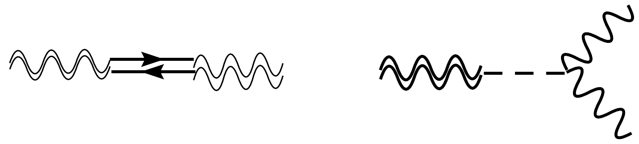


Figure 4.1: The diagrams describing the anomaly pole in the dispersive approach. Fig. (a) depicts the singularity of the spectral density $\rho(s)$ as a spacetime process. Fig. (b) describes the anomalous pole part of the interaction via the exchange of a pole.

these interactions, extracted from a direct perturbative computation, by a painstaking analysis of anomaly graphs under general kinematical conditions, and not just by a dispersive approach. The dispersive approach allows to connect this behaviour of the spectral density to a very specific infrared configuration.

4.2.1 The kinematics of an anomaly pole

In our conventions we will denote with p and q the outgoing momenta of the two photons/gluons and with k the incoming momentum of the graviton. $s \equiv (p+q)^2$ denotes the invariant mass of the external graviton line. A computation of the spectral density $\rho(s)$ of the TJJ amplitude in QED shows that this takes the form $\rho(s) \sim \delta(s)$. The configuration responsible for the appearance of a pole is illustrated in Fig. 4.1 (a). It describes the decay of a graviton line into two on-shell photons. The decay is mediated by a collinear and on-shell fermion-antifermion pair and can be interpreted as a spacetime process. The corresponding interaction vertex, described as the exchange of a pole, is instead shown in Fig. 4.1 (b). The actual process depicted in Fig. 4.1 (a) is obtained at diagrammatic level by setting on-shell the fermion/antifermion pair attached to the graviton line. This configuration, present in the spectral density of the diagram only for on-shell photons, generates a pole contribution which can be shown to be coupled in the infrared. This means that if we compute the residue of the amplitude for $s \to 0$ we find that it is non-vanishing. In the general expression of the vertex, a similar configuration is extracted in the high energy limit, not by a dispersive analysis, but by an explicit (off-shell) computation of the diagrams. Clearly, the pole, in this second case, has a vanishing residue as $s \to 0$, but is nevertheless a signature of the anomaly at high energy. Either for virtual or for real photons, a direct computation of the vertex allows to extract the pole term, without having to rely on a dispersive analysis. This point has been illustrated in our previous computations of the chiral anomaly vertex [40] and in the computation of the TJJ vertex for QED [52]. The identification of this singularity in the case of QCD is in perfect agreement with those previous results.

4.2.2 The single pole from Δ_4

In the case of the gravitational effective action, the appearance of the inverse of Δ_4 operator seems to be hard to reconcile with the simpler $1/\Box$ interaction which is predicted by the perturbative analysis of the TJJ correlator, which manifests a single anomaly pole. In [51], Giannotti and Mottola show step by step how a single pole emerges from this quartic operator, by using the auxiliary field formulation of the same effective action. Clearly, more computations are needed in order to show that the nonlocal effective action consistently does justice of *all* the poles (of second order and higher) which should be present in the perturbative expansion. Obviously, the perturbative computations - being either based on dispersion theory or on complete evaluations of the vertices, as in our case - become rather hard as we increase the number of external lines of the corresponding perturbative correlator. For instance, this check becomes almost impossible for correlators of the form $TTTT$ or higher, due to the appearance of a very large number of tensor structure in the reduction to scalar form of the tensor Feynman integrals. In the case of TJJ the computation is still manageable, since it does not require Feynman integrals beyond rank-4.

Expanding around flat space, the local formulation of Riegert's action, as shown in [51, 96], can be rewritten in the form

$$S_{anom}[g, A] \to -\frac{c}{6} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'}\, R_x \, \Box^{-1}_{x,x'} \, [F_{\alpha\beta} F^{\alpha\beta}]_{x'} \,, \tag{4.6}$$

which is valid to first order in the fluctuation of the metric around a flat background, denoted as $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \qquad\qquad \kappa = \sqrt{16\pi G_N} \tag{4.7}$$

with G_N being the 4-dimensional Newton's constant. The formulation in terms of auxiliary fields of this axion gives

$$S_{anom}[g, A; \varphi, \psi'] = \int d^4x \sqrt{-g} \left[-\psi' \Box \varphi - \frac{R}{3} \psi' + \frac{c}{2} F_{\alpha\beta} F^{\alpha\beta} \varphi \right] , \tag{4.8}$$

where ϕ and ψ are the auxiliary scalar fields. They satisfy the equations

$$\psi' \equiv b \Box \psi \,, \tag{4.9}$$

$$\Box \psi' = \frac{c}{2} F_{\alpha\beta} F^{\alpha\beta} \,, \tag{4.10}$$

$$\Box \varphi = -\frac{R}{3} \,. \tag{4.11}$$

In order to make contact with the TJJ amplitude, one needs the expression of the energy momentum extracted from (4.8) to leading order in $h_{\mu\nu}$, or, equivalently, from (4.6) that can be shown to take the form

$$T^{\mu\nu}_{anom}(z) = \frac{c}{3}\left(g^{\mu\nu}\Box - \partial^\mu\partial^\nu\right)_z \int d^4x' \, \Box^{-1}_{z,x'} \left[F_{\alpha\beta}F^{\alpha\beta}\right]_{x'} . \tag{4.12}$$

Notice that $T^{\mu\nu}_{anom}$ is the expression of the energy momentum tensor of the theory in the background of the gravitational and gauge fields. We recall, in fact, that in the QED case, for instance, the energy momentum tensor of the theory is split into the free fermionic part T_f, the interacting fermion-photon part T_{fp} and the photon contribution T_{ph} which are given by

$$T^{\mu\nu}_f = -i\bar\psi\gamma^{(\mu}\overset{\leftrightarrow}{\partial}{}^{\nu)}\psi + g^{\mu\nu}(i\bar\psi\gamma^\lambda\overset{\leftrightarrow}{\partial}_\lambda\psi - m\bar\psi\psi), \tag{4.13}$$

$$T^{\mu\nu}_{fp} = -\,eJ^{(\mu}A^{\nu)} + eg^{\mu\nu}J^\lambda A_\lambda\,, \tag{4.14}$$

and

$$T^{\mu\nu}_{ph} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho}, \tag{4.15}$$

where the current is defined as

$$J^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x)\,. \tag{4.16}$$

The connected components of TJJ can be obtained directly from the quantum average of T_p, defined as the sum of the fermion contribution and its interaction part with the photon field,

$$T^{\mu\nu}_p \equiv T^{\mu\nu}_f + T^{\mu\nu}_{fp}\,. \tag{4.17}$$

In the formalism of the background fields, the TJJ correlator then can be extracted from the defining functional integral

$$\begin{aligned}\langle T^{\mu\nu}_p(z)\rangle_A &\equiv \int D\psi D\bar\psi \; T^{\mu\nu}_p(z)\; e^{i\int d^4x\mathcal{L} + \int J\cdot A(x)d^4x} \\ &= \langle T^{\mu\nu}_p \, e^{i\int d^4x\, J\cdot A(x)}\rangle\end{aligned} \tag{4.18}$$

via two functional derivatives respect to the background field A_μ and generates the effective action

$$\Gamma^{\mu\nu\alpha\beta}(z;x,y) \equiv \left.\frac{\delta^2\,\langle T^{\mu\nu}_p(z)\rangle_A}{\delta A_\alpha(x)\delta A_\beta(y)}\right|_{A=0} = \Gamma^{\mu\nu\alpha\beta}_{anom} + \tilde\Gamma^{\mu\nu\alpha\beta}. \tag{4.19}$$

We have separated in (4.19) the pole contribution Γ_{anom} from the rest of the amplitude ($\tilde\Gamma$), which does not contribute to the trace part. Notice that Γ_{anom}, derived from either the classical generating functional (4.12) given by Riegert's action or from the direct perturbative expansion of (4.19), should nevertheless coincide, for the pole term not to be a spurious artifact of the variational solution. In particular, a computation performed in QED shows that the pole term extracted from T_{anom} via functional differentiation

$$\Gamma^{\mu\nu\alpha\beta}_{anom}(p,q) = \int d^4x \int d^4y\, e^{ip\cdot x+iq\cdot y}\, \frac{\delta^2 T^{\mu\nu}_{anom}(0)}{\delta A_\alpha(x) A_\beta(y)} = \frac{e^2}{18\pi^2}\frac{1}{k^2}\left(g^{\mu\nu}k^2 - k^\mu k^\nu\right) u^{\alpha\beta}(p,q) \tag{4.20}$$

with

$$u^{\alpha\beta}(p,q) \equiv (p\cdot q)\, g^{\alpha\beta} - q^\alpha\, p^\beta\,, \tag{4.21}$$

indeed coincides with the result of the perturbative expansion, as defined from the first term on the rhs of (4.19). Thus, the entire contribution to the anomaly is extracted form T_{anom} as

$$g_{\mu\nu}T^{\mu\nu}_{anom} = cF_{\alpha\beta}F^{\alpha\beta} = -\frac{e^2}{24\pi^2}F_{\alpha\beta}F^{\alpha\beta}. \tag{4.22}$$

As we have already mentioned, the full action (4.2), varied several times with respect to the background metric $g_{\mu\nu}$ and/or the background gauge fields A_α gives those parts of the correlators of higher order, such as $\langle TTT...JJ\rangle$ and $\langle TTT...\rangle$, which contribute to the trace anomaly. In particular, the anomalous contributions of the T^nJJ's vertices are obtained by varying the local action both respect to the metric and to the gauge fields.

4.3 The energy momentum tensor and the Ward identities

Moving to the QCD case, we introduce the definition of the QCD energy-momentum tensor, which is given by

$$\begin{aligned} T_{\mu\nu} &= -g_{\mu\nu}\mathcal{L}_{QCD} - F^a_{\mu\rho}F^{a\rho}_\nu - \frac{1}{\xi}g_{\mu\nu}\partial^\rho(A^a_\rho\partial^\sigma A^a_\sigma) + \frac{1}{\xi}(A^a_\nu\partial_\mu(\partial^\sigma A^a_\sigma) + A^a_\mu\partial_\nu(\partial^\sigma A^a_\sigma)) \\ &+ \frac{i}{4}\Big[\overline{\psi}\gamma_\mu(\overrightarrow{\partial}_\nu - igT^aA^a_\nu)\psi - \overline{\psi}(\overleftarrow{\partial}_\nu + igT^aA^a_\nu)\gamma_\mu\psi + \overline{\psi}\gamma_\nu(\overrightarrow{\partial}_\mu - igT^aA^a_\mu)\psi \\ &- \overline{\psi}(\overleftarrow{\partial}_\mu + igT^aA^a_\mu)\gamma_\nu\psi\Big] + \partial_\mu\overline{\omega}^a(\partial_\nu\omega^a - gf^{abc}A^c_\nu\omega^b) + \partial_\nu\overline{\omega}^a(\partial_\mu\omega^a - gf^{abc}A^c_\mu\omega^b), \end{aligned} \tag{4.23}$$

where $F^a_{\mu\nu}$ is the non-abelian field strength of the gauge field A

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{4.24}$$

and we have denoted with ω^a the Faddeev-Popov ghosts and with $\overline{\omega}^a$ the antighosts, while ξ is the gauge-fixing parameter. The T^a's are the gauge group generators in the fermion representation and f^{abc} are the antisymmetric structure constants. For later use, it is convenient to isolate the gauge-fixing and ghost dependent contributions from the entire tensor

$$\begin{aligned} T^{g.f.}_{\mu\nu} &= \frac{1}{\xi}\left[A^a_\nu \partial_\mu(\partial\cdot A^a) + A^a_\mu \partial_\nu(\partial\cdot A^a)\right] - \frac{1}{\xi} g_{\mu\nu}\left[-\frac{1}{2}(\partial\cdot A)^2 + \partial^\rho(A^a_\rho \partial\cdot A^a)\right], \quad (4.25)\\ T^{gh}_{\mu\nu} &= \partial_\mu \bar{\omega}^a D^{ab}_\nu \omega^b + \partial_\nu \bar{\omega}^a D^{ab}_\mu \omega^b - g_{\mu\nu}\partial^\rho \bar{\omega}^a D^{ab}_\rho \omega^b. \quad (4.26)\end{aligned}$$

The coupling of QCD to gravity in the weak gravitational field limit is given by the interaction Lagrangian

$$\mathcal{L}_{int} = -\frac{1}{2}\kappa\, h^{\mu\nu} T_{\mu\nu}. \quad (4.27)$$

Notice that $T_{\mu\nu}$ as defined in Eq. (4.23) is symmetric, while traceless for a massless theory. The symmetric expression can be easily found as suggested in [105], by coupling the theory to gravity and then defining it via a functional derivative with respect to the metric, recovering (4.23) in the flat spacetime case.

The conservation equation of the energy-momentum tensor takes the following form off-shell [106, 107]

$$\begin{aligned} \partial^\mu T_{\mu\nu} &= -\frac{\delta S}{\delta\psi}\partial_\nu\psi - \partial_\nu\bar{\psi}\frac{\delta S}{\delta\bar{\psi}} + \frac{1}{2}\partial^\mu\left(\frac{\delta S}{\delta\psi}\sigma_{\mu\nu}\psi - \bar{\psi}\sigma_{\mu\nu}\frac{\delta S}{\delta\bar{\psi}}\right) - \partial_\nu A^a_\mu \frac{\delta S}{\delta A^a_\mu}\\ &+ \partial_\mu\left(A^a_\nu \frac{\delta S}{\delta A^a_\mu}\right) - \frac{\delta S}{\delta\omega^a}\partial_\nu\omega^a - \partial_\nu\bar{\omega}^a\frac{\delta S}{\delta\bar{\omega}^a} \quad (4.28)\end{aligned}$$

where $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$. It is indeed conserved by using the equations of motion of the ghost, antighost and fermion/antifermion fields. The off-shell relation is particularly useful, since it can be inserted into the functional integral in order to derive some of the Ward identities satisfied by the correlator. In fact, the implications of the conservation of the energy-momentum tensor on the Green's functions can be exploited through the generating functional, obviously defined as

$$\begin{aligned} Z[J,\eta,\bar{\eta},\chi,\bar{\chi},h] = \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\omega\,\mathcal{D}\bar{\omega}\,\exp\Big\{ i\int d^4x\,(\mathcal{L} + J_\mu A^\mu\\ +\bar{\eta}\psi + \bar{\psi}\eta + \bar{\chi}\omega + \bar{\omega}\chi + h_{\mu\nu}T^{\mu\nu})\Big\}, \quad (4.29)\end{aligned}$$

where $\mathcal{L}$ is the standard QCD action and we have added the coupling of the energy-momentum tensor of the theory to the background gravitational field $h_{\mu\nu}$, which is the typical expression needed in the study of QCD coupled to gravity with a linear deviation from the flat metric. We have denoted with $J, \eta, \bar{\eta}, \chi, \bar{\chi}$ the sources of the gauge field A (J), the source of the fermion and antifermion fields ($\bar{\eta}, \eta$) and of the ghost and antighost fields ($\bar{\chi}$, χ) respectively. The generating functional W of the connected Green's functions is, as usual, denoted by

$$\exp i\, W[J,\eta,\bar{\eta},\chi,\bar{\chi},h] = \frac{Z[J,\eta,\bar{\eta},\chi,\bar{\chi},h]}{Z[0,0,0,0,0,0]} \tag{4.30}$$

(normalized to the vacuum functional) and the effective action, defined as the generating functional Γ of the 1-particle irreducible and truncated amplitudes. This is obviously obtained from W by a Legendre transformation respect to all the sources, except, in our case, $h_{\mu\nu}$, which is taken as a background external field

$$\Gamma[A_c,\bar{\psi}_c,\psi_c,\bar{\omega}_c,\omega_c,h] = W[J,\eta,\bar{\eta},\chi,\bar{\chi},h] - \int d^4x\left(J_\mu A^\mu_c + \bar{\eta}\psi_c + \bar{\psi}_c\eta + \bar{\chi}\omega_c + \bar{\omega}_c\chi\right). \tag{4.31}$$

The source fields are eliminated from the right hand side of Eq. (4.31) inverting the relations

$$A^\mu_c = \frac{\delta W}{\delta J_\mu}, \qquad \psi_c = \frac{\delta W}{\delta\bar{\eta}}, \qquad \bar{\psi}_c = \frac{\delta W}{\delta\eta}, \qquad \omega_c = \frac{\delta W}{\delta\bar{\chi}}, \qquad \bar{\omega}_c = \frac{\delta W}{\delta\chi} \tag{4.32}$$

so that the functional derivatives of the effective action Γ with respect to its independent variables are

$$\frac{\delta\Gamma}{\delta A^\mu_c} = -J_\mu, \qquad \frac{\delta\Gamma}{\delta\psi_c} = -\bar{\eta}, \qquad \frac{\delta\Gamma}{\delta\bar{\psi}_c} = -\eta, \qquad \frac{\delta\Gamma}{\delta\omega_c} = -\bar{\chi}, \qquad \frac{\delta\Gamma}{\delta\bar{\omega}_c} = -\chi, \tag{4.33}$$

and for the source $h_{\mu\nu}$ we have instead

$$\frac{\delta\Gamma}{\delta h_{\mu\nu}} = \frac{\delta W}{\delta h_{\mu\nu}}. \tag{4.34}$$

The conservation of the energy-momentum tensor summarized in Eq. (4.28) in terms of classical fields, can be re-expressed in a functional form by a differentiation of W with respect to $h_{\mu\nu}$ and the use of Eq. (4.28) under the functional integral. We obtain

$$\begin{aligned}\partial_\mu\frac{\delta W}{\delta h_{\mu\nu}} &= \bar{\eta}\,\partial_\nu\frac{\delta W}{\delta\bar{\eta}} + \partial_\nu\frac{\delta W}{\delta\eta}\eta - \frac{1}{2}\partial^\mu\left(\bar{\eta}\sigma_{\mu\nu}\frac{\delta W}{\delta\bar{\eta}} - \frac{\delta W}{\delta\eta}\sigma_{\mu\nu}\eta\right)\\ &+ \partial_\nu\frac{\delta W}{\delta J_\mu}J_\mu - \partial_\mu\left(\frac{\delta W}{\delta J_\mu}J_\nu\right) + \bar{\chi}\partial_\nu\frac{\delta W}{\delta\bar{\chi}} + \partial_\nu\frac{\delta W}{\delta\chi}\chi\,,\end{aligned} \tag{4.35}$$

and finally, for the one particle irreducible generating functional, this gives

$$\begin{aligned}\partial_\mu\frac{\delta\Gamma}{\delta h_{\mu\nu}} &= -\frac{\delta\Gamma}{\delta\psi_c}\partial^\nu\psi_c - \partial^\nu\bar{\psi}_c\frac{\delta\Gamma}{\delta\bar{\psi}_c} + \frac{1}{2}\partial_\mu\left(\frac{\delta\Gamma}{\delta\psi_c}\sigma^{\mu\nu}\psi_c - \bar{\psi}_c\sigma^{\mu\nu}\frac{\delta\Gamma}{\delta\bar{\psi}_c}\right)\\ &- \partial^\nu A^\mu_c\frac{\delta\Gamma}{\delta A^\mu_c} + \partial^\mu\left(A^\nu_c\frac{\delta\Gamma}{\delta A^\mu_c}\right) - \frac{\delta\Gamma}{\delta\omega_c}\partial^\nu\omega_c - \partial^\nu\bar{\omega}_c\frac{\delta\Gamma}{\delta\bar{\omega}_c},\end{aligned} \tag{4.36}$$

obtained from Eq. (4.35) with the help of Eqs. (4.32 - 4.34). We summarize below the relevant Ward identities that can be used in order to fix the expression of the correlator.

- **Single derivative general Ward identity**

  The Ward identities describing the conservation of the energy-momentum tensor for the one-particle irreducible Green's functions with an insertion of $T_{\mu\nu}$ can be obtained from the functional equation (4.36) by taking functional derivatives with respect to the classical fields. For example, the Ward identity for the graviton - gluon gluon vertex is obtained by differentiating Eq. (4.36) with respect to $A^a_{c\,\alpha}(x_1)$ and $A^b_{c\,\beta}(x_2)$ and then setting all the external fields to zero

$$\partial^\mu \langle T_{\mu\nu}(x) A^a_\alpha(x_1) A^b_\beta(x_2)\rangle_{trunc} = -\partial_\nu\, \delta^4(x_1 - x)\, D^{-1}_{\alpha\beta}(x_2, x) - \partial_\nu\, \delta^4(x_2 - x)\, D^{-1}_{\alpha\beta}(x_1, x) + \partial^\mu \Big(g_{\alpha\nu}\, \delta^4(x_1 - x)\, D^{-1}_{\beta\mu}(x_2, x) + g_{\beta\nu}\, \delta^4(x_2 - x)\, D^{-1}_{\alpha\mu}(x_1, x) \Big) \tag{4.37}$$

  where $D^{-1}_{\alpha\beta}(x_1, x_2)$ is the inverse gluon propagator defined as

$$D^{-1}_{\alpha\beta}(x_1, x_2) = \langle A_\alpha(x_1) A_\beta(x_2)\rangle_{trunc} = \frac{\delta^2 \Gamma}{\delta A^\alpha_c(x_1) \delta A^\beta_c(x_2)} \tag{4.38}$$

  and where we have omitted, for simplicity, both the colour indices and the symbol of the T-product. The first Ward identity (4.37) becomes

$$k^\mu \langle T_{\mu\nu}(k) A_\alpha(p) A_\beta(q)\rangle_{trunc} = q_\mu D^{-1}_{\alpha\mu}(p) g_{\beta\nu} + p_\mu D^{-1}_{\beta\mu}(q) g_{\alpha\nu} - q_\nu D^{-1}_{\alpha\beta}(p) - p_\nu D^{-1}_{\alpha\beta}(q)\,. \tag{4.39}$$

- **Trace Ward identity at zero momentum transfer**

  It is possible to extract a Ward identity for the trace of the energy-momentum tensor for the same correlation function using just Eq. (4.39). In fact, differentiating it with respect to p_μ (or q_μ) and then evaluating the resulting expression at zero momentum transfer ($p = -q$) we obtain the Ward identity in d spacetime dimensions

$$\langle T^\mu_\mu(0) A_\alpha(p) A_\beta(-p)\rangle_{trunc} = \left(2 - d + p \cdot \frac{\partial}{\partial p}\right) D^{-1}_{\alpha\beta}(p) \tag{4.40}$$

  where the number 2 counts the number of external gluon lines. For $d = 4$ and using the transversality of the one-particle irreducible self-energy, namely

$$D^{-1}_{\alpha\beta}(p) = (p^2 g_{\alpha\beta} - p_\alpha q_\beta)\Pi(p^2), \tag{4.41}$$

  the Ward identity in Eq. (4.40) simplifies to

$$\langle T^\mu_\mu(0) A_\alpha(p) A_\beta(-p)\rangle_{trunc} = 2p^2 (p^2 g_{\alpha\beta} - p_\alpha q_\beta) \frac{d\Pi}{dp^2}(p^2). \tag{4.42}$$

The trace Ward identity in Eq. (4.40) at zero momentum transfer can also be explicitly related to the β function and the anomalous dimensions of the renormalized theory. These enter through the renormalization group equation for the two-point function of the gluon. Defining the beta function and the anomalous dimensions as

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}, \qquad \gamma(g) = \mu \frac{\partial}{\partial \mu} \log \sqrt{Z_A}, \qquad m\, \gamma_m(g) = \mu \frac{\partial m}{\partial \mu} \tag{4.43}$$

and denoting with Z_A the wave function renormalization constant of the gluon field, with g the renormalized coupling, and with m the renormalized mass, the trace Ward identity can be related to these functions by the relation

$$\langle T^{\mu}_{\mu}(0) A_\alpha(p) A_\beta(-p)\rangle_{trunc} = \left[\beta(g)\frac{\partial}{\partial g} - 2\gamma(g) + m(\gamma_m(g) - 1)\frac{\partial}{\partial m}\right] D^{-1}_{\alpha\beta}(p). \tag{4.44}$$

- **Two-derivatives Ward identity via BRST symmetry**

We can exploit the BRST symmetry of the gauge-fixed lagrangian in order to derive some generalized Ward (Slavnov-Taylor) identities. We start by computing the BRST variation of the energy-momentum tensor, which is given by

$$\delta A^a_\mu = \lambda D^{ab}_\mu \omega^b, \tag{4.45}$$
$$\delta \omega^a = -\frac{1}{2} g \lambda f^{abc} \omega^b \omega^c, \tag{4.46}$$
$$\delta \bar\omega^a = -\frac{1}{\xi}(\partial^\mu A^a_\mu)\lambda, \tag{4.47}$$
$$\delta \psi = i g \lambda \omega^a t^a \psi, \tag{4.48}$$
$$\delta \bar\psi = -i g \bar\psi t^a \lambda \omega^a, \tag{4.49}$$

where λ is an infinitesimal Grassmann parameter.
A careful analysis of the energy-momentum tensor presented in Eq. (4.23) shows that the fermionic and the gauge part are gauge invariant and therefore invariant also under BRST. Instead the gauge-fixing and the ghost contributions must be studied in more detail. Using the transformation equations (4.45) and (4.47) in (4.26) one can prove the two identities

$$\lambda\, T^{g.f.}_{\mu\nu} = -A^a_\nu \partial_\mu \delta\bar\omega^a - A^a_\mu \partial_\nu \delta\bar\omega^a + g_{\mu\nu}\left[\frac{1}{2}\partial\cdot A^a \delta\bar\omega^a + A^a_\rho \partial^\rho \delta\bar\omega^a\right], \tag{4.50}$$
$$\lambda\, T^{gh}_{\mu\nu} = -\partial_\mu \bar\omega^a \delta A^a_\nu - \partial_\nu \bar\omega^a \delta A^a_\mu + g_{\mu\nu} \partial^\rho \bar\omega^a \delta A^a_\rho, \tag{4.51}$$

which show that the ghost and the gauge-fixing parts of the energy-momentum tensor (times the anticommuting factor λ) can be written as an appropriate BRST variation of ghost/antighost and gauge contributions. Their sum, instead, can be expressed as the BRST variation of a certain operator plus an extra term which vanishes when using the ghost equations of motion

$$\lambda\left(T^{g.f.}_{\mu\nu}+T^{gh}_{\mu\nu}\right) = \delta\left[-\partial_\mu\bar{\omega}^a A^a_\nu-\partial_\nu\bar{\omega}^a A^a_\mu+g_{\mu\nu}\left(A^a_\rho\partial_\rho\bar{\omega}^a+\frac{1}{2}\partial\cdot A^a\omega^a\right)\right] + g_{\mu\nu}\frac{1}{2}\lambda\bar{\omega}^a\partial^\rho D^{ab}_\rho\omega^b, \qquad (4.52)$$

which shows explicitly the structure of the gauge-variant terms in the energy-momentum tensor. Using the nilpotency of the BRST operator ($\delta^2=0$), the BRST variation of $T_{\mu\nu}$ is given by

$$\delta T_{\mu\nu}=\delta(T^{g.f.}_{\mu\nu}+T^{gh}_{\mu\nu})=\frac{\lambda}{\xi}\left[A^a_\mu\partial_\nu\partial^\rho D^{ab}_\rho\omega^b+A^a_\nu\partial_\mu\partial^\rho D^{ab}_\rho\omega^b-g_{\mu\nu}\partial^\sigma(A^a_\sigma\partial^\rho D^{ab}_\rho\omega^b)\right], \qquad (4.53)$$

where it is straightforward to recognize the equation of motion of the ghost field on its right-hand side. Using this last relation, we are able to derive some constraints on the Green's functions involving insertions of the energy-momentum tensor. In particular, we are interested in some identities satisfied by the correlator $\langle T_{\mu\nu}A^a_\alpha A^b_\beta\rangle$ in order to define it unambiguously. For this purpose, it is convenient to choose an appropriate Green's function, in our case this is given by $\langle T_{\mu\nu}\partial^\alpha A^a_\alpha\bar{\omega}^b\rangle$, and then exploit its BRST invariance to obtain

$$\delta\langle T_{\mu\nu}\partial^\alpha A^a_\alpha\bar{\omega}^b\rangle=\langle\delta T_{\mu\nu}\partial^\alpha A^a_\alpha\bar{\omega}^b\rangle+\lambda\langle T_{\mu\nu}\partial^\alpha D^{ac}_\alpha\omega^c\bar{\omega}^b\rangle-\frac{\lambda}{\xi}\langle T_{\mu\nu}\partial^\alpha A^a_\alpha\partial^\beta A^b_\beta\rangle=0, \qquad (4.54)$$

where the first two correlators, built with operators proportional to the equations of motion, contribute only with disconnected amplitudes, that are not part of the one-particle irreducible vertex function. From Eq. (4.54) we obtain the identity

$$\partial^\alpha_{x_1}\partial^\beta_{x_2}\langle T_{\mu\nu}(x)A^a_\alpha(x_1)A^b_\beta(x_2)\rangle_{trunc}=0, \qquad (4.55)$$

which in momentum space becomes

$$p^\alpha q^\beta\langle T_{\mu\nu}(k)A^a_\alpha(p)A^b_\beta(q)\rangle_{trunc}=0. \qquad (4.56)$$

A subtlety in these types of derivations concerns the role played by the commutators, which are generated because of the T-product and can be ignored only if they vanish. In general, in fact, the derivatives are naively taken out of the correlator, in order to arrive at Eq. (4.56) and this can generate an error. In this case, due to the presence of an energy momentum tensor, the evaluation of these terms is rather involved. For this reason one needs to perform an explicit check of Eq. (4.56) to ensure the consistency of the formal result in a suitable regularization scheme. As we are going to show in the next sections, these three Ward identities turn out to be satisfied in dimensional regularization.

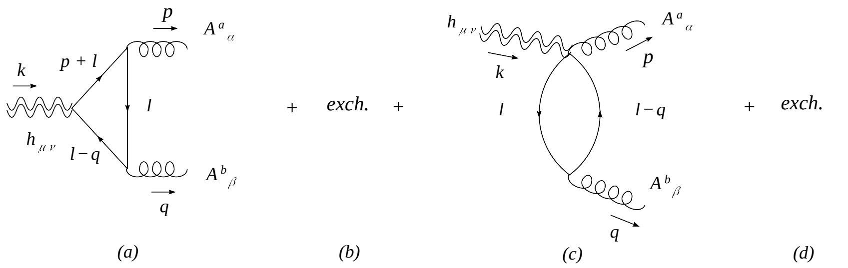


Figure 4.2: The fermionic contributions with a graviton $h_{\mu\nu}$ in the initial state and two gluons A^a_α, A^b_β in the final state.

4.4 The perturbative expansion

The perturbative expansion is obtained by taking into account all the diagrams depicted in Figs. 4.2, 4.3, 4.4, where an incoming graviton appears in the initial state and two gluons with momenta p and q characterize the final state. The different contributions to the total amplitude are identified by the nature of the internal lines and are computed with the aid of the Feynman rules defined in Appendix A.11. Each amplitude is denoted by Γ, with a superscript in square brackets indicating the figure of the corresponding diagram.

The contributions with a massive fermion running in the loop are depicted in Fig. 4.2; for the triangle in Fig. 4.2a we obtain

$$
\begin{aligned}
-i\frac{\kappa}{2}\Gamma^{[\mathbf{2a}]\,ab}_{\mu\nu\alpha\beta}(p,q) \;=\; & -\frac{\kappa}{2}g^2\,\mathrm{tr}(T^bT^a)\int\frac{d^4l}{(2\pi)^4}\,\cdot \\
& \cdot\mathrm{tr}\left\{V'_{\mu\nu}(l-q,l+p)\frac{1}{\not l-\not q-m}\gamma_\beta\frac{1}{\not l-m}\gamma_\alpha\frac{1}{\not l+\not p-m}\right\}
\end{aligned}
\tag{4.57}
$$

where the color factor is given by $\mathrm{tr}(T^bT^a)=\frac{1}{2}\delta^{ab}$; the diagram in Fig. 4.2c contributes as

$$
-i\frac{\kappa}{2}\Gamma^{[\mathbf{2c}]\,ab}_{\mu\nu\alpha\beta}(p,q) = -\frac{\kappa}{2}g^2\,\mathrm{tr}(T^aT^b)\int\frac{d^4l}{(2\pi)^4}\mathrm{tr}\left\{W'_{\mu\nu\alpha}\frac{1}{\not l-\not q-m}\gamma_\beta\frac{1}{\not l-m}\right\}, \tag{4.58}
$$

with the vertices $V'_{\mu\nu}(l-q,l+p)$ and $W'_{\mu\nu\alpha}$ defined in Appendix A.11, Eqs. (A.138) and (A.141) respectively. The remaining diagrams in Fig. 4.2 are obtained by exchanging $\alpha\leftrightarrow\beta$ and $p\leftrightarrow q$

$$
-i\frac{\kappa}{2}\Gamma^{[\mathbf{2b}]\,ab}_{\mu\nu\alpha\beta}(p,q) \;=\; -i\frac{\kappa}{2}\Gamma^{[\mathbf{2a}]\,ab}_{\mu\nu\alpha\beta}(p,q)\bigg|_{\substack{\alpha\leftrightarrow\beta\\ p\leftrightarrow q}}\,, \tag{4.59}
$$

$$
-i\frac{\kappa}{2}\Gamma^{[\mathbf{2d}]\,ab}_{\mu\nu\alpha\beta}(p,q) \;=\; -i\frac{\kappa}{2}\Gamma^{[\mathbf{2c}]\,ab}_{\mu\nu\alpha\beta}(p,q)\bigg|_{\substack{\alpha\leftrightarrow\beta\\ p\leftrightarrow q}}\,. \tag{4.60}
$$

Moving to the gauge sector we find the four contributions in Fig. 4.3: the first one with a

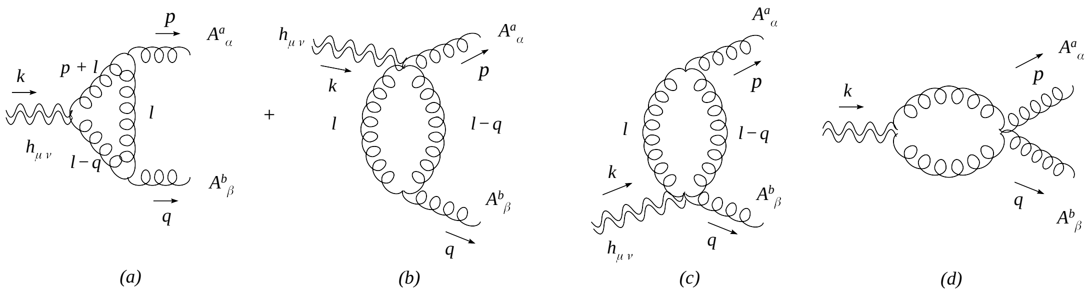


Figure 4.3: The gauge contributions with a graviton $h_{\mu\nu}$ in the initial state and two gluons A^a_α, A^b_β in the final state.

triangular topology is given by

$$
- i\frac{\kappa}{2}\,\Gamma^{[\mathbf{3a}]\,ab}_{\mu\nu\alpha\beta}(p,q) = -\frac{\kappa}{2}g^2 f^{ade}f^{bde}\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2\,(l+p)^2\,(l-q)^2}\bigg[V^{Ggg}_{\mu\nu\rho\sigma}(l-q,-l-p)\,\times
$$
$$
V^3_{\tau\sigma\alpha}(-l,l+p,-p)\;V^3_{\rho\tau\beta}(-l+q,l,-q)\bigg], \quad (4.61)
$$

where the color factor is $f^{ade}f^{bde} = C_A\,\delta^{ab}$. Those in Figs. 4.3b and 4.3c, containing gluon loops attached to the graviton vertex, are called "t-bubbles" and can be obtained one from the other by the exchange of $\alpha \leftrightarrow \beta$ and $p \leftrightarrow q$. The first "t-bubble" is given by

$$
- i\frac{\kappa}{2}\,\Gamma^{[\mathbf{3b}]\,ab}_{\mu\nu\alpha\beta}(p,q) = -\frac{1}{2}\frac{\kappa}{2}g^2 f^{ade}f^{bde}\int\frac{d^4l}{(2\pi)^4}\frac{V^{Gggg}_{\mu\nu\rho\sigma\beta}(-l,l-p,-q)\,V^3_{\rho\alpha\sigma}(k,-p,-l+p)}{l^2\,(l-p)^2} \quad (4.62)
$$

which is multiplied by an additional symmetry factor $\frac{1}{2}$. There is another similar contribution obtained from the previous one after exchanging $\alpha \leftrightarrow \beta$ and $p \leftrightarrow q$

$$
- i\frac{\kappa}{2}\,\Gamma^{[\mathbf{3c}]\,ab}_{\mu\nu\alpha\beta}(p,q) = -i\frac{\kappa}{2}\,\Gamma^{[\mathbf{3b}]\,ab}_{\mu\nu\alpha\beta}(p,q)\bigg|_{\substack{\alpha\leftrightarrow\beta\\ p\leftrightarrow q}}\,. \quad (4.63)
$$

The last diagram with gluons running in the loop is the one in Fig. 4.3d which is given by

$$
- i\frac{\kappa}{2}\,\Gamma^{[\mathbf{3d}]ab}_{\mu\nu\alpha\beta}(p,q) = \frac{1}{2}\frac{\kappa}{2}g^2\int\frac{d^4l}{(2\pi)^4}\frac{V^{Ggg}_{\mu\nu\rho\sigma}(-l,l-p-q)\,\delta^{df}\;V^{4\,abcd}_{\rho\alpha\sigma\beta}}{l^2\,(l-p-q)^2}, \quad (4.64)
$$

where V^4 is the four gluon vertex defined as

$$
\begin{aligned}
- ig^2 V^{4\,abcd}_{\mu\nu\rho\sigma} \;=\; & -ig^2\Big[f^{abe}f^{cde}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}) + f^{ace}f^{bde}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\nu\rho}) \\
& + f^{ade}f^{bce}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\rho}g_{\nu\sigma})\Big]
\end{aligned} \quad (4.65)
$$

and therefore

$$
\delta^{df}\,V^{4\,abcd}_{\rho\alpha\sigma\beta} \;=\; -C_A\delta^{ab}\tilde{V}^4_{\rho\alpha\sigma\beta} = -C_A\delta^{ab}\left(g_{\alpha\sigma}g_{\beta\rho}+g_{\alpha\rho}g_{\beta\sigma}-2g_{\alpha\beta}g_{\sigma\rho}\right), \quad (4.66)
$$

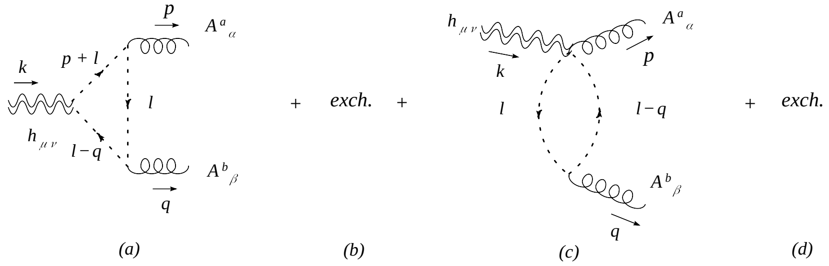


Figure 4.4: The ghost contributions with a graviton $h_{\mu\nu}$ in the initial state and two gluons A^a_α, A^b_β in the final state.

so that the amplitude in Eq. (4.64) becomes

$$-\,i\frac{\kappa}{2}\,\Gamma^{[\mathbf{3d}]\,ab}_{\mu\nu\alpha\beta}(p,q) = -\frac{1}{2}\frac{\kappa}{2}g^2 C_A\delta^{ab}\int\frac{d^4l}{(2\pi)^4}\frac{V^{Ggg}_{\mu\nu\rho\sigma}(-l,l-p-q)\,\tilde{V}^4_{\rho\alpha\sigma\beta}}{l^2\,(l-p-q)^2}. \tag{4.67}$$

In the expression above we have explicitly isolated the color factor $C_A\delta^{ab}$ and the symmetry factor $\frac{1}{2}$.

Finally, the ghost contributions shown in Fig. 4.4 are given by the sum of

$$-\,i\frac{\kappa}{2}\,\Gamma^{[\mathbf{4a}]\,ab}_{\mu\nu\alpha\beta}(p,q) = -\frac{\kappa}{2}g^2 f^{ade}f^{bde}\int\frac{d^4l}{(2\pi)^4}\frac{C_{\mu\nu\rho\sigma}(l-q)^\rho(l+p)^\sigma l_\alpha(l-q)_\beta}{l^2\,(l+p)^2\,(l-q)^2} \tag{4.68}$$

for the triangle diagram in Fig. 4.4a and

$$-\,i\frac{\kappa}{2}\,\Gamma^{[\mathbf{4b}]\,ab}_{\mu\nu\alpha\beta}(p,q) = \frac{\kappa}{2}g^2 f^{ade}f^{bde}\int\frac{d^4l}{(2\pi)^4}\frac{C_{\mu\nu\alpha\sigma}l^\sigma(l-q)_\beta}{l^2\,(l-q)^2} \tag{4.69}$$

for the "T-bubble" diagram shown in Fig. 4.4c. The two exchanged diagrams are obtained from those in Eqs. (4.68) and (4.69) with the usual replacement $\alpha\leftrightarrow\beta$ and $p\leftrightarrow q$.

$$-\,i\,\frac{\kappa}{2}\,\Gamma^{[\mathbf{4b}]\,ab}_{\mu\nu\alpha\beta}(p,q) \;=\; -i\,\frac{\kappa}{2}\,\Gamma^{[\mathbf{4a}]\,ab}_{\mu\nu\alpha\beta}(p,q)\Big|_{\substack{\alpha\leftrightarrow\beta\\ p\leftrightarrow q}}\,, \tag{4.70}$$

$$-i\,\frac{\kappa}{2}\,\Gamma^{[\mathbf{4d}]\,ab}_{\mu\nu\alpha\beta}(p,q) \;=\; -i\,\frac{\kappa}{2}\,\Gamma^{[\mathbf{4c}]\,ab}_{\mu\nu\alpha\beta}(p,q)\Big|_{\substack{\alpha\leftrightarrow\beta\\ p\leftrightarrow q}}\,. \tag{4.71}$$

Having identified the different sectors we obtain the total amplitude for quarks, denoted by a "q" subscript

$$\Gamma^{ab}_{q,\,\mu\nu\alpha\beta}(p,q) = \Gamma^{[\mathbf{2a}]\,ab}_{\mu\nu\alpha\beta}(p,q) + \Gamma^{[\mathbf{2b}]\,ab}_{\mu\nu\alpha\beta}(p,q) + \Gamma^{[\mathbf{2c}]\,ab}_{\mu\nu\alpha\beta}(p,q) + \Gamma^{[\mathbf{2d}]\,ab}_{\mu\nu\alpha\beta}(p,q) \tag{4.72}$$

and the one for gluons and ghosts as

$$\Gamma^{ab}_{g,\,\mu\nu\alpha\beta}(p,q) = \sum_{j=3,4}\Big[\Gamma^{[\mathbf{ja}]\,ab}_{\mu\nu\alpha\beta}(p,q) + \Gamma^{[\mathbf{jb}]\,ab}_{\mu\nu\alpha\beta}(p,q) + \Gamma^{[\mathbf{jc}]\,ab}_{\mu\nu\alpha\beta}(p,q) + \Gamma^{[\mathbf{jd}]\,ab}_{\mu\nu\alpha\beta}(p,q)\Big]\,. \tag{4.73}$$

4.5 The on-shell $\langle TAA\rangle$ correlator, pole terms and form factors

We proceed with a classification of all the diagrams contributing to the on-shell vertex, starting from the gauge invariant subset of diagrams that involve fermion loops and then moving to the second set, the one relative to gluons and ghosts. The analysis follows rather closely the method presented in the case of QED in previous works [51, 52], with a classification of all the relevant tensor structures which can be generated using the 43 monomials built out of the 2 of the 3 external momenta of the triangle diagram and the metric tensor $g_{\mu\nu}$. In general, one can proceed with the identification of a subset of these tensor structure which allow to formulate the final expression in a manageable form. The fermionic triangle diagrams, which define one of the two gauge invariant subsets of the entire correlator, can be given in a simplified form also for off mass-shell external momenta, in terms of 13 form factors as in [51, 52] while the structure of the gluon contributions are more involved. Some drastic semplifications take place in the on-shell case, where only 3 form factors - both in the quark and fermion sectors - are necessary to describe the final result.

We write the whole amplitude $\Gamma^{\mu\nu\alpha\beta}(p,q)$ as

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \Gamma_q^{\mu\nu\alpha\beta}(p,q) + \Gamma_g^{\mu\nu\alpha\beta}(p,q), \tag{4.74}$$

referring respectively to the contributions with quarks (Γ_q) and with gluons/ghosts (Γ_g) in Eqs. (4.72) and (4.73). We have omitted the color indices for simplicity. The amplitude Γ is expressed in terms of 3 tensor structures and 3 form factors renormalized in the $\overline{MS}$ scheme

$$\Gamma_{q/g}^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{3} \Phi_{i\,q/g}(s,0,0,m^2)\,\delta^{ab}\,\phi_i^{\mu\nu\alpha\beta}(p,q)\,. \tag{4.75}$$

One comment concerning the choice of this basis is in order. The 3 form factors are more easily identified in the fermion sector after performing the on-shell limit of the off-shell amplitude, where the 13 form factors introduced in [51, 52] for QED simplify into the 3 tensor structures that will be given below. It is then observed that the tensor structure of the gluon sector, originally expressed in terms of the 43 monomials of [51, 52], can be arranged consistently in terms of these 3 reduced structures.

The tensor basis on which we expand the on-shell vertex is given by

$$\phi_1^{\mu\nu\alpha\beta}(p,q) = (s\,g^{\mu\nu} - k^\mu k^\nu)\,u^{\alpha\beta}(p,q), \tag{4.76}$$

$$\phi_2^{\mu\nu\alpha\beta}(p,q) = -2\,u^{\alpha\beta}(p,q)\,[s\,g^{\mu\nu} + 2(p^\mu\,p^\nu + q^\mu\,q^\nu) - 4\,(p^\mu\,q^\nu + q^\mu\,p^\nu)]\,, \tag{4.77}$$

$$\begin{aligned}\phi_3^{\mu\nu\alpha\beta}(p,q) = {} & \big(p^\mu q^\nu + p^\nu q^\mu\big)g^{\alpha\beta} + \frac{s}{2}\left(g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu}\right) \\ & - g^{\mu\nu}\left(\frac{s}{2}g^{\alpha\beta} - q^\alpha p^\beta\right) - \left(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu\right)q^\alpha - \big(g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu\big)p^\beta,\end{aligned} \tag{4.78}$$

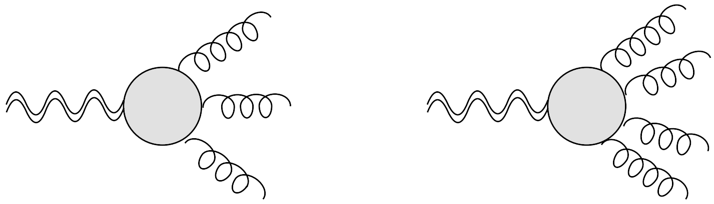


Figure 4.5: Higher order contributions to the anomaly pole involved in the covariantization of the graviton/2-gluons amplitude.

where $u^{\alpha\beta}(p,q)$ has been defined in Eq. (4.21). The form factors $\Phi_i(s, s_1, s_2, m^2)$ have as entry variables, beside $s = (p+q)^2$, the virtualities of the two gluons $s_1 = p^2$ and $s_2 = q^2$.
In the on-shell case only 3 invariant amplitudes contribute, which for the quark loop amplitude are given by

$$\Phi_{1\,q}(s,\,0,\,0,\,m^2) = -\frac{g^2}{36\pi^2 s} + \frac{g^2 m^2}{6\pi^2 s^2} - \frac{g^2\,m^2}{6\pi^2 s}\mathcal{C}_0(s,0,0,m^2)\left[\frac{1}{2} - \frac{2m^2}{s}\right], \tag{4.79}$$

$$\Phi_{2\,q}(s,\,0,\,0,\,m^2) = -\frac{g^2}{288\pi^2 s} - \frac{g^2 m^2}{24\pi^2 s^2} - \frac{g^2\,m^2}{8\pi^2 s^2}\mathcal{D}(s,0,0,m^2) - \frac{g^2\,m^2}{12\pi^2 s}\mathcal{C}_0(s,0,0,m^2)\left[\frac{1}{2} + \frac{m^2}{s}\right], \tag{4.80}$$

$$\Phi_{3\,q}(s,\,0,\,0,\,m^2) = \frac{11 g^2}{288\pi^2} + \frac{g^2 m^2}{8\pi^2 s} + g^2\mathcal{C}_0(s,0,0,m^2)\left[\frac{m^4}{4\pi^2 s} + \frac{m^2}{8\pi^2}\right] + \frac{5\,g^2\,m^2}{24\pi^2 s}\mathcal{D}(s,0,0,m^2) + \frac{g^2}{24\pi^2}\mathcal{B}_0^{\overline{MS}}(s,m^2), \tag{4.81}$$

where the on-shell scalar integrals $\mathcal{D}(s,0,0,m^2)$, $\mathcal{C}_0(s,0,0,m^2)$ and $B_0^{\overline{MS}}(s,m^2)$ are computed in Appendix A.2. In the massless limit the amplitude $\Gamma_q^{\mu\nu\alpha\beta}(p,q)$ takes a simpler expression and the previous form factors become

$$\Phi_{1\,q}(s,0,0,0) = -\frac{g^2}{36\pi^2 s}, \tag{4.82}$$

$$\Phi_{2\,q}(s,0,0,0) = -\frac{g^2}{288\pi^2\, s}, \tag{4.83}$$

$$\Phi_{3\,q}(s,0,0,0) = -\frac{g^2}{288\pi^2}\left[12 L_s - 35\right], \tag{4.84}$$

where

$$L_s \equiv \log\left(-\frac{s}{\mu^2}\right) \qquad s < 0. \tag{4.85}$$

In the gluon sector the computation of $\Gamma_g^{\mu\nu\alpha\beta}(p,q)$ is performed analogously by using dimensional regularization with modified minimal subtraction ($\overline{MS}$) and we obtain for on-shell gluons

$$\Gamma_g^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{3} \Phi_{i\,g}(s,0,0)\,\delta^{ab}\,\phi_i^{\mu\nu\alpha\beta}(p,q)\,, \tag{4.86}$$

where the form factors obtained from the explicit computation are

$$\Phi_{1\,g}(s,0,0) = \frac{11\,g^2}{72\pi^2\,s}\,C_A, \tag{4.87}$$

$$\Phi_{2\,g}(s,0,0) = \frac{g^2}{288\pi^2\,s}\,C_A, \tag{4.88}$$

$$\Phi_{3\,g}(s,0,0) = -g^2\,C_A\left[\frac{65}{288\pi^2} + \frac{11}{48\pi^2}\mathcal{B}_0^{\overline{MS}}(s,0) - \frac{1}{8\pi^2}\mathcal{B}_0^{\overline{MS}}(0,0) + \frac{s}{8\pi^2}\,\mathcal{C}_0(s,0,0,0)\right]. \tag{4.89}$$

The renormalized scalar integrals can be found in Appendix A.2.

The full on-shell vertex, which is the sum of the quark and pure gauge contributions, can be decomposed by using the same three tensor structures $\phi_i^{\mu\nu\alpha\beta}$ appearing in the expansion of $\Gamma_q^{\mu\nu\alpha\beta}(p,q)$ and $\Gamma_g^{\mu\nu\alpha\beta}(p,q)$

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \Gamma_g^{\mu\nu\alpha\beta}(p,q) + \Gamma_q^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{3} \Phi_i(s,0,0)\,\delta^{ab}\,\phi_i^{\mu\nu\alpha\beta}(p,q)\,, \tag{4.90}$$

with form factors defined as

$$\Phi_i(s,0,0) = \Phi_{i,\,g}(s,0,0) + \sum_{j=1}^{n_f} \Phi_{i,\,q}(s,0,0,m_j^2), \tag{4.91}$$

where the sum runs over the n_f quark flavors. In particular we have

$$\Phi_1(s,0,0) = -\frac{g^2}{72\pi^2\,s}\,(2n_f - 11C_A) + \frac{g^2}{6\pi^2}\sum_{i=1}^{n_f} m_i^2\left\{\frac{1}{s^2} - \frac{1}{2s}\mathcal{C}_0(s,0,0,m_i^2)\left[1 - \frac{4m_i^2}{s}\right]\right\}, \tag{4.92}$$

$$\begin{aligned}\Phi_2(s,0,0) &= -\frac{g^2}{288\pi^2\,s}\,(n_f - C_A) \\ &- \frac{g^2}{24\pi^2}\sum_{i=1}^{n_f} m_i^2\left\{\frac{1}{s^2} + \frac{3}{s^2}\mathcal{D}(s,0,0,m_i^2) + \frac{1}{s}\mathcal{C}_0(s,0,0,m_i^2)\,\left[1 + \frac{2m_i^2}{s}\right]\right\},\end{aligned} \tag{4.93}$$

$$\begin{aligned}\Phi_3(s,0,0) &= \frac{g^2}{288\pi^2}\,(11n_f - 65C_A) - \frac{g^2\,C_A}{8\pi^2}\left[\frac{11}{6}\mathcal{B}_0^{\overline{MS}}(s,0) - \mathcal{B}_0^{\overline{MS}}(0,0) + s\,\mathcal{C}_0(s,0,0,0)\right] \\ &+ \frac{g^2}{8\pi^2}\sum_{i=1}^{n_f}\left\{\frac{1}{3}\mathcal{B}_0^{\overline{MS}}(s,m_i^2) + m_i^2\left[\frac{1}{s} + \frac{5}{3s}\mathcal{D}(s,0,0,m_i^2)\right.\right. \\ &\qquad\left.\left. + \,\mathcal{C}_0(s,0,0,m_i^2)\,\left[1 + \frac{2m_i^2}{s}\right]\right]\right\},\end{aligned} \tag{4.94}$$

with $C_A = N_C$ and the scalar integrals defined in Appendix A.2. Notice the appearance in the total amplitude of the $1/s$ pole in Φ_1, which is present both in the quark and in the gluon sectors, and which saturates the contribution to the trace anomaly in the massless limit. In this case the entire trace anomaly is just proportional to this component, which becomes

$$\Phi_1(s,0,0) = -\frac{g^2}{72\pi^2\, s}\,(2n_f - 11C_A)\,. \tag{4.95}$$

The correlator $\Gamma^{\mu\nu\alpha\beta}(p,q)$, computed using dimensional regularization, satisfies all the Ward identities defined in the previous sections. Notice that the two-derivatives Ward identity introduced in Eq. (4.56)

$$p_\alpha q_\beta\, \Gamma^{\mu\nu\alpha\beta}(p,q) = 0, \tag{4.96}$$

derived from the BRST symmetry of the QCD Lagrangian, is straightforwardly satisfied by the on-shell amplitude. This is easily seen from the tensor decomposition introduced in Eq. (4.75) because all the tensors fulfill the condition

$$p_\alpha q_\beta\, \phi_1^{\mu\nu\alpha\beta}(p,q) = 0. \tag{4.97}$$

Furthermore, we have checked at one-loop order the validity of the single derivative Ward identity given in Eq. (4.39) and describing the conservation of the energy-momentum tensor. Using the transversality of the two-point gluon function Eq. (4.39) this gives

$$\begin{aligned} k_\mu\, \Gamma^{\mu\nu\alpha\beta}(p,q) = {} & \Big(q^\nu\, p^\alpha\, p^\beta - q^\nu\, g^{\alpha\beta}\, p^2 + g^{\nu\beta}\, q^\alpha\, p^2 - g^{\nu\beta}\, p^\alpha\, p\cdot q\Big)\; \Pi(p^2) \\ & + \Big(p^\nu\, q^\alpha\, q^\beta - p^\nu\, g^{\alpha\beta}\, q^2 + g^{\nu\alpha}\, p^\beta\, q^2 - g^{\nu\alpha}\, q^\beta\, p\cdot q\Big)\; \Pi(q^2), \end{aligned} \tag{4.98}$$

where the renormalized gluon self energies are defined as

$$\begin{aligned} \Pi(p^2) &= \frac{g^2\, C_A\, \delta^{ab}}{144\,\pi^2}\, \Big(15\, \mathcal{B}_0^{\overline{MS}}(p^2,0) - 2\Big) \\ &+ \frac{g^2\,\delta^{ab}}{72\,\pi^2 p^2} \sum_{i=1}^{n_f} \Big[6\,\mathcal{A}_0^{\overline{MS}}(m_i^2) + p^2 - 6\, m_i^2 - 3\,\mathcal{B}_0^{\overline{MS}}(p^2,m_i^2)\,\big(2\,m_i^2 + p^2\big)\Big]\,. \end{aligned} \tag{4.99}$$

The QCD β function can be related to the residue of the pole and can be easily computed starting from the amplitude $\Gamma^{\mu\nu\alpha\beta}(p,q)$ for on-shell external lines and in the conformal limit

$$g_{\mu\nu}\, \Gamma^{\mu\nu\alpha\beta}(p,q) = 3\, s\, \Phi_1(s;0,0,0)\, u^{\alpha\beta}(p,q) = -2\, \frac{\beta(g)}{g}\, u^{\alpha\beta}(p,q), \tag{4.100}$$

with the QCD β function given by

$$\beta(g) = \frac{g^3}{16\pi^2}\left(-\frac{11}{3}\, C_A + \frac{2}{3}\, n_f\right). \tag{4.101}$$

As we have already mentioned, after contracting the metric tensor $g_{\mu\nu}$ with the whole amplitude Γ, only the tensor structure $\phi_1^{\mu\nu\alpha\beta}(p,q)$ contributes to the anomaly, being the remaining ones traceless, with a contribution entirely given by $\Phi_1|_{m=0}$ in Eq. (4.92), i.e. Eq. (4.95). In the massive fermion case, the anomalous contribution are corrected by terms proportional to the fermion mass m and represent an explicit breaking of scale invariance. From a direct computation we can also extract quite straightforwardly the effective action, which is given by

$$\begin{aligned} S_{pole} &= -\frac{c}{6}\int d^4x\, d^4y\, R^{(1)}(x)\,\square^{-1}(x,y)\, F^a_{\alpha\beta}\, F^{a\,\alpha\beta} \\ &= \frac{1}{3}\,\frac{g^3}{16\pi^2}\left(-\frac{11}{3}\,C_A+\frac{2}{3}\,n_f\right)\int d^4x\, d^4y\, R^{(1)}(x)\,\square^{-1}(x,y)\, F_{\alpha\beta}F^{\alpha\beta} \end{aligned} \tag{4.102}$$

and is in agreement with Eq. (4.6), derived from the nonlocal gravitational action. Here $R^{(1)}$ denotes the linearized expression of the Ricci scalar

$$R^{(1)}_x \equiv \partial^x_\mu\,\partial^x_\nu\, h^{\mu\nu} - \square\, h, \qquad h = \eta_{\mu\nu}\, h^{\mu\nu} \tag{4.103}$$

and the constant c is related to the non-abelian β function as

$$c = -2\,\frac{\beta(g)}{g}. \tag{4.104}$$

Notice that the contribution coming from TJJ generates the abelian part of the non-abelian field strength, while extra contributions (proportional to extra factors of g and g^2) are expected from the $TJJJ$ and $TJJJJ$ diagrams (see Fig. 4.5). This situation is analogous to that of the gauge anomaly, where one needs to render gauge covariant the anomalous amplitude given by the triangle diagram. In that case the gauge covariant expression is obtained by adding to the AVV vertex also the $AVVV$ and $AVVVV$ diagrams, with 3 and 4 external gauge lines, respectively.

4.6 Comments

The appearance of massless degrees of freedom in the effective action describing the coupling of gravity to the gauge fields is rather intriguing, and is an aspect that will require further analysis.

The nonlocal structure of the action that contributes to the trace anomaly, which is entirely reproduced, within the local description, by two auxiliary scalar fields, seems to indicate that the effective dynamics of the coupling between gravity and matter might be controlled, at least in part, by these degrees of freedom. As we have just mentioned, however, this point requires a dedicated study and for this specific reason our conclusions remain open ended.

Our computation, however, being general, allows also the identification of other massless contributions to the effective action which are surely bound to play a role in the physical S-matrix. They appear in form factors such as Φ_2 (Eq. 4.93) and Φ_3 (Eq. 4.94) which do not contribute to the trace, but are nevertheless part of the 1-loop effective action mediated by the triangle graph.

There are also some other comments, at this point, which are in order. Notice that while the isolation of the pole in the fermion sector indeed requires a massless fermion limit, as obvious from the structure of Γ_q, the other gauge invariant sector, described by Γ_g, is obviously not affected by this limit, being the corresponding form factors mass independent. This obviously does not imply necessarily that the gluon pole, which survives the extrapolation to the massless limit, is coupled in the physical S-matrix.

Building on considerations of this nature, in particular on the possible significance of massless effective degrees of freedom, the role of the trace anomaly in establishing the effective interaction of gravity with matter has been reconsidered [100, 101]. The explicit goal of this approach has been to trying to bypass the existing hierarchy problem between the value of the expected vacuum energy density ($\rho \sim (10^{-3}\mathrm{eV})^4$), well-described by a cosmological constant, and the Planck mass ($\rho \sim M_P^4$), which is a fundamental issue in contemporary cosmology that has not found yet a convincing explanation. In fact, it has been known for a long time that free massless particles contribute to the anomaly by an insignificant amount ($T^\mu_\mu \sim H_0^4$), proportional to the fourth power of the current Hubble rate, which is far too small as a value to solve the dark energy problem, due to the fact that we are living in a flat universe. However, it has been suggested that this small value for the vacuum energy density, originally attributed to the anomaly, could be raised to the expected one if the gravitational effective action is characterized by some effective nonlocality. In this case the contribution due to the trace anomaly could be modified as [108]

$$T^\mu_\mu \sim H_0 \, \Lambda^3_{QCD} \sim (10^{-3}\mathrm{eV})^4, \tag{4.105}$$

where Λ_{QCD} is the QCD scale, which is tantalizingly close to the estimated value. While this proposal and similar others are clearly not the only possible solutions of the dark energy problem (similar values of the vacuum energy can be obtained, for instance, using axions misaligned at the electroweak scale [109] and in several other ways) they share the positive feature of being characterized by few minimal assumptions. If so, one could envision a solution of the problem of the origin of dark energy without the need to enlarge the Standard Model spectrum with yet unknown particles and symmetries. Crucial, in these types of approaches, appears to be the role played by the effective scalar fields in the anomalous effective action, which are present in the local formulation of Riegert's action, together with their possible boundary conditions.

4.7 Conclusions

One of the standing issues of the anomalous effective action describing the interaction of a non-abelian theory to gravity is a test of its consistency with the standard perturbative approach. Thus, variational solutions of the effective action controlled by the trace anomaly should be reproduced by the perturbative expansion. Building on previous analysis in QED and contained in the previous chapters, here we have shown that also in the non-abelian case there is a perfect match between the two approaches. This implies that the interaction of gravity with a non-abelian gauge theory, mediated by the trace anomaly, indeed can be reformulated in terms of auxiliary scalar degrees of freedom, in analogy to the abelian case. We have proven this result by an explicit computation. Our findings indicate that this feature is typical of each gauge invariant subsector of the non-abelian TJJ amplitude, a result which is likely to hold also for singularities of higher order. These are expected to be present in correlators with a larger number of energy momentum insertions.

Chapter 5

Anomaly cancellation by pole subtraction and ghost instabilities

5.1 Introduction

The goal of this chapter is to stress on some (and unique) features of this subtraction from a perturbative perspective, in particular on the issues left open - at field theory level - and which have not yet found a satisfactory answer. Two different approaches appear in the description of the mechanism of anomaly cancellation, involving either a counterterm in the form of a pole subtraction [33, 34], or a Wess-Zumino term (see for instance [35]). This goes under the name - rather generically - of the Green-Schwarz mechanism (GS) in four dimensional field theory.
These two forms of the mechanism at the level of the 1-particle irreducible (1PI) effective action are, obviously, not equivalent, and the issue of their completeness, from a field theory point of view, is still open. For instance, axionic shift symmetries, which are present in some formulations of gauged supergravities, have been investigated using a Wess-Zumino approach [110, 111]. On the other hand, the subtraction of the anomaly pole in superspace - which is the one that we will mostly address in this note - has also been introduced as a possible way to give consistency to the effective action, in the presence of quantum anomalies. At the same time, a large amount of work along the years has addressed the problem of anomaly cancellation in matter-coupled supergravities using, at least in some cases, the subtraction mechanism. These studies have been and are focused on the role of Kähler and sigma model anomalies [34, 112, 113, 114] and on their implications in anomaly-mediated supersymmetry breaking [115].

5.1.1 Open issues

We point out that there are two challenges to the understanding of the subtraction mechanism in field theory. They are related 1) to the presence of ghosts in the spectrum of anomalous theories after the subtraction and 2) to the question whether a simple pole subtraction can actually erase the trace anomaly, in case also this needs to be cancelled. This second point is rather subtle since in supergravities the gauging combines several different symmetries, by requiring the invariance of the complete action under a combination of scaling symmetries (super-Weyl) together with ordinary Kähler transformations in addition to a $U(1)_R$ gauge symmetry.

A third issue concerns the relation between anomaly induced actions, which are derived by a solution of the anomaly equation, and the complete perturbative action obtained from a direct (and complete) diagrammatic approach. Both methods determine effective actions which are characterized by anomaly poles, the second approach being, obviously, more complete. Explicit computations, in fact, allow to understand the significance of the anomaly poles also as specific ultraviolet (UV) contributions, emerging from the perturbative expansion in the large energy limit. This point, as we are going to explain below, allows to put into the right context the meaning of the subtraction mechanism, which should be part of a UV completion.

All these issues have some implications for supersymmetric Yang Mills theories when these are coupled to conformal supergravity or to the various (old and new) multiplets of Poincare supergravities, due to the emergence of an infrared instability at perturbative level, induced by the mechanism. This can be identified by a direct analysis of the Coleman-Weinberg potential of the corrected theory, which shows the presence at 1-loop level of a ghost condensate.

Therefore, a true understanding of the mechanism of anomaly mediation and/or cancellation, to be significant at phenomenological level, has to address the role of the axion-ghost system and of the scalar-ghost system which, as we are going to explain, are introduced by these subtractions. Our simplified analysis has the role to stress the essential features of the pole subtraction, using very simple examples, but coming to conclusions which are, in fact, quite general. As we are going to show, much of the problem arises due to the nature of these pole counterterms in perturbation theory. The lifting of this approach to superspace, while necessary, complicates considerably the matter, especially since chiral gauge anomalies and trace anomalies may be jointly involved in the cancellation. This may happen if the Kähler symmetry has physical significance and needs to be preserved [112].

5.2 Removing the chiral gauge anomaly by an axion or by a pole subtraction

The simplest Lagrangians that in field theory realize the Wess-Zumino version of the mechanism can be written down quite straightforwardly, starting, for instance, with a single anomalous $U(1)_B$ model. It is defined as

$$L = \bar{\psi}(i\partial + gB\gamma_5)\psi - \frac{1}{4}F_B^2 + \langle \Delta_{BBB} BBB \rangle + c_1 \frac{b}{M} F_B \wedge F_B \tag{5.1}$$

and contains one chiral fermion, which indeed introduces an anomaly at quantum level. A discussion of this action is given in [22]. We have included in its structure the $\langle \Delta_{BBB} BBB \rangle$ interaction, which represents the contribution from the triangle diagram [16]. We can fix the counterterm c_1 from the requirement of gauge invariance, balancing the anomalous variation of the anomaly diagram with the variation of the axion counterterm. The axion undergoes a local shift under a gauge transformation

$$\delta b = M\theta_B(x) \qquad\qquad \delta B_\mu = \partial_\mu \theta_B(x) \tag{5.2}$$

where $\theta_B(x)$ parameterizes a gauge transformation. The Lagrangian implements in a simple form the GS mechanism (via an asymptotic axion b) and is obviously generalizable to supersymmetry via a shifting supermultiplet (see for instance [35] and [116] for a theoretical and phenomenological discussions in the supersymmetric case). As we have already mentioned, there is no equivalence between the pole subtraction mechanism and the Wess-Zumino counterterm, and these approaches are sometime not clearly distinguished in the literature. This difference, at the level of the 1-particle irreducible effective action, is indeed substantial.

The model Lagrangian introduced in (5.1) has some pitfalls, the first of them being the absence of a kinetic term for the axion. We can try to avoid the problem by introducing a kinetic term in a gauge invariant form. There is only one possibility, the Stückelberg mass term, obtaining the modified action

$$L = \bar{\psi}(i\partial + gB\gamma_5)\psi - \frac{1}{4}F_B^2 + \langle \Delta_{BBB} BBB \rangle + c_1 \frac{b}{M} F_B \wedge F_B + \frac{1}{2}\left(\partial_\mu b - MB_\mu\right)^2 . \tag{5.3}$$

This Lagrangian has a typical $Mb\,\partial B$ interaction that one could try to remove via a gauge fixing. In fact, one can do so and investigate the behaviour of the perturbative expansion in such a gauge (of R_ξ type). These studies have been performed in [16]. The theory describes consistently the mechanism of anomaly cancellation up to a certain scale, which is essentially the Stückelberg Mass M, since there is, indeed, a unitarity bound. There is a second limitation of this type of action, coming directly from gauge invariance. In fact one could choose a gauge in which b is set to vanish, and the theory would turn out to be equivalent to a massive Yang Mills theory coupled to a chiral fermion. For this reason, this action should necessarily be viewed as an approximate description of a more general one. This could be deduced starting from an anomaly free theory and decoupling even a single chiral fermion from the functional integral [29]. It has been shown that the effective action obtained by this decoupling is indeed corrected by an infinite number of higher dimensional operators. In this respect, the Lagrangian given in (5.3) has a unitary completion, at least in a field theory sense. Notice that b can be thought of as the phase of an extra Higgs field (complex scalar) having decoupled its modulus. For this reason, Lagrangians of this type are sufficient to describe the leading behaviour of the effective action in a $1/M$ expansion.

A second version of the mechanism is described instead by the second (nonlocal) Lagrangian

$$L = \bar{\psi}(i\partial + gB\gamma_5)\psi - \frac{1}{4}F_B^2 + \langle\Delta_{BBB}BBB\rangle + c_2\partial B\frac{1}{\square}F_B\tilde{F}_B \tag{5.4}$$

where the term $\partial B\frac{1}{\square}F_B\tilde{F}_B$ is the anomaly pole. It does not take much to realize that the cancellations corresponding to (5.3) and (5.4) allow to restore gauge invariance of the effective action. In general, extra counterterms can also be added to these types of actions in the presence of at least two gauge simmetries, in the form of Chern-Simons (CS) interactions. In the case that we consider the only possible anomaly is the consistent one, given the symmetry. For all practical purposes, CS interactions simply allow to re-distribute the partial anomalies (a_i) on a given leg of a diagram, keeping their sum fixed ($a_1 + a_2 + a_3 = a_n$). In the case of a theory with two U(1)'s (e.g. $U(1)_A \times U(1)_B$) with A vector-like and B axial-vector-like, terms such as ($AB \wedge F_B, AB \wedge F_A$) allow to move from the consistent to the covariant form of the anomaly. In any case, the discussion of CS interactions is not relevant for our goals and it will be omitted.

This second version of the mechanism, realized via (5.4), introduces one additional degree of freedom compared to (5.3). As we are going to show, this extra degree of freedom is an anomaly ghost. In fact, the Lagrangian (5.4) admits a different (local) formulation, now in terms of two extra pseudoscalars of the form

$$\begin{aligned}\mathcal{L} &= \overline{\psi}\,(i\,\not\partial + e\,\not B\gamma_5)\,\psi - \frac{1}{4}F_B^2 + \langle\Delta_{BBB}BBB\rangle + c_3 F_B \wedge F_B(a+b) \\ &\quad + \frac{1}{2}\left(\partial_\mu b - M_1 B_\mu\right)^2 - \frac{1}{2}\left(\partial_\mu a - M_1 B_\mu\right)^2 ,\end{aligned} \tag{5.5}$$

where both a and b shift as in (5.2). The equivalence between (5.4) and (5.5) can be proven directly from the functional integral, integrating out both a and b, which gives two gaussian integrations. Notice that b has a positive kinetic term and a is ghost-like.

There is a third equivalent formulation of the same action (5.5) which can be defined with the inclusion of a kinetic mixing between the two pseudoscalars. This has been given for QED (with a single fermion) coupled to an external axial-vector field $\mathcal{B}_\mu$ [51] and takes the form

$$\mathcal{L} = \partial_\mu \eta \, \partial^\mu \chi - \chi \, \partial \mathcal{B} + \frac{e^2}{8\pi^2} \, \eta \, F \, \tilde{F}, \tag{5.6}$$

where F is the field strength of the photon A_μ while $\mathcal{B}_\mu$ takes the role of a source. An anomaly pole is indeed induced by the $\mathcal{B}AA$ anomaly vertex. It is quite straightforward to relate (5.5) and (5.6). This can be obtained by the field redefinitions

$$\eta = \frac{(a+b)}{M}, \tag{5.7}$$

$$\chi = M(a+b), \tag{5.8}$$

showing that indeed a mixing term is equivalent to the presence of either an anomaly pole or to two pseudoscalars in the spectrum of the theory, one of them being a ghost. It is obvious that the pole subtraction in superspace does exactly the same thing, in a rather unobvious way.

5.2.1 The anomaly pole and the trace anomaly

The appearance of an anomaly pole in the perturbative expansion is not limited to the chiral anomaly. To clarify this point, let's denote with k the incoming momentum of the anomalous gauge current or of the graviton and with p and q the outgoing momenta of the two vector gauge bosons.

Similar singularities appear in explicit computations of the correlation functions for the trace anomaly in the absence of any second scale in the loop, involving one insertion of the energy momentum tensor (T) on 2-point functions of gauge fields (VV'), the TVV' correlator. By a second scale we refer either to a fermion mass term m in the anomaly loop, or to any of the two virtualities s_1 and s_2 ($s_1 \equiv p^2$, $s_2 \equiv q^2$) of the two gauge currents. With the term "first scale" in the loop we refer to the virtuality of the graviton s ($s \equiv k^2$), or, in the case of the chiral anomaly, the virtuality of the axial-vector current. This is the scale that as s goes to zero (with $k^\mu \to 0$, soft infrared (IR) limit) or as s goes to infinity (i.e. k^μ goes to infinity with a large invariant mass) controls the effects of the anomaly on the trilinear vertex. In fact the TVV' correlator takes a role quite similar to that of the corresponding AVV diagram of the chiral gauge anomaly. Surprisingly, this correlator has never been computed explicitly until recently in QED, QCD and the Standard Model. In the case of QED, for instance, the effective action takes the form [51] [52, 54]

$$S_{anom}[g, A] \to -\frac{c}{6} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'} \, R_x \, \Box^{-1}_{x,x'} \, [F_{\alpha\beta} F^{\alpha\beta}]_{x'} \,, \tag{5.9}$$

($c = -e^2/(24\pi^2)$) which is valid to first order in the fluctuation of the metric around a flat background, denoted as $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \qquad \kappa = \sqrt{16\pi G_N}, \tag{5.10}$$

with G_N being the 4-dimensional Newton's constant. The pole emerges from a single form factor evaluated in momentum space. If we denote with $\Gamma_{\mu\nu\alpha\beta} \equiv \langle T_{\mu\nu} V_\alpha V_\beta \rangle$ the correlation function responsible for the trace anomaly, this takes the form

$$\Gamma^{\mu\nu\alpha\beta} \sim \frac{1}{k^2}(g^{\mu\nu}k^2 - k^\mu k^\nu)u^{\alpha\beta}(p,q) + ... \tag{5.11}$$

where $u^{\alpha\beta}(p,q)$ is a tensor structure obtained by functional differentiation of the FF term of the trace anomaly Fourier transformed to momentum space,

$$u^{\alpha\beta}(p,q) = -\frac{1}{4}\int d^4x d^4y e^{ip\cdot x + iq\cdot y}\frac{\delta^2 F_{\mu\nu}F^{\mu\nu}}{\delta V_\alpha(x)\delta V_\beta(y)}. \tag{5.12}$$

The ellypsis refer to terms which are traceless. This relation is the analogous of the anomaly pole expression

$$\Delta^{\lambda\mu\nu}_{AVV} = a_n \frac{k^\lambda}{k^2}\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta + ... \tag{5.13}$$

for the chiral anomaly, with a_n being the anomaly. The pole structure above is usually called a Dolgov-Zakharov pole (DZ), which is IR coupled only in the absence of any second scale in an anomaly diagram. It is important to remark that only in this case (i.e. for two on shell vector lines and massless particles in the loop) the cancellation between an anomaly diagram and the subtraction countertem is identical. There is no identical cancellation under any other circumstance. For this obvious reason, in the presence of any second scale in the anomaly loop, the anomaly cancellation mechanism amounts to an "oversubtraction".

The meaning of this last term can be clarified quite simply. In fact we just recall that in the case of the chiral anomaly, the pole subtraction can be absorbed into a redefiniton of the anomaly vertex - this is not the case for the Wess-Zumino cancellation with a single axion (b) [24] - which now satisfies regular Ward identities (i.e. non anomalous) on each of its three external legs. This redefined vertex, however, now has a pole which is infrared coupled for *any* virtuality of the external vector lines, a feature which is unique among all the known vertices in local quantum field theory and, in particular, in the Standard Model. We will come back to this point in the next sections, trying to address the issue in the case of the chiral anomaly vertex.

As in the case of the chiral anomaly pole, also for the trace anomaly two auxiliary fields allow to re-express in a local form the corresponding nonlocal action (5.9) which takes the form [51]

$$S_{anom}[g, A; \varphi, \psi'] = \int d^4x \sqrt{-g}\left[-\psi'\Box\varphi - \frac{R}{3}\psi' + \frac{c}{2}F_{\alpha\beta}F^{\alpha\beta}\varphi\right], \tag{5.14}$$

where ϕ and ψ' are auxiliary scalar fields. Also in this case one can perform the same changes of variables as in Eqs. (5.7,5.8) and remove the kinetic mixing from this Lagrangian. Notice that the two auxiliary fields, in this case, are scalars. One of the two degrees of freedom is indeed a ghost. It is then clear that the subtraction of a anomaly pole induces into the theory some ghosts which are supposed to cancel those present in the trilinear anomalous vertices. As we are going to show, simple arguments in perturbation theory show that as soon as eq. 5.4 is used in the computation of quantum corrections, one discovers the presence of an infrared instability. For this we need to use the local version of (5.4), but before moving to that discussion we briefly comment on some of the main features of a pole subtraction in superspace.

5.2.2 The superconformal case and the gauging to gravity

Several puzzles emerge as soon as we put together the pieces of our previous discussion and frame it into a supersymmetric context (see [117, 118] for an overview).

When we come to analyze a super Yang-Mills theory, the trace anomaly, the gamma-trace of the supersymmetric current and the anomaly of the $U(1)_R$ current are part of the same anomaly supermultiplet $(T^\mu_\mu, \gamma \cdot s, \partial J_5)$ [119]. In this case the supermultiplet describes the radiative breaking of the superconformal symmetry. In particular, the presence of an anomaly pole for the axial-vector $U(1)_R$ global current indeed implies that a similar pole should appear in the correlation functions involving the insertion of either an energy-momentum tensor - or of the supersymmetric current - on two vector currents. This result is necessary for a consistent formulation of the anomaly-free effective action in superspace. Indeed, explicit computations support this picture to lowest order in the case of the trace anomaly, being obviously true (and to all orders) for the $U(1)_R$ anomaly.

The gauging of such an anomaly multiplet to gravity, for instance via a conformal multiplet $(g_{\mu\nu}, \psi_\mu, B_\mu)$ containing a graviton, a gravitino and an axial-vector gauge field, indeed produces an anomaly. In this case the energy momentum tensor couples to gravity $(g_{\mu\nu})$, the supersymmetric current couples to the gravitino background (ψ_μ) and the anomalous $U(1)_R$ current couples to the axial-vector gauge boson B_μ. Diffeomorphism invariance gives the standard conservation conditions for T^μ_ν and the spinor current s_μ $(\nabla_\mu T^{\mu\nu} = 0, \nabla_\mu s^\mu = 0)$, but the super-Weyl and $U(1)_R$ symmetry of the theory $((T^\mu_\mu = 0, \gamma \cdot s = 0, \partial J_5 = 0)$ are radiatively broken (see also [120, 121, 122] for related studies). It is obvious that the cancellation of the superconformal anomaly can't be obtained by using a single pole in superspace, given the different nature of the chiral and trace anomalies.

Anomaly induced actions [115] for $N = 1$ matter coupled supergravities carry both the signature of the breaking of scale invariance and of gauge invariance under Super-Weyl-Kähler transformations of the effective action, as shown by the presence both of the 1) $R\,\Box^{-1}FF$ and of the 2) $\partial B\,\Box^{-1}F\tilde{F}$ terms in the effective action, with R being the scalar curvature [115] [33].

While the appearance of the second term is, in a way, obvious, since it is generated by the Dolgov-Zakharov (DZ) anomaly pole present in the AVV diagram in superspace[37], the first one is far from being obvious since its identification requires a rather involved computation of the full correlator, not carried out until recently [51, 52, 54]. Similar poles emerge in the same vertices of the Standard Model, so far computed in the case of the neutral currents [53, 55]. It is then amusing that the lifting to superspace of the DZ pole of the $U(1)_R$ current, induces a similar pole in the correlator responsible for the trace anomaly.

It is however clear that the $R\,\Box^{-1}FF$ result is just valid to lowest order ($O(G_N g^2)$) in Newton's constant G_N and gauge coupling g. Indeed, in general, the structure of the anomaly-induced effective action for the trace anomaly is expected to be far more involved compared to the simple pole result. For instance, this action should describe the structure of the singularities of anomalous correlators with any number of insertions of the energy momentum tensor and two photons ($T^n VV$).

For obvious reasons, explicit checks of the corresponding effective action using perturbation theory - as the number of external graviton lines grows - becomes increasingly difficult to handle. The TVV correlator is the first (leading) contribution to this infinite sum of correlators in which the anomalous gravitational effective action is expanded. One proposal for the effective action is due to Riegert [83], which has been successfully tested, so far only for the TVV case, by two independent groups [51, 52, 54].

Given the presence of a quartic operator in Riegert's nonlocal action, the proof that this action contains a single pole to lowest order (in the TVV vertex), once expanded around flat space, has been given in [51] and provides the basis for the discussion of the anomalous effective action (5.14) in terms of massless auxiliary fields.

This shows that the ghost appearing in the trace anomaly is a genuine result which is extracted in two ways: 1) by integration of the anomaly and 2) by a direct perturbative computation using dispersion theory [51] or the complete evaluation of the diagrammatic expansion [52, 54].

5.3 Features of an anomaly pole and oversubtractions

Once we allow a pole solution of the anomalous Ward identities (see [92] for a general discussion) of a certain correlator, we need to define the kinematical range in which this solution is reproduced in perturbation theory, since explicit computations show that the tensor decompositions of anomaly diagrams are not unique. We start with the case of the AVV diagram. For simplicity, we will still denote with k the incoming momentum on the axial-vector line, and use symmetric expressions ($k_1 \equiv p$, $k_2 \equiv q$) for the two outgoing momenta of the vector lines. $s \equiv k^2$ denotes the virtuality of the momentum of the axial-vector current. We have the standard parameterization due to Rosenberg [41]

$$\begin{aligned}\Delta_0^{\lambda\mu\nu} &= A_1(k_1,k_2)\varepsilon[k_1,\mu,\nu,\lambda] + A_2(k_1,k_2)\varepsilon[k_2,\mu,\nu,\lambda] + A_3(k_1,k_2)\varepsilon[k_1,k_2,\mu,\lambda]{k_1}^{\nu} \\ &+ A_4(k_1,k_2)\varepsilon[k_1,k_2,\mu,\lambda]k_2^{\nu} + A_5(k_1,k_2)\varepsilon[k_1,k_2,\nu,\lambda]k_1^{\mu} + A_6(k_1,k_2)\varepsilon[k_1,k_2,\nu,\lambda]k_2^{\mu}.\end{aligned} \tag{5.15}$$

This parameterization is not always the most convenient. For instance, if one wants to study the mechanism of pole subtraction, it is convenient to use Schouten's relation and re-express Rosenberg's expression in an alternative form. A second decomposition of the anomaly graph into longitudinal and transverse form factors [42] is possible. It has been shown [40] that this representation is equivalent to the Rosenberg expression [41] (see the discussion in [123]). It takes the form

$$W^{\lambda\mu\nu} = \frac{1}{8\pi^2}\left[W^{L\,\lambda\mu\nu} - W^{T\,\lambda\mu\nu}\right], \tag{5.16}$$

where the longitudinal component

$$W^{L\,\lambda\mu\nu} = w_L\, k^{\lambda}\varepsilon[\mu,\nu,k_1,k_2] \tag{5.17}$$

(with $w_L = -4i/s$) describes the anomaly pole, while the transverse contributions take the form

$$\begin{aligned}{W^T}_{\lambda\mu\nu}(k_1,k_2) &= w_T^{(+)}\left(k^2,k_1^2,k_2^2\right)\, t^{(+)}_{\lambda\mu\nu}(k_1,k_2) + w_T^{(-)}\left(k^2,k_1^2,k_2^2\right)\, t^{(-)}_{\lambda\mu\nu}(k_1,k_2) \\ &+ \widetilde{w}_T^{(-)}\left(k^2,k_1^2,k_2^2\right)\, \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1,k_2),\end{aligned} \tag{5.18}$$

with the transverse tensors given by

$$\begin{aligned}t^{(+)}_{\lambda\mu\nu}(k_1,k_2) &= k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] - k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,(k_1-k_2)] \\ &\qquad + \frac{k_1^2+k_2^2-k^2}{k^2}\;k_\lambda\;\varepsilon[\mu,\nu,k_1,k_2]\;, \\ t^{(-)}_{\lambda\mu\nu}(k_1,k_2) &= \left[(k_1-k_2)_\lambda - \frac{k_1^2-k_2^2}{k^2}\;k_\lambda\right]\varepsilon[\mu,\nu,k_1,k_2] \\ \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1,k_2) &= k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] + k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,k].\end{aligned} \tag{5.19}$$

One should notice the presence of pole-like singularities in both the L and the T components proportional to s, which clearly invalidate the separation as s goes to zero. The presence of such singularities is also the signal that in the absence of any extra scale beside s, the two terms (L/T) reduce to a single structure.

To illustrate this point, let's consider in fact the case $s_1 = s_2 = 0$. In this case the two nonzero form factors are w_L and $w_T^{(+)}$

$$w_L(s,0,0) = w_T^{(+)}(s,0,0) = -\frac{4i}{s}, \tag{5.20}$$

$$w_T^{(-)}(s,0,0) = \tilde{w}_T^{(-)}(s,0,0) = 0. \tag{5.21}$$

The only contributions to the anomaly vertex come from the longitudinal W_L component and by $t^{(+)}_{\lambda\mu\nu}$, the second one being irrelevant when the two vector lines are set on-shell. Therefore, the parameterization reduces only to the longitudinal contribution, and generates, correctly, the anomaly pole. This is essentially the only case in which the pole is IR coupled, since with the inclusion of any other scale in the vertex (beside s), this structure, although present, does not have the right IR limit. However, this is not the end of the story, since there is a second kinematical configuration where the pole-like $1/s$ component becomes significant, and this involves the UV limit. In fact, we are allowed to perform a large s limit, in any direction away from the light cone, and observe the persistence of a $1/s$ component related to the anomaly. Notice that - differently from the case in which the two vector lines are on-shell - in this limit there is no redundancy between the longitudinal and transverse structure of the L/T decomposition (the two structures are independent), and the $1/s$ behaviour is indeed a genuine (irreducible) part of the amplitude.

Indeed, we can repeat the same analysis for the case in which at least one of the three scales (m, s_1, s_2) is non-vanishing. Let's suppose, for instance, that only m is non-zero. In this case we obtain (with $w_L(s_1, s_2, s, m^2) = W_L(0,0,s,m^2)$)

$$w_L(0,0,s,m^2) = -\frac{4i}{s}\left[1 + \frac{m^2}{s}\log^2\left(\frac{a_3+1}{a_3-1}\right)\right], \tag{5.22}$$

$$w_T^{(+)}(0,0,s,m^2) = \frac{4i}{s}\left[3 + \frac{m^2}{s}\log^2\left(\frac{a_3+1}{a_3-1}\right) - a_3\log\left(\frac{a_3+1}{a_3-1}\right)\right], \tag{5.23}$$

$$w_T^{(-)}(0,0,s,m^2) = \tilde{w}_T^{(-)}(0,0,s,m^2) = 0, \qquad a_3 = \sqrt{1 - \frac{4m^2}{s}}. \tag{5.24}$$

It is straightforward to verify that there is no residue for the $1/s$ pole term contained in w_L. This involves a cancellation between the two terms present in w_L, the constant and the logarithmic ($\sim \log^2$) term.

We conclude that the coupling of the pole in the infrared is controlled - in the absence of any other scale except s in the diagram - by the $1/s$ component of W_L. This structure indeed saturates the anomaly. As soon as any other scale is generated, there is no IR coupling of this invariant amplitude, although it is formally present in the L/T decomposition. It is then clear that, if other scales are also present, we are still formally allowed to restore the Ward identities of the anomalous vertex by a subtraction of W_L (which is what the GS mechanism does), but, by doing so, we have generated a vertex which is unique in its IR properties respect to any trilinear gauge vertex of the Standard Model. We refer to this situation as to an "oversubtraction" which can be potentially dangerous in the context of perturbative unitarity. This occurs whenever we move off-shell on the external lines (with s_1 or s_2 nonzero) or include a massive exchange in the loop, while still allowing an ordinary GS subtraction.

A final comment, in this section, is due for the second (and independent) region where the W_L contribution plays a role, which is the UV region. Notice that in the UV, being the external virtualities and mass negligible compared to the large value of s, we are again approaching the "pole dominance" typical of an IR ($m, s_1, s_2 \sim 0$) amplitude. It is instructive to perform a large s limit of the massive form factors given in (5.24), obtaining

$$w_L = -\frac{4\,i}{s} - \frac{4\,i\,m^2}{s^2}\log\left(-\frac{s}{m^2}\right) + O(m^3), \tag{5.25}$$

$$w_T^{(+)}(s,0,0,m^2) = \frac{12\,i}{s} - \frac{4\,i}{s}\log\left(-\frac{s}{m^2}\right) + \frac{4\,i\,m^2}{s^2}\left[2 + \log\left(\frac{s^2}{m^4}\right) - \log^2\left(-\frac{s}{m^2}\right)\right] + O(m^3). \tag{5.26}$$

The result above is susceptible of a simple intepretation. The anomalous contribution can be uniquely attributed to the pole in W_L, and the anomalous Ward identities are corrected by suppressed terms of the form m^2/s which include logarithms of the same ratio. Differently from the $s \to 0$ case, in this limit of large s there is no "overlap" between the two L/T tensor structures, and one can unambiguously attribute the anomalous contribution to W_L. This is the second - unequivocally distinct - region where the anomalous $1/s$ contribution appears. It is somehow a misnomer, since there is no residue to compute in this case, but this contribution can still be called an "anomaly pole", since it is a manifestation of the anomaly and saturates the anomalous Ward identities as s grows large. It is then clear which are the open issues typical of the mechanism of pole subtraction. If viewed as an asymptotic statement, we then should look for a completion of this mechanism. On the other hand, if we insist that the subtraction represents the only logical way to erase the anomalous variation of the action, then we are bound to face the issue of oversubtraction that we have mentioned before.

5.4 Quantifying the oversubtraction of an anomaly pole

For the reasons mentioned above, one can ask the question whether there is a completion of the GS mechanism - viewed as a pole subtraction - in order to avoid possible problems with the new (corrected) effective action in the infrared.

The simplest possibility is to cancel identically the anomaly vertex and not just to restore its Ward identities under any kinematical configurations, which is what the pole subtraction does. We are going to do it using as a reference the ordinary cancellation via charge assignment, which allows to generate a complete unitary theory. However, we will be separating the contribution to the cancellation which can be attributed to the exchange of the pseudoscalars, from the rest, with the residual interaction fixed by the condition of complete vanishing of the vertex. The residual terms, not included in the pole subtraction, could be attributed to the dynamics of the completion theory (e.g. a string theory), but can be quantified in a definite form, as we are going to show, also in ordinary field theory.

Thus, let's consider a theory with a single chiral fermion with vector and axial-vector gauged interactions and the corresponding AVV diagram. A similar analysis can be done for the AAA diagram of the same model.

We have seen that in this diagram any configuration - except for the on-shell case $(m, s_1, s_2 = 0)$ of the two V lines - does not allow an identical cancellation of this diagram by a pole counterterm. It amounts, therefore, to an oversubtraction, as we have explained above. We denote this vertex by $W^{\lambda\mu\nu}(m, s, s_1, s_2)$ and using a standard Pauli-Villars regularization procedure, we subtract the same amplitude with a generic fermion of mass M in the loop. We obtain, in a simplified notation, the regulated amplitude

$$W_R = W(m) - W(M) \tag{5.27}$$

which is obviously finite and satisfies ordinary Ward identities of the form

$$k_\lambda W_R^{\lambda\mu\nu} = 2mW^{\nu\lambda}(m) - 2MW^{\nu\lambda}(M). \tag{5.28}$$

Obviously, in a standard Pauli-Villars regularization one could send M to infinity, recuperating the anomaly contribution from the $2MW(M)$ term (up to a sign). At this point we re-express each of the amplitudes in terms of a pole plus the transverse contributions obtaining

$$W_R = (W_L(m, s_1, s_2)) + W_T(m, s_1, s_2)) - (W_L(M, s_1, s_2) + W_T(M, s, s_1, s_2))\,. \tag{5.29}$$

Notice that each of $W_L(m, s_1, s_2)$ and $W_L(M, s_1, s_2)$ are made of an anomaly pole plus mass correction terms.

Eq. (5.29) can be decomposed in terms of w_L and w_T, showing that W_R is free of anomaly poles, leaving some extra contributions both in the L and T parts which are mass dependent. However, W_R simplifies remarkably if the mass of the subtracted fermion is zero (M=0), since the anomaly diagram has no correction on the longitudinal structure W_L. In this specific case we obtain

$$W_R = \left(W_L'(m, s_1, s_2)) + W_T(m, s_1, s_2)\right) - W_T(s, s_1, s_2), \tag{5.30}$$

where W'_L denotes the L component of the diagram for the physical fermion with the subtraction of the anomaly pole. The interpretation of equation (5.30) is now obvious. Had we performed a pole subtraction on an AVV diagram, $W(m, s_1, s_2)$, the result would have been given just by the first two terms in the round bracket, causing on oversubtraction. This is corrected by the second term $W_T(s, s_1, s_2)$ which performs the unitarization of the vertex at any scales. We stress once more that this unitarization is obtained from field theory arguments and does not necessarily correspond to the unitarization that a nonlocal completion theory, such as a string theory, should perform on the subtraction.

We have gone through this argument to show that if the subtraction of a pole can be understood as a procedure which can be, eventually, unitarized in some way, then we can obviously give a coherent interpretation of the complete mechanism. This would allow us to attribute the subtraction of the pole term to one interaction, for instance to the exchange of an axion-ghost couple, while, at the same time, extra terms, not directly related to axionic contributions, would be involved in the extra correction. In the example that we have described, this extra term is given by $W_T(s, s_1, s_2)$, whose explicit expression, in this case, is known [40].

It is clear that there is a way out and a possible answer to the unitarization of the chiral anomaly pole, but it may not be so in the case of the trace anomaly. It appears obvious that such a procedure is bound to fail in the trace anomaly case, unless extra contributions to the running of the beta function will manage to induce a conformal phase. In this respect, while a coherent formulation of a pole subtraction in superspace treats the trace and the chiral anomaly components of an anomaly supermultiplet equally, in practice one can't ignore the different nature of the two anomalies. This may pose severe constraints on the coupling of superanomaly multiplets to gravity, since the mechanism of cancellation of the anomaly, if realized by a pole subtractions in superspace, is not satisfactory. Pole-like contributions appear indeed both in the case of chiral and trace anomaly diagrams. However, the anomalous effective action generated by the insertion of arbitrary powers of the energy momentum tensor on correlators of gauge currents is far more involved. It may not be completely saturated just by a pole to all orders, even in the weak field limit of the external gravitational field.

5.5 Conclusions

There is an incomplete understanding of the effective action which emerge at low energy from string theory and which involves a GS mechanism. It should be realized that this discussion is not just of formal nature, since it involves some issues which are of fundamental interest. First among them is the possible role played by the GS axion in the cosmology of the early

Universe. The appearance of an axion is, in fact, the crucial feature of the anomaly cancellation mechanism also in its realization in terms of a pole subtraction. The superspace formulation of the subtraction is not so obvious for Kähler anomalies, given the different nature of the chiral and conformal anomalies which are involved in combination in this subtraction.

Our analysis, clearly, is far from being conclusive, but it raises, we believe, some points which should motivate further discussions. Taken frontally, the subtraction of an anomaly pole to ensure the cancellation of some of the anomalies in a certain theory is the correct thing to do. At the same time, however, it leaves some issues of consistency wide open. In fact, this approach could be possibly correct only in the on-shell case. By rewriting the nonlocal action into a local form, using a formulation with two extra degrees of freedom, one ghost and one axion, one indeed finds that the effective action breaks the Lorentz symmetry. In these effective actions the dynamical generation of the breaking is, in fact, rather economical. There is indeed a signal of vacuum instability in theories corrected by a pole subtraction, which seems to indicate that the ghost can be taken out of the physical spectrum, leaving for the rest a theory which could be potentially useful but in a nontrivial vacuum.

Studies of gravity expanded around nontrivial background of ghosts are at the center of an increasing theoretical interest [124, 125] as are studies of the breaking of the Lorentz symmetry in brane models [126, 127]. Certainly, our comprehension of the vacuum structure of these theories on more physical grounds, especially in the presence of gravity multiplets, will probably require a big effort.

Chapter 6

Trilinear gauge interactions in extensions of the Standard Model with anomalous abelian symmetries

6.1 Introduction

Models of intersecting branes (see [128] for an overview) have been under an intense theoretical scrutiny in the last several years. The motivations for studying this class of theories are manifolds, being them obtained from special vacua of string theory, for instance from the orientifold construction [7, 19, 129, 130]. Their generic gauge structure is of the form $SU(3) \times SU(2) \times U(1)_Y \times U(1)^p$, where the symmetry of the Standard Model (SM) is enlarged with a certain number of extra abelian factors (p). Several phenomenological studies [15, 16, 17, 20, 23, 131] have allowed to characterize their general structure, whose string origin has been analyzed at an increasing level of detail [132, 133, 134] down to more direct issues, connected with their realization as viable theories beyond the SM. Related studies of the Stückelberg field [2, 11, 135, 136, 137] in a non-anomalous context have clarified this mechanism of mass generation and analyzed some of its implications at colliders both in the SM and in its supersymmetric extensions.

In scenarios with extra dimensions where the interplay between anomaly cancellations in the bulk and on the boundary branes is critical for their consistency, very similar models could be obtained following the construction of [138, 139], with a suitable generalization in order to generate at low energy a non abelian gauge structure.

Specifically, the role played by the extra $U(1)$'s at low energy in theories of this type after electroweak symmetry breaking has been addressed in [15, 16, 17], where some of the quantum features of their effective action have been clarified. These, for instance, concern the phases of these models, from their defining phase, the Stückelberg phase, being the anomalous $U(1)$ broken at low energy but with a gauge symmetry restored by shifting (Stückelberg) axions, down to the electroweak phase - or Higgs-Stückelberg phase, (HS) - where the vev's of the Higgs of the SM combine with the Stückelberg axions to produce a physical axion [15] and a certain number of goldstone modes. The axion in the low energy effective action is interesting both for collider physics and for cosmology [23], working as a modified Peccei-Quinn (PQ) axion. In this respect some interesting proposals to explain an anomaly in gamma ray propagation as seen by MAGIC [140, 141] using a pseudoscalar (axion-like) has been presented recently, while more experimental searches of effects of this type are planned for the future by several collaborations using Cerenkov telescopes (see [140, 141] for more details and references). Other interesting revisitations of the traditional Weinberg-Wilczek axion [142, 143] to evade the astrophysical constraints and in the context of Grand Unification/mirror worlds [144] may well deserve attention in the future and be analyzed within the framework that we outline below. At the same time, comparisons between anomalous and non anomalous string constructions of models with extra Z's should also be part of this analysis [8, 9, 145, 146].

The presence of axion-like particles in effective theories is, in general, connected to an anomalous gauge structure, but for reasons which may be rather different and completely unrelated, as discussed in [23]. For the rest, though, the study of the perturbative expansion in theories of this type is rather general and shows some interesting features that deserve a careful analysis. In [16, 17] several steps in the analysis of the perturbative expansion have been performed. In particular it has been shown how to organize the loop expansion in a gauge-invariant way in $1/M_1$, where M_1 is the Stückelberg mass. A way to address this point is to use a typical R_ξ gauge and follow the pattern of cancellation of the gauge parameter in order to characterize it. This has been done up to 3-loop level in a simple $U(1) \times U(1)$ model where one of the two $U(1)$'s is anomalous.

The Stückelberg symmetry is responsible for rendering the anomalous gauge bosons massive (with a mass M_1) before electroweak symmetry breaking. A second scale M controls the interaction of the axions with the gauge fields but is related to the first by a condition of gauge invariance in the effective action [23]. In general, for a theory with several $U(1)$'s, there is an independent mass scale for each Stückelberg field.

In the case of a complete extension of the SM incorporating anomalous $U(1)$'s, all the neutral current sectors, except for the photon current, acquire an anomalous contribution that modifies the trilinear (chiral) gauge interactions. For the Z gauge boson this anomalous component decouples as M_1 gets large, though it remains unspecified. For instance, in theories containing extra dimensions it could even be of the order of 10 TeV's or so, in general being of the order of $1/R$, where R is the radius of compactification. In other constructions [7, 19] based on toroidal compactifications with branes wrapping around the extra dimensions, their masses and couplings are expressed in terms of a string scale M_s and of the integers characterizing the wrappings [20]. Beside the presence of the extra neutral currents, which are common to all the models with extra abelian gauge structures, here, in addition, the presence of chiral anomalies leaves some of the trilinear interactions to contribute even in the massless fermion (chiral) limit, a feature which is completely absent in the SM, since in the chiral limit these vertices vanish.

As we are going to see, the analysis of these vertices is quite delicate, since their behaviour is essentially controlled by the mass differences within a given fermion generation [17], and for this reason they are sensitive both to spontaneous and to chiral symmetry breaking. The combined role played by these sources of breaking is not unexpected, since any pseudoscalar induced in an anomalous theory feels both the structure of the QCD vacuum and of the electroweak sector, as in the case of the Peccei-Quinn (PQ) axion. In this chapter we are going to proceed with a general analysis of these vertices, extending the discussion in [17]. The analysis performed here is organized as follows.

After a brief summary on the structure of the effective action, which has been included to make our treatment self-contained, we analyze the Slavnov-Taylor identities of the theory, focusing our attention on the trilinear gauge boson vertices. Then we characterize the structure of the $Z\gamma\gamma$ and $ZZ\gamma$ vertices away from the chiral limit, extending the discussion presented in [17]. In particular we clarify when the CS terms can be absorbed by a re-distribution of the anomaly before moving away from the chiral limit. In models containing several anomalous $U(1)$'s different theories are identified by the different partial anomalies associated to the trilinear gauge interactions involving at least three extra Z's. In this case the CS terms are genuine components which are specific for a given model and are accompanied by a specific set of axion counterterms. Symmetric distributions of the partial anomalies are sufficient to exclude all the CS terms, but these particular assignments may not be general enough.

Away from the chiral limit, we show how the mass dependence of the vertices is affected by the external Ward identity, which is a generic feature of anomalous interactions for nonzero fermion masses. This point is worked out using chiral projectors and counting the mass insertions into each vertex. On the basis of this study we are able to formulate general and simple rules which allow to handle quite straightforwardly all the vertices of the theory. We conclude with some phenomenological comments concerning the possibility of future studies of these theories at the LHC. In an appendix we present the Faddeev-Popov Lagrangian of the model, which has not been given before, and that can be useful for further studies of these theories.

6.1.1 Construction of the effective action

The construction of the effective action, from the field theory point of view, proceeds as follows [15, 17].

One introduces a set of counterterms in the form of CS and WZ operators and requires that the effective action is gauge invariant at 1-loop. Each anomalous $U(1)$ is accompanied by an axion, and every gauge variation of the anomalous gauge field can be cancelled by the corresponding WZ term. The remaining anomalous gauge variations are cancelled by CS counterterms. A list of typical vertices and counterterms is shown in Fig. 6.1. We consider the simplest anomalous

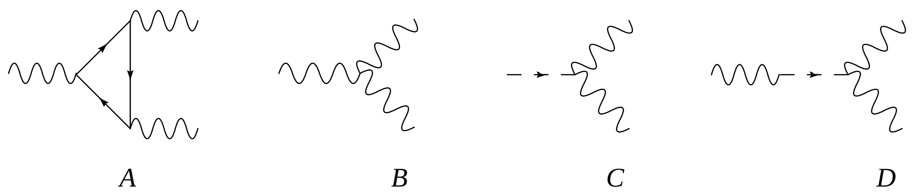


Figure 6.1: Counterterms allowed in the low energy effective action in the chiral limit: anomalous contributions (A), CS interaction (B), WZ term (C) and $B - b$ mixing contribution (D). In particular the bilinear mixing of the axions with the gauge fields is vanishing only for on-shell vertices and is removed in the R_ξ gauge in the WZ case. A discussion of this term and its role in the GS mechanism can be found in [30].

extension of the SM with a gauge structure of the form $SU(3) \times SU(2) \times U(1)_Y \times U(1)_B$ model with a single anomalous $U(1)_B$. The anomalous contributions are those involving the B gauge boson and involve the trilinear (triangle) vertices BBB, BYY, BBY, BWW and BGG, where W's and the G's are the $SU(2)$ and $SU(3)$ gauge bosons respectively. All the remaining trilinear interactions mediated by fermions are anomaly-free and therefore vanish in the massless limit. Therefore the axion (b) associated to B appears in abelian counterterms of the form $bF_B \wedge F_B, bF_B \wedge F_Y, bF_Y \wedge F_Y$ and in the analogous non-abelian ones $bTrW \wedge W$ and $bTrG \wedge G$. In the absence of a kinetic term for the axion b, its role is unclear: it allows to "cancel" the anomaly but can be gauged away. As emphasized by Preskill [22], the role of the WZ term is, at this stage, just to allow a consistent power counting in the perturbative expansion, hinting that an anomalous theory is non-renormalizable, but, for the rest, unitary below a certain scale. Theories of this type are in fact characterized by a unitarity bound since local a counterterm is not sufficient to erase the bad high energy behaviour of the anomaly [30]. Although the structure of the vertices constructed in this chapter is identified using the WZ effective action at the lowest order (using only the axion counterterm), their extension to the Green-Schwarz case is straightforward. In this second case the vertices here defined need to be modified with the addition of extra massless poles on the external gauge lines.

The b field remains unphysical even in the presence of a Stückelberg mass term for the B field, $\sim (\partial b - MB)^2$ since the gauge freedom remains and it is then natural to interpret b as a Nambu-Goldstone mode. In a physical gauge it can be set to vanish.

Things change drastically when the B field mixes with the other scalars of the Higgs sector of the theory. In this case a linear combination of b and the remaining CP-odd phases (goldstones) of the Higgs doublets becomes physical and is called the axi-Higgs. This happens only in specific potentials characterized also by a global $U(1)_{PQ}$ symmetry (V_{PQ}) [15] which are, however, sufficiently general. In the absence of Higgs-axion mixing the CP odd goldstone modes of the broken theory, after electroweak symmetry breaking, are just linear combinations of the Stückelberg and of the goldstone mode of the Higgs potential and no physical axion appears in the spectrum. For potentials that allow a physical axion, even in the massless case, the axion mass can be lifted by the QCD vacuum due to instanton effects exactly as for the Peccei-Quinn axion, but now the spectrum allows an axion-like particle.

6.1.2 Anomaly cancellation in the interaction eigenstate basis

The anomalies of the model are cancelled in the interaction eigenstate basis of (b, A_Y, B, W) and the CS and WZ terms are fixed at this stage. The B field is massive and mixes with the axion, but the gauge symmetry is still intact. The Ward identities of the theory for the triangle diagrams assume a nontrivial form due to the $B\partial b$ mixing. In the case of on-shell trilinear vertices one can show that these mixing terms vanish.

The CS counterterms are necessary in order to cancel the gauge variations of the Y, W and G gauge bosons in anomalous diagrams involving the interaction with B. These are the diagrams mentioned before. The role of these terms is to render vector-like at 1-loop all the currents which become anomalous in the interaction with the B gauge boson. For instance, in a triangle such as YBB, the $A_Y B \wedge F_B$ CS term effectively "moves" the chiral projector from the Y vertex to the B vertex symmetrically on the two B's, assigning the anomalies to the B vertices. These will then be cancelled by the axion b via a suitable WZ term ($bF_B \wedge F_Y$).

The effective action has the structure given by

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{an} + \mathcal{S}_{WZ} + \mathcal{S}_{CS} \tag{6.1}$$

where $\mathcal{S}_0$ is the classical action. It is a canonical gauge theory with dimension-4 operators whose explicit structure can be found in [17]. In Eq. (6.1) the anomalous contributions coming from the 1-loop triangle diagrams involving abelian and non-abelian gauge interactions are summarized by the expression

$$\begin{aligned}\mathcal{S}_{an} &= \frac{1}{2!}\langle T_{BWW}\,BWW\rangle + \frac{1}{2!}\langle T_{BGG}\,BGG\rangle + \frac{1}{3!}\langle T_{BBB}\,BBB\rangle \\ &+ \frac{1}{2!}\langle T_{BYY}\,BYY\rangle + \frac{1}{2!}\langle T_{YBB}\,YBB\rangle,\end{aligned} \tag{6.2}$$

where the symbols $\langle\rangle$ denote integration [16]. In the same notations the Wess Zumino (WZ) counterterms are given by

$$\begin{aligned}\mathcal{S}_{WZ} &= \frac{C_{BB}}{M}\langle b\,F_B\wedge F_B\rangle + \frac{C_{YY}}{M}\langle b\,F_Y\wedge F_Y\rangle + \frac{C_{YB}}{M}\langle b\,F_Y\wedge F_B\rangle \\ &+ \frac{F}{M}\langle b\,Tr[F^W\wedge F^W]\rangle + \frac{D}{M}\langle b\,Tr[F^G\wedge F^G]\rangle,\end{aligned} \tag{6.3}$$

and the gauge dependent CS abelian and non abelian counterterms [133, 134] needed to cancel the mixed anomalies involving a B line with any other gauge interaction of the SM take the form

$$\mathcal{S}_{CS} = d_1\langle BY\wedge F_Y\rangle + d_2\langle YB\wedge F_B\rangle + c_1\langle\epsilon^{\mu\nu\rho\sigma}B_\mu C^{SU(2)}_{\nu\rho\sigma}\rangle + c_2\langle\epsilon^{\mu\nu\rho\sigma}B_\mu C^{SU(3)}_{\nu\rho\sigma}\rangle. \tag{6.4}$$

Explicitly

$$\langle T_{BWW}BWW\rangle \equiv \int dx\,dy\,dz T^{\lambda\mu\nu,ij}_{BWW}(z,x,y)B^\lambda(z)W^\mu_i(x)W^\nu_j(y) \tag{6.5}$$

and so on.

The non-abelian CS forms are given by

$$C^{SU(2)}_{\mu\nu\rho} = \frac{1}{6}\left[W^i_\mu\left(F^W_{i,\nu\rho} + \frac{1}{3}g_2\,\varepsilon^{ijk}W^j_\nu W^k_\rho\right) + cyclic\right], \tag{6.6}$$

$$C^{SU(3)}_{\mu\nu\rho} = \frac{1}{6}\left[G^a_\mu\left(F^G_{a,\nu\rho} + \frac{1}{3}g_3\,f^{abc}G^b_\nu G^c_\rho\right) + cyclic\right]. \tag{6.7}$$

In our conventions, the field strengths are defined as

$$F^W_{i,\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g_2\varepsilon_{ijk}W^j_\mu W^k_\nu = \hat{F}^W_{i,\mu\nu} - g_2\varepsilon_{ijk}W^j_\mu W^k_\nu \tag{6.8}$$

$$F^G_{a,\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_3 f_{abc}G^b_\mu G^c_\nu = \hat{F}^G_{a,\mu\nu} - g_3 f_{abc}G^b_\mu G^c_\nu, \tag{6.9}$$

whose variations under non-abelian gauge transformations are

$$\delta_{SU(2)}C^{SU(2)}_{\mu\nu\rho} = \frac{1}{6}\left[\partial_\mu\theta^i\,(\hat{F}^W_{i,\nu\rho}) + cyclic\right], \tag{6.10}$$

$$\delta_{SU(3)}C^{SU(3)}_{\mu\nu\rho} = \frac{1}{6}\left[\partial_\mu\vartheta^a\,(\hat{F}^G_{a,\nu\rho}) + cyclic\right], \tag{6.11}$$

where $\hat{F}$ denotes the "abelian" part of the non-abelian field strength.

Coming to the formal definition of the effective action, interpreted as the generator of the 1-particle irreducible diagrams with external classical fields, this is defined, as usual, as a linear combination of correlation functions with an arbitrary number of external lines of the form A_Y, B, W, G, that we will denote conventionally as $\mathcal{W}(Y, B, W)$. It is given by

$$\begin{aligned} W[Y,B,W,G] &= \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{i^{n_1+n_2}}{n_1!n_2!}\int dx_1...dx_{n_1}dy_1...dy_{n_2}T^{\lambda_1...\lambda_{n_1}\mu_1...\mu_{n_2}}(x_1...x_{n_1},y_1...y_{n_2}) \\ &\qquad B^{\lambda_1}(x_1)...B^{\lambda_{n_1}}(x_{n_1})A_{Y\mu_1}(y_1)...A_{Y\mu_{n_2}}(y_{n_2}) + ... \end{aligned}$$

where we have explicitly written only its abelian part and the ellipsis refer to the additional non abelian or mixed (abelian/non-abelian) contributions. We will be using the invariance of the effective action under re-parameterizations of the external fields to obtain information on the trilinear vertices of the theory away from the chiral limit. Before coming to that point, however, we show how to fix the structure of the counterterms exploiting its BRST symmetry. This will allow to derive simple STI's for the action involving the anomalous vertices.

6.2 BRST conditions in the Stückelberg and HS phases

We show in this section how to fix the counterterms of the effective action by imposing directly the STI's on its anomalous vertices in the two broken phases of the theory, thereby removing the Higgs-axion mixing of the low energy effective theory. As we have already mentioned, the Lagrangian of the Stückelberg phase contains a coupling of the Stückelberg field to the gauge field which is typical of a goldstone mode. In [16, 17] this mixing has been removed and the WZ counterterms have been computed in a particular gauge, which is a typical R_ξ gauge with $\xi = 1$. Here we start by showing that this way of fixing the counterterms is equivalent to require that the trilinear interactions of the theory in the Stückelberg phase satisfy a generalized Ward identity (STI).

After electroweak symmetry breaking, in general one would be needing a second gauge choice, since the new breaking would again re-introduce bilinear derivative couplings of the new goldstones to the gauge fields. So the question to ask is if the STI's of the first phase, which fix completely the counterterms of the theory and remove the b-B mixing, are compatible with the STI's of the second phase, when we remove the coupling of the gauge bosons to their goldstones. The reason for asking these questions is obvious: it is convenient to fix the counterterms once and for all in the effective Lagrangians and this can be more easily done in the Stückelberg phase or in the HS phase depending on whether we need the effective action either expressed in terms of interactions or of mass eigenstates respectively. In both cases we need generalized Ward identities which are *local*. The presence of bilinear mixings on the external lines of the

3-point functions would render the analysis of these interactions more complex and essentially non-local.

This point is also essential in our identification of the effective vertices of the physical gauge bosons since, as we will discuss below, the definition of these vertices is entirely based on the possibility of parameterizing the anomalous effective action, at the same time, in the interaction base and in the mass eigenstate basis. We need these mixing terms to disappear in both cases. This happens, as we are going to show, if both in the Stückeberg phase and in the HS phase we perform a gauge choice of R_ξ type (we will choose $\xi = 1$). These technical points are easier to analyze in a simple abelian model, following the lines of [16]. In this model the B is a vector-axial vector ($\mathbf{V} - \mathbf{A}$) anomalous gauge boson and A is vector-like and anomaly-free.

We will show that in this model we can fix the counterterms in the first phase, having removed the b-B mixing and then proceed to determine the effective action in the HS phase, with its STI's which continue to be valid also in this phase.

Let's illustrate this point in some detail. We recall that for an ordinary (non abelian) gauge theory in the exact (non-broken) phase the derivation of the conditions of BRST invariance follow from the well known BRST variations in the R_ξ gauge

$$\delta_{BRST}\, A^a_\mu \equiv sA^a_\mu = \omega \mathcal{D}^{ab}_\mu c_b \tag{6.12}$$

$$\delta_{BRST}\, c^a \equiv sc^a = -\frac{1}{2}\omega g f^{abc} c_b c_c \tag{6.13}$$

$$\delta_{BRST}\, \bar{c}^{\,a} \equiv s\bar{c}^{\,a} = \frac{\omega}{\xi}\partial_\mu A^{\mu\, a}. \tag{6.14}$$

These involve the non-abelian gauge field A^a_μ, the ghost (c^a) and antighost ($\bar{c}^a$) fields, with ω being a Grassmann parameter. We will be interested in trilinear correlators whose STI's are arrested at 1-loop level and which involve anomalous diagrams. For instance we could use the invariance of a specific correlator ($\bar{c}AA$) under a BRST transformation in order to obtain the generalized WI's for trilinear gauge interactions

$$s\,\langle 0|T\,\bar{c}^a(x)A^b_\nu(y)A^c_\rho(z)|0\rangle = 0. \tag{6.15}$$

These are obtained from the relations (6.14) rather straightforwardly

$$\begin{aligned} s\,\langle 0|T\,\bar{c}^a(x)A^b_\nu(y)A^c_\rho(z)|0\rangle &= \langle 0|T\,(s\bar{c}^a(x))A^b_\nu(y)A^c_\rho(z)|0\rangle + \\ +\langle 0|T\,\bar{c}^a(x)(sA^b_\nu(y))A^c_\rho(z)|0\rangle &+ \langle 0|T\,\bar{c}^a(x)A^b_\nu(y)(sA^c_\rho(z))|0\rangle = 0. \end{aligned} \tag{6.16}$$

In fact, by using Eqs. (6.12) and (6.14) we obtain

$$\begin{aligned} s\,\langle 0|T\,\bar{c}^a(x)A^b_\nu(y)A^c_\rho(z)|0\rangle &= \frac{1}{\xi}\langle 0|T\,\omega\partial_\mu A^{\mu\, a}A^b_\nu(y)A^c_\rho(z)|0\rangle + \\ +\langle 0|T\,\bar{c}^a(x)\omega\mathcal{D}^{bl}_\nu c_l(y)A^c_\rho(z)|0\rangle &+ \langle 0|T\,\bar{c}^a(x)A^b_\nu(y)\omega\mathcal{D}^{cm}_\rho c_m(z)|0\rangle = 0. \end{aligned} \tag{6.17}$$

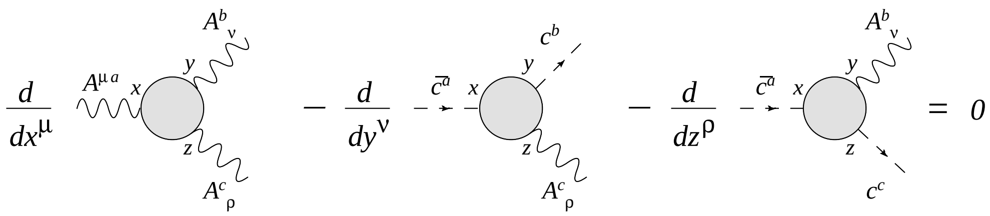


Figure 6.2: Graphical representation of Eq. (6.19) at any perturbative order.

Choosing $\xi = 1$ we get

$$\begin{aligned} & \frac{\partial}{\partial x^\mu}\langle 0|T\, A^{\mu\, a}(x) A^b_\nu(y) A^c_\rho(z)|0\rangle \\ + \quad & \langle 0|T\, \bar{c}^a(x)[\delta^{bl}\partial_\nu - g f^{bld} A_{\nu\, d}(y)] c_l(y) A^c_\rho(z)|0\rangle \\ + \quad & \langle 0|T\, \bar{c}^a(x) A^b_\nu(y)[\delta^{cm}\partial_\rho - g f^{cmr} A_{\rho\, r}(z)] c_m(z)|0\rangle = 0. \end{aligned} \tag{6.18}$$

The two fields $A_{\nu\, d}(y) c_l(y)$ e $A_{\rho\, r}(z) c_m(z)$ on the same spacetime point do not contribute on-shell and integrating by parts on the second and third term we obtain

$$\begin{aligned} \frac{\partial}{\partial x^\mu}\langle 0|T\, A^{\mu\, a} A^b_\nu(y) A^c_\rho(z)|0\rangle - \frac{\partial}{\partial y^\nu}\langle 0|T\, \bar{c}^a(x) c^b(y) A^c_\rho(z)|0\rangle \\ -\frac{\partial}{\partial z^\rho}\langle 0|T\, \bar{c}^a(x) A^b_\nu(y) c^c(z)|0\rangle = 0, \end{aligned} \tag{6.19}$$

which is described diagrammatically in Fig. 6.2. Let's now focus our attention on the A-B model of [16] where we have an anomalous generator Y_B. This model describes quite well many of the properties of the abelian sector of the general model discussed in [17] with a single anomalous $U(1)$. It is an ordinary gauge theory of the form $U(1)_A \times U(1)_B$ with B made massive at tree level by the Stückelberg term

$$\mathcal{L}_{St} = \frac{1}{2}(\partial_\mu b + M_1 B_\mu)^2. \tag{6.20}$$

This term introduces a mixing $M_1 B_\mu \partial^\mu b$ which signals the presence of a broken phase in the theory. Introducing the gauge fixing Lagrangian

$$\mathcal{L}_{gf} = -\frac{1}{2\xi_B}(\mathcal{F}^S_B[B_\mu])^2, \tag{6.21}$$

$$\mathcal{F}^S_B[B_\mu] \equiv \partial_\mu B^\mu - \xi_B M_1 b, \tag{6.22}$$

we obtain the partial contributions (mass term plus gauge fixing term) to the total action

$$\mathcal{L}_{St} + \mathcal{L}_{gf} = \frac{1}{2}\left[(\partial_\mu b)^2 + M_1^2 B_\mu B^\mu - (\partial_\mu B^\mu)^2 - \xi_B M_1^2 b^2\right] \tag{6.23}$$

and the corresponding Faddeev-Popov Lagrangian

$$\mathcal{L}_{FP} = \bar{c}_B \, \frac{\delta \mathcal{F}_B}{\delta \theta_B} \, c_B = \bar{c}_B \left[\partial_\mu \frac{\delta B^\mu}{\delta \theta_B} - \xi_B M_1 \, \frac{\delta b}{\delta \theta_B} \right] c_B, \tag{6.24}$$

with c_B and $\bar{c}_B$ are the anticommuting ghost/antighosts fields. It can be written as

$$\mathcal{L}_{FP} = \bar{c}_B \, (\Box + \xi_B M_1^2) \, c_B, \tag{6.25}$$

having used the shift of the axion under a gauge transformation

$$\delta b = -M_1 \theta. \tag{6.26}$$

In the following we will choose $\xi_B = 1$. The anomalous sector is described by

$$\begin{aligned} \mathcal{S}_{an} &= \mathcal{S}_1 + \mathcal{S}_3 && (6.27) \\ \mathcal{S}_1 &= \int dx \, dy \, dz \, \left(\frac{g_B \, g_A^2}{2!} \, T^{\lambda\mu\nu}_{\mathbf{AVV}}(x,y,z) B_\lambda(z) A_\mu(x) A_\nu(y) \right) && (6.28) \\ \mathcal{S}_3 &= \int dx \, dy \, dz \, \left(\frac{g_B^3}{3!} \, T^{\lambda\mu\nu}_{\mathbf{AAA}}(x,y,z) B_\lambda(z) B_\mu(x) B_\nu(y) \right), && (6.29) \end{aligned}$$

where we have collected all the anomalous diagrams of the form (**AVV** and **AAA**) and whose gauge variations are

$$\begin{aligned} \frac{1}{2!} \delta_B \left[T_{\mathbf{AVV}} BAA \right] &= \frac{i}{2!} a_3(\beta) \frac{1}{4} \left[F_A \wedge F_A \theta_B \right] && (6.30) \\ \frac{1}{3!} \delta_B \left[T_{\mathbf{AAA}} BBB \right] &= \frac{i}{3!} \frac{a_n}{3} \frac{3}{4} \langle F_B \wedge F_B \theta_B \rangle, && (6.31) \end{aligned}$$

having left open the choice over the parameterization of the loop momentum, denoted by the presence of the arbitrary parameter β with

$$a_3(\beta) = -\frac{i}{4\pi^2} + \frac{i}{2\pi^2}\beta \qquad a_3 \equiv \frac{a_n}{3} = -\frac{i}{6\pi^2}, \tag{6.32}$$

while

$$\frac{1}{2!} \delta_A \left[T_{\mathbf{AVV}} BAA \right] = \frac{i}{2!} a_1(\beta) \frac{2}{4} \left[F_B \wedge F_A \theta_A \right]. \tag{6.33}$$

We have the following equations for the anomalous variations

$$\begin{aligned} \delta_B \mathcal{L}_{an} &= \frac{i g_B g_A^2}{2!} \, a_3(\beta) \frac{1}{4} F_A \wedge F_A \theta_B + \frac{i g_B^3}{3!} \, \frac{a_n}{3} \frac{3}{4} F_B \wedge F_B \theta_B && (6.34) \\ \delta_A \mathcal{L}_{an} &= \frac{i g_B g_A^2}{2!} \, a_1(\beta) \frac{2}{4} F_B \wedge F_A \theta_A, && (6.35) \end{aligned}$$

while $\mathcal{L}_{b,c}$, the axionic contributions (Wess-Zumino terms) needed to restore the gauge symmetry violated at 1-loop level, are given by

$$\mathcal{L}_b = \frac{C_{AA}}{M} b \, F_A \wedge F_A + \frac{C_{BB}}{M} b \, F_B \wedge F_B. \tag{6.36}$$

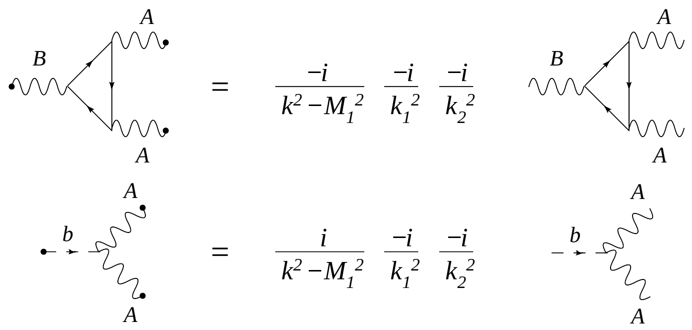


Figure 6.3: Relation between a correlator with non amputated external lines (left) used in a STI and an amputated one (right) used in the effective action for a triangle vertex and for a CS term.

The gauge invariance on A requires that $\beta = -1/2 \equiv \beta_0$ and is equivalent to a vector current conservation (CVC) condition. By imposing gauge invariance under B gauge transformations, on the other hand, we obtain

$$\delta_B \left(\mathcal{L}_b + \mathcal{L}_{an}\right) = 0 \tag{6.37}$$

which implies that

$$C_{AA} = \frac{i\, g_B g_A^2}{2!} \frac{1}{4}\, a_3(\beta_0)\, \frac{M}{M_1}, \qquad C_{BB} = \frac{i g_B^3}{3!} \frac{1}{4}\, a_n\, \frac{M}{M_1}. \tag{6.38}$$

This procedure, as we are going to show, is equivalent to the imposition of the STI on the corresponding anomalous vertices of the effective action. In fact the counterterms C_{AA} and C_{BB} can be determined formally from a BRST analysis.

In fact, the BRST variations of the model are defined as

$$\delta_{BRST}\, B_\mu = \omega \partial_\mu c_B \tag{6.39}$$

$$\delta_{BRST}\, b = -\omega M_1 c_B \tag{6.40}$$

$$\delta_{BRST}\, A_\mu = \omega\, \partial_\mu c_A \tag{6.41}$$

$$\delta_{BRST}\, c_B = 0 \tag{6.42}$$

$$\delta_{BRST}\, \bar{c}_B = \frac{\omega}{\xi_B} \mathcal{F}_B^S = \frac{\omega}{\xi_B} (\partial_\mu B^\mu - \xi_B M_1 b). \tag{6.43}$$

To derive constraints on the 3-linear interactions involving 2 abelian (vector-like) and one vector-axial vector gauge field, that we will encounter in our analysis below, we require the BRST invariance of a specific correlator such as

$$\delta_{BRST}\, \langle 0 | T\, \bar{c}_B(z) A_\mu(x) A_\nu(y) | 0 \rangle = 0, \tag{6.44}$$

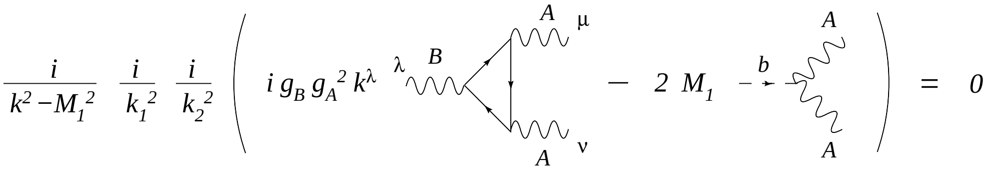


Figure 6.4: Representation in terms of Feynman diagrams in momentum space of the Slavnov-Taylor identity obtained in the Stückelberg phase for the anomalous triangle BAA. Here we deal with correlators with non-amputated external lines. A CS term has been absorbed to ensure the conserved vector current (CVC) conditions on the A lines.

(Fig. 6.3 shows the difference between the non-amputated and the amputated correlators) and applying the BRST operator we obtain

$$\frac{\omega}{\xi_B}\langle 0|T\,[\partial_\lambda B^\lambda(z)-\xi_B M_1 b(z)]A_\mu(x)A_\nu(y)|0\rangle \;+\; \langle 0|T\,\bar{c}_B(z)\omega\partial_\mu c_A(x)A_\nu(y)|0\rangle \;+\; \langle 0|T\,\bar{c}_B(z)A_\mu(x)\omega\partial_\nu c_A(y)|0\rangle = 0, \tag{6.45}$$

with the last two terms being trivially zero. Choosing $\xi_B = 1$ we obtain the STI (see Fig. 6.4) involving only the WZ term and the anomalous triangle diagram BAA. This reads

$$\frac{\partial}{\partial z^\lambda}\langle 0|T\,B^\lambda(z)A_\mu(x)A_\nu(y)|0\rangle - M_1\langle 0|T\,b(z)A_\mu(x)A_\nu(y)|0\rangle = 0. \tag{6.46}$$

A similar STI holds for the BBB vertex and its counterterm

$$\frac{\partial}{\partial z^\lambda}\langle 0|T\,B^\lambda(z)B_\mu(x)B_\nu(y)|0\rangle - M_1\langle 0|T\,b(z)B_\mu(x)B_\nu(y)|0\rangle = 0. \tag{6.47}$$

These two equations can be rendered explicit. For instance, to extract from (6.46) the corresponding expression in momentum space and the constraint on C_{AA}, we work at the lowest order in the perturbative expansion obtaining

$$\frac{1}{2!}\frac{\partial}{\partial z^\lambda}\langle 0|T\,B^\lambda(z)A_\mu(x)A_\nu(y)\,[J_5 B]\,[JA]^2\,|0\rangle - M_1\langle 0|T\,b(z)A_\mu(x)A_\nu(y)\,[bF_A\wedge F_A]\,|0\rangle = 0, \tag{6.48}$$

where we have introduced the notation [ ] to denote the spacetime integration of the vector (J) and axial current (J_5) to their corresponding gauge fields

$$J\,A = -g_A\bar{\psi}\gamma^\mu\psi A_\mu, \tag{6.49}$$

$$J_5\,B = -g_B\bar{\psi}\gamma^\mu\gamma^5\psi B_\mu \tag{6.50}$$

$$\tilde{J}_5\,G_B = 2ig_B\frac{m_f}{M_B}\bar{\psi}\gamma^5\psi G_B\,, \tag{6.51}$$

where M_B is the mass of the B gauge boson in the Higgs-Stückelberg phase that we will analyze in the next sections.

In momentum space the STI represented in Fig. 6.4 becomes ($\xi_B = 1$)

$$\frac{1}{2!}\,2\,\left[ik^{\lambda'}\right]\left[-\frac{ig_{\lambda\lambda'}}{k^2-M_1^2}\right]\left[-\frac{ig_{\mu\mu'}}{k_1^2}\right]\left[-\frac{ig_{\nu\nu'}}{k_2^2}\right]\left[-g_B g_A^2\right]\,\Delta^{\lambda\mu\nu}(k_1,k_2)$$
$$-\quad 2\,M_1\left[\frac{i}{k^2-M_1^2}\right]\left[-\frac{ig_{\mu\mu'}}{k_1^2}\right]\left[-\frac{ig_{\nu\nu'}}{k_2^2}\right]\,V_A^{\mu\nu}(k_1,k_2)=0, \tag{6.52}$$

where the factor $\frac{1}{2!}$ comes from the presence in the effective action of a diagram with 2 identical external lines, in this case two A gauge bosons, and the factor 2, present in both terms, comes from the possible contractions with the external fields. Using in (6.52) the corresponding anomaly equation

$$k_\lambda \Delta^{\lambda\mu\nu}(k_1,k_2) = a_3(\beta_0)\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta} \tag{6.53}$$

and the expression of the vertex $V_A^{\mu\nu}(k_1,k_2)$

$$V_A^{\mu\nu}(k_1,k_2) = \frac{4C_{AA}}{M}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta} \tag{6.54}$$

we obtain

$$\left[\frac{i}{k^2-M_1^2}\right]\left[-\frac{ig_{\mu\mu'}}{k_1^2}\right]\left[-\frac{ig_{\nu\nu'}}{k_2^2}\right]\left[i\,g_B g_A^2 a_3(\beta_0)\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta} - 2\,M_1\frac{4C_{AA}}{M}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\right] = 0, \tag{6.55}$$

from which we get

$$i\,g_B g_A^2 a_3(\beta_0) = 2\,M_1\frac{4C_{AA}}{M} \qquad \Rightarrow \qquad C_{AA} = \frac{i\,g_B g_A^2}{2}\frac{1}{4}\,a_3(\beta_0)\,\frac{M}{M_1}. \tag{6.56}$$

This condition determines C_{AA} at the same value as before in (6.38), using the constraints of gauge invariance, having brought the anomaly on the B vertex ($\beta_0 = -1/2$).

In the case of the second STI given in (6.47), expanding this equation at the lowest relevant order we get

$$\frac{1}{3!}\frac{\partial}{\partial z^\lambda}\langle 0|T\,B^\lambda(z)B_\mu(x)B_\nu(y)\,[J_5 B]^3\,|0\rangle - M_1\langle 0|T\,b(z)B_\mu(x)B_\nu(y)\,[bF_B\wedge F_B]\,|0\rangle = 0. \tag{6.57}$$

Also in this case, setting $\xi_B = 1$, we re-express (6.57) as

$$\frac{1}{3!}\,3!\,\left[ik^{\lambda'}\right]\left[-\frac{ig_{\lambda\lambda'}}{k^2-M_1^2}\right]\left[-\frac{ig_{\mu\mu'}}{k_1^2-M_1^2}\right]\left[-\frac{ig_{\nu\nu'}}{k_2^2-M_1^2}\right]\left[-g_B^3\right]\,\Delta^{\lambda\mu\nu}(k_1,k_2)$$
$$-\quad 2\,M_1\left[\frac{i}{k^2-M_1^2}\right]\left[-\frac{ig_{\mu\mu'}}{k_1^2-M_1^2}\right]\left[-\frac{ig_{\nu\nu'}}{k_2^2-M_1^2}\right]\,V_B^{\mu\nu}(k_1,k_2)=0, \tag{6.58}$$

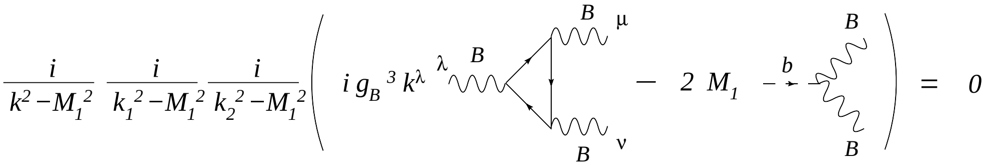


Figure 6.5: Diagrammatic representation of (6.59) in the Stückelberg phase, determining the counterterm C_{BB}.

where, similarly to BAA, the factor $\frac{1}{3!}$ comes from the 3 identical gauge B bosons on the external lines, the coefficient 3! in the first term counts all the contractions between the vertex $\Delta^{\lambda\mu\nu}$ and the propagators of the B gauge bosons, while the coefficient 2 comes from the contractions of $V_B^{\mu\nu}$ with the external lines. From Eq. (6.58) we get

$$\left[\frac{i}{k^2 - M_1^2}\right]\left[-\frac{i g_{\mu\mu'}}{k_1^2 - M_1^2}\right]\left[-\frac{i g_{\nu\nu'}}{k_2^2 - M_1^2}\right]\left[i g_B^3\, k_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) - 2\, M_1 V_B^{\mu\nu}(k_1, k_2)\right] = 0\,, \tag{6.59}$$

as depicted in Fig. 6.5.

The anomaly equation for BBB distributes the total anomaly a_n equally among the three B vertices, therefore

$$k_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{a_n}{3}\epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}, \tag{6.60}$$

and for the $V_B^{\mu\nu}(k_1, k_2)$ vertex we have

$$V_B^{\mu\nu}(k_1, k_2) = \frac{4C_{BB}}{M}\epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}. \tag{6.61}$$

Inserting (6.60), (6.61) into (6.59) we obtain

$$i\, g_B^3\, \frac{a_n}{3} = 2\, M_1\, \frac{4C_{BB}}{M} \qquad \Rightarrow \qquad C_{BB} = \frac{i\, g_B^3}{2}\frac{1}{4}\,\frac{a_n}{3}\,\frac{M}{M_1}, \tag{6.62}$$

in agreement with (6.38). Therefore we have shown that if we gauge-fix the effective Lagrangian in the Stückelberg phase to remove the b-B mixing and fix the CS counterterms so that the anomalous variations of the trilinear vertices are absent, we are actually imposing generalized Ward identities or STI's on the effective action. On this gauge-fixed axion the b-B mixing is completely absent also off-shell and the structure of the trilinear vertices is rather simple. We need to check that these STI's are compatible with those obtained after electroweak symmetry breaking, so that the mixing is absent off-shell also in the physical basis.

6.2.1 The Higgs-Stückelberg phase (HS)

Now consider the same effective action of the previous model after electroweak symmetry breaking. If we interpret the gauge-fixed action derived above as a completely determined theory where the counterterms have been found by the procedure that we have just illustrated, once we expand the fields around the Higgs vacuum we encounter a new mixing of the goldstones with the gauge fields. Due to Higgs-axion mixing [16] the goldstones of this theory are extracted by a suitable rotation that allows to separate physical from unphysical degrees of freedom. In fact the Stückelberg is decomposed into a physical axi-Higgs and a genuine goldstone. It is then natural to ask whether we could have just worked out the Lagrangian *directly* in this phase by keeping the coefficients in front of the counterterms of the theory free, and had them fixed by imposing directly generalized WI's in this phase, bypassing completely the first construction. As we are now going to show in this model the counterterms are determined consistently also in this case at the same values given before.

Let's see how this happens. In this phase the mixing that needs to be eliminated is of the form $B^\mu \partial_\mu G_B$, where G_B is the goldstone of the HS phase. In this case we use the gauge-fixing Lagrangian

$$\mathcal{L}_{gf} = -\frac{1}{2\xi_B}(\mathcal{F}_B^H)^2 = -\frac{1}{2\xi_B}\left(\partial_\mu B^\mu - \xi_B M_B G_B\right), \tag{6.63}$$

and the BRST transformation of the antighost field $\bar{c}_B$ is given by

$$\delta_{BRST}\,\bar{c}_B = \frac{\omega}{\xi_B}\mathcal{F}_B^H = \frac{\omega}{\xi_B}\left(\partial_\mu B^\mu - \xi_B M_B G_B\right). \tag{6.64}$$

Also in this case we use the 3-point function in Eq. (6.38) and $\xi_B = 1$ to obtain the STI

$$\frac{\partial}{\partial z^\lambda}\langle 0|T\, B^\lambda(z) A_\mu(x) A_\nu(y)|0\rangle - M_B\langle 0|T\, G_B(z) A_\mu(x) A_\nu(y)|0\rangle = 0. \tag{6.65}$$

To get insight into this equation we expand perturbatively (6.65) and obtain

$$\begin{aligned}&\frac{1}{2!}\frac{\partial}{\partial z^\lambda}\langle 0|T\, B^\lambda(z) A_\mu(x) A_\nu(y)\,[J_5 B]\,[JA]^2\,|0\rangle\\ &- M_B\,\langle 0|T\, G_B(z) A_\mu(x) A_\nu(y)\,[G_B F_A\wedge F_A]\,|0\rangle\\ &- M_B\,\langle 0|T\, G_B(z) A_\mu(x) A_\nu(y)\left[\tilde{J}_5 G_B\right][JA]^2\,|0\rangle = 0,\end{aligned} \tag{6.66}$$

where the first term is the usual triangle diagram with the BAA gauge bosons on the external lines, the second is a WZ vertex with G_B on the exernal line and the third term, which is absent in the Stückelberg phase, is a triangle diagram involving the G_B gauge boson that couples to the fermions by a Yukawa coupling (see Fig. 6.6). In the Stückelberg phase there is no analogue of this third contribution in the cancellation of the anomalies for this vertex, since b does not couple to the fermions.

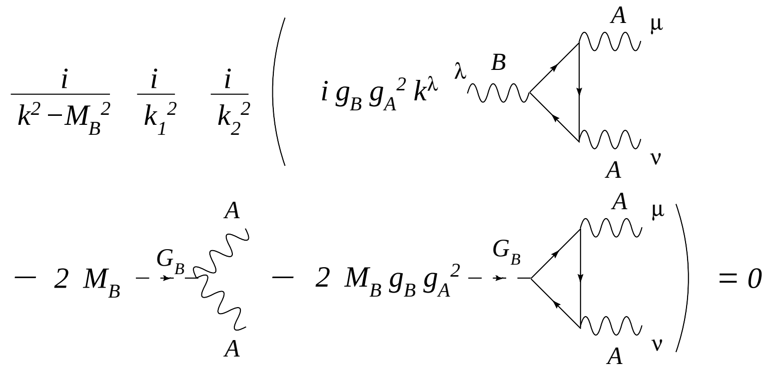


Figure 6.6: Diagrammatic representation of Eq. (6.66) in the HS phase, determining the counterterm C_{AA}. A CS term has been absorbed by the CVC conditions on the A gauge bosons.

Notice that the STI now contains a vertex derived from the $bF_A \wedge F_A$ counterterm, but projected on the interaction $G_B F_A \wedge F_A$ via the factor M_1/M_B. This factor is generated by the rotation matrix that allows the change of variables $(\phi_2, b) \to (\chi_B, G_B)$ and is given by

$$U = \begin{pmatrix} -\cos\theta_B & \sin\theta_B \\ \sin\theta_B & \cos\theta_B \end{pmatrix} \tag{6.67}$$

with $\theta_B = \arccos(M_1/M_B) = \arcsin(q_B g_B v/M_B)$. We recall [16] that the axion b can be expressed as a linear combination of the rotated fields χ and G_B of the form

$$b = \alpha_1 \chi_B + \alpha_2 G_B = \frac{q_B g_B v}{M_B}\chi_B + \frac{M_1}{M_B} G_B, \tag{6.68}$$

where χ is the physical axion and G_B the Goldstone boson; we also recall that the gauge field B_μ gets its mass M_B through the combined Higgs-Stückelberg mechanism

$$M_B = \sqrt{M_1^2 + (q_B g_B v)^2}. \tag{6.69}$$

Now we express the STI given in (6.66) choosing $\xi_B = 1$

$$\begin{aligned} & \frac{1}{2!}\, 2 \left[ik^{\lambda'}\right] \left[-\frac{i g_{\lambda\lambda'}}{k^2 - M_B^2}\right] \left[-\frac{i g_{\mu\mu'}}{k_1^2}\right] \left[-\frac{i g_{\nu\nu'}}{k_2^2}\right] \left[-g_B g_A^2\right] \Delta^{\lambda\mu\nu}(m_f, k_1, k_2) \\ - \quad & M_B \left[\frac{i}{k^2 - M_B^2}\right] \left[-\frac{i g_{\mu\mu'}}{k_1^2}\right] \left[-\frac{i g_{\nu\nu'}}{k_2^2}\right] \left\{ 2\, \frac{M_1}{M_B}\, V_A^{\mu\nu}(k_1, k_2) \right. \\ & \left. + \frac{1}{2!}\, 2\, i\, g_B g_A^2 \left(2i\, \frac{m_f}{M_B}\right) \Delta^{\mu\nu}_{G_B AA}(m_f, k_1, k_2) \right\} = 0, \end{aligned} \tag{6.70}$$

where the $[G_B F_A \wedge F_A]$ interaction has been obtained from the $[bF_A \wedge F_A]$ vertex by projecting the b field on the field G_B, and the coefficient $2im_f/M_B$ comes from the coupling of G_B with the massive fermions [16]. The remaining coefficient M_1/M_B rotates the $V_A^{\mu\nu}(k_1,k_2)$ vertex as in Eq. (6.70).

Replacing in (6.70) the WI obtained for a massive AVV vertex

$$k_\lambda \Delta^{\lambda\mu\nu}(\beta, m_f, k_1, k_2) \;=\; a_3(\beta)\varepsilon^{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta + 2m_f\Delta^{\mu\nu}(m_f, k_1, k_2), \tag{6.71}$$

where

$$\Delta^{\mu\nu}(m_f, k_1, k_2) = m_f \varepsilon^{\alpha\beta\mu\nu} k_{1,\alpha} k_{2,\beta}\left(\frac{1}{2\pi^2}\right) I(m_f)$$
$$I(m_f) \equiv -\int_0^1 \int_0^{1-x} dxdy \frac{1}{m_f^2 + (x-1)xk_1^2 + (y-1)yk_2^2 - 2xyk_1\cdot k_2}, \tag{6.72}$$

and the expression for the $V_A^{\mu\nu}(k_1,k_2)$ vertex

$$V_A^{\mu\nu}(k_1,k_2) = \frac{4C_{AA}}{M}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}, \tag{6.73}$$

we get

$$\left[\frac{ig_{\lambda\lambda'}}{k^2 - M_B^2}\right]\left[\frac{ig_{\mu\mu'}}{k_1^2}\right]\left[\frac{ig_{\nu\nu'}}{k_2^2}\right]\Big\{ i\, g_B g_A^2\, a_3(\beta_0)\,\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}$$
$$+2\, i\, g_B g_A^2\, m_f\, \Delta^{\mu\nu}(m_f,k_1,k_2) - 2\, M_B\, \frac{4C_{AA}}{M}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}$$
$$-2\, i g_B g_A^2\, M_B\, \frac{m_f}{M_B}\, \Delta^{\mu\nu}_{G_B AA}(m_f,k_1,k_2)\Big\} = 0. \tag{6.74}$$

Since $\Delta^{\mu\nu}_{G_B AA} = \Delta^{\mu\nu}$, Eq. (6.74) yields the same condition obtained by fixing C_{AA} in the Stückelberg phase, that is

$$i\, g_B g_A^2 a_3(\beta_0) = 2\, M_1 \frac{4C_{AA}}{M} \qquad \Rightarrow \qquad C_{AA} = \frac{i\, g_B g_A^2}{2}\frac{1}{4}\, a_3(\beta_0)\,\frac{M}{M_1}. \tag{6.75}$$

A similar STI can be derived for the BBB vertex in this phase, obtaining

$$\frac{\partial}{\partial z^\lambda}\langle 0|T\, B^\lambda(z)B_\mu(x)B_\nu(y)|0\rangle - M_B\langle 0|T\, G_B(z)B_\mu(x)B_\nu(y)|0\rangle = 0. \tag{6.76}$$

Expanding perturbatively (6.76) we obtain

$$\frac{1}{3!}\frac{\partial}{\partial z^\lambda}\langle 0|T\, B^\lambda(z)B_\mu(x)B_\nu(y)\,[J_5 B]^3\,|0\rangle$$
$$-\, M_B\,\langle 0|T\, G_B(z)B_\mu(x)B_\nu(y)\,[G_B F_B\wedge F_B]\,|0\rangle$$
$$-\, M_B\,\langle 0|T\, G_B(z)B_\mu(x)B_\nu(y)\left[\tilde{J}_5 G_B\right][J_5 B]^2\,|0\rangle = 0, \tag{6.77}$$

that gives

$$\frac{1}{3!}\,3!\,\left[ik^{\lambda'}\right]\left[-\frac{ig_{\lambda\lambda'}}{k^2-M_B^2}\right]\left[-\frac{ig_{\mu\mu'}}{k_1^2-M_B^2}\right]\left[-\frac{ig_{\nu\nu'}}{k_2^2-M_B^2}\right]\left[-g_B^3\right]\Delta^{\lambda\mu\nu}(m_f,k_1,k_2)$$
$$-M_B\left[\frac{i}{k^2-M_B^2}\right]\left[-\frac{ig_{\mu\mu'}}{k_1^2-M_B^2}\right]\left[-\frac{ig_{\nu\nu'}}{k_2^2-M_B^2}\right]\left\{2\,\frac{M_1}{M_B}\,V_B^{\mu\nu}(k_1,k_2)\right.$$
$$\left.+\frac{1}{2!}\,2\,i\,g_B^3\left(2i\,\frac{m_f}{M_B}\right)\Delta^{\mu\nu}_{G_BBB}(m_f,k_1,k_2)\right\}=0, \tag{6.78}$$

where we have defined

$$\Delta^{\mu\nu}_{G_BBB}=\int\frac{d^4q}{(2\pi)^4}\frac{Tr\left[\gamma^5(/q-/k+m_f)\gamma^\nu\gamma^5(/q-/k_1+m_f)\gamma^\mu\gamma^5(/q+m_f)\right]}{\left[q^2-m_f^2\right]\left[(q-k)^2-m_f^2\right]\left[(q-k_1)^2-m_f^2\right]}$$
$$+\{\mu\leftrightarrow\nu,k_1\leftrightarrow k_2\}\;. \tag{6.79}$$

Since this contribution is finite, it gives

$$\Delta^{\mu\nu}_{G_BBB}=2\int\frac{d^4q}{(2\pi)^4}\int_0^1\int_0^{1-x}dxdy\frac{2m4i\varepsilon^{\mu\nu\alpha\beta}k_{1,\alpha}k_{2,\beta}}{\left[q^2-k_2^2(y-1)y-k_1^2(x-1)x+2xy-m_f^2\right]^3} \tag{6.80}$$

and we obtain again

$$\Delta^{\mu\nu}_{G_BBB}=\Delta^{\mu\nu}=\varepsilon^{\alpha\beta\mu\nu}k_{1,\alpha}k_{2,\beta}m_f\left(\frac{1}{2\pi^2}\right)I(m_f)\,, \tag{6.81}$$

Using the anomaly equations in the chirally broken phase

$$k_\lambda\Delta_3^{\lambda\mu\nu}(k_1,k_2)\;\;=\;\;\frac{a_n}{3}\varepsilon^{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta+2m_f\Delta^{\mu\nu} \tag{6.82}$$

and the expression of the vertex

$$V_B^{\mu\nu}(k_1,k_2)=\frac{4C_{BB}}{M}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}, \tag{6.83}$$

we obtain

$$C_{BB}=\frac{i\,g_B^3}{2}\frac{1}{4}\,\frac{a_n}{3}\,\frac{M}{M_1}. \tag{6.84}$$

Expanding to the lowest nontrivial order this identity we obtain

$$i\left(\frac{a_n}{3}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}+2m_f\Delta^{\mu\nu}\right)-2M_B\left(\frac{4}{M}C_{BB}\frac{M_1}{M_B}\right)\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}-M_B\left(2i\frac{m_f}{M_B}\right)\Delta^{\mu\nu}_{G_BBB}=0, \tag{6.85}$$

which can be easily solved for C_{BB}, thereby determining C_{BB} exactly at the same value inferred from the Stückelberg phase, as discussed above.

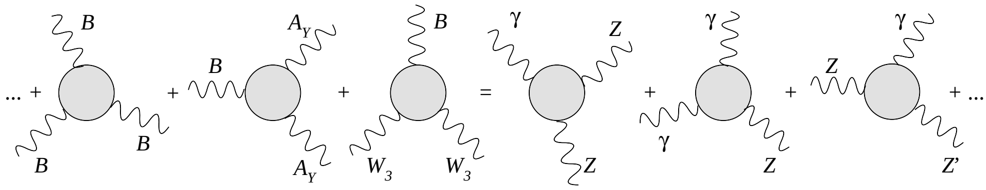


Figure 6.7: The anomalous effective action in the two basis in the R_ξ gauge where we have eliminated the mixings on the external lines in both basis.

6.2.2 Slavnov-Taylor Identities and BRST symmetry in the complete model

It is obvious, from the analysis presented above, that a similar treatment is possible also in the non-abelian case, though the explicit analysis is more complex. The objective of this investigation, however, is by now clear: we need to connect the anomalous effective action of the general model in the interaction basis and in the mass eigenstate basis keeping into account that *both* phases are broken phases. In Fig. 6.7 this point is shown pictorially. In both cases the bilinear mixings of the goldstones with the corresponding gauge fields, $Z\partial G_Z, Z'\partial G_{Z'}$ have been removed and the counterterms in the eigenstate basis have been fixed as in [17], where we have just shown it for the A-B model. Equivalently, we can fix the counterterms in the HS phase by imposing the STI's directly at this stage, thereby defining the anomalous effective action plus WZ terms completely. For this we need the BRST transformation of the fundamental fields. As usual, in the gauge sector these can be obtained by replacing the gauge parameter in their gauge variations with the corresponding ghost fields times a Grassmann parameter ω. Denoting by s the BRST operator, these are given by

$$sA^\gamma_\mu = \omega\,\partial_\mu c_\gamma + i\,O^A_{11}\,g_2\,\omega\left(c^- W^+_\mu - c^+ W^-_\mu\right), \tag{6.86}$$

$$sZ_\mu = \omega\,\partial_\mu c_Z + i\,O^A_{21}\,g_2\,\omega\left(c^- W^+_\mu - c^+ W^-_\mu\right), \tag{6.87}$$

$$sZ'_\mu = \omega\,\partial_\mu c_{Z'} + i\,O^A_{31}\,g_2\,\omega\left(c^- W^+_\mu - c^+ W^-_\mu\right) \tag{6.88}$$

$$\begin{aligned} s\,W^+_\mu &= \omega\,\partial_\mu c^+ - ig_2 W^+_\mu\,\omega\left(O^A_{11}c_\gamma + O^A_{21}c_Z + O^A_{31}c_{Z'}\right) \\ &+ ig_2\left(O^A_{11}A_{\gamma\mu} + O^A_{21}Z_\mu + O^A_{31}Z'_\mu\right)\omega c^+, \end{aligned} \tag{6.89}$$

$$\begin{aligned} sW^-_\mu &= \omega\partial_\mu c^- + ig_2 W^-_\mu\omega\left(O^A_{11}c_\gamma + O^A_{21}c_Z + O^A_{31}c_{Z'}\right) \\ &- ig_2\left(O^A_{11}A_{\gamma\mu} + O^A_{21}Z_\mu + O^A_{31}Z'_\mu\right)\omega c^-, \end{aligned} \tag{6.90}$$

where the O^A_{ij} are matrix elements defined exactly as in Eq. (6.111) below. To determine the transformations rules for the ghost/antighost fields we recall that the gauge-fixing Lagrangians in the R_ξ gauge are given by

$$
\begin{aligned}
\mathcal{L}_{gf}^{Z} &= -\frac{1}{2\xi_Z}\mathcal{F}[Z,G^Z]^2 = -\frac{1}{2\xi_Z}(\partial_\mu Z^\mu - \xi_Z M_Z G^Z)^2, && (6.91)\\
\mathcal{L}_{gf}^{Z'} &= -\frac{1}{2\xi_{Z'}}\mathcal{F}[Z',G^{Z'}]^2 = -\frac{1}{2\xi_{Z'}}(\partial_\mu Z'^\mu - \xi_{Z'} M_{Z'} G^{Z'})^2, && (6.92)\\
\mathcal{L}_{gf}^{A_\gamma} &= -\frac{1}{2\xi_A}\mathcal{F}[A_\gamma]^2 = -\frac{1}{2\xi_A}(\partial_\mu A_\gamma^\mu)^2, && (6.93)\\
\mathcal{L}_{gf}^{W} &= -\frac{1}{\xi_W}\mathcal{F}[W^+,G^+]\mathcal{F}[W^-,G^-] = \\
&= -\frac{1}{\xi_W}(\partial_\mu W^{+\mu} + i\xi_W M_W G^+)(\partial_\mu W^{-\mu} - i\xi_W M_W G^-), && (6.94)
\end{aligned}
$$

where G^Z, $G^{Z'}$, G^+ and G^- are the goldstones of Z, Z', W^+ and W^- respectively.

In particular, the FP (ghost) part of the Lagrangian is canonically given by

$$\mathcal{L}_{FP} = -\bar{c}^a \frac{\delta \mathcal{F}^a[Z,z]}{\delta\theta^b} c^b, \tag{6.95}$$

where the sum over a and b runs over the fields Z, Z', A_γ, W^+ e W^- and is explicitly given in the appendix. For the BRST variations of the antighosts we obtain

$$s\,\bar{c}_a = -\frac{i}{\xi_a}\,\omega\,\mathcal{F}^a \qquad\qquad a = Z, Z', \gamma, +, - \tag{6.96}$$

and in particular

$$
\begin{aligned}
s\,\bar{c}_Z &= -\frac{i}{\xi_Z}\,\omega\,\left(\partial_\mu Z^\mu - \xi_Z M_Z G^Z\right) && (6.97)\\
s\,\bar{c}_{Z'} &= -\frac{i}{\xi_{Z'}}\,\omega\,\left(\partial_\mu Z'^\mu - \xi_{Z'} M_{Z'} G^{Z'}\right) && (6.98)\\
s\,\bar{c}_\gamma &= -\frac{i}{\xi_\gamma}\,\omega\,\left(\partial_\mu A_\gamma^\mu\right) && (6.99)\\
s\,\bar{c}_+ &= -\frac{i}{\xi_W}\,\omega\,\left(\partial_\mu W^{+\mu} + i\xi_W M_W G^+\right) && (6.100)\\
s\,\bar{c}_- &= -\frac{i}{\xi_W}\,\omega\,\left(\partial_\mu W^{-\mu} - i\xi_W M_W G^-\right), && (6.101)
\end{aligned}
$$

giving typically the STI

$$\frac{\partial}{\partial z^\lambda}\langle 0|T\,Z^\lambda(z)A_\mu(x)A_\nu(y)|0\rangle - M_Z\langle 0|T\,G_Z(z)A_\mu(x)A_\nu(y)|0\rangle = 0, \tag{6.102}$$

and a similar one for the Z' gauge boson.

We pause for a moment to emphasize the difference between this STI and the corresponding one in the SM. In this latter case the structure of the STI is

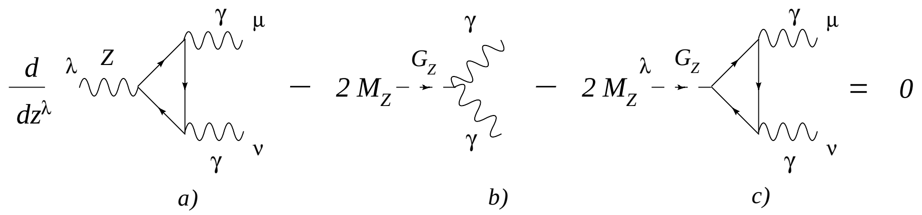


Figure 6.8: The general STI for the $Z\gamma\gamma$ vertex in our anomalous model away from the chiral limit. The analogous STI for the SM case consists of only diagrams a) and c).

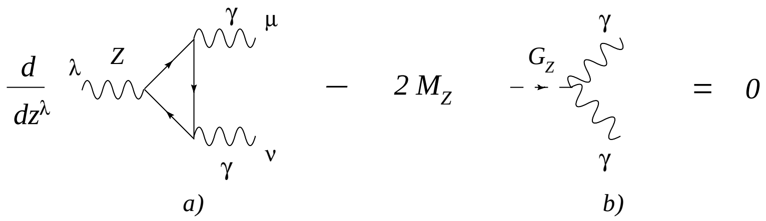


Figure 6.9: The STI for the $Z\gamma\gamma$ vertex for our anomalous model and in the chiral phase. The analogous STI in the SM consists of only diagram a).

$$
\begin{aligned}
k_\rho\, G^{\rho\nu\mu} &= (k_1+k_2)_\rho\, G^{\rho\nu\mu} \\
&= \frac{e^2 g}{\pi^2\cos\theta_W}\sum_f g_A^f Q_f^2\, \epsilon^{\nu\mu\alpha\beta} k_{1\alpha}k_{2\beta}\left[-m_f^2\int_0^1 dx_1\int_0^{1-x_1} dx_2\,\frac{1}{\Delta}\right],
\end{aligned}
\tag{6.103}
$$

where $G^{\rho\nu\mu}$ is the gauge boson vertex, which is shown pictorially in Fig. 6.8 (diagrams a and c). Notice that the goldstone contribution is the factor in square brackets in the expression above, being the coupling of the Goldstone proportional to m_f^2/M_Z. In the chiral limit the STI of the $Z\gamma\gamma$ vertex of the Standard Model becomes an ordinary Ward identity, as in the photon case. In Fig. 6.8 the modification due to the presence of the WZ term is evident. In fact, expanding (6.102) in the anomalous case we have

$$
\begin{aligned}
k_\rho\, G^{\rho\nu\mu} &= (k_1+k_2)_\rho\, G^{\rho\nu\mu} \\
&= \frac{e^2 g}{\pi^2\cos\theta_W}\sum_f g_A^f Q_f^2\, \epsilon^{\nu\mu\alpha\beta} k_{1\alpha}k_{2\beta}\left[\frac{1}{2}-m_f^2\int_0^1 dx_1\int_0^{1-x_1} dx_2\,\frac{1}{\Delta}\right],
\end{aligned}
\tag{6.104}
$$

where the first term in the square brackets is now the WZ contribution and the second the usual goldstone contribution, as in the SM case. Notice that the factor $\sum_f g_A^f Q_f^2$ is in fact proportional to the total chiral asymmetry of the Z vertex, which is mass independent and appears as a factor in front of the WZ counterterm. In the chiral limit the anomalous STI is represented in Fig. 6.9. At this point we are ready to proceed with a more general analysis of the trilinear gauge interactions to derive the expressions of all the anomalous vertices of a given theory in the mass eigenstate basis and *away from the chiral limit*. The reason for stressing this aspect has to do with the way the chiral symmetry breaking effects appear in the SM and in the anomalous models. In particular, we will start by extending the analysis presented in [17] for the derivation of the $Z\gamma\gamma$ vertex, which is here presented in far more detail. Compared to [17] we show some unobvious features of the derivation which are essential in order to formulate general rules for the computation of these vertices. We rotate the fields from the interaction eigenstate basis to the physical basis and the CS counterterms are partly absorbed and the anomaly is moved from the anomaly-free gauge boson vertices to the anomalous ones. This analysis is then extended to other trilinear vertices and we finally provide general rules to handle these types of interactions for a generic number of $U(1)$'s.

Before we come to the analysis of this vertex, we recall that the neutral current sector of the model is defined as [17]

$$-\mathcal{L}_{NC} = \overline{\psi}_f \gamma^\mu \mathcal{F} \psi_f, \tag{6.105}$$

with

$$\mathcal{F} = g_2 W^3_\mu T^3 + g_Y Y A^Y_\mu + g_B Y_B B_\mu \tag{6.106}$$

expressed in the interaction eigenstate basis. Equivalently it can be re-expressed as

$$\mathcal{F} \;=\; g_Z Q_Z Z_\mu + g_{Z'} Q_{Z'} Z'_\mu + e\, Q A^\gamma_\mu, \tag{6.107}$$

where $Q = T^3 + Y$. The physical fields A^γ, Z, Z' and W_3, A^Y, B are related by the rotation matrix O^A to the interaction eigenstates

$$\begin{pmatrix} A^\gamma \\ Z \\ Z' \end{pmatrix} = O^A \begin{pmatrix} W_3 \\ A^Y \\ B \end{pmatrix} \tag{6.108}$$

or equivalently

$$\begin{aligned} W^3_\mu &= O^A_{W_3\gamma} A^\gamma_\mu + O^A_{W_3 Z} Z_\mu + O^A_{W_3 Z'} Z'_\mu && (6.109)\\ A^Y_\mu &= O^A_{Y\gamma} A^\gamma_\mu + O^A_{YZ} Z_\mu + O^A_{YZ'} Z'_\mu && (6.110)\\ B_\mu &= O^A_{BZ} Z_\mu + O^A_{BZ'} Z'_\mu. && (6.111)\end{aligned}$$

Substituting these transformations in the expression of the bosonic operator $\mathcal{F}$ and reading the coefficients of the fields Z_μ, Z'_μ and A^γ_μ we obtain this set of relations for the coupling constants

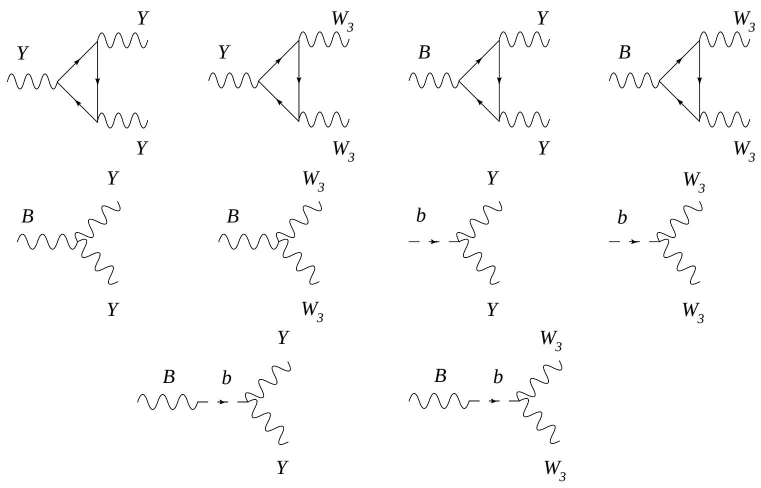


Figure 6.10: All the triangle diagrams and the possible CS and WZ counterterms present in the model (chiral phase). Not all these diagrams project on $Z \to \gamma\gamma$ in the mass eigenstate basis.

and the generators in the two basis, given here in a chiral form

$$
\begin{aligned}
g_Z Q_Z^L &= g_2 T^{3L} O^A_{W_3 Z} + g_Y Y^L O^A_{YZ} + g_B Y^L_B O^A_{BZ} && (6.112)\\
g_Z Q_Z^R &= g_Y Y^R O^A_{YZ} + g_B Y^R_B O^A_{BZ} && (6.113)\\
g_{Z'} Q_{Z'}^L &= g_2 T^{3L} O^A_{W_3 Z'} + g_Y Y^L O^A_{YZ'} + g_B Y^L_B O^A_{BZ'} && (6.114)\\
g_{Z'} Q_{Z'}^R &= g_Y Y^R O^A_{YZ'} + g_B Y^R_B O^A_{BZ'} && (6.115)\\
e Q^L &= g_2 T^{3L} O^A_{W_3 A} + g_Y Y^L O^A_{YA} = g_Y Y^R O^A_{YA} = e Q^R. && (6.116)
\end{aligned}
$$

6.3 General analysis of the $Z\gamma\gamma$ vertex

Let's now come to a brief analysis of this vertex, stressing on the general features of its derivation, which has not been detailed in [17]. In particular we highlight the general approach to follow in order to derive these vertices and apply it to the case when several anomalous $U(1)$'s are present. We will exploit the invariance of the anomalous part of the effective action under transformations of the external classical fields. This is illustrated in Fig. 6.7. More formally we can set

$$W_{anom}(B, W, A_Y) = W_{anom}(Z, Z', A_\gamma), \tag{6.117}$$

where we limit our analysis to the anomalous contributions.

In the chiral phase, the triangle diagrams projecting on this vertex are the following: YYY, YW_3W_3, BYY and BW_3W_3. They are represented in Fig. 6.10, where we have added the corresponding counterterms.

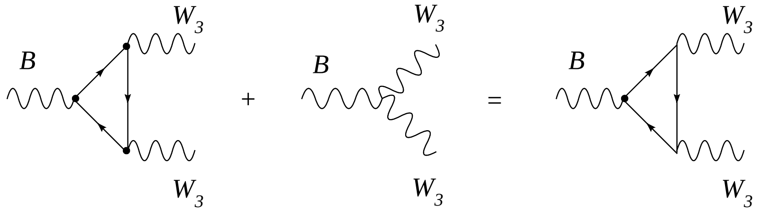


Figure 6.11: The routing of the anomaly and the absorption of the CS term into the anomalous B gauge boson. The anomaly is distributed among the vertices with the black dot.

The first two are SM-like and hence anomaly-free by charge assignment. The diagrams involving the B gauge boson are typical of these models, are anomalous, and require suitable counterterms in order to cancel their anomalies. All the possible counterterms are shown in Fig. 6.10. The WZ terms of the form bYY or bW_3W_3 will project both on a $G_Z\gamma\gamma$ and a $\chi\gamma\gamma$ interactions, the first one being relevant for the STI of the vertex. The main issue to be addressed is that of the distribution of the anomaly among the triangular vertices. These points have been discussed in [16] and [17] working in the chiral limit, when the fermion masses are removed from the diagrams.

The procedure can follow, equivalently, two directions: we can start from the BYW_3 basis and project onto the vertices $Z\gamma\gamma, ZZ\gamma...$, rotating the fields (not the charges) or, equivalently, start from the $Z, Z'\gamma$ basis and rotate the charges (but not the fields) and the generators onto the interaction eigenstate basis BYW_3. We obtain two equivalent descriptions of the various vertices. In the interaction basis the CS terms are absorbed and the anomaly is moved from the Y or W vertices into the B vertex, where it is cancelled by the axion (see Fig. 6.11). This is the meaning of the STI's shown above. Therefore it is clear that most of the CS terms do not appear explicitly if we use this approach. On the other hand, if we work in the mass eigenstate basis they can be kept explicit, but one has to be careful because in this case also the remaining vertices containing the generator of the electric charge $Q \sim Y + T_3$ have partial anomalies. The two approaches, as we are going to see, can be combined in a very economical way in some special cases, for instance for the $Z\gamma\gamma$ vertex, where one can attach all the anomaly to the Z gauge boson and add only the $G_Z\gamma\gamma$ counterterm. Similarly, for other interactions such as the $ZZ\gamma$ vertex, the total anomaly has to be equally distributed between the two Z's, since only the B generator carries an anomaly in the chiral limit, if we choose to absorb the CS terms. For other vertices such as ZZZ' etc, all the vertices contribute to the total anomaly and their partial contributions can be identified by decomposing the corresponding triangle in the YBW_3 basis with some CS terms left over.

6.4 The $\langle Z_l\gamma\gamma\rangle$ vertex

In this section we begin our technical discussion of the method. Since the most general case is encountered when at least 3 anomalous $U(1)$'s are present in the theory, we will consider for definiteness a model with three of them, say $B_j = \{B_1, B_2, B_3\}$. We can write the field transformation from interaction eigenstates basis to the mass eigenstates basis as

$$W_3 = O^A_{W_3\gamma}A_\gamma + \sum_{l=0}^{3} O^A_{W_3 Z_l} Z_l \tag{6.118}$$

$$Y = O^A_{Y\gamma}A_\gamma + \sum_{l=0}^{3} O^A_{Y Z_l} Z_l \tag{6.119}$$

$$B_j = O^A_{B_j\gamma}A_\gamma + \sum_{l=0}^{3} O^A_{B_j Z_l} Z_l, \tag{6.120}$$

with $j = 1, 2, 3$, where for $l = 0$ we have the Z_0 belonging to the SM and Z_1, Z_2, Z_3 are the anomalous ones. As in [17] we rotate the external field of the anomalous interactions from one base to the other, selecting the projections over the $Z_l\gamma\gamma$ vertex (the ellipsis indicate additional contributions that have no projection on the vertex that we consider)

$$\frac{1}{3!}Tr\left[Q_Y^3\right]\langle YYY\rangle = \frac{1}{3!}Tr\left[Q_Y^3\right] R^{YYY}_{Z_l\gamma\gamma}\langle Z_l\gamma\gamma\rangle + \ldots \tag{6.121}$$

$$\frac{1}{2!}Tr\left[Q_Y T_3^2\right]\langle YWW\rangle = \frac{1}{2!}Tr\left[Q_Y T_3^2\right] R^{YWW}_{Z_l\gamma\gamma}\langle Z_l\gamma\gamma\rangle + \ldots \tag{6.122}$$

$$\frac{1}{2!}Tr\left[Q_{B_j} Q_Y^2\right]\langle B_j YY\rangle = \frac{1}{2!}Tr\left[Q_{B_j} Q_Y^2\right] R^{B_j YY}_{Z_l\gamma\gamma}\langle Z_l\gamma\gamma\rangle + \ldots \tag{6.123}$$

$$\frac{1}{2!}Tr\left[Q_{B_j} T_3^2\right]\langle B_j WW\rangle = \frac{1}{2!}Tr\left[Q_{B_j} T_3^2\right] R^{WWB_j}_{Z_l\gamma\gamma}\langle Z_l\gamma\gamma\rangle + \ldots \tag{6.124}$$

where the rotation coefficients $R^{YYY}_{Z_l\gamma\gamma}, R^{YWW}_{Z_l\gamma\gamma}, R^{B_iYY}_{Z_l\gamma\gamma}, R^{B_iWW}_{Z_l\gamma\gamma}$ containing several products of the elements of the rotation matrix O^A are given by

$$R^{YYY}_{Z_l\gamma\gamma} = 3\left[(O^A)_{YZ_l}(O^A)^2_{Y\gamma}\right] \tag{6.125}$$

$$R^{YWW}_{Z_l\gamma\gamma} = \left[2(O^A)_{W_3\gamma}(O^A)_{YZ_l}(O^A)_{Y\gamma} + (O^A)^2_{W_3\gamma}(O^A)_{YZ_l}\right] \tag{6.126}$$

$$R^{WWW}_{Z_l\gamma\gamma} = \left[3(O^A)_{B_iZ_l}(O^A)^2_{W_3\gamma}\right] \tag{6.127}$$

$$R^{YYW}_{Z_l\gamma\gamma} = \left[2(O^A)_{YZ_l}(O^A)_{Y\gamma}(O^A)_{W_3\gamma} + (O^A)_{W_3Z_l}(O^A)^2_{Y\gamma}\right] \tag{6.128}$$

$$R^{B_iYY}_{Z_l\gamma\gamma} = (O^A)^2_{Y\gamma}(O^A)_{B_iZ_l} \tag{6.129}$$

$$R^{B_iWW}_{Z_l\gamma\gamma} = \left[(O^A)^2_{W_3\gamma}(O^A)_{B_iZ_l}\right] \tag{6.130}$$

$$R^{B_iYW}_{Z_l\gamma\gamma} = \left[2(O^A)_{B_iZ_l}(O^A)_{W_3\gamma}(O^A)_{Y\gamma}\right] \tag{6.131}$$

It is important to note that in the chiral phase the YYY and YWW contributions vanish because of the SM charge assignment. As we move to the $m_f \neq 0$ phase we must include (together with YYY and YWW) the other contributions listed below

$$\frac{1}{3!}Tr\left[Q_W^3\right]\langle WWW\rangle = \frac{1}{3!}Tr\left[T_3^3\right]R^{WWW}_{Z_l\gamma\gamma}\langle Z_l\gamma\gamma\rangle + \ldots \tag{6.132}$$

$$Tr\left[Q_{B_j}Q_Y T_3\right]\langle B_j YW\rangle = Tr\left[Q_{B_j}Q_Y T_3\right]R^{B_jYW}_{Z_l\gamma\gamma}\langle Z_l\gamma\gamma\rangle + \ldots \tag{6.133}$$

$$\frac{1}{2!}Tr\left[Q_Y^2 T_3\right]\langle YYW\rangle = \frac{1}{2!}Tr\left[Q_Y^2 T_3\right]R^{YYW}_{Z_l\gamma\gamma}\langle Z_l\gamma\gamma\rangle + \ldots \tag{6.134}$$

More details on the approach will be given below. For the moment we just mention that the structure of the CS term can be computed by rotating the WZ counterterms into the physical basis, having started with a symmetric distribution of the anomaly in all the triangle diagrams. The CS terms in this case take the form

$$V_{CS} = \frac{a_n}{3}\varepsilon^{\lambda\mu\nu\alpha}(k_{1,\alpha} - k_{2,\alpha})\frac{1}{8}\sum_j\sum_f\left[g_{B_j}g_Y^2\theta_f^{B_jYY}R^{B_jYY}_{Z_l\gamma\gamma} + g_{B_j}g_2^2\theta_f^{B_jWW}R^{B_jWW}_{Z_l\gamma\gamma}\right]Z_l^\lambda A_\gamma^\mu A_\gamma^\nu\,, \tag{6.135}$$

and they are rotated into the physical basis together with the anomalous interactions [17]. We have defined the following chiral asymmetries

$$\theta_f^{B_jYY} = Q^L_{B_j,f}(Q^L_{Y,f})^2 - Q^R_{B_j,f}(Q^R_{Y,f})^2 \tag{6.136}$$

$$\theta_f^{B_jWW} = Q^L_{B_j,f}(T^3_{L,f})^2\,. \tag{6.137}$$

We can show that the equations of the vertices in the momentum space can be obtained following a procedure similar to the case of a single $U(1)$ [17], that we are now going to generalize. In particular we will try to absorb all the CS terms that we can, getting as close as possible to the SM result. This is in general possible for diagrams that have specific Bose symmetries or conserved electromagnetic currents, but some of the details of this construction are quite subtle especially as we move away from the chiral limit.

6.4.1 Decomposition of the $Z_l\gamma\gamma$ vertex

As we have mentioned, the anomalous effective action, composed of the triangle diagrams plus their CS counterterms can be expressed either in the base of the mass eigenstates or in that of the interaction eigenstates.

We start by keeping all the pieces of the 1-loop effective action in the interaction basis in the $m_f \neq 0$ phase and rotate the external (classical) fields on the physical basis taking all the contribution to the $\langle Z_l\gamma\gamma\rangle$ vertex.

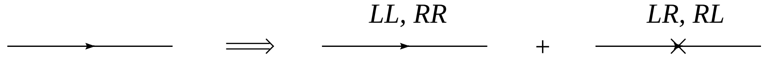


Figure 6.12: Chiral decomposition of the fermionic propagator after a mass insertion.

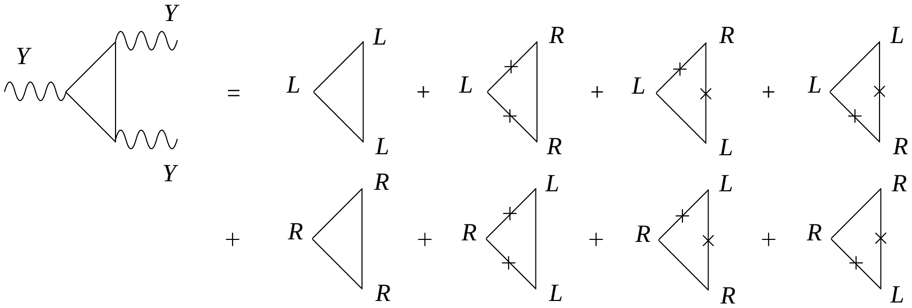


Figure 6.13: Chiral triangle contributions to the YYY vertex. The same decomposition holds for the B_iYY case.

A given vertex is first decomposed into its chiral contributions and then rotated into the physical gauge boson eigenstates. For instance, let's start with the non anomalous YYY vertex see Figs. (6.12,6.13). Actually, in this specific case the sums over each fermion generation are actually zero in the chiral limit, but we will impose this condition at the end and prefer to follow the general treatment as for other (anomalous) vertices. We write this vertex in terms of chiral projectors (L/R), where $L/R \equiv 1 \mp \gamma_5$, and the diagrams contain a massive fermion of mass m_f. The structure of the vertex is

$$\langle LLL\rangle|_{m_f\neq 0} = \int \frac{d^4q}{(2\pi)^4}\frac{Tr\left[(\not{q}+m_f)\gamma^\lambda P_L(\not{q}+\not{k}+m_f)\gamma^\nu P_L(\not{q}+\not{k}_1+m_f)\gamma^\mu P_L\right]}{(q^2-m_f^2)\left[(q+k)^2-m_f^2\right]\left[(q+k_1)^2-m_f^2\right]} + exch. \tag{6.138}$$

The vertices of the form LLR, RRL, and so on, are obtained from the expression above just by substituting the corresponding chiral projectors. Notice that for loops of fixed chirality we have no mass contributions from the trace in the numerator and we easily derive the identity

$$\langle LLL\rangle|_{m_f\neq 0} = -\langle RRR\rangle|_{m_f\neq 0}. \tag{6.139}$$

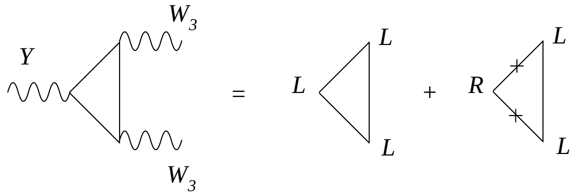


Figure 6.14: Chiral triangle contributions to the YWW vertex. The same decomposition holds for the B_iWW case.

At this point we start decomposing each diagram in the interaction basis

$$\begin{aligned}\langle YYY\rangle\, g_Y^3\, Tr[Q_Y^3] \;=\; & \sum_f \Big[g_Y^3 (Q_{Y,f}^L)^3 \langle LLL\rangle^{\lambda\mu\nu} + g_Y^3 (Q_{Y,f}^R)^3 \langle RRR\rangle^{\lambda\mu\nu} \\ & + g_Y^3\, Q_{Y,f}^L\, (Q_{Y,f}^R)^2 \langle LRR\rangle^{\lambda\mu\nu} + g_Y^3\, Q_{Y,f}^L\, Q_{Y,f}^R\, Q_{Y,f}^L \langle LRL\rangle^{\lambda\mu\nu} \\ & + g_Y^3\, (Q_{Y,f}^L)^2\, Q_{Y,f}^R \langle LLR\rangle^{\lambda\mu\nu} + g_Y^3\, Q_{Y,f}^R\, (Q_{Y,f}^L)^2 \langle RLL\rangle^{\lambda\mu\nu} \\ & + +g_Y^3\, Q_{Y,f}^R\, Q_{Y,f}^L\, Q_{Y,f}^R \langle RLR\rangle^{\lambda\mu\nu} + g_Y^3 (Q_{Y,f}^R)^2 Q_{Y,f}^L \langle RRL\rangle^{\lambda\mu\nu} \Big] \frac{1}{8} Z_l^\lambda A_\gamma^\mu A_\gamma^\nu R_{Z_l\gamma\gamma}^{YYY} + \ldots \end{aligned} \tag{6.140}$$

where the factor of 1/8 comes from the chiral projectors and the dots indicate all the other contributions of the type $Z_l Z_m \gamma, Z_l Z_m Z_r$ and so on, which do not contribute to the $Z_l\gamma\gamma$ vertex. This *projection* contains chirality conserving and chirality flipping terms. The two combinations which are chirally conserving are LLL and RRR while the remaining ones need to have 2 chirality flips to be nonzero (ex. LLR or RRL) and are therefore proportional to m_f^2.

We repeat this procedure for all the other vertices in the interaction eigenstate basis that project on the vertex we are interested in. For instance, in the case of the $\langle YWW\rangle$ vertex the structure is simpler because the generator associated to W_3 is left-chiral (Fig. 6.14)

$$\begin{aligned}\langle YWW\rangle\, g_Y g_2^2\, Tr[Q_Y (T^3)^2] = & \sum_f \Big[g_Y g_2^2 Q_{Y,f}^L (T_{L,f}^3)^2 \langle LLL\rangle^{\lambda\mu\nu} \\ & + g_Y g_2^2 Q_{Y,f}^R (T_{L,f}^3)^2 \langle RLL\rangle^{\lambda\mu\nu} \Big] \frac{1}{8} Z_l^\lambda A_\gamma^\mu A_\gamma^\nu R_{Z_l\gamma\gamma}^{YWW} + \ldots \end{aligned} \tag{6.141}$$

Similarly, all the pieces B_iYY and B_iWW for $i = 1, 2, 3$, give the projections

$$\begin{aligned}\langle B_iYY\rangle\, g_B\, g_Y^2\, Tr[Q_{B_i}\, Q_Y^2] \;=\; & \sum_f \Big[g_{B_i}\, g_Y^2\, Q_{B_i,f}^L\, (Q_{Y,f}^L)^2 \langle LLL\rangle^{\lambda\mu\nu} \\ & + g_{B_i}\, g_Y^2\, Q_{B_i,f}^R\, (Q_{Y,f}^R)^2 \langle RRR\rangle^{\lambda\mu\nu} + g_{B_i}\, g_Y^2\, Q_{B_i,f}^L\, (Q_{Y,f}^R)^2 \langle LRR\rangle^{\lambda\mu\nu} \\ & + g_{B_i}\, g_Y^2\, Q_{B_i,f}^L\, Q_{Y,f}^R\, Q_{Y,f}^L \langle LRL\rangle^{\lambda\mu\nu} + g_{B_i}\, g_Y^2\, Q_{B_i,f}^L\, Q_{Y,f}^L\, Q_{Y,f}^R \langle LLR\rangle^{\lambda\mu\nu} \\ & + g_{B_i}\, g_Y^2\, Q_{Y,f}^R\, (Q_{Y,f}^L)^2 \langle RLL\rangle^{\lambda\mu\nu} + g_{B_i}\, g_Y^2\, Q_{B_i,f}^R\, Q_{Y,f}^L\, Q_{Y,f}^R \langle RLR\rangle^{\lambda\mu\nu} \\ & + g_{B_i} g_Y^2 Q_{B_i,f}^R Q_{Y,f}^R Q_{Y,f}^L \langle RRL\rangle^{\lambda\mu\nu} \Big] \frac{1}{8} Z_l^\lambda A_\gamma^\mu A_\gamma^\nu R^{B_iYY} + \ldots \end{aligned} \tag{6.142}$$

and

$$\begin{aligned}\langle B_iWW\rangle\, g_Y g_2^2\, Tr[Q_{B_i}(T^3)^2] &= \sum_f \Big[g_{B_i}g_2^2 Q^L_{B_i,f}(T^3_{L,f})^2\langle LLL\rangle^{\lambda\mu\nu} \\ &+ g_{B_i}g_2^2 Q^R_{B_i,f}(T^3_{L,f})^2\langle RLL\rangle^{\lambda\mu\nu}\Big]\frac{1}{8}Z_l^\lambda A^\mu_\gamma A^\nu_\gamma R^{B_iWW}_{Z_l\gamma\gamma}+\dots\end{aligned} \tag{6.143}$$

We obtain similar expressions for the terms WWW, YYW, B_iYW, etc. which appear in the $m_f \neq 0$ phase.

The $m_f = 0$ phase

To proceed with the analysis of the amplitude we start from the chirally symmetric phase ($m_f = 0$). The terms of mixed chirality (such as $\langle LRR\rangle$ and so on) vanish in this limit, leaving only the chiral preserving interactions LLL and RRR. In this limit we can formally impose the relation

$$\langle LLL\rangle^{\lambda\mu\nu}(m_f = 0) = -4\Delta_{AAA}(0) \tag{6.144}$$

that will be used extensively throughout the chapter. This relation or other similar relations are just the starting point of the entire construction. The final expressions of the anomalous vertices are obtained using the generalized Ward identities of the theory. What really defines the theories are the distributions of the partial anomalies. We will attach an equal anomaly on each axial-vector vertex in diagrams of the form AAA and we will compensate this equal distribution with additional CS interactions - so to bring these diagrams to the desired form AVV or VAV or VVA - whenever a non anomalous $U(1)$ appears at a given vertex. For models where a single anomalous $U(1)$ is present this does not bring-in any ambiguity. For instance, conservation of the Y current in B_iYY will allow us to move the anomaly from the Y's to the B_i vertices and this is implicitly done using a CS term. We say that this procedure is allowing us to *absorb* a CS interaction. Moving to the YYY vertex, this vanishes identically in the chiral limit since we factorize left- and right-handed modes for each generation by an anomaly-free charge assignment

$$(YYY): \qquad g_Y^3 Tr[Q_Y^3] = 0, \tag{6.145}$$

$$(YWW): \qquad g_Y g_2^2 Tr[Q_Y(T_3^L)^2] = 0. \tag{6.146}$$

At this point we pause to show how the re-distribution of the anomaly goes in the case at hand. We have the contribution

$$V_{CS}^{B_iYY} = d_i\langle B_iY\wedge F_Y\rangle \tag{6.147}$$

and the BRST conditions in the Stückelberg phase give

$$d_i = -i g_{B_i} g_Y^2 \frac{2}{3} a_n D_{B_i YY}; \qquad D_{B_i YY} = \frac{1}{8} Tr[Q_{B_i} Q_Y^2]. \tag{6.148}$$

Also these terms are projected on the vertex to give

$$\begin{aligned} V_{CS}^{B_i YY} &= d_i \langle B_i Y \wedge F_Y \rangle = (-i) d_i \varepsilon^{\lambda\mu\nu\alpha}(k_{1\alpha} - k_{2\alpha}) \left[(O^A)^2_{Y\gamma} (O^A)_{B_i Z_l} \right] Z_l^\lambda A_\gamma^\mu A_\gamma^\nu + \dots \\ V_{CS}^{B_i WW} &= c_i \langle \varepsilon^{\mu\nu\rho\sigma} B_{\mu,i} C^{Abelian}_{\nu\rho\sigma} \rangle = (-i) c_i \varepsilon^{\lambda\mu\nu\alpha}(k_{1\alpha} - k_{2\alpha}) \left[(O^A)^2_{W_3\gamma} (O^A)_{B_i Z_l} \right] Z_l^\lambda A_\gamma^\mu A_\gamma^\nu + \dots \end{aligned} \tag{6.149}$$

In general, a vertex such as $B_i YY$ is changed into an **AVV**, while vertices of the form YBB and YB_iB_j which appear in the computation of the γZZ $\gamma Z_l Z_m$ interactions are changed into **VAV** + **VVA**. This procedure is summarized by the equations

$$\Delta^{\lambda\mu\nu}_{AAA}(m_f = 0, k_1, k_2) - \frac{a_n}{3} \varepsilon^{\lambda\mu\nu\alpha}(k_{1,\alpha} - k_{2,\alpha}) = \Delta^{\lambda\mu\nu}_{AVV}(m_f = 0, k_1, k_2) \tag{6.150}$$

$$\begin{aligned} \Delta^{\mu\nu\lambda}_{AAA}(m_f = 0, k_2, -k) - \frac{a_n}{3} \varepsilon^{\mu\nu\lambda\alpha}(k_{1,\alpha} + 2k_{2,\alpha}) &= \Delta^{\mu\nu\lambda}_{AVV}(m_f = 0, k_2, -k) \\ &= \Delta^{\lambda\mu\nu}_{VAV}(m_f = 0, k_1, k_2) \end{aligned} \tag{6.151}$$

$$\begin{aligned} \Delta^{\nu\lambda\mu}_{AAA}(m_f = 0, -k, k_1) - \frac{a_n}{3} \varepsilon^{\nu\lambda\mu\alpha}(-2k_{1,\alpha} - k_{2,\alpha}) &= \Delta^{\nu\lambda\mu}_{AVV}(m_f = 0, -k, k_1) \\ &= \Delta^{\lambda\mu\nu}_{VVA}(m_f = 0, k_1, k_2) \end{aligned} \tag{6.152}$$

$$\begin{aligned} \Delta^{\lambda\mu\nu}_{AAA}(m_f = 0, k_1, k_2) + \frac{a_n}{6} \varepsilon^{\lambda\mu\nu\alpha}(k_{1,\alpha} - k_{2,\alpha}) &= \\ & \frac{1}{2} \left[(\Delta^{\lambda\mu\nu}_{VAV}(m_f = 0, k_1, k_2) + \Delta^{\lambda\mu\nu}_{VVA}(m_f = 0, k_1, k_2) \right], \end{aligned} \tag{6.153}$$

where the last relation can be proved in a simple way by summing the second and the third contributions. Defining $k_3^\lambda = -k^\lambda$, one can combine together the **AAA** plus the counterterms into a unique expression for each case

$$\begin{aligned} \mathbf{V}^{\lambda\mu\nu}_{B_i YY} &= 4 D_{B_i YY}\, g_{B_i} g_Y^2\, \Delta^{\lambda\mu\nu}_{\mathbf{AAA}}(k_1, k_2) + D_{B_i YY}\, g_{B_i} g_Y^2 \frac{i}{\pi^2} \frac{2}{3} \epsilon^{\lambda\mu\nu\sigma}(k_1 - k_2)_\sigma \\ \mathbf{V}^{\mu\nu\lambda}_{Y B_i Y} &= 4 D_{B_i YY}\, g_{B_i} g_Y^2\, \Delta^{\mu\nu\lambda}_{\mathbf{AAA}}(k_2, k_3) + D_{B_i YY}\, g_{B_i} g_Y^2 \frac{i}{\pi^2} \frac{2}{3} \epsilon^{\mu\nu\lambda\sigma}(k_2 - k_3)_\sigma \\ \mathbf{V}^{\nu\lambda\mu}_{YY B_i} &= 4 D_{B_i YY}\, g_{B_i} g_Y^2\, \Delta^{\nu\lambda\mu}_{\mathbf{AAA}}(k_3, k1) + D_{B_i YY}\, g_{B_i} g_Y^2 \frac{i}{\pi^2} \frac{2}{3} \epsilon^{\nu\lambda\mu\sigma}(k_3 - k_1)_\sigma \\ \mathbf{V}^{\lambda\mu\nu}_{Y B_i B_j} &= 4 D_{Y B_i B_j}\, g_Y g_{B_i} g_{B_j}\, \Delta^{\lambda\mu\nu}_{\mathbf{AAA}}(k_1, k_2) - D_{Y B_i B_j}\, g_Y g_{B_i} g_{B_j} \frac{i}{\pi^2} \frac{1}{3} \epsilon^{\lambda\mu\nu\sigma}(k_1 - k_2)_\sigma, \end{aligned} \tag{6.154}$$

where we have rotated them onto the $Z_l\gamma\gamma$ vertex. For the non abelian case (WB_iW and WWB_i), the calculation is similar, so we omit the details.

Finally the anomalous contributions plus the CS interactions are given by

$$\begin{aligned}
&\langle B_iYY\rangle|_{m_f=0}+\langle B_iWW\rangle|_{m_f=0}=\\
&\quad+g_{B_i}g_Y^2\sum_f\left[Q^L_{B_i,f}(Q^L_{Y,f})^2-Q^R_{B_i,f}(Q^R_{Y,f})^2\right]\frac{1}{2}\Delta^{\lambda\mu\nu}_{AAA}(0)R^{B_iYY}_{Z_l\gamma\gamma}Z_l^\lambda A^\mu_\gamma A^\nu_\gamma\\
&\quad+g_{B_i}g_2^2\sum_f Q^L_{B_i,f}(T^3_{L,f})^2\frac{1}{2}\Delta_{AAA}(0)^{\lambda\mu\nu}R^{B_iWW}_{Z_l\gamma\gamma}Z_l^\lambda A^\mu_\gamma A^\nu_\gamma\\
&\quad-i\left[g_{B_i}g_Y^2\frac{4}{3}a_nD_{B_iYY}R^{B_iYY}_{Z_l\gamma\gamma}+g_{B_i}g_2^2\frac{4}{3}a_nD^{(L)}_{B_i}R^{B_iWW}_{Z_l\gamma\gamma}\right]\varepsilon^{\lambda\mu\nu\alpha}\left(k_{1,\alpha}-k_{2,\alpha}\right)Z_l^\lambda A^\mu_\gamma A^\nu_\gamma,
\end{aligned} \tag{6.155}$$

which allows to move the anomaly on the axial current and we simply get

$$\begin{aligned}
\langle Z_l\gamma\gamma\rangle|_{m_f=0}=&\sum_i g_{B_i}g_Y^2\sum_f\left[Q^L_{B_i,f}(Q^L_{Y,f})^2-Q^R_{B_i,f}(Q^R_{Y,f})^2\right]\frac{1}{2}\Delta^{\lambda\mu\nu}_{AVV}(0)R^{B_iYY}_{Z_l\gamma\gamma}Z_l^\lambda A^\mu_\gamma A^\nu_\gamma\\
&+\sum_i g_{B_i}g_2^2\sum_f Q^L_{B_i,f}(T^3_{L,f})^2\frac{1}{2}\Delta^{\lambda\mu\nu}_{AVV}(0)R^{B_iWW}_{Z_l\gamma\gamma}Z_l^\lambda A^\mu_\gamma A^\nu_\gamma\,,
\end{aligned} \tag{6.156}$$

where we transfer all the anomaly on the vertex labelled by the λ index, obtaining that the Ward identities on the photons are satisfied.

At this point, it is convenient to introduce the chiral asymmetry

$$\theta^{YB_iB_j}_f=\left[(Q^L_{Y,f})(Q^L_{B_i,f})(Q^L_{B_j,f})-(Q^R_{Y,f})(Q^R_{B_i,f})(Q^R_{B_j,f})\right] \tag{6.157}$$

and express the coefficients in front of the CS counterterms as follows

$$D_{B_iYY}=-\frac{1}{8}\sum_f\theta^{B_iYY}_f \tag{6.158}$$

$$D_{B_iWW}=-\frac{1}{8}\sum_f\theta^{B_iWW}_f \tag{6.159}$$

$$D_{YB_iB_j}=-\frac{1}{8}\sum_f\theta^{YB_iB_j}_f\,. \tag{6.160}$$

After some manipulations we obtain the expression of the $\langle Z_l\gamma\gamma\rangle$ vertex in the $m_f=0$ phase which is given by

$$\langle Z_l\gamma\gamma\rangle|_{m_f=0}=-\frac{1}{2}\Delta^{\lambda\mu\nu}_{AVV}(0)Z_l^\lambda A^\mu_\gamma A^\nu_\gamma\sum_i\sum_f\left[g_{B_i}g_Y^2\theta^{B_iYY}_fR^{B_iYY}_{Z_l\gamma\gamma}+g_{B_i}g_2^2\theta^{B_iWW}_fR^{B_iWW}_{Z_l\gamma\gamma}\right], \tag{6.161}$$

where for $\Delta_{AVV}(0)$ we write

$$\begin{aligned}\Delta_{AVV}(0)^{\lambda\mu\nu}(k_1,k_2,0) = &\frac{1}{\pi^2}\int_0^1 dx\int_0^{1-x} dy\frac{1}{\Delta(0)}\\ &\left\{\varepsilon[k_1,\lambda,\mu,\nu]\left[y(y-1)k_2^2 - xyk_1\cdot k_2\right]\right.\\ &+\varepsilon[k_2,\lambda,\mu,\nu]\left[x(1-x)k_1^2 + xyk_1\cdot k_2\right]\\ &+\varepsilon[k_1,k_2,\lambda,\nu]\left[x(x-1)k_1^\mu - xyk_2^\mu\right]\\ &\left.+\varepsilon[k_1,k_2,\lambda,\mu]\left[xyk_1^\nu + (1-y)yk_2^\nu\right]\right\},\end{aligned} \tag{6.162}$$

$$\Delta(0) = x(x-1)k_1^2 + y(y-1)k_2^2 - 2xyk_1\cdot k_2. \tag{6.163}$$

At this stage we should keep in mind that if all the external particles are on-shell, the total amplitude vanishes because of the Landau-Yang theorem. In other words the Z_l's can't decay on shell into two on-shell photons. However it is possible to have two on-shell photons if the initial state is characterized by an anomalous process as well, such as gluon fusion. This does not contradict the Landau-Yang theorem since the Z-pole disappears [30] in the presence of an anomalous Z' exchange [30].

6.4.2 The $m_f \neq 0$ phase

Now we move to the analysis of the vertices away from the chiral limit. Also in this case we separate the mass-dependent from the mass-independent contributions.

Chirality preserving vertices

We start analyzing the vertices away from the chiral limit by separating the chiral preserving contributions from the remaining ones. The general expression of LLL is given by

$$\begin{aligned}\langle LLL\rangle|_{m_f\neq 0} = &A_1\varepsilon[k_1,\lambda,\mu,\nu] + A_2\varepsilon[k_2,\lambda,\mu,\nu] + A_3k_1^\nu\varepsilon[k_1,k_2,\lambda,\mu] + A_4k_2^\nu\varepsilon[k_1,k_2,\lambda,\mu]\\ &+A_5k_1^\mu\varepsilon[k_1,k_2,\lambda,\nu] + A_6k_2^\mu\varepsilon[k_1,k_2,\lambda,\nu]\end{aligned} \tag{6.164}$$

where we have removed, for simplicity, the dependence on the charges and the coupling constants.

The divergent structures A_1 and A_2 are given by

$$\begin{aligned}A_1 &= 8i\left[\mathcal{I}_{30}(k_1,k_2) - \mathcal{I}_{20}(k_1,k_2)\right]k_1^2 + 16i\left[\mathcal{I}_{11}(k_1,k_2) - \mathcal{I}_{21}(k_1,k_2)\right]k_1\cdot k_2\\ &+ 8i\left[\mathcal{I}_{01}(k_1,k_2) - \mathcal{I}_{02}(k_1,k_2) + \mathcal{I}_{12}(k_1,k_2)\right]k_2^2 + 4i\left[3\mathcal{D}_{10}(k_1,k_2) - 2\mathcal{D}_{00}(k_1,k_2)\right]\end{aligned} \tag{6.165}$$

where

$$\mathcal{I}_{st}(k_1,k_2)=\int_0^1 dx\int_0^{1-x}dy\int\frac{d^4q}{(2\pi)^4}\frac{x^s y^t}{\left[q^2-x(1-x)k_1^2-y(1-y)k_2^2-2xyk_1\cdot k_2+m_f^2\right]^3}\tag{6.166}$$

$$\mathcal{D}_{st}(k_1,k_2)=\int_0^1 dx\int_0^{1-x}dy\int\frac{d^4q}{(2\pi)^4}\frac{q^2x^s y^t}{\left[q^2-x(1-x)k_1^2-y(1-y)k_2^2-2xyk_1\cdot k_2+m_f^2\right]^3}.\tag{6.167}$$

and one can verify that $A_1(k_1,k_2)=-A_2(k_2,k_1)$. All the mass dependence is contained only in the denominators of the propagators appearing in the Feynman parametrization.

The finite structures $A_3\dots A_6$ are the following

$$A_3(k_1,k_2) = -16i\mathcal{I}_{11}(k_1,k_2)=-A_6(k_2,k_1)\tag{6.168}$$

$$A_4(k_1,k_2) = 16i\left[\mathcal{I}_{02}(k_1,k_2)-\mathcal{I}_{01}(k_1,k_2)\right]=-A_5(k_2,k_1)\tag{6.169}$$

where still we need to perform the trivial finite integrals over the momentum q.

The decomposition of $\langle LLL\rangle_f$ into massless and massive components gives

$$\langle LLL\rangle_f = \langle LLL(m_f\neq 0)\rangle-\langle LLL\rangle(0)\tag{6.170}$$

$$\langle LLL\rangle(0) = \langle LLL(m_f=0)\rangle\tag{6.171}$$

$$\langle LLL(m_f\neq 0)\rangle = \langle LLL\rangle_f+\langle LLL\rangle(0),\tag{6.172}$$

where we have isolated the massless contributions. As we have seen before, the CS terms act only on the massless part of the triangles (having used Eq. (6.144)) and reproduce the massless contribution calculated in Eq. (6.161). Since the mass terms are proportional to the tensors $\varepsilon[k_1,\lambda,\mu,\nu]$ and $\varepsilon[k_2,\lambda,\mu,\nu]$ they can be included in the singular structures A_1 and A_2 of $\langle LLL\rangle|_{m_f\neq 0}$

$$\begin{aligned}\bar{A}_1 = A_1 &+ im_f^2(Q^R_{Y,f})^2(Q^L_{Y,f})\left[-8\mathcal{I}_{00}(q^2,k_1,k_2)+24\mathcal{I}_{10}(q^2,k_1,k_2)\right]\\
&+im_f^2(Q^L_{Y,f})^2(Q^R_{Y,f})\left[8\mathcal{I}_{00}(q^2,k_1,k_2)-24\mathcal{I}_{10}(q^2,k_1,k_2)\right]\\
&-8im_f^2Q^R_{Y,f}(T^L_{3,f})^2\mathcal{I}_{10}(q^2,k_1,k_2)\\
&-im_f^2\sum_i Q^R_{B_i,f}Q^L_{Y,f}Q^R_{Y,f}\left[8\mathcal{I}_{10}(q^2,k_1,k_2)+4\mathcal{I}_{00}(q^2,k_1,k_2)\right]\\
&+im_f^2\sum_i Q^L_{B_i,f}Q^R_{Y,f}Q^L_{Y,f}\left[8\mathcal{I}_{10}(q^2,k_1,k_2)+4\mathcal{I}_{00}(q^2,k_1,k_2)\right]\\
&-8im_f^2\sum_i Q^R_{B_i,f}(Q^L_{Y,f})^2\mathcal{I}_{10}(q^2,k_1,k_2)+8im_f^2\sum_i Q^L_{B_i,f}(Q^R_{Y,f})^2\mathcal{I}_{10}(q^2,k_1,k_2)\\
&-8im_f^2\sum_i Q^R_{B_i,f}(T^L_{3,f})^2\mathcal{I}_{10}(q^2,k_1,k_2).\end{aligned}\tag{6.173}$$

At this point we have to consider also the chirality flipping terms. For simplicity we discuss only the case of the YYY vertex, the others being similar.

Chirality flipping vertices

These contributions are extracted rather straighforwardly and contribute to the total vertex amplitude with mass corrections that modify A_1 and A_2. We discuss this point first for the $\langle YYY\rangle$, and then quote the result for the entire contribution to $Z\gamma\gamma$.

For YYY we obtain

$$\begin{aligned}(Q^R_{Y,f})^2(Q^L_{Y,f})\left[\langle RRL\rangle+\langle LRR\rangle+\langle RLR\rangle\right]=&\\ (Q^R_{Y,f})^2(Q^L_{Y,f})\,\big[8im_f^2\mathcal{I}_{00}(k_1,k_2)\left(\varepsilon[k_2,\lambda,\mu,\nu]-\varepsilon[k_1,\lambda,\mu,\nu]\right)&\\ +24im_f^2\left(\mathcal{I}_{10}(k_1,k_2)\varepsilon[k_1,\lambda,\mu,\nu]-\mathcal{I}_{01}(k_1,k_2)\varepsilon[k_2,\lambda,\mu,\nu]\right)\big]\,,&\end{aligned}\tag{6.174}$$

and the analysis can be extended to the other trilinear contributions and can be simplified using the relations

$$\left[\langle RRL\rangle+\langle LRR\rangle+\langle RLR\rangle\right]=-\left[\langle LLR\rangle+\langle RLL\rangle+\langle LRL\rangle\right].\tag{6.175}$$

The final result is given by

$$\begin{aligned}\text{mass terms}=\;&im_f^2g_Y^3(Q^R_{Y,f})^2(Q^L_{Y,f})\left[8\mathcal{I}_{00}(k_1,k_2)\left(\varepsilon[k_2,\lambda,\mu,\nu]-\varepsilon[k_1,\lambda,\mu,\nu]\right)\right.\\ &\left.+24\left(\mathcal{I}_{10}(k_1,k_2)\varepsilon[k_1,\lambda,\mu,\nu]-\mathcal{I}_{01}(k_1,k_2)\varepsilon[k_2,\lambda,\mu,\nu]\right)\right]\\ &-im_f^2g_Y^3(Q^R_{Y,f})^2(Q^L_{Y,f})\left[8\mathcal{I}_{00}(k_1,k_2)\left(\varepsilon[k_2,\lambda,\mu,\nu]-\varepsilon[k_1,\lambda,\mu,\nu]\right)\right.\\ &\left.+24\left(\mathcal{I}_{10}(k_1,k_2)\varepsilon[k_1,\lambda,\mu,\nu]-\mathcal{I}_{01}(k_1,k_2)\varepsilon[k_2,\lambda,\mu,\nu]\right)\right]\\ &+8im_f^2g_Yg_2^2Q^R_{Y,f}(T^L_{3,f})^2\left(\mathcal{I}_{01}(k_1,k_2)\varepsilon[k_2,\lambda,\mu,\nu]-\mathcal{I}_{10}(k_1,k_2)\varepsilon[k_1,\lambda,\mu,\nu]\right)\\ &+im_f^2\sum_i g_{B_i}g_Y^2Q^L_{B_i,f}Q^R_{Y,f}Q^L_{Y,f}\left[(8\mathcal{I}_{01}(q^2,k_1,k_2)-4\mathcal{I}_{00}(k_1,k_2))\varepsilon[k_2,\lambda,\mu,\nu]\right.\\ &\quad\left.+(8\mathcal{I}_{10}(k_1,k_2)+4\mathcal{I}_{00}(k_1,k_2))\varepsilon[k_1,\lambda,\mu,\nu]\right]\\ &-im_f^2\sum_i g_{B_i}g_Y^2Q^R_{B_i,f}Q^L_{Y,f}Q^R_{Y,f}\left[(8\mathcal{I}_{01}(k_1,k_2)-4\mathcal{I}_{00}(k_1,k_2))\varepsilon[k_2,\lambda,\mu,\nu]\right.\\ &\quad\left.+(8\mathcal{I}_{10}(k_1,k_2)+4\mathcal{I}_{00}(k_1,k_2))\varepsilon[k_1,\lambda,\mu,\nu]\right]\\ &+im_f^2\sum_i g_{B_i}g_Y^2Q^R_{B_i,f}(Q^L_{Y,f})^2 8\left(\mathcal{I}_{01}(k_1,k_2)\varepsilon[k_2,\lambda,\mu,\nu]-\mathcal{I}_{10}(k_1,k_2)\varepsilon[k_1,\lambda,\mu,\nu]\right)\\ &-im_f^2\sum_i g_{B_i}g_Y^2Q^L_{B_i,f}(Q^R_{Y,f})^2 8\left(\mathcal{I}_{01}(k_1,k_2)\varepsilon[k_2,\lambda,\mu,\nu]-\mathcal{I}_{10}(k_1,k_2)\varepsilon[k_1,\lambda,\mu,\nu]\right)\\ &+8im_f^2\sum_i g_{B_i}g_2^2Q^R_{B_i,f}(T^L_{3,f})^2\left(\mathcal{I}_{01}(k_1,k_2)\varepsilon[k_2,\lambda,\mu,\nu]-\mathcal{I}_{10}(k_1,k_2)\varepsilon[k_1,\lambda,\mu,\nu]\right)\end{aligned}\tag{6.176}$$

and is finite. To conclude our derivation in this special case, we can summarize our findings as follows.

In a triangle diagram of the form, say, AVV, if we impose a vector Ward identity on the two V lines we redefine the divergent invariant amplitudes A_1 and A_2 ($A_2 = -A_1$) in terms of the remaining amplitudes $A_3, ..., A_6$, which are convergent. The chirality flip contributions such as LLR turn out to be finite, but are proportional to A_1 and A_2, and disappear once we impose the WI's on the V lines. This observation clarifies why in the $Z\gamma\gamma$ vertex of the SM the mass dependence of the numerators disappears and the traces can be computed as in the chiral limit. Including the mass dependent contributions we obtain (see Fig. 6.15 for the $m_f \neq 0$ phase)

$$\begin{aligned}
\langle Z_l\gamma\gamma\rangle|_{m_f\neq 0} &= \langle Z_l\gamma\gamma\rangle|_{m_f=0} - \sum_f \frac{1}{8}\langle LLL\rangle_f^{\lambda\mu\nu}\left\{g_Y^3\theta_f^{YYY}\bar{R}^{YYY}_{Z_l\gamma\gamma} + g_2^3\theta_f^{WWW}\bar{R}^{WWW}_{Z_l\gamma\gamma}\right.\\
&+g_2^2 g_Y\theta_f^{YWW}R^{YWW}_{Z_l\gamma\gamma} + g_2 g_Y^2\theta_f^{YYW}R^{YYW}_{Z_l\gamma\gamma} + \sum_i g_{B_i}g_2 g_Y\theta_f^{B_iYW}R^{B_iYW}_{Z_l\gamma\gamma}\\
&\left.+\sum_i g_{B_i}g_Y^2\theta_f^{B_iYY}R^{B_iYY}_{Z_l\gamma\gamma} + \sum_i g_{B_i}g_2^2\theta_f^{B_iWW}R^{B_iWW}_{Z_l\gamma\gamma}\right\} Z_l^\lambda A_\gamma^\mu A_\gamma^\nu\\
&+m_f^2 \text{ (chirally flipped terms)} \qquad (6.177)
\end{aligned}$$

where $\langle LLL\rangle_f^{\lambda\mu\nu}$ is now defined by Eqs. (6.170-6.172). In Eq. (6.177) we have also defined the following chiral asymmetries

$$\theta_f^{WWW} = (T^3_{L,f})^3 \qquad (6.178)$$

$$\theta_f^{YYW} = \left[(Q^L_{Y,f})^2 T^3_{L,f}\right] \qquad (6.179)$$

$$\theta_f^{B_iYW} = \left[Q^{B_i,f}Q^L_{Y,f}T^3_{L,f}\right] \qquad (6.180)$$

It is important to note that Eq. (6.177) is still expressed as in Rosenberg (see [16, 41]), with the usual finite cubic terms in the momenta k_1 and k_2, the two singular invariant amplitudes (A_1 and A_2) and the mass contributions.

At this stage, to get the physical amplitude, we must impose e.m. current conservation on the external photons

$$\begin{aligned}
k_1^\mu\langle Z_l\gamma\gamma\rangle|^{\lambda\mu\nu}_{m_f\neq 0} &= 0\\
k_2^\nu\langle Z_l\gamma\gamma\rangle|^{\lambda\mu\nu}_{m_f\neq 0} &= 0\,. \qquad (6.181)
\end{aligned}$$

Using these conditions, again we can re-express the coefficient $\bar{A}_1, \bar{A}_2$ in terms of $A_3, \ldots, A_6$ and we drop the explicit mass dependence in the numerators of the expression of the physical amplitude.

Thus, applying the Ward identities on the triangle $\langle LLL\rangle_f$, it reduces to the combination $\Delta_{AVV}(m_f)-\Delta_{AVV}(0)$ which must be added to the first term in the curly brackets of Eq. (6.177), thereby giving our final result for the physical amplitude

$$\begin{aligned}\langle Z_l\gamma\gamma\rangle|_{m_f\neq 0} &= -\frac{1}{2}Z_l^\lambda A_\gamma^\mu A_\gamma^\nu\sum_f\left[g_Y^3\theta_f^{YYY}\bar{R}^{YYY}_{Z_l\gamma\gamma}+g_2^3\theta_f^{WWW}\bar{R}^{WWW}_{Z_l\gamma\gamma}+g_Y g_2^2\theta_f^{YWW}R^{YWW}_{Z_l\gamma\gamma}\right.\\ &+ g_Y^2 g_2\theta_f^{YYW}R^{YYW}_{Z_l\gamma\gamma}+\sum_i g_{B_i}g_Y g_2\theta_f^{B_iYW}R^{B_iYW}_{Z_l\gamma\gamma}\\ &+ \left.\sum_i g_{B_i}g_Y^2\theta_f^{B_iYY}R^{B_iYY}_{Z_l\gamma\gamma}+g_{B_i}g_2^2\theta_f^{B_iWW}R^{B_iWW}_{Z_l\gamma\gamma}\right]\Delta^{\lambda\mu\nu}_{AVV}(m_f\neq 0).\end{aligned} \tag{6.182}$$

We have defined

$$\bar{R}^{YYY}_{Z_l\gamma\gamma}=(O^A)_{YZ_l}(O^A)^2_{Y\gamma},\qquad \bar{R}^{WWW}_{Z_l\gamma\gamma}=(O^A)_{W_3Z_l}(O^A)^2_{W_3\gamma}, \tag{6.183}$$

and the triangle $\Delta_{AVV}(m_f\neq 0)$ is given by

$$\begin{aligned}\Delta_{AVV}(m_f\neq 0,k_1,k_2)^{\lambda\mu\nu} &= \frac{1}{\pi^2}\int_0^1 dx\int_0^{1-x}dy\frac{1}{\Delta(m_f)}\cdot\\ &\cdot\left\{\varepsilon[k_1,\lambda,\mu,\nu]\left[y\,(y-1)k_2^2-x\,y\,k_1\cdot k_2\right]\cdot+\varepsilon[k_2,\lambda,\mu,\nu]\left[x\,(1-x)\,k_1^2+x\,y\,k_1\cdot k_2\right]\right.\\ &\left.+\varepsilon[k_1,k_2,\lambda,\nu]\,\left[x\,(x-1)k_1^\mu-x\,y\,k_2^\mu\right]+\varepsilon[k_1,k_2,\lambda,\mu]\left[x\,y\,k_1^\nu+(1-y)yk_2^\nu\right]\right\},\end{aligned} \tag{6.184}$$

$$\Delta(m_f) = m_f^2+x(x-1)\,k_1^2+y(y-1)\,k_2^2-2\,x\,y\,k_1\cdot k_2\,. \tag{6.185}$$

The SM limit

It is straightforward to obtain the corresponding expression in the SM from the previous result. As usual we obtain, beside the tensor structures of the Rosenberg expansion, all the chirally flipped terms which are proportional to a mass term times a tensor $k^\alpha_{1,2}\varepsilon[\alpha,\lambda,\mu,\nu]$. As we have seen before in the previous sections all these terms can be re-absorbed once we impose the conservation of the electromagnetic current.

Then, setting the anomalous pieces to zero by taking $g_{B_i}\to 0$, we are left with the usual Z boson ($Z_l\to Z$), and we have

$$\begin{aligned}\langle Z\gamma\gamma\rangle|_{m_f\neq 0} &= -g_Z e^2\sum_f\left[Q_Z^{L,f}(Q_f^L)^2-Q_Z^{R,f}(Q_f^R)^2\right]\frac{1}{2}\Delta^{\lambda\mu\nu}_{AVV}(m_f\neq 0)Z^\lambda A_\gamma^\mu A_\gamma^\nu\\ &= -\sum_f\frac{1}{2}\Delta^{\lambda\mu\nu}_{AVV}(m_f\neq 0)\left\{g_Y^3\theta_f^{YYY}\bar{R}^{YYY}+g_2^2 g_Y\theta_f^{YWW}R^{YWW}_{Z\gamma\gamma}\right.\\ &+ \left.g_2^3\theta_f^{WWW}\bar{R}^{WWW}_{Z\gamma\gamma}+g_Y^2 g_2\theta_f^{YYW}R^{YYW}_{Z\gamma\gamma}\right\}Z^\lambda A_\gamma^\mu A_\gamma^\nu,\end{aligned} \tag{6.186}$$

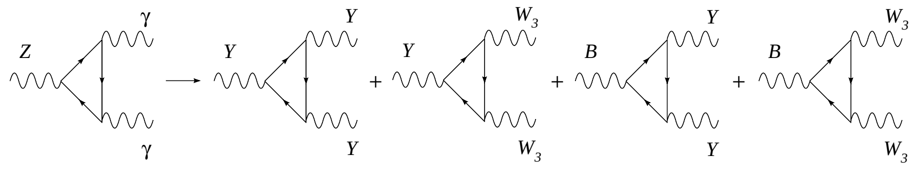


Figure 6.15: Interaction basis contributions to the $Z\gamma\gamma$ vertex. In the SM only the first two diagrams survive. The CS terms, in this case, are absorbed so that only the B vertex is anomalous. In the chiral limit in the SM the first two diagrams vanish.

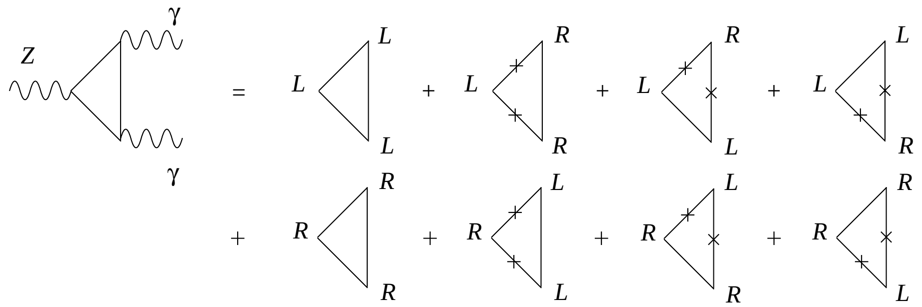


Figure 6.16: Chiral triangle contributions to the $Z\gamma\gamma$ vertex.

where the coefficients $\bar{R}^{YYY}_{Z\gamma\gamma}, \bar{R}^{WWW}_{Z\gamma\gamma}$ are defined in the previous section. It is not difficult to recognize that in the first line we have

$$\langle Z\gamma\gamma\rangle|_{m_f\neq 0} = -g_Z e^2 \frac{1}{2}\sum_f (Q_f)^2 \left[Q_Z^{L,f} - Q_Z^{R,f}\right] \Delta^{\lambda\mu\nu}_{AVV}(m_f \neq 0) Z^\lambda A^\mu_\gamma A^\nu_\gamma \tag{6.187}$$

and since

$$\begin{aligned} \left[Q_Z^{L,f} - Q_Z^{R,f}\right] &= 2 g^Z_{A,f} \\ g_Z &\approx \frac{g_2}{\cos\theta_W} \end{aligned} \tag{6.188}$$

finally we obtain

$$\langle Z\gamma\gamma\rangle|_{m_f\neq 0} = -\frac{g_2}{\cos\theta_W} e^2 \sum_f (Q_f)^2 g^Z_{A,f} \Delta^{\lambda\mu\nu}_{AVV}(m_f \neq 0) Z^\lambda A^\mu_\gamma A^\nu_\gamma\,, \tag{6.189}$$

which is exactly the SM vertex [147].

6.5 The γZZ vertex

Before coming to analyze the most general cases involving two or three anomalous Z's, it is more convenient to start with the γZZ interaction with two identical Z's in the final state and use the result in this simpler case for the general analysis.

6.5.1 The vertex in the chiral limit

We proceed in the same manner as before. In the $m_f = 0$ phase, the terms in the interaction eigenstates basis we need to consider are

$$\frac{1}{3!}Tr\left[Q_Y^3\right]\langle YYY\rangle = \frac{1}{3!}Tr\left[Q_Y^3\right]\left[3(O^A_{YZ})^2 O^A_{Y\gamma}\right]\langle\gamma ZZ\rangle + \dots \quad (6.190)$$

$$\frac{1}{2!}Tr\left[Q_Y T_3^2\right]\langle YWW\rangle = \frac{1}{2!}Tr\left[Q_Y T_3^2\right]\left[2O^A_{WZ}O^A_{W\gamma}O^A_{YZ} + (O^A_{WZ})^2 O^A_{Y\gamma}\right]\langle\gamma ZZ\rangle + \dots \quad (6.191)$$

$$\frac{1}{2!}Tr\left[Q_Y Q_B^2\right]\langle YBB\rangle = \frac{1}{2!}Tr\left[Q_Y Q_B^2\right]\left[O^A_{Y\gamma}(O^A_{BZ})^2\right]\langle\gamma ZZ\rangle + \dots \quad (6.192)$$

$$\frac{1}{2!}Tr\left[Q_B Q_Y^2\right]\langle BYY\rangle = \frac{1}{2!}Tr\left[Q_B Q_Y^2\right]\left[2O^A_{BZ}O^A_{YZ}O^A_{Y\gamma}\right]\langle\gamma ZZ\rangle + \dots \quad (6.193)$$

$$\frac{1}{2!}Tr\left[Q_B T_3^2\right]\langle BWW\rangle = \frac{1}{2!}Tr\left[Q_B T_3^2\right]\left[2O^A_{BZ}O^A_{WZ}O^A_{W\gamma}\right]\langle\gamma ZZ\rangle + \dots \quad (6.194)$$

We define for future reference the following expressions for the rotation matrices

$$R^{YYY}_{\gamma ZZ} = \left[3(O^A_{YZ})^2 O^A_{Y\gamma}\right] \quad (6.195)$$

$$R^{WWW}_{\gamma ZZ} = \left[3(O^A_{W_3Z})^2 O^A_{W_3\gamma}\right] \quad (6.196)$$

$$R^{WYY}_{\gamma ZZ} = \left[2O^A_{W_3Z}O^A_{Y\gamma}O^A_{YZ} + (O^A_{W_3\gamma})(O^A_{YZ})^2\right] \quad (6.197)$$

$$R^{YWW}_{\gamma ZZ} = \left[2O^A_{W_3Z}O^A_{W_3\gamma}O^A_{YZ} + (O^A_{W_3Z})^2 O^A_{Y\gamma}\right] \quad (6.198)$$

$$R^{BYY}_{\gamma ZZ} = \left[2O^A_{BZ}O^A_{YZ}O^A_{Y\gamma}\right] \quad (6.199)$$

$$R^{BBY}_{\gamma ZZ} = \left[O^A_{Y\gamma}(O^A_{BZ})^2\right] \quad (6.200)$$

$$R^{BBW}_{\gamma ZZ} = \left[O^A_{W_3\gamma}(O^A_{BZ})^2\right] \quad (6.201)$$

$$R^{BWW}_{\gamma ZZ} = \left[2O^A_{BZ}O^A_{W_3Z}O^A_{W_3\gamma}\right] \quad (6.202)$$

$$R^{BYW}_{\gamma ZZ} = \left[O^A_{BZ}O^A_{W_3Z}O^A_{Y\gamma} + O^A_{BZ}O^A_{W_3\gamma}O^A_{YZ}\right]. \quad (6.203)$$

The chiral decomposition proceeds similarly to the case of $Z\gamma\gamma$ (see Fig. 6.16). Also in this situation the tensor $\langle LLL\rangle_f^{\lambda\mu\nu}$ is characterized by the two independent momenta $k_{1,\mu}$ and $k_{2,\nu}$ of the two outgoing Z's. Since the LLL triangle is still ill-defined, we must distribute the anomaly in a certain way. This is driven by the symmetry of the theory, and in this case the STI's play a crucial role even in the ($m_f = 0$) unbroken chiral phase of the theory. In order to define the $\langle LLL\rangle^{\lambda\mu\nu}|_{m_f=0}$ diagram we choose a symmetric assignment of the anomaly

$$k_{1,\mu}\langle LLL\rangle^{\lambda\mu\nu}|_{m_f=0} = \frac{a_n}{3}\varepsilon[k_1,k_2,\lambda,\nu] \quad (6.204)$$

$$k_{2,\nu}\langle LLL\rangle^{\lambda\mu\nu}|_{m_f=0} = -\frac{a_n}{3}\varepsilon[k_1,k_2,\lambda,\mu] \quad (6.205)$$

$$k_{\lambda}\langle LLL\rangle^{\lambda\mu\nu}|_{m_f=0} = \frac{a_n}{3}\varepsilon[k_1,k_2,\mu,\nu]\,. \quad (6.206)$$

These conditions together with the Bose symmetry on the two Z's

$$\langle LLL\rangle^{\lambda\mu\nu}|_{m_f=0}(k,k_1,k_2) = \langle LLL\rangle^{\lambda\nu\mu}|_{m_f=0}(k,k_2,k_1) \tag{6.207}$$

allow us to remove the singular coefficients proportional to the two linear tensor structures of the amplitude. The complete tensor structure of the γZZ vertex in this case can be written in terms of the usual invariant amplitudes $A_1, ...A_6$

$$A_3 = -16\left(\mathcal{I}_{10}(k_1,k_2) - \mathcal{I}_{20}(k_1,k_2)\right) \tag{6.208}$$

$$A_4 = +16\mathcal{I}_{11}(k_1,k_2) \tag{6.209}$$

$$A_5 = -16\mathcal{I}_{11}(k_1,k_2) \tag{6.210}$$

$$A_6 = -16\left(\mathcal{I}_{01}(k_1,k_2) - \mathcal{I}_{02}(k_1,k_2)\right) \tag{6.211}$$

$$A_1 = -k_1\cdot k_2 A_5 - k_2^2 A_6 + \frac{a_n}{3} \tag{6.212}$$

$$A_2 = -k_1\cdot k_2 A_4 - k_1^2 A_3 - \frac{a_n}{3}. \tag{6.213}$$

We have the constraints

$$k_\lambda \langle LLL\rangle^{\lambda\mu\nu}|_{m_f=0} = \frac{a_n}{3}\varepsilon\left[k_1,k_2,\mu,\nu\right] \Rightarrow A_1 - A_2 = \frac{a_n}{3} \tag{6.214}$$

and the relation written in Eq. (6.144). In this case the CS terms coming from the Lagrangian in the interaction eigenstates basis are defined as follows

$$\begin{aligned} V_{CS} = \sum_f \Bigg\{ & -g_B\, g_Y^2 \frac{1}{8}\theta_f^{YBY} R_{\gamma ZZ}^{YBY} \frac{a_n}{3}\varepsilon^{\mu\nu\lambda\alpha}(k_{2,\alpha} - k_{3,\alpha}) \\ & - g_B g_Y^2 \frac{1}{8}\theta_f^{YYB} R_{\gamma ZZ}^{YYB} \frac{a_n}{3}\varepsilon^{\nu\lambda\mu\alpha}(k_{3,\alpha} - k_{1,\alpha}) + g_Y g_B^2 \frac{1}{8}\theta_f^{YBB} R_{ZZ\gamma}^{YBB} \frac{a_n}{6}\varepsilon^{\lambda\mu\nu\alpha}(k_{1,\alpha} - k_{2,\alpha}) \\ & - g_B g_2^2 \frac{1}{8}\theta_f^{WBW} R_{ZZ\gamma}^{WBW} \frac{a_n}{3}\varepsilon^{\mu\nu\lambda\alpha}(k_{2,\alpha} - k_{3,\alpha}) - g_B g_2^2 \frac{1}{8}\theta_f^{WWB} R_{ZZ\gamma}^{WWB} \frac{a_n}{3}\varepsilon^{\nu\lambda\mu\alpha}(k_{3,\alpha} - k_{1,\alpha}) \Bigg\}. \end{aligned} \tag{6.215}$$

Then, collecting all the terms, the expression in the $m_f = 0$ phase for the γZZ process can be written as

$$\begin{aligned} \langle \gamma ZZ\rangle|_{m_f=0} = -\frac{1}{2} A_\gamma^\lambda Z^\mu Z^\nu \sum_f \Bigg\{ & g_B g_Y^2 \theta_f^{YBY} R_{\gamma ZZ}^{YBY}\left[\Delta_{AAA}^{\mu\nu\lambda}(0) - \frac{a_n}{3}\varepsilon^{\mu\nu\lambda\alpha}(k_{2,\alpha} - k_{3,\alpha})\right] \\ & + g_B g_Y^2 \theta_f^{YYB} R_{\gamma ZZ}^{YYB}\left[\Delta_{AAA}^{\nu\lambda\mu}(0) - \frac{a_n}{3}\varepsilon^{\nu\lambda\mu\alpha}(k_{3,\alpha} - k_{1,\alpha})\right] \\ & + g_Y g_B^2 \theta_f^{YBB} R_{ZZ\gamma}^{YBB}\left[\Delta_{AAA}^{\lambda\mu\nu}(0) + \frac{a_n}{6}\varepsilon^{\lambda\mu\nu\alpha}(k_{1,\alpha} - k_{2,\alpha})\right] \\ & + g_B g_2^2 \theta_f^{WBW} R_{ZZ\gamma}^{WBW}\left[\Delta_{AAA}^{\mu\nu\lambda}(0) - \frac{a_n}{3}\varepsilon^{\mu\nu\lambda\alpha}(k_{2,\alpha} - k_{3,\alpha})\right] \\ & + g_B g_2^2 \theta_f^{WWB} R_{ZZ\gamma}^{WWB}\left[\Delta_{AAA}^{\nu\lambda\mu}(0) - \frac{a_n}{3}\varepsilon^{\nu\lambda\mu\alpha}(k_{3,\alpha} - k_{1,\alpha})\right]\Bigg\}, \end{aligned} \tag{6.216}$$

and after some manipulations, we obtain

$$\langle\gamma ZZ\rangle|_{m_f=0} = -\frac{1}{2}\left[\Delta^{\lambda\mu\nu}_{VAV}(0) + \Delta^{\lambda\mu\nu}_{VVA}(0)\right] A^{\lambda}_{\gamma} Z^{\mu} Z^{\nu} \sum_f \{g_B g_Y^2 \theta_f^{BYY} R^{BYY} + g_Y g_B^2 \theta_f^{YBB} \bar{R}^{YBB} + g_B g_2^2 \theta_f^{BWW} R^{BWW}\}, \tag{6.217}$$

where we have used

$$\theta_f^{YBB} = Q^L_{Y,f}(Q^L_{B,f})^2 - Q^R_{Y,f}(Q^R_{B,f})^2 \tag{6.218}$$

$$\bar{R}^{BBY}_{\gamma ZZ} = \frac{1}{2} R^{BBY}_{\gamma ZZ}. \tag{6.219}$$

If we define

$$T^{\lambda\mu\nu}(0) = \left[\Delta^{\lambda\mu\nu}_{VAV}(0) + \Delta^{\lambda\mu\nu}_{VVA}(0)\right] \tag{6.220}$$

we can write an explicit expression for $T^{\lambda\mu\nu}$, which is given by

$$\begin{aligned} T^{\lambda\mu\nu}(0) = \frac{1}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{1}{\Delta(0)} \Big\{ & \varepsilon^{\alpha\lambda\mu\nu} k_{1,\alpha}\left[(1-x)x k_1^2 + y(y-1)k_2^2\right] \\ & + \varepsilon^{\alpha\lambda\mu\nu} k_{2,\alpha}\left[(1-x)x k_1^2 + y(y-1)k_2^2\right] \\ & + \varepsilon[k_1, k_2, \lambda, \nu]\left[2(x-1)x k_{1,\mu} - 2xy k_{2,\mu}\right] \\ & + \varepsilon[k_1, k_2, \lambda, \mu]\left[2(1-y)y k_{2,\nu} + 2xy k_{1,\nu}\right]\Big\}, \end{aligned} \tag{6.221}$$

and it is straightforward to observe that the electromagnetic current conservation is satisfied on the photon line

$$\begin{aligned} k_{1,\mu} T^{\lambda\mu\nu} &= \frac{1}{2\pi^2}\varepsilon\left[k_1, k_2, \lambda, \nu\right] \\ k_{2,\nu} T^{\lambda\mu\nu} &= -\frac{1}{2\pi^2}\varepsilon\left[k_1, k_2, \lambda, \mu\right] \\ (k_{1,\lambda} + k_{2,\lambda}) T^{\lambda\mu\nu} &= 0. \end{aligned} \tag{6.222}$$

6.5.2 γZZ: The $m_f \neq 0$ phase

In the $m_f \neq 0$ phase we must add to the previous chirally conserved contributions all the chirally flipped interactions of the type $\langle LLR\rangle$ and similar, which are proportional to m_f^2. As we have already seen in the $Z\gamma\gamma$ case, all the mass terms have a tensor structure of the type $m_f^2 \varepsilon^{\alpha\lambda\mu\nu} k_{1,2,\alpha}$ and we can always define the coefficients $\bar{A}_1$ and $\bar{A}_2$ so that they include all the mass terms. Again, they are expressed in terms of the finite quantities $A_3, \ldots, A_6$ by imposing the physical restriction, i.e. the e.m. current conservation on the photon line, and the anomalous Ward identities on the two Z's lines. Since the CS interactions act only on the massless part of the triangles, they are absorbed by splitting the tensor $\langle LLL\rangle^{\lambda\mu\nu}$ as

$$\begin{aligned}\langle LLL\rangle^{\lambda\mu\nu}|_f &= \langle LLL\rangle^{\lambda\mu\nu}|_{m_f=0} \;+\; \langle LLL\rangle^{\lambda\mu\nu}(m_f);\\ \langle LLL\rangle^{\lambda\mu\nu}(m_f) &= \langle LLL\rangle^{\lambda\mu\nu}|_{m_f\neq 0} \;-\; \langle LLL\rangle^{\lambda\mu\nu}|_{m_f=0}.\end{aligned} \tag{6.223}$$

Then, the structure of the amplitude will be

$$\begin{aligned}\frac{1}{2!}\langle\gamma ZZ\rangle|_{m_f\neq 0} = {}& \bar{A}_1\varepsilon[k_1,\lambda,\mu,\nu] + \bar{A}_2\varepsilon[k_2,\lambda,\mu,\nu] + A_3 k_1^\mu\varepsilon[k_1,k_2,\lambda,\nu]\\ &+A_4 k_2^\mu\varepsilon[k_1,k_2,\lambda,\nu] + A_5 k_1^\nu\varepsilon[k_1,k_2,\lambda,\mu] + A_6 k_2^\nu\varepsilon[k_1,k_2,\lambda,\nu]\end{aligned} \tag{6.224}$$

and using the explicit expressions of the coefficients we obtain

$$\begin{aligned}\langle\gamma ZZ\rangle|_{m_f\neq 0} = -\sum_f \Big[& g_Y^3\theta_f^{YYY}\bar{R}_{\gamma ZZ}^{YYY} + g_2^3\theta_f^{WWW}\bar{R}_{\gamma ZZ}^{WWW}\\ & +g_Y g_2^2\theta_f^{YWW}R_{\gamma ZZ}^{YWW} + g_Y^2 g_2\theta_f^{YYW}R_{\gamma ZZ}^{YYW}\\ & +g_B g_Y^2\theta_f^{BYY}R_{\gamma ZZ}^{BYY} + g_Y g_B^2\theta_f^{YBB}\bar{R}_{\gamma ZZ}^{YBB}\\ & +g_B^2 g_2\theta_f^{WBB}\bar{R}_{\gamma ZZ}^{WBB} + g_B g_2^2\theta_f^{BWW}R_{\gamma ZZ}^{BWW}\\ & +g_B^2 g_2 g_Y\theta_f^{BYW}R_{\gamma ZZ}^{BYW}\Big]\frac{1}{2}T^{\lambda\mu\nu}(m_f\neq 0)A_\gamma Z^\mu Z^\nu\,,\end{aligned} \tag{6.225}$$

where we have defined

$$\begin{aligned}T^{\lambda\mu\nu}(m_f\neq 0) &= \left[\Delta_{VAV}^{\lambda\mu\nu}(m_f\neq 0) + \Delta_{VVA}^{\lambda\mu\nu}(m_f\neq 0)\right]\,,\\ \theta_f^{WBB} &= (Q_{B,f}^L)^2 T_{L,f}^3\,,\\ \bar{R}_{\gamma ZZ}^{WBB} &= \frac{1}{2}R_{\gamma ZZ}^{WBB}\,,\end{aligned} \tag{6.226}$$

with

$$\begin{aligned}T^{\lambda\mu\nu}(m_f\neq 0) = \frac{1}{\pi^2}\int_0^1 dx\int_0^{1-x}dy\frac{1}{\Delta(m_f)}\Big\{ & \varepsilon^{\alpha\lambda\mu\nu}k_{1,\alpha}\left[(1-x)xk_1^2 - y(1-y)k_2^2\right]\\ & +\varepsilon^{\alpha\lambda\mu\nu}k_{2,\alpha}\left[(1-x)xk_1^2 - y(1-y)k_2^2\right]\\ & +\varepsilon[k_1,k_2,\lambda,\nu]\left[2(x-1)xk_{1,\mu} - 2xyk_{2,\mu}\right]\\ & +\varepsilon[k_1,k_2,\lambda,\mu]\left[2(1-y)yk_{2,\nu} + 2xyk_{2,\mu}\right]\Big\}\,.\end{aligned} \tag{6.227}$$

We can immediately see that the expected broken Ward identities

$$\begin{aligned}k_{1,\mu}T^{\lambda\mu\nu} &= \frac{1}{\pi^2}\varepsilon\left[k_1,k_2,\lambda,\nu\right]\left\{\frac{1}{2} - m_f^2\int_0^1 dx\int_0^{1-x}dy\frac{1}{\Delta(m_f)}\right\}\\ k_{2,\nu}T^{\lambda\mu\nu} &= -\frac{1}{\pi^2}\varepsilon\left[k_1,k_2,\lambda,\nu\right]\left\{\frac{1}{2} - m_f^2\int_0^1 dx\int_0^{1-x}dy\frac{1}{\Delta(m_f)}\right\}\\ (k_{1,\lambda}+k_{2,\lambda})T^{\lambda\mu\nu} &= 0\end{aligned} \tag{6.228}$$

are indeed satisfied.

6.6 Trilinear interactions in multiple $U(1)$ models

Building on the computation of the $Z\gamma\gamma$ and γZZ presented in the sections above, we formulate here some general prescriptions that can be used in the analysis of anomalous abelian models when several $U(1)$'s are present and which help to simplify the process of building the structure of the anomalous vertices in the mass eigenstates basis. The general case is already encountered when the anomalous gauge structure contains three anomalous $U(1)$'s besides the usual gauge group of the SM. We prefer to work with this specific choice in order to simplify the formalism, though the discussion and the results are valid in general.

We denote respectively with W_3, A_Y, B_1, B_2, B_3 the weak, the hypercharge gauge boson and their 3 anomalous partners. At this point we consider the anomalous triangle diagrams of the model and observe that we can either

1) distribute the anomaly equally among all the corresponding generators $(T_3, Y, Y_{B_1}, Y_{B_2}, Y_{B_3})$ and compensate for the violation of the Ward identity on the non anomalous vertices with suitable CS interactions

or

2) re-define the trilinear vertices *ab initio* so that some partial anomalies are removed from the $Y - W_3$ generators in the diagrams containing mixed anomalies. Also in this case some CS counterterms may remain.

We recall that the anomaly-free generators are not accompanied by axions. The difference between the first and the second method is in the treatment of the CS terms: in the first case they all appear explicitly as separate contributions, while in the second one they can be absorbed, at least in part, into the definition of the vertices. In one case or the other the final result is the same. In particular one has to be careful on how to handle the distribution of the partial anomalies (in the physical basis) especially when a certain vertex does not have any Bose symmetry, such as for three different gauge bosons, and this is not constrained by specific relations. In this section we will go back again to the examples that we have discussed in detail above and illustrate how to proceed in the most general case.

Consider the $Z\gamma\gamma$ case in the chiral limit. For instance, a vertex of the form B_2YY will be projected into the $Z\gamma\gamma$ vertex with a combination of rotation matrices of the form $R^{B_2YY}_{Z\gamma\gamma}$, generating a partial contribution which is typically of the form $\langle LLL\rangle R^{B_2YY}_{Z\gamma\gamma}$. At this point, in the B_2YY diagram, which is interpreted as a $\langle LLL\rangle \sim \Delta_{AAA}$ contribution, we move the anomaly on the B_2-vertex by absorbing one CS term, thereby changing the $\langle LLL\rangle$ vertex into an **AVV** vertex.

We do the same for all the trilinear contributions such as B_3YY, B_1WW and so on, similarly to what we have discussed in the previous sections. For instance B_3YY, which is also proportional to an **AAA** diagram, is turned into an **AVV** diagram by a suitable CS term. The $Z\gamma\gamma$ is identified by adding up all the projections. This is the second approach.

The alternative procedure, which is the basic content of the first prescription mentioned above, consists in keeping the B_2YY vertex as an **AAA** vertex, while the CS counterterm, which is needed to remove the anomaly from the Y vertex, has to be kept separate. Also in this case the contribution of B_2YY to $Z\gamma\gamma$ is of the form $\langle LLL\rangle R^{B_2YY}_{Z\gamma\gamma}$, with $\langle LLL\rangle \sim \Delta_{AAA}$, and the CS term that accompanies this contribution is also rotated into the same $Z\gamma\gamma$ vertex.

Using the second approach in the final construction of the $Z\gamma\gamma$ vertex we add up all the projections and obtain as a result a single AVV diagram, as one would have naively expect using QED Ward identities on the photon lines. Instead, following the first we are forced to describe the same vertex as a sum of two contributions: a fermionic triangle (which has partial anomalies on the two photon lines) plus the CS counterterm, the sum of which is again of the form **AVV**.

However, when possible, it is convenient to use a single diagram to describe a certain interaction, especially if the vertex has specific Bose symmetries, as in the case of the $Z\gamma\gamma$ vertex.

For instance, we could have easily inferred the result in the $Z\gamma\gamma$ case with no difficulty at all, since the partial anomaly on the photon lines is zero and the total anomaly, which is a constant, has to be necessarily attached to the Z line and not to the photons.

A similar result holds for the ZZZ vertex where the anomaly has to be assigned symmetrically. Notice that, in prescription 2) when several extra $U(1)$'s are present, the vertices in the interaction eigenstate basis such as $B_1B_2B_3$ or $B_1B_1B_2$ should be kept in their **AAA** form, since the presence of axions (b_1, b_2, b_3) is sufficient to guarantee the gauge invariance of each anomalous gauge boson line.

A final example concerns the case when 3 different anomalous gauge bosons are present, for instance $ZZ'Z''$. In this case the distribution of the partial anomalies can be easily inferred by combining all the projections of the trilinear vertices $B_1YY, B_1WW, B_1B_2B_3, B_1B_2B_3, B_2B_3B_3$... etc. into $ZZ'Z''$. The absorption of the CS terms here is also straightforward, since vertices such as B_1YY, YB_1Y and YYB_1 are rewritten as **AVV**, **VAV** and **VVA** contributions respectively. On the other hand, terms such as $B_2B_1B_1$ or $B_1B_2B_3$ are kept in their **AAA** form with an equal share of partial anomalies. Notice that in this case the final vertex, also in the second approach where the CS terms are partially absorbed, does not result in a single diagram as in the $Z\gamma\gamma$ case, but in a combination of several contributions.

6.6.1 Moving away from the chiral limit with several anomalous $U(1)$'s

Chiral symmetry breaking, as we have seen in the examples discussed before, introduces a higher level of complications in the analysis of these vertices. Also in this case we try to find a prescription to fix the trilinear anomalous gauge interactions away from the chiral limit. As we have seen from the treatment of the previous sections, the presence of mass terms in any triangle graph is confined to the denominator of their Feynman parameterization, once the Ward identities are imposed on each vertex. This implies that all the mixed terms of the form LLR or RRL containing quadratic mass insertions can be omitted in any diagram and the final result for any anomalous contributions such as $B_1B_2B_3$ or B_1YY involves only an $\langle LLL\rangle$ fermionic triangle where the mass from the Dirac traces is removed.

For instance, let's consider again the derivation of the γZZ vertex in this case. We project the trilinear gauge interactions of the effective action written in the eigenstate basis into the γZZ vertex (see Fig. 6.17) as before and, typically, we encounter vertices such as YB_1Y or B_1YY (and so on) that need to be rotated. We remove the masses from the numerator of these vertices and reduce each of them to a standard $\langle LLL\rangle$ form, having omitted the mixing terms LLR, RRL, etc. Also in this case a vertex such as B_1YY is turned into an **AVV** by absorbing a corresponding CS interaction, while its broken Ward identities will be of the form

$$\begin{aligned}
k_{1\mu}\Delta^{\lambda\mu\nu}(\beta,k_1,k_2) &= 0\\
k_{2\nu}\Delta^{\lambda\mu\nu}(\beta,k_1,k_2) &= 0\\
k_{\lambda}\Delta^{\lambda\mu\nu}(\beta,k_1,k_2) &= a_n(\beta)\varepsilon^{\mu\nu\alpha\beta}k_1^{\alpha}k_2^{\beta}+2m_f\Delta^{\mu\nu},
\end{aligned}\tag{6.229}$$

with a broken WI on the **A** line and exact ones on the remaining **V** lines corresponding to the two Y generators. Similarly, when we consider the projection of a term such as $B_1B_2B_3$ into the $Z'Z''Z$ vertex, we impose a symmetric distribution of the anomaly and broken WI's on the three external lines

$$\begin{aligned}
k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) &= \frac{a_n}{3}\varepsilon^{\lambda\nu\alpha\beta}k_1^{\alpha}k_2^{\beta}+2m_f\Delta^{\lambda\nu},\\
k_{2\nu}\Delta^{\lambda\mu\nu}(k_1,k_2) &= \frac{a_n}{3}\varepsilon^{\lambda\mu\alpha\beta}k_2^{\alpha}k_1^{\beta}+2m_f\Delta^{\lambda\mu},\\
k_{\lambda}\Delta^{\lambda\mu\nu}(k_1,k_2) &= \frac{a_n}{3}\varepsilon^{\mu\nu\alpha\beta}k_1^{\alpha}k_2^{\beta}+2m_f\Delta^{\mu\nu}.
\end{aligned}\tag{6.230}$$

The total vertex is therefore obtained by adding up all these projections together with 3 CS contributions to redistribute the anomalies. Next we are going to discuss the explicit way of doing this.

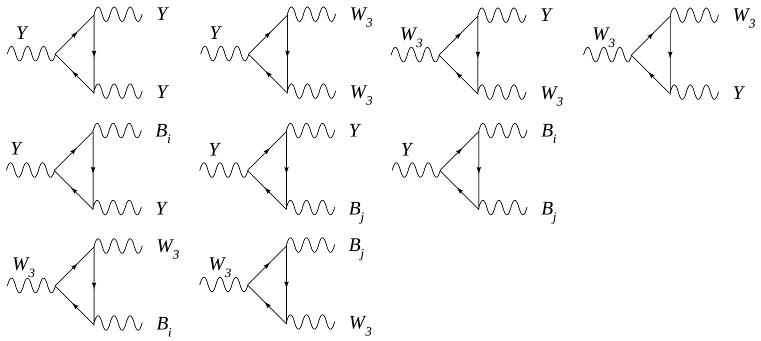


Figure 6.17: Triangle contributions to the $\langle\gamma Z_l Z_m\rangle$ vertex in the chiral phase. Notice that the first four contributions vanish because of the SM charge assignment.

6.7 The $\langle\gamma Z_l Z_m\rangle$ vertex

At this stage we can generalize the construction of $\langle\gamma ZZ\rangle$ to a general $\langle\gamma Z_l Z_m\rangle$ vertex. The contributions coming from the interaction eigenstates basis to the $\langle\gamma Z_l Z_m\rangle$ in the chiral limit are given by

$$
\begin{aligned}
&\frac{1}{3!}Tr\left[Q_Y^3\right]\langle YYY\rangle = \frac{1}{3!}Tr\left[Q_Y^3\right]R^{YYY}_{\gamma Z_l Z_m}\langle\gamma Z_l Z_m\rangle + \ldots\\
&\frac{1}{2!}Tr\left[Q_Y T_3^2\right]\langle YWW\rangle = \frac{1}{2!}Tr\left[Q_Y T_3^2\right]R^{YWW}_{\gamma Z_l Z_m}\langle\gamma Z_l Z_m\rangle + \ldots\\
&\frac{1}{2!}Tr\left[Q_Y T_3^2\right]\langle WYW\rangle = \frac{1}{2!}Tr\left[Q_Y T_3^2\right]R^{WYW}_{\gamma Z_l Z_m}\langle\gamma Z_l Z_m\rangle + \ldots\\
&\frac{1}{2!}Tr\left[Q_Y T_3^2\right]\langle WWY\rangle = \frac{1}{2!}Tr\left[Q_Y T_3^2\right]R^{WWY}_{\gamma Z_l Z_m}\langle\gamma Z_l Z_m\rangle + \ldots\\
&\frac{1}{2!}Tr\left[Q_{B_j} T_3^2\right]\langle WB_jW\rangle = \frac{1}{2!}Tr\left[Q_{B_j} T_3^2\right]R^{WB_jW}_{\gamma Z_l Z_m}\langle\gamma Z_l Z_m\rangle + \ldots\\
&\frac{1}{2!}Tr\left[Q_{B_j} T_3^2\right]\langle WWB_j\rangle = \frac{1}{2!}Tr\left[Q_{B_j} T_3^2\right]R^{WWB_j}_{\gamma Z_l Z_m}\langle\gamma Z_l Z_m\rangle + \ldots\\
&\frac{1}{2!}Tr\left[Q_{B_j} Q_Y^2\right]\langle YB_jY\rangle = \frac{1}{2!}Tr\left[Q_{B_j} Q_Y^2\right]R^{YB_jY}_{\gamma Z_l Z_m}\langle\gamma Z_l Z_m\rangle + \ldots\\
&\frac{1}{2!}Tr\left[Q_{B_j} Q_Y^2\right]\langle YYB_j\rangle = \frac{1}{2!}Tr\left[Q_{B_j} Q_Y^2\right]R^{YYB_j}_{\gamma Z_l Z_m}\langle\gamma Z_l Z_m\rangle + \ldots\\
&Tr\left[Q_Y Q_{B_j} Q_{B_k}\right]\langle YB_jB_k\rangle = Tr\left[Q_Y Q_{B_j} Q_{B_k}\right]R^{YB_jB_k}_{\gamma Z_l Z_m}\langle\gamma Z_l Z_m\rangle + \ldots
\end{aligned}
\tag{6.231}
$$

and they are pictured in Fig. 6.17. The rotation matrices are defined as

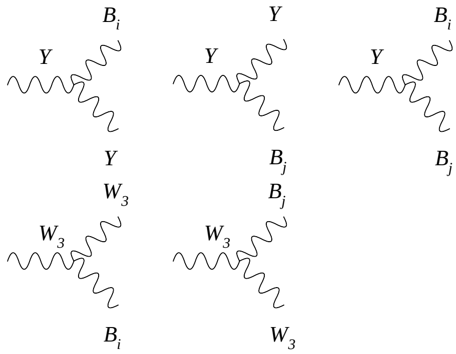


Figure 6.18: Chern-Simons counterterms of the $\langle\gamma Z_l Z_m\rangle$ vertex.

$$
\begin{aligned}
R^{YYY}_{\gamma Z_l Z_m} &= \left[3 O^A_{Y Z_l} O^A_{Y Z_m} O^A_{Y\gamma}\right] \\
R^{WWW}_{\gamma Z_l Z_m} &= \left[3 O^A_{W_3 Z_l} O^A_{W_3 Z_m} O^A_{W_3\gamma}\right] \\
R^{YWW}_{\gamma Z_l Z_m} &= \left[O^A_{W Z_l} O^A_{W\gamma} O^A_{Y Z_m} + O^A_{W Z_m} O^A_{W\gamma} O^A_{Y Z_l} + O^A_{W Z_l} O^A_{W Z_m} O^A_{Y\gamma}\right] \\
R^{WYY}_{\gamma Z_l Z_m} &= \left[(O^A_{W_3 Z_l} O^A_{Y Z_m} + O^A_{W_3 Z_m} O^A_{Y Z_l}) O^A_{Y\gamma} + O^A_{W_3\gamma} O^A_{Y Z_m} O^A_{Y Z_l}\right] \\
R^{B_j YY}_{\gamma Z_l Z_m} &= \left[O^A_{B_j Z_l} O^A_{Y Z_m} O^A_{Y\gamma} + O^A_{B_j Z_m} O^A_{Y Z_l} O^A_{Y\gamma}\right] \\
R^{B_j YW}_{\gamma Z_l Z_m} &= \left[(O^A_{B_j Z_l} O^A_{Y Z_m} + O^A_{B_j Z_m} O^A_{Y Z_l}) O^A_{W_3\gamma} + (O^A_{B_j Z_m} O^A_{W_3 Z_l} + O^A_{B_j Z_l} O^A_{W_3 Z_m}) O^A_{Y\gamma}\right] \\
R^{Y B_i B_j}_{\gamma Z_l Z_m} &= \left[(O^A_{B_i Z_l} O^A_{B_j Z_m} + O^A_{B_i Z_m} O^A_{B_j Z_l}) O^A_{Y\gamma}\right] \\
R^{W B_i B_j}_{\gamma Z_l Z_m} &= \left[(O^A_{B_i Z_l} O^A_{B_j Z_m} + O^A_{B_i Z_m} O^A_{B_j Z_l}) O^A_{W_3\gamma}\right] \\
R^{B_j WW}_{\gamma Z_l Z_m} &= \left[O^A_{B_j Z_l} O^A_{W Z_m} O^A_{W\gamma} + O^A_{B_j Z_m} O^A_{W Z_l} O^A_{W\gamma}\right]
\end{aligned}
\tag{6.232}
$$

while all the possible CS counterterms are listed in Fig. 6.18 and their explicit expression in the rotated basis is given by

$$
\begin{aligned}
V_{CS,lm} = \sum_f \Bigg\{ & -\sum_i \frac{1}{8}\theta_f^{Y B_i Y} \frac{a_n}{3} \varepsilon^{\lambda\mu\nu\alpha}(k_{2,\alpha} - k_{3,\alpha}) R^{Y B_i Y}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m \\
& -\sum_j \frac{1}{8}\theta_f^{Y Y B_j} \frac{a_n}{3} \varepsilon^{\lambda\mu\nu\alpha}(k_{3,\alpha} - k_{1,\alpha}) R^{Y Y B_j}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m \\
& +\sum_{i,j} \frac{1}{8}\theta_f^{Y B_i B_j} \frac{a_n}{6} \varepsilon^{\lambda\mu\nu\alpha}(k_{1,\alpha} - k_{2,\alpha}) R^{Y B_i B_j}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m \\
& -\sum_i \frac{1}{8}\theta_f^{W B_i W} \frac{a_n}{3} \varepsilon^{\lambda\mu\nu\alpha}(k_{2,\alpha} - k_{3,\alpha}) R^{W B_i W}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m \\
& -\sum_j \frac{1}{8}\theta_f^{W W B_j} \frac{a_n}{3} \varepsilon^{\lambda\mu\nu\alpha}(k_{3,\alpha} - k_{1,\alpha}) R^{W W B_j}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m \Bigg\},
\end{aligned}
\tag{6.233}
$$

where we have defined $k_{3,\alpha} = -k_\alpha$, with $k_\alpha = (k_1 + k_2)_\alpha$ the incoming momenta of the triangle. Using Eq. (6.154) it is easy to write the expression of the amplitude for the $\langle\gamma Z_l Z_m\rangle$ interaction in the $m_f = 0$ phase, and to separate the chiral components exactly as we have done for the $\langle\gamma ZZ\rangle$ vertex. Again, the tensorial structure that we can factorize out is $\langle LLL\rangle^{\lambda\mu\nu}(0)$

$$\begin{aligned}\langle\gamma Z_l Z_m\rangle|_{m_f=0} &= \sum_f \frac{1}{8}\langle LLL\rangle^{\lambda\mu\nu}(0) A^\lambda_\gamma Z^\mu_l Z^\nu_m \Bigg\{ \sum_i g_Y^2 g_{B_i} \theta_f^{YB_iY} R^{YB_iY}_{\gamma Z_l Z_m} \\ &+ \sum_j g_Y^2 g_{B_j} \theta_f^{YYB_j} R^{YYB_j}_{\gamma Z_l Z_m} + \sum_{i,j} g_Y g_{B_i} g_{B_j} \theta_f^{YB_iB_j} R^{YB_iB_j}_{\gamma Z_l Z_m} \\ &+ \sum_i g_2^2 g_{B_i} \theta_f^{WB_iW} R^{WB_iW}_{\gamma Z_l Z_m} + \sum_j g_2^2 g_{B_j} \theta_f^{WWB_j} R^{WWB_j}_{\gamma Z_l Z_m} \Bigg\} .\end{aligned} \tag{6.234}$$

Also in this case we use Eq. (6.144) and proceed from a symmetric distribution of the anomalies and absorb the equations the CS interactions so to obtain

$$\begin{aligned}-\langle\gamma Z_l Z_m\rangle|_{m_f=0} = &\sum_i g_Y^2 g_{B_i} \sum_f \frac{1}{2}\theta_f^{YB_iY} \Delta^{\lambda\mu\nu}_{VAV}(0) R^{YB_iY}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m \\ &+ \sum_j g_Y^2 g_{B_j} \sum_f \frac{1}{2}\theta_f^{YYB_j} \Delta^{\lambda\mu\nu}_{VVA}(0) R^{YYB_j}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m \\ &+ \sum_{i,j} g_Y g_{B_i} g_{B_j} \sum_f \theta_f^{YB_iB_j} \frac{1}{2}\left[\Delta^{\lambda\mu\nu}_{VAV}(0) + \Delta^{\lambda\mu\nu}_{VVA}(0)\right] R^{YB_iB_j}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m \\ &+ \sum_i g_2^2 g_{B_i} \sum_f \theta_f^{WB_iW} \frac{1}{2}\Delta^{\lambda\mu\nu}_{VAV}(0) R^{WB_iW}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m \\ &+ \sum_j g_2^2 g_{B_j} \sum_f \theta_f^{WWB_j} \frac{1}{2}\Delta^{\lambda\mu\nu}_{VVA}(0) R^{WWB_j}_{\gamma Z_l Z_m} A^\lambda_\gamma Z^\mu_l Z^\nu_m .\end{aligned} \tag{6.235}$$

At this point one can readily observe that a simple rearrangement of the summations over the i, j index leads us to factor out the structure **VAV** plus **VVA** since we have the same rotation matrices. Finally, in the $m_f = 0$ phase we have

$$\begin{aligned}\langle\gamma Z_l Z_m\rangle|_{m_f=0} = -\sum_f \frac{1}{2}\left[\Delta^{\lambda\mu\nu}_{VAV}(0) + \Delta^{\lambda\mu\nu}_{VVA}(0)\right] A^\lambda_\gamma Z^\mu_l Z^\nu_m \times \\ \sum_i \left\{ g_Y^2 g_{B_i} \theta_f^{B_iYY} R^{YYB_i}_{\gamma Z_l Z_m} + \sum_j g_Y g_{B_i} g_{B_j} \theta_f^{YB_iB_j} R^{YB_iB_j}_{\gamma Z_l Z_m} + g_2^2 g_{B_i} \theta_f^{WWB_i} R^{WWB_i}_{\gamma Z_l Z_m} \right\} .\end{aligned} \tag{6.236}$$

If the CS terms are instead not absorbed we have

$$\langle\gamma Z_l Z_m\rangle|_{m_f=0} = V_{CS,lm} - \sum_f \frac{1}{2}\Delta^{\lambda\mu\nu}_{AAA}(0)A^\lambda_\gamma Z^\mu_l Z^\nu_m \times$$
$$\sum_i \left\{ g_Y^2 g_{B_i}\theta_f^{B_iYY}R^{YYB_i}_{\gamma Z_l Z_m} + \sum_j g_Y g_{B_i} g_{B_j}\theta_f^{YB_iB_j}R^{YB_iB_j}_{\gamma Z_l Z_m} + g_2^2 g_{B_i}\theta_f^{WWB_i}R^{WWB_i}_{\gamma Z_l Z_m}\right\}, \tag{6.237}$$

which is equivalent to that obtained in (6.236).

6.7.1 Amplitude in the $m_f \neq 0$ phase

Once we have fixed the structure of the triangle in the $m_f = 0$ phase, its extension to the massive case can be obtained using the relation

$$\langle LLL\rangle(m_f \neq 0) = -\left[\Delta_{AVV}(m_f \neq 0) + \Delta_{VAV}(m_f \neq 0) + \Delta_{VVA}(m_f \neq 0) + \Delta_{AAA}(m_f \neq 0)\right] \tag{6.238}$$

and the expression of the vertex will be

$$\begin{aligned}\langle\gamma Z_l Z_m\rangle|_{m_f\neq 0} = \frac{1}{8}\sum_f \langle LLL\rangle^{\lambda\mu\nu}(m_f\neq 0)A^\lambda_\gamma Z^\mu_l Z^\nu_m &\left\{g_Y^3\theta_f^{YYY}R^{YYY}_{\gamma Z_l Z_m}\right.\\
&+g_2^3\theta_f^{WWW}R^{WWW}_{\gamma Z_l Z_m} + g_Y g_2^2\theta_f^{YWW}R^{YWW}_{\gamma Z_l Z_m}\\
&+g_Y^2 g_2\theta_f^{WYY}R^{WYY}_{\gamma Z_l Z_m} + \sum_i g_Y^2 g_{B_i}\theta_f^{YYB_i}R^{YYB_i}_{\gamma Z_l Z_m}\\
&+\sum_i g_Y g_2 g_{B_i}\theta_f^{B_iYW}R^{B_iYW}_{\gamma Z_l Z_m} + \sum_{i,j} g_Y g_{B_i} g_{B_j}\theta_f^{YB_iB_j}R^{YB_iB_j}_{\gamma Z_l Z_m}\\
&\left.+\sum_{i,j} g_2 g_{B_i} g_{B_j}\theta_f^{WB_iB_j}R^{WB_iB_j}_{\gamma Z_l Z_m} + \sum_i g_2^2 g_{B_i}\theta_f^{WWB_i}R^{WWB_i}_{\gamma Z_l Z_m}\right\}\\
&+m_f^2\left[\langle LRL\rangle + \langle RRL\rangle + \ldots\right].\end{aligned} \tag{6.239}$$

By imposing the following broken Ward identities on the tensor structure

$$k_1^\mu\left(\langle\gamma Z_l Z_m\rangle^{\lambda\mu\nu} + V^{\lambda\mu\nu}_{CS}\right) = \frac{a_n}{2}\varepsilon^{\lambda\nu\alpha\beta}k_{1,\alpha}k_{2,\beta} + 2m_f\Delta^{\lambda\nu} \tag{6.240}$$
$$k_2^\nu\left(\langle\gamma Z_l Z_m\rangle^{\lambda\mu\nu} + V^{\lambda\mu\nu}_{CS}\right) = -\frac{a_n}{2}\varepsilon^{\lambda\mu\alpha\beta}k_{1,\alpha}k_{2,\beta} - 2m_f\Delta^{\lambda\mu} \tag{6.241}$$
$$k^\lambda\left(\langle\gamma Z_l Z_m\rangle^{\lambda\mu\nu} + V^{\lambda\mu\nu}_{CS}\right) = 0 \tag{6.242}$$

we arrange all the mass terms into the coefficients $\bar{A}_1$ and $\bar{A}_2$ of the Rosenberg parametrization of $\langle LLL\rangle^{\lambda\mu\nu}$ and we absorbe all the singular pieces. Since all the CS interactions act only on the massless part of the **LLL** structure, we are left with an expression which is similar to Eq. (6.235) but with the addition of the triangle contributions coming from the Standard Model where the mass is contained only in the denominators. Organizing all the partial contributions we arrive at the final expression in which the structure **VAV** plus **VVA** is factorized out

$$
\begin{aligned}
\langle \gamma Z_l Z_m \rangle|_{m_f \neq 0} = & -\sum_f \frac{1}{2}\left[\Delta^{\lambda\mu\nu}_{VAV}(m_f \neq 0) + \Delta^{\lambda\mu\nu}_{VVA}(m_f \neq 0)\right] A^\lambda_\gamma Z^\mu_l Z^\nu_m \cdot \\
& \cdot \Big\{ g_Y^3 \theta_f^{YYY} \bar{R}^{YYY}_{\gamma Z_l Z_m} + g_2^3 \theta_f^{WWW} \bar{R}^{WWW}_{\gamma Z_l Z_m} \\
& + g_Y g_2^2 \theta_f^{YWW} R^{YWW}_{\gamma Z_l Z_m} + g_Y^2 g_2 \theta_f^{WYY} R^{WYY}_{\gamma Z_l Z_m} \\
& + \sum_i g_Y^2 g_{B_i} \theta_f^{B_i YY} R^{B_i YY}_{\gamma Z_l Z_m} + \sum_i g_Y g_2 g_{B_i} \theta_f^{B_i YW} R^{B_i YW}_{\gamma Z_l Z_m} \\
& + \sum_{i,j} g_Y g_{B_i} g_{B_j} \theta_f^{Y B_i B_j} R^{Y B_i B_j}_{\gamma Z_l Z_m} + \sum_{i,j} g_2 g_{B_i} g_{B_j} \theta_f^{W B_i B_j} R^{W B_i B_j}_{\gamma Z_l Z_m} \\
& + \sum_i g_2^2 g_{B_i} \theta_f^{WW B_i} R^{B_i WW}_{\gamma Z_l Z_m} \Big\} . \qquad (6.243)
\end{aligned}
$$

6.8 The $\langle Z_l Z_m Z_r \rangle$ vertex

Moving to the more general trilinear vertex is rather straightforward. We can easily identify all the contributions coming from the interaction eigenstates basis to the $\langle Z_l Z_m Z_r \rangle$. In the chiral limit these are

$$
\begin{aligned}
\frac{1}{3!} Tr\left[Q_Y^3\right] \langle YYY \rangle &= \frac{1}{3!} Tr\left[Q_Y^3\right] R^{YYY}_{Z_l Z_m Z_r} \langle Z_l Z_m Z_r \rangle + \dots \\
\frac{1}{2!} Tr\left[Q_Y T_3^2\right] \langle YWW \rangle &= \frac{1}{2!} Tr\left[Q_Y T_3^2\right] R^{YWW}_{Z_l Z_m Z_r} \langle Z_l Z_m Z_r \rangle + \dots \\
\frac{1}{2!} Tr\left[Q_Y T_3^2\right] \langle WYW \rangle &= \frac{1}{2!} Tr\left[Q_Y T_3^2\right] R^{WYW}_{Z_l Z_m Z_r} \langle Z_l Z_m Z_r \rangle + \dots \\
\frac{1}{2!} Tr\left[Q_Y T_3^2\right] \langle WWY \rangle &= \frac{1}{2!} Tr\left[Q_Y T_3^2\right] R^{WWY}_{Z_l Z_m Z_r} \langle Z_l Z_m Z_r \rangle + \dots \\
\frac{1}{2!} Tr\left[Q_{B_j} T_3^2\right] \langle B_j WW \rangle &= \frac{1}{2!} Tr\left[Q_{B_j} T_3^2\right] R^{B_j WW}_{Z_l Z_m Z_r} \langle Z_l Z_m Z_r \rangle + \dots \\
\frac{1}{2!} Tr\left[Q_{B_j} T_3^2\right] \langle W B_j W \rangle &= \frac{1}{2!} Tr\left[Q_{B_j} T_3^2\right] R^{W B_j W}_{Z_l Z_m Z_r} \langle Z_l Z_m Z_r \rangle + \dots \\
\frac{1}{2!} Tr\left[Q_{B_j} T_3^2\right] \langle WW B_j \rangle &= \frac{1}{2!} Tr\left[Q_{B_j} T_3^2\right] R^{WW B_j}_{Z_l Z_m Z_r} \langle Z_l Z_m Z_r \rangle + \dots \\
\frac{1}{2!} Tr\left[Q_{B_j} Q_Y^2\right] \langle B_j YY \rangle &= \frac{1}{2!} Tr\left[Q_{B_j} Q_Y^2\right] R^{B_j YY}_{Z_l Z_m Z_r} \langle Z_l Z_m Z_r \rangle + \dots \\
\frac{1}{2!} Tr\left[Q_{B_j} Q_Y^2\right] \langle Y B_j Y \rangle &= \frac{1}{2!} Tr\left[Q_{B_j} Q_Y^2\right] R^{Y B_j Y}_{Z_l Z_m Z_r} \langle Z_l Z_m Z_r \rangle + \dots
\end{aligned}
\qquad (6.244)
$$

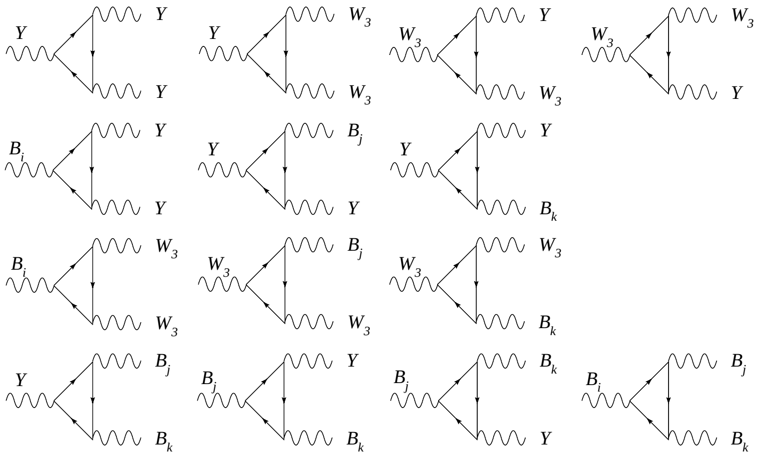


Figure 6.19: Triangle contributions to the $\langle Z_l Z_m Z_r \rangle$ vertex. As before, in the $m_f = 0$ phase all the SM contributions vanish because of the charge assignment.

$$
\begin{aligned}
\frac{1}{2!}Tr\left[Q_{B_j}Q_Y^2\right]\langle YYB_j\rangle &= \frac{1}{2!}Tr\left[Q_{B_j}Q_Y^2\right]R^{YYB_j}_{Z_lZ_mZ_r}\langle Z_lZ_mZ_r\rangle + \dots \\
Tr\left[Q_YQ_{B_j}Q_{B_k}\right]\langle YB_jB_k\rangle &= Tr\left[Q_YQ_{B_j}Q_{B_k}\right]R^{YB_jB_k}_{Z_lZ_mZ_r}\langle Z_lZ_mZ_r\rangle + \dots \\
Tr\left[Q_YQ_{B_j}Q_{B_k}\right]\langle B_jYB_k\rangle &= Tr\left[Q_YQ_{B_j}Q_{B_k}\right]R^{B_jYB_k}_{Z_lZ_mZ_r}\langle Z_lZ_mZ_r\rangle + \dots \\
Tr\left[Q_YQ_{B_j}Q_{B_k}\right]\langle B_jB_kY\rangle &= Tr\left[Q_YQ_{B_j}Q_{B_k}\right]R^{B_jB_kY}_{Z_lZ_mZ_r}\langle Z_lZ_mZ_r\rangle + \dots \\
Tr\left[Q_{B_i}Q_{B_j}Q_{B_k}\right]\langle B_iB_jB_k\rangle &= Tr\left[Q_{B_i}Q_{B_j}Q_{B_k}\right]R^{B_iB_jB_k}_{Z_lZ_mZ_r}\langle Z_lZ_mZ_r\rangle + \dots
\end{aligned}
\tag{6.245}
$$

and are listed in Fig. 6.19. The rotation matrices, in this case, are defined as

$$
\begin{aligned}
R^{YYY}_{Z_lZ_mZ_r} &= \left[3O^A_{YZ_l}O^A_{YZ_m}O^A_{YZ_r}\right] \\
R^{WWW}_{Z_lZ_mZ_r} &= \left[3O^A_{W_3Z_l}O^A_{W_3Z_m}O^A_{W_3Z_r}\right] \\
R^{YWW}_{Z_lZ_mZ_r} &= \left[O^A_{YZ_l}O^A_{WZ_m}O^A_{WZ_r} + O^A_{YZ_m}O^A_{WZ_l}O^A_{WZ_r} + O^A_{YZ_r}O^A_{WZ_l}O^A_{WZ_m}\right] \\
R^{WYY}_{Z_lZ_mZ_r} &= \left[O^A_{W_3Z_l}O^A_{YZ_m}O^A_{YZ_r} + O^A_{W_3Z_m}O^A_{YZ_l}O^A_{YZ_r} + O^A_{W_3Z_r}O^A_{YZ_l}O^A_{YZ_m}\right] \\
R^{B_jYY}_{Z_lZ_mZ_r} &= \left[O^A_{B_jZ_l}O^A_{YZ_m}O^A_{YZ_r} + O^A_{B_jZ_m}O^A_{YZ_l}O^A_{YZ_r} + O^A_{B_jZ_r}O^A_{YZ_m}O^A_{YZ_l}\right] \\
R^{B_jYW}_{Z_lZ_mZ_r} &= \Big[O^A_{B_jZ_l}(O^A_{YZ_m}O^A_{W_3Z_r} + O^A_{YZ_r}O^A_{W_3Z_m}) + O^A_{B_jZ_m}(O^A_{YZ_l}O^A_{W_3Z_r} + O^A_{W_3Z_l}O^A_{YZ_r}) \\
&\quad + O^A_{B_jZ_r}(O^A_{YZ_m}O^A_{W_3Z_l} + O^A_{YZ_l}O^A_{W_3Z_m})\Big] \\
R^{B_jB_kY}_{Z_lZ_mZ_r} &= \Big[(O^A_{B_jZ_m}O^A_{B_kZ_r} + O^A_{B_jZ_r}O^A_{B_kZ_m})O^A_{YZ_l} + (O^A_{B_jZ_r}O^A_{B_kZ_l} + O^A_{B_jZ_l}O^A_{B_kZ_r})O^A_{YZ_m} \\
&\quad + (O^A_{B_jZ_l}O^A_{B_kZ_m} + O^A_{B_jZ_m}O^A_{B_kZ_l})O^A_{YZ_r}\Big]
\end{aligned}
\tag{6.246}
$$

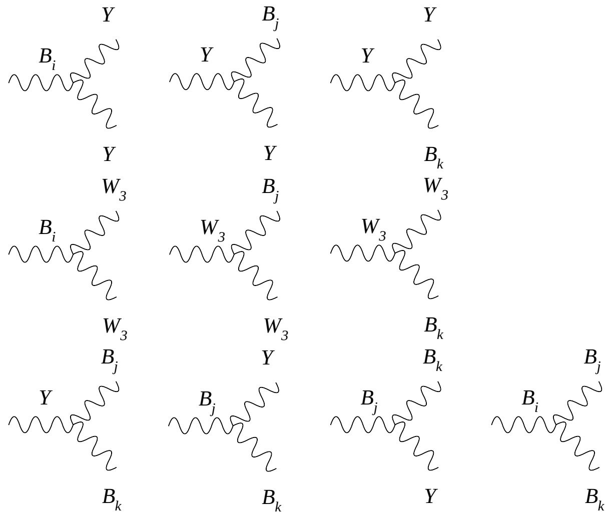


Figure 6.20: Chern-Simons contributions to the $\langle Z_l Z_m Z_r \rangle$ vertex

$$
\begin{aligned}
R^{B_jB_kW}_{Z_lZ_mZ_r} &= \Big[(O^A_{B_jZ_m}O^A_{B_kZ_r} + O^A_{B_jZ_r}O^A_{B_kZ_m})O^A_{W_3Z_l} + (O^A_{B_jZ_r}O^A_{B_kZ_l} + O^A_{B_jZ_l}O^A_{B_kZ_r})O^A_{W_3Z_m} \\
&\quad +(O^A_{B_jZ_l}O^A_{B_kZ_m} + O^A_{B_jZ_m}O^A_{B_kZ_l})O^A_{W_3Z_r}\Big] \\
R^{B_jWW}_{Z_lZ_mZ_r} &= \Big[O^A_{B_jZ_l}O^A_{W_3Z_m}O^A_{W_3Z_r} + O^A_{B_jZ_m}O^A_{W_3Z_l}O^A_{W_3Z_r} + O^A_{B_jZ_r}O^A_{W_3Z_m}O^A_{W_3Z_l}\Big] \\
R^{B_iB_jB_k}_{Z_lZ_mZ_r} &= \Big[(O^A_{B_jZ_m}O^A_{B_kZ_r} + O^A_{B_jZ_r}O^A_{B_kZ_m})O^A_{B_iZ_l} + (O^A_{B_jZ_r}O^A_{B_kZ_l} + O^A_{B_jZ_l}O^A_{B_kZ_r})O^A_{B_iZ_m} \\
&\quad +(O^A_{B_jZ_l}O^A_{B_kZ_m} + O^A_{B_jZ_m}O^A_{B_kZ_l})O^A_{B_iZ_r}\Big]. \qquad (6.247)
\end{aligned}
$$

Regarding the CS interactions (see Fig. (6.20)), we observe that we have a CS term corresponding to the anomalous vertex of the type $B_iB_jB_k$ which is non-zero, and we can formally write this trilinear interaction as

$$
\begin{aligned}
V^{ijk}_{CS,\,lmr} &= g_{B_i}g_{B_j}g_{B_k}a_n\theta^{ijk}_{lmr}R^{ijk}_{lmr}Z^\lambda_l Z^\mu_m Z^\nu_r\left[\kappa_i\left(\varepsilon[k_1,\lambda,\mu,\nu] - \varepsilon[k_2,\lambda,\mu,\nu]\right)\right. \\
&\quad \left. +\kappa_j\left(\varepsilon[k_2,\lambda,\mu,\nu] - \varepsilon[k_3,\lambda,\mu,\nu]\right) + \kappa_k\left(\varepsilon[k_3,\lambda,\mu,\nu] - \varepsilon[k_1,\lambda,\mu,\nu]\right)\right], \qquad (6.248)
\end{aligned}
$$

where for brevity we have defined $R^{ijk}_{lmr} = R^{B_iB_jB_k}_{Z_mZ_lZ_r}$, and so on.

The coefficients θ^{ijk}_{lmr} are the charge asymmetries, and the coefficients $\kappa_{i,j,k}$, are real numbers that tell us how the anomaly will be distributed on the **AAA** triangles. Both are driven by the generalized Ward identities of the theory. In this generalized case the CS interactions are not all re-absorbed in the definition of the fermionic triangles. In fact in this case there is no symmetry in the diagram that forces a symmetric assignment of the anomaly, and the CS terms in the $B_iB_jB_k$ interaction can re-distribute the partial anomalies. In this case the expression of the $B_iB_jB_k$ vertex in the momentum space is given by

$$\begin{aligned}\mathbf{V}^{\lambda\mu\nu}_{B_iB_jB_k} &= 4D_{B_iB_jB_k}\, g_{B_i}g_{B_j}g_{B_k}\, \Delta^{\lambda\mu\nu}_{\mathbf{AAA}}(m_f=0,k_1,k_2) \\ &+D_{B_iB_jB_k}\, g_{B_i}g_{B_j}g_{B_k}\frac{i}{\pi^2}\left[\frac{2\kappa_i}{9}\varepsilon^{\lambda\mu\nu\alpha}(k_{1,\alpha}-k_{2,\alpha})\right. \\ &\left.+\frac{2\kappa_j}{9}\varepsilon^{\lambda\mu\nu\alpha}(k_{2,\alpha}-k_{3,\alpha})+\frac{2\kappa_k}{9}\varepsilon^{\lambda\mu\nu\alpha}(k_{3,\alpha}-k_{1,\alpha})\right]. \end{aligned} \tag{6.249}$$

We recall that in the treatment of YB_jB_k and other similar triangles we still have two contributions for each triangle, due to the two orientations of the fermion number in the loop, so that our previous expression, obtained for the case of the YBB vertex, still holds. Also in this case we are allowed to absorb the CS interaction in the anomalous vertex. On the other hand, for the $B_iB_jB_k$ vertex we have

$$\begin{aligned}&3\Delta^{\lambda\mu\nu}_{AAA}(0,k_1,k_2)-\frac{a_n^i}{3}\varepsilon^{\lambda\mu\nu\alpha}(k_{1,\alpha}-k_{2,\alpha})-\frac{a_n^j}{3}\varepsilon^{\lambda\mu\nu\alpha}(k_{2,\alpha}-k_{3,\alpha})-\frac{a_n^k}{3}\varepsilon^{\lambda\mu\nu\alpha}(k_{3,\alpha}-k_{1,\alpha}) \\ &=3\Delta^{\lambda\mu\nu}_{A_iA_jA_k}(0,k_1,k_2)\,,\end{aligned} \tag{6.250}$$

where we have used the notation $\Delta(m_f=0,k_1,k_2)=\Delta(0,k_1,k_2)$ and $a_n^i=\kappa^i a_n$. Using these equations we can write the $\langle Z_lZ_mZ_r\rangle$ triangle in the following way

$$\begin{aligned}&\langle Z_lZ_mZ_r\rangle|_{m_f=0} = -\frac{1}{3}\left[\Delta^{\lambda\mu\nu}_{VAV}(0)+\Delta^{\lambda\mu\nu}_{VVA}(0)+\Delta^{\lambda\mu\nu}_{AVV}(0)\right]Z_l^\lambda Z_m^\mu Z_r^\nu\,\times \\ &\sum_f\sum_i\left\{g_Y^2g_{B_i}\theta_f^{YYB_i}R^{YYB_i}_{Z_lZ_mZ_r}+\sum_j g_Yg_{B_i}g_{B_j}\theta_f^{B_iB_jY}R^{YB_jB_k}_{Z_lZ_mZ_r}+g_{B_i}g_2^2\theta_f^{B_iWW}R^{B_iWW}_{Z_lZ_mZ_r}\right\} \\ &+\sum_f\sum_{i,j,k}g_{B_i}g_{B_j}g_{B_k}\theta_f^{B_iB_jB_k}\frac{1}{2}\Delta^{\lambda\mu\nu}_{A_iA_jA_k}(0)R^{B_iB_jB_k}_{Z_lZ_mZ_r}Z_l^\lambda Z_m^\mu Z_r^\nu.\end{aligned} \tag{6.251}$$

From this last result we can observe that the anomaly distribution on the last piece is, in general, not of the type $\Delta^{\lambda\mu\nu}_{AAA}(0)$, i.e. symmetric. If we want to factorize out a $\Delta^{\lambda\mu\nu}_{AAA}(0)$ triangle, we should think of this amplitude as a factorized $\Delta^{\lambda\mu\nu}_{AAA}(0)$ contribution plus an external suitable CS interaction which is not re-absorbed and such that it changes the partial anomalies from the symmetric distribution $\Delta^{\lambda\mu\nu}_{AAA}(0)$ to the non-symmetric one $\Delta^{\lambda\mu\nu}_{A_iA_jA_k}(0)$. These two points of view are completely equivalent and give the same result.

Finally, the analytic expression for each tensor contribution in the $m_f=0$ phase is given below. The **AVV** vertex has been shown in Eq. (6.163) while for **VAV** we have

$$\begin{aligned}\Delta^{\lambda\mu\nu}_{VAV}(0) &= \frac{1}{\pi^2}\int_0^1 dx\int_0^{1-x}dy\frac{1}{\Delta(0)}\left\{\varepsilon[k_1,\lambda,\mu,\nu](k_2\cdot k_2y(y-1)-xyk_1\cdot k_2)\right. \\ &+\varepsilon[k_2,\lambda,\mu,\nu](k_2\cdot k_2y(y-1)-xyk_1\cdot k_2) \\ &+\varepsilon[k_1,k_2,\lambda,\nu](k_1^\mu x(x-1)-xyk_2^\mu) \\ &\left.+\varepsilon[k_1,k_2,\lambda,\mu](k_2^\nu y(1-y)+xyk_1^\nu)\right\}\,,\end{aligned} \tag{6.252}$$

where the denominator is defined as $\Delta(0) = k_1^2(x-1)x + y(y-1)k_2^2 + 2xyk_1 \cdot k_2$.

Then, for the **VVA** contribution we obtain

$$\begin{aligned}
\Delta^{\lambda\mu\nu}_{VVA}(0) = \frac{1}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{1}{\Delta(0)} \{&\varepsilon[k_1,\lambda,\mu,\nu](k_1\cdot k_1 x(1-x) + xyk_1\cdot k_2) \\
&+\varepsilon[k_2,\lambda,\mu,\nu](k_1\cdot k_1 x(1-x) + xyk_1\cdot k_2) \\
&+\varepsilon[k_1,k_2,\lambda,\nu](k_1^\mu x(x-1) - xyk_2^\mu) \\
&+\varepsilon[k_1,k_2,\lambda,\mu](k_2^\nu y(1-y) + xyk_1^\nu)\} \ ,
\end{aligned} \tag{6.253}$$

and finally the contribution for **AAA** is $\Delta_{AAA}(0) = 1/3(\Delta_{AVV}(0) + \Delta_{VAV}(0) + \Delta_{VVA}(0))$

$$\begin{aligned}
\Delta^{\lambda\mu\nu}_{AAA}(0) = \frac{1}{3\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{1}{\Delta(0)} \{&\varepsilon[k_1,\lambda,\mu,\nu]\left(2y(y-1)k_2^2 - xyk_1\cdot k_2 + x(1-x)k_1^2\right) \\
&+\varepsilon[k_2,\lambda,\mu,\nu]\left(2(1-x)xk_1^2 + xyk_1\cdot k_2 + y(y-1)k_2^2\right) \\
&+\varepsilon[k_1,k_2,\lambda,\nu](k_1^\mu x(x-1) - xyk_2^\mu) \\
&+\varepsilon[k_1,k_2,\lambda,\mu](k_2^\nu y(1-y) + xyk_1^\nu)\} \,.
\end{aligned} \tag{6.254}$$

6.9 The $m_f \neq 0$ phase of the $\langle Z_l Z_m Z_r \rangle$ triangle

To obtain the contribution in the $m_f \neq 0$ phase we must include again all the contributions $\langle YYY \rangle$ and $\langle YWW \rangle$ coming from the SM. Since the final tensor structure of the triangle is driven by the STI's, we start by assuming the following symmetric distribution of the anomalies on the Δ_{AAA} triangle

$$\begin{aligned}
k_1^\mu \Delta^{\lambda\mu\nu}_{AAA}(m_f \neq 0, k_1, k_2) &= \frac{a_n}{3}\varepsilon^{\lambda\nu\alpha\beta}k_{1\alpha}k_{2\beta} + 2m_f\frac{1}{3}\Delta^{\lambda\nu} \\
k_2^\nu \Delta^{\lambda\mu\nu}_{AAA}(m_f \neq 0, k_1, k_2) &= -\frac{a_n}{3}\varepsilon^{\lambda\mu\alpha\beta}k_{1\alpha}k_{2\beta} - 2m_f\frac{1}{3}\Delta^{\lambda\mu} \\
k^\lambda \Delta^{\lambda\mu\nu}_{AAA}(m_f \neq 0, k_1, k_2) &= \frac{a_n}{3}\varepsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta} + 2m_f\frac{1}{3}\Delta^{\mu\nu} \ ,
\end{aligned} \tag{6.255}$$

where

$$\Delta^{\lambda\nu} = -\frac{m_f}{\pi^2}\varepsilon^{\lambda\nu\alpha\beta}k_{1\alpha}k_{2\beta}\int_0^1\int_0^{1-x} dxdy\frac{1}{\Delta(m_f)}. \tag{6.256}$$

These relations define the **AAA** structure in the massive case. The explicit form of this

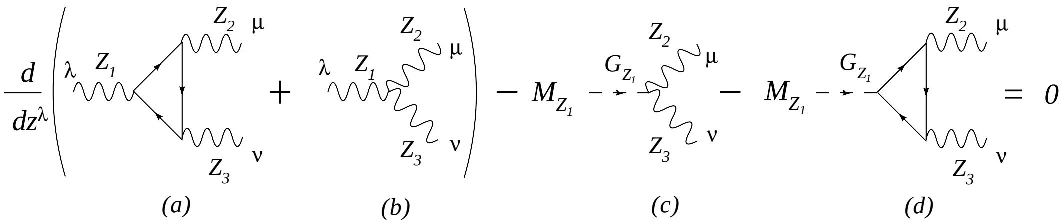


Figure 6.21: STI for the Z_1 vertex in a trilinear anomalous vertex with several $U(1)$'s. The CS counterterm is not absorbed and redistributes the anomaly according to the specific model.

triangle is given by

$$
\begin{aligned}
\Delta^{\lambda\mu\nu}_{AAA}(m_f \neq 0) = \frac{1}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{1}{\Delta(m_f)} \{ \\
\varepsilon[k_1,\lambda,\mu,\nu]\left[-\frac{\Delta(m_f)-m_f^2}{3} + k_2\cdot k_2 y(y-1) - xyk_1\cdot k_2\right] \\
+\varepsilon[k_2,\lambda,\mu,\nu]\left[\frac{\Delta(m_f)-m_f^2}{3} - k_1\cdot k_1 x(x-1) + xyk_1\cdot k_2\right] \\
+\varepsilon[k_1,k_2,\lambda,\nu](k_1^\mu x(x-1) - xyk_2^\mu) \\
+\varepsilon[k_1,k_2,\lambda,\mu](k_2^\nu y(1-y) + xyk_1^\nu)\}\ ,
\end{aligned}
\tag{6.257}
$$

where $\Delta(m_f) = m_f^2 + (y-1)yk_2^2 + (x-1)xk_1^2 - 2xyk_1\cdot k_2$.

Then, the final expression in the $m_f \neq 0$ phase is

$$
\begin{aligned}
\langle Z_l Z_m Z_r\rangle|_{m_f\neq 0} = -Z_l^\lambda Z_m^\mu Z_r^\nu \times \sum_f \Delta^{\lambda\mu\nu}_{AAA}(m_f\neq 0)\sum_i \{g_Y^3\theta_f^{YYY}R^{YYY}_{Z_lZ_mZ_r} \\
+g_2^3\theta_f^{WWW}R^{WWW}_{Z_lZ_mZ_r} + g_Y g_2^2\theta_f^{YWW}R^{YWW}_{Z_lZ_mZ_r} + g_Y^2 g_2\theta_f^{YYW}R^{YYW}_{Z_lZ_mZ_r} \\
+g_Y^2 g_{B_i}\theta_f^{YYB_i}R^{YYB_i}_{Z_lZ_mZ_r} + g_Y g_2 g_{B_i}\theta_f^{B_iYW}R^{B_iYW}_{Z_lZ_mZ_r} \\
+\sum_j g_Y g_{B_i} g_{B_j}\theta_f^{B_iB_jY}R^{YB_jB_k}_{Z_lZ_mZ_r} + \sum_j g_2 g_{B_i} g_{B_j}\theta_f^{B_iB_jW}R^{B_jB_kW}_{Z_lZ_mZ_r} \\
\left.+g_{B_i}g_2^2\theta_f^{B_iWW}R^{B_iWW}_{Z_lZ_mZ_r} + \sum_{j,k} g_{B_i}g_{B_j}g_{B_k}\theta_f^{B_iB_jB_k}R^{B_iB_jB_k}_{Z_lZ_mZ_r}\right\} + V_{CS}\,.
\end{aligned}
\tag{6.258}
$$

The diagrammatic structure of the STI for this general vertex is shown in Fig. 6.21, where an irreducible CS vertex (the second contribution in the bracket) is now present.

6.10 Discussions

The possibility of detecting anomalous gauge interactions at the LHC remains an interesting avenue that requires further analysis. The topic is clearly very interesting and may be a way to shed light on physics beyond the SM in a rather simple framework, though, at a hadron collider these studies are naturally classified as difficult ones. There are some points, however, that need clarification when anomalous contributions are taken into account. The first concerns the real mechanism of cancellation of the anomalies, if it is not realized by a charge assignment, and in particular whether it is of GS or of WZ type. In the two cases the high energy behaviour of a certain class of processes is rather different, and the WZ theory, which induces an axion-like particle in the spectrum, is in practice an effective theory with a unitarity bound, which has now been quantified [30]. The second point concerns the size of these anomalous interactions compared against the QCD background, which needs to be determined to next-to-next-to-leading-order (NNLO) in the strong coupling, at least for those processes involving anomalous gluon interactions with the extra Z'. These points are under investigations and we hope to return with some quantitative predictions in the near future.

6.11 Conclusions

In this chapter we have analyzed those trilinear gauge interactions that appear in the context of anomalous abelian extensions of the SM with several extra $U(1)$'s. We have discussed the defining conditions on the effective action, starting from the Stückelberg phase of this model, down to the electroweak phase, where Higgs-axion mixing takes place. In particular, we have shown that it is possible to simplify the study of the model in a suitable gauge, where the Higgs-axion mixing is removed from the effective action. The theory is conveniently defined, after electroweak symmetry breaking, by a set of generalized Ward identities and the counterterms can be fixed in any of the two phases. We have also derived the expressions of these vertices using the equivalence of the effective action in the interaction and in the mass eigenstate basis, and used this result to formulate general rules for the computation of the vertices which allow to simplify this construction. Using the various anomalous models that have been constructed in the previous literature in the last decade or so, it is now possible to explicitly proceed with a more direct phenomenological analysis of these theories, which remain an interesting avenue for future experimental searches of anomalous gauge interactions at the LHC.

Chapter 7

Conclusions and perspectives

We have presented in this thesis several analysis of anomalous correlators involving chiral and trace anomalies, with the intent of providing a more complete theoretical description of the corresponding effective actions in which they appear.

One of the main results of our analysis has been the discovery of anomaly poles in perturbation theory in the trace anomaly diagrams for QCD and in illustrating their similarity to those already known in the chiral anomaly. Our work has extended previous analysis by Giannotti and Mottola in QED [51] and has shown that anomaly poles are the common signatures of these types of anomalies. The poles, in both cases, can be coupled or decoupled in the IR, as we have shown in our technical discussions. Obviously, this result raises important questions concerning the significance and the implications of massless scalar degrees of freedom in gravity. In fact, the possible significance of these effective degrees of freedom widely discussed in this work is still open with implications that involve both particle physics and cosmology.
Other possible extensions of this line of research concerns the case of anomaly mediation in supersymmetric theories. Our results strongly suggest that the anomaly supermultiplet in super-Yang Mills theory is completely characterized by its anomaly poles. In turn, this suggests that a pole should be present also in the gamma-trace of the supersymmetric current.
This raises compelling issues in regards to the consistency of the gauging of these multiplets to supergravities, as we have discussed at length in Chapter 5.
Coming to the mechanism of anomaly cancellation using a pole subtraction, our interpretation of the pole contributions also as an ultraviolet component, which is inferred from the light-cone dominance of the correlators at high energy, seems to indicate that this version of the mechanism of cancellation should be viewed as an ultraviolet procedure. On the other hand, the use of an asymptotic axion for anomaly cancellation is most likely to be consistent in the infrared, given the presence of a unitarity bound in the formulation of anomalous theories corrected by

Wess-Zumino terms [30].

As we have discussed in chaper 6, one of the most direct way to test experimentally in the infrared the appearance of anomalous gauge symmetries is in the study of trilinear gauge interactions, which should be viable at the LHC. For this reason, we have investigated the general structure of these contributions in the neutral currents sector, analysis which should be combined with those of anomalous extra Z' gauge bosons, in channels such as Drell-Yan and in the production of direct photons. Some of these issues have been studied in a related work of us, and we refer to [6] for more details on this point.

Appendix A

Appendix

A.1 Poles and residui for massive gauge bosons

We are interested in the limit

c) $s_1 = s_2 = M^2 \qquad s \neq 0 \qquad m = 0.$

In this case only few simplifications occur in the complete expressions of the amplitudes A_i since the only surviving symmetry is the one between s_1 and s_2 and no momentum is set to zero. The expansion of the three point function is the most general one and the invariant amplitudes are given by

$$A_1(s, M^2, M^2) = -\frac{i}{4\pi^2} \tag{A.1}$$

$$\begin{aligned} A_3(s, M^2, M^2) &= -\frac{2\,i\,M^4}{\pi^2 s^2 \left(s - 4M^2\right)^2}\, \Phi_M\, \left(s - M^2\right) \\ &\quad -\frac{i}{2\pi^2 s \left(s - 4M^2\right)^2} \left[s^2 - 6sM^2 + 2\left(2M^2 + s\right)\log\left[\frac{M^2}{s}\right] M^2 + 8M^4\right] \end{aligned} \tag{A.2}$$

$$\begin{aligned} A_4(s, M^2, M^2) &= \frac{iM^2}{\pi^2 s^2 \left(s - 4M^2\right)^2} \Phi_M \left(s^2 - 3sM^2 + 2M^4\right) \\ &\quad + \frac{i}{2\pi^2 s \left(s - 4M^2\right)^2} \left[2sM^2 + \left(s^2 - 4M^4\right)\log\left(\frac{M^2}{s}\right) - 8M^4\right], \end{aligned} \tag{A.3}$$

with the functions $\Phi(x, y)$ and $\lambda(x, y)$ defined in this specific case by

$$\Phi_M \equiv \Phi(\frac{M^2}{s}, \frac{M^2}{s}) = \frac{1}{\lambda_M}\left[\log^2\left(\frac{2M^2}{s(\lambda_M + 1) - 2M^2}\right) + 4\text{Li}_2\left(\frac{2M^2}{-s(\lambda_M + 1) + 2M^2}\right) + \frac{\pi^2}{3}\right], \tag{A.4}$$

$$\lambda_M \equiv \lambda(M^2/s, M^2/s) = \sqrt{1 - \frac{4M^2}{s}}, \tag{A.5}$$

as in Eqs. (1.20,1.21), with $x = y = M^2/s$.

As usual, a symmetric configuration of this type yields

$$A_2(s, M^2, M^2) = -A_1(s, M^2, M^2), \tag{A.6}$$

$$A_5(s, M^2, M^2) = -A_4(s, M^2, M^2), \tag{A.7}$$

$$A_6(s, M^2, M^2) = -A_3(s, M^2, M^2) \tag{A.8}$$

and in the total amplitude only few simplifications occur

$$\Delta^{\lambda\mu\nu}(s, M^2, M^2) = A_3(s, M^2, M^2)\, \eta_3^{\lambda\mu\nu}(k_1, k_2) + A_4(s, M^2, M^2)\, \eta_4^{\lambda\mu\nu}(k_1, k_2) + A_5(s, M^2, M^2)\, \eta_5^{\lambda\mu\nu}(k_1, k_2) + A_6(s, M^2, M^2)\, \eta_6^{\lambda\mu\nu}(k_1, k_2). \tag{A.9}$$

The analysis of the spurious pole at $s = 0$ requires the analytic continuation in the euclidean region ($s < 0$) according to the $i\eta$ prescription: $s \to s + i\eta$, $M^2 \to M^2 + i\eta$. In this case the only trascendental functions requiring the analytic regularizations are the logarithmic ones, the dilogarithm being well-definite since

$$\frac{2M^2}{-s(\lambda_M + 1) + 2M^2} < 1 \qquad \text{for}\, s < 0. \tag{A.10}$$

Then we substitute

$$\log\left[\frac{M^2}{s} - i\eta\right] \to \log\left[-\frac{M^2}{s}\right] - i\pi \qquad \text{for}\, s < 0 \tag{A.11}$$

$$\log\left[\frac{2M^2}{-2M^2 + s + s\lambda} - i\eta\right] \to \log\left[-\frac{2M^2}{-2M^2 + s + s\lambda}\right] - i\pi \qquad \text{for}\, s < 0 \tag{A.12}$$

into the expressions of $A_3(s, M^2, M^2)$ and $A_4(s, M^2, M^2)$ and perform the limit for $s \to 0$. We obtain

$$\lim_{s\to 0} s\, A_i(s, M^2, M^2) = 0 \qquad\qquad i = 3, \ldots, 6 \tag{A.13}$$

and also

$$\lim_{s\to 0} s\, \Delta^{\lambda\mu\nu}(s, M^2, M^2) = 0, \tag{A.14}$$

showing that in the presence of external massive gauge lines the triangle amplitude $\Delta^{\lambda\mu\nu}$ exhibits no poles. This can be confirmed by a parallel analysis based on the L/T parameterization whose coefficients are

$$w_L(s, M^2, M^2) = -\frac{4i}{s}, \tag{A.15}$$

$$w_T^{(+)}(s, M^2, M^2) = \frac{4i}{(s - 4M^2)^2}\left[(s + 2M^2)\log\left[\frac{M^2}{s}\right] + \frac{2M^2(s - M^2)}{s}\Phi_M\right] + \frac{4i}{s - 4M^2}, \tag{A.16}$$

$$w_T^{(-)}(s, M^2, M^2) = \tilde{w}_T^{(-)}(s, M^2, M^2) = 0. \tag{A.17}$$

Combining the previous results, the whole amplitude becomes

$$W^{\lambda\mu\nu}(s, M^2, M^2) = \frac{1}{8\pi^2}\left[w_L(s, M^2, M^2)\, k^\lambda \varepsilon[\mu, \nu, k_1, k_2] - w_T^{(+)}(s, M^2, M^2)\, t^{(+)}_{\lambda\mu\nu}(k_1, k_2)\right]. \tag{A.18}$$

At this point we perform the same analytic continuations discussed above, shown in Eqs. (A.11) and (A.12) and take the limits

$$\lim_{s\to 0} s\, w_L(s, M^2, M^2) = -\,4\,i \tag{A.19}$$

$$\lim_{s\to 0} s\, w_T^{(+)}(s, M^2, M^2 2)\, t^{(+)}_{\lambda\mu\nu}(k_1, k_2) = -\,4\,i \tag{A.20}$$

which, in combination, give a vanishing residue also in this parameterization

$$\lim_{s\to 0} s\; W^{\lambda\mu\nu}(s, M^2, M^2) = 0. \tag{A.21}$$

When the mass of the fermion in the loop is non vanishing, $m \neq 0$, we consider cases *d*), *e*) and *f*). We take the appropriate limits starting from the expressions in Eq. (1.32-1.34) obtaining

d) $k_1^2 = 0 \qquad k_2^2 \neq 0 \qquad k^2 \neq 0 \qquad m \neq 0$

$$A_1(s, 0, s_2, m^2) = -\frac{i}{4\pi^2} + \frac{s_2}{4\pi^4\,(s - s_2)}\, D_2 - \frac{m^2}{2\pi^4}\, \bar{C}_0, \tag{A.22}$$

$$A_2(s, 0, s_2, m^2) = \frac{i}{4\pi^2} + \frac{s_2}{4\pi^4\,(s - s_2)}\, D_2 + \frac{m^2}{2\pi^4}\, \bar{C}_0, \tag{A.23}$$

$$\begin{aligned} A_3(s, 0, s_2, m^2) &= -A_6(s, 0, s_2, m^2) = \\ &-\frac{i}{2\pi^2\,(s - s_2)} - \frac{s_2}{2\pi^4\,(s - s_2)^2}\, D_2 - \frac{m^2}{\pi^4\,(s - s_2)}\, \bar{C}_0, \end{aligned} \tag{A.24}$$

$$A_4(s, 0, s_2, m^2) = \frac{1}{2\pi^4\,(s - s_2)}\, D_2, \tag{A.25}$$

$$\begin{aligned} A_5(s, 0, s_2, m^2) &= -\frac{s_2}{\pi^4 (s + s_2)^2}\left(s - 2m^2\right) \bar{C}_0 - \frac{(s + s_2)}{2\pi^4 (s - s_2)^2}\, \bar{D}_1 \\ &+ \frac{(2s + s_2) s_2}{\pi^4 (s_2 - s)^3}\, D_2 - \frac{i s_2}{\pi^2 (s - s_2)^2}, \end{aligned} \tag{A.26}$$

where D_2 is defined in Eq. (1.38), while $\bar{D}_1$ and $\bar{C}_0$ are the two $s_1 \to 0$ limits of D_1 and $C_0(s_1, s_2, s, m^2)$ respectively, that is

$$\bar{D}_1 \equiv \lim_{s_1\to 0} D_1(s, s_1, m^2) = i\pi^2\left[2 - a_3 \log\frac{a_3 + 1}{a_3 - 1}\right], \tag{A.27}$$

$$\bar{C}_0 \equiv \lim_{s_1\to 0} C_0(s, s_1, s_2, m^2) = -\frac{i\pi^2}{2(s - s_2)}\left[\log^2\frac{a_2 + 1}{a_2 - 1} - \log^2\frac{a_3 + 1}{a_3 - 1}\right]. \tag{A.28}$$

The coefficients of the w's in the L/T formulation, in this case, are

$$w_L(s,0,s_2,m^2) = -\frac{4i}{s} - \frac{8m^2}{\pi^2 s}\,\bar{C}_0, \tag{A.29}$$

$$w_T^{(+)}(s,0,s_2,m^2) = \frac{1}{\pi^2(s-s_2)^2}\Bigg[4i\pi^2 s + 2(s+s_2)\,\bar{D}_1 + 4\,s\left(2\,m^2+s_2\right)\,\bar{C}_0 + \frac{2\left(s^2+4s_2 s+s_2^2\right)}{s-s_2}\,D_2\Bigg], \tag{A.30}$$

$$w_T^{(-)}(s,0,s_2,m^2) = -\frac{1}{\pi^2(s-s_2)^2}\Bigg[4i\pi^2 s + 2(s+s_2)\,\bar{D}_1 + 4\,s_2\left(2\,m^2+s\right)\,\bar{C}_0 + \frac{2\left(s^2-6s_2 s-s_2^2\right)}{s-s_2}\,D_2\Bigg], \tag{A.31}$$

$$\tilde{w}_T^{(-)}(s,0,s_2,m^2) = \frac{1}{\pi^2(s-s_2)^2}\Bigg[4i\pi^2 s_2 + 2(s+s_2)\,\bar{D}_1 + 4\,s_2\left(2\,m^2+s\right)\,\bar{C}_0 + \frac{2\left(-s^2+6s_2 s+s_2^2\right)}{s-s_2}\,D_2\Bigg]. \tag{A.32}$$

Furthermore, in the case in which the massive amplitude has both external vector lines on-shell

e) $k_1^2 = 0 \qquad k_2^2 = 0 \qquad k^2 \neq 0 \qquad m \neq 0$

one obtains

$$A_1(0,0,s,m^2) = -\frac{i}{4\pi^2}\left(1+\frac{m^2}{s}\log^2\frac{a_3+1}{a_3-1}\right), \tag{A.33}$$

$$A_3(0,0,s,m^2) = -A_6(0,0,s,m^2) = -\frac{i}{2\pi^2 s}\left(1+\frac{m^2}{s}\log^2\frac{a_3+1}{a_3-1}\right), \tag{A.34}$$

$$A_4(0,0,s,m^2) = -\frac{i}{2\pi^2 s}\left(a_3\log\frac{a_3+1}{a_3-1}-2\right). \tag{A.35}$$

These simple results are obtained with a limiting procedure, starting from the scalar triangle diagram with off-shell external lines and involves the function $\Phi(x,y)$ [148] already encountered in the explicit expression of the Rosenberg parameterization [41]. Instead, for the L/T parameterization we obtain

$$w_L(0,0,s,m^2) = -\frac{4i}{s}\left[1+\frac{m^2}{s}\log^2\left(\frac{a_3+1}{a_3-1}\right)\right], \tag{A.36}$$

$$w_T^{(+)}(0,0,s,m^2) = \frac{4i}{s}\left[3+\frac{m^2}{s}\log^2\left(\frac{a_3+1}{a_3-1}\right) - a_3\log\left(\frac{a_3+1}{a_3-1}\right)\right], \tag{A.37}$$

$$w_T^{(-)}(0,0,s,m^2) = \tilde{w}_T^{(-)}(0,0,s,m^2) = 0. \tag{A.38}$$

Finally, the particles can be on-shell and both of mass M and in this case we obtain

f) $k_1^2 = M^2 \qquad k_2^2 = M^2 \qquad k^2 \neq 0 \qquad m \neq 0$

$$A_1(M^2, M^2, s, m^2) = -\frac{i}{4\pi^2} - \frac{m^2}{2\pi^4} C_0, \tag{A.39}$$

$$\begin{aligned} A_3(M^2, M^2, s, m^2) &= \frac{1}{\pi^4 s\,(s-4M^2)} \left[\frac{i\pi^2}{2} \left(2M^2 - s\right) - \frac{\left(2M^2+s\right) M^2}{s-4M^2} D_M \right. \\ &+ \left. \left(\frac{2M^4(M^2-s)}{s-4M^2} - m^2(s-2M^2) \right) C_0 \right], \end{aligned} \tag{A.40}$$

$$\begin{aligned} A_4(M^2, M^2, s, m^2) &= \frac{1}{\pi^4 s\,(s-4M^2)} \left[i\pi^2 M^2 + \frac{s^2-4M^4}{2(s-4M^2)} D_M \right. \\ &+ \left. \left(\frac{M^2(2M^4-3M^2 s+s^2)}{s-4M^2} + 2m^2 M^2 \right) C_0 \right]. \end{aligned} \tag{A.41}$$

In the previous expressions we have denoted by C_0 the complete expression $C_0(s_1, s_2, s, m^2)$ in Eq. (1.39) computed at $s_1 = s_2 = M^2$. In addition to this we have defined

$$D_M(M^2, s, m^2) \equiv B_0(k^2, m^2) - B_0(M^2, m^2) = i\pi^2 \left[a_M \log\frac{a_M+1}{a_M-1} - a_3 \log\frac{a_3+1}{a_3-1} \right], \tag{A.42}$$

$$a_M = \sqrt{1-\frac{4m^2}{M^2}}, \qquad a_3 = \sqrt{1-\frac{4m^2}{s}}. \tag{A.43}$$

Similarly, the expressions of the w's invariant amplitudes in the L/T parameterization for the massive triangle amplitude are given by

$$w_L(s, m^2) = -\frac{4i}{s} - \frac{8m^2}{\pi^2 s} C_0, \tag{A.44}$$

$$w_T^{(+)}(s, m^2, M^2) = \frac{1}{\pi^2(s-4M^2)} \left[4i\pi^2 + \frac{4(s+2M^2)}{s-4M^2} D_M + \left(8m^2 + \frac{8M^2(s-M^2)}{s-4M^2} \right) C_0 \right], \tag{A.45}$$

$$w_T^{(-)}(s, m^2, M^2) = \tilde{w}_T^{(-)}(s, m^2, M^2) = 0. \tag{A.46}$$

A.2 Definitions and conventions for the scalar integrals

We collect in this appendix all the scalar integrals involved in this computation. To set all our conventions, we start with the definition of the one-point function, or massive tadpole $\mathcal{A}_0(m^2)$, the massive bubble $\mathcal{B}_0(s,m^2)$ and the massive three-point function $\mathcal{C}_0(s,s_1,s_2,m^2)$

$$\mathcal{A}_0(m^2) = \frac{1}{i\pi^2}\int d^n l\,\frac{1}{l^2-m^2} = m^2\left[\frac{1}{\bar\epsilon}+1-\log\left(\frac{m^2}{\mu^2}\right)\right], \tag{A.47}$$

$$\begin{aligned}\mathcal{B}_0(k^2,m^2) &= \frac{1}{i\pi^2}\int d^n l\,\frac{1}{(l^2-m^2)\,((l-k)^2-m^2)}\\ &= \frac{1}{\bar\epsilon}+2-\log\left(\frac{m^2}{\mu^2}\right)-a_3\log\left(\frac{a_3+1}{a_3-1}\right),\end{aligned} \tag{A.48}$$

$$\begin{aligned}\mathcal{C}_0(s,s_1,s_2,m^2) &= \frac{1}{i\pi^2}\int d^n l\,\frac{1}{(l^2-m^2)\,((l-q)^2-m^2)\,((l+p)^2-m^2)}\\ &= -\frac{1}{\sqrt{\sigma}}\sum_{i=1}^{3}\left[Li_2\frac{b_i-1}{a_i+b_i}-Li_2\frac{-b_i-1}{a_i-b_i}+Li_2\frac{-b_i+1}{a_i-b_i}-Li_2\frac{b_i+1}{a_i+b_i}\right],\end{aligned} \tag{A.49}$$

with

$$a_i=\sqrt{1-\frac{4m^2}{s_i}}\qquad\qquad b_i=\frac{-s_i+s_j+s_k}{\sqrt{\sigma}}, \tag{A.50}$$

where $s_3=s$ and in the last equation $i=1,2,3$ and $j,k\neq i$.
The one-point and two-point functions written before in $n=4-2\,\epsilon$ are divergent in dimensional regularization with the singular parts given by

$$\mathcal{A}_0(m^2)^{sing.}\to\frac{1}{\bar\epsilon}\,m^2,\qquad\qquad \mathcal{B}_0(s,m^2)^{sing.}\to\frac{1}{\bar\epsilon}, \tag{A.51}$$

with

$$\frac{1}{\bar\epsilon}=\frac{1}{\epsilon}-\gamma-\ln\pi \tag{A.52}$$

We use two finite combinations of scalar functions given by

$$\mathcal{B}_0(s,m^2)\,m^2-\mathcal{A}_0(m^2)=m^2\left[1-a_3\log\frac{a_3+1}{a_3-1}\right], \tag{A.53}$$

$$\mathcal{D}_i\equiv\mathcal{D}_i(s,s_i,m^2)=\mathcal{B}_0(s,m^2)-\mathcal{B}_0(s_i,m^2)=\left[a_i\log\frac{a_i+1}{a_i-1}-a_3\log\frac{a_3+1}{a_3-1}\right]\qquad i=1,2. \tag{A.54}$$

The scalar integrals $\mathcal{C}_0(s,0,0,m^2)$ and $\mathcal{D}(s,0,0,m^2)$ are the $\{s_1\to 0,\ s_2\to 0\}$ limits of the generic functions $\mathcal{C}_0(s,s_1,s_2,m^2)$ and $\mathcal{D}_1(s,s_1,m^2)$

$$\mathcal{C}_0(s,0,0,m^2) = \frac{1}{2s}\log^2\frac{a_3+1}{a_3-1}, \tag{A.55}$$

$$\mathcal{D}(s,0,0,m^2) = \mathcal{D}_1(s,0,m^2)=\mathcal{D}_2(s,0,m^2)=\left[2-a_3\log\frac{a_3+1}{a_3-1}\right]. \tag{A.56}$$

The master integrals denoted by $\mathcal{B}_0(s,0)$, $\mathcal{D}_i(s,s_i,0)$ $(i=1,2)$ and $\mathcal{C}_0(s,s_1,s_2,0)$ are consistently redefined for $m=0$ (and $s<0$) as

$$\mathcal{B}_0(s,0) = \left[\frac{1}{\bar{\epsilon}} - \log\left(-\frac{s}{\mu^2}\right) + 2\right], \tag{A.57}$$

$$D_i(s,s_i,0) = \mathcal{B}_0(s,0) - \mathcal{B}_0(s_i,0) = \log\left(\frac{s_i}{s}\right), \qquad i=1,2 \tag{A.58}$$

$$\mathcal{C}_0(s,s_1,s_2,0) = \frac{1}{s}\Phi(x,y), \tag{A.59}$$

where μ is the renormalization scale and the function $\Phi(x,y)$ is defined as [67]

$$\Phi(x,y) \;=\; \frac{1}{\lambda}\left\{2[Li_2(-\rho x) + Li_2(-\rho y)] + \ln\frac{y}{x}\ln\frac{1+\rho y}{1+\rho x} + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\right\}, \tag{A.60}$$

with

$$\lambda(x,y) = \sqrt{\Delta}, \qquad \Delta = (1-x-y)^2 - 4xy, \rho(x,y) = 2(1-x-y+\lambda)^{-1}, \tag{A.61}$$

$$x = \frac{s_1}{s}, \qquad y = \frac{s_2}{s}. \tag{A.62}$$

The singularities in $1/\bar{\epsilon}$ and the dependence on the renormalization scale μ cancel out when taking into account the difference of two functions $\mathcal{B}_0$, so that the $\mathcal{D}_i$'s are well-defined; the three-point master integral is convergent.

A.3 Alternative conditions on the correlator in the massless case

As we have mentioned, one can follow an entirely different approach in order to fix the expression of the correlator. This is based on the requirement that the trace anomaly satisfies a well known operatorial relation which is imposed on the matrix elements at nonzero momentum. Specifically we proceed as follows, and illustrate this point in the massless limit. We impose the value of the trace anomaly as a defining condition on the whole amplitude, so that the (new) request c') will be

c') the non-zero anomaly trace in the massless limit.

As the first two conditions a) and b), respectively the $\{\mu \leftrightarrow \nu\}$ symmetry and the vector current conservation, remain the same as before, we continue illustrating the modifications due to this approach from this point on. The third condition is given by

$$g_{\mu\nu}\,\Gamma^{\mu\nu\alpha\beta}(p,q) = c\,u^{\alpha\beta}(p,q), \tag{A.63}$$

where c is related to the usual QED β-function as $c = -\frac{2\beta}{e}$. The resulting system is

$$
Eq.\ (\text{A.63}) \Rightarrow \begin{cases} 4\,\frac{A_{41}}{p\cdot q} - A_7 + 2\,A_9 - A_{12} = 0, \\ c + 4\,A_{37} + 4\,A_{42} + A_4\,p\cdot p - 2\,A_6\,p\cdot p \\ \qquad\qquad +2\,A_{11}\,p\cdot q + 2\,A_{14}\,q\cdot q + A_{16}\,q\cdot q = 0, \end{cases} \tag{A.64}
$$

whose solutions for A_{41} and A_{37} read as

$$
A_{41} = \frac{p\cdot q}{4}\,(A_7 - 2\,A_9 + A_{12}) \tag{A.65}
$$

$$
A_{37} + A_{42} = \frac{1}{4}\left[-c - A_4\,p\cdot p + 2\,A_6\,p\cdot p - 2\,A_{11}\,p\cdot q - 2\,A_{14} q\cdot q - A_{16}\,q\cdot q)\right]. \tag{A.66}
$$

As seen from the last equation, the second solution returns the sum of two UV divergent amplitudes, A_{37} and A_{42}. However, an explicit computation shows that in the explicit mapping between the two sets of A_i and F_i these two amplitudes appear in such a way that their divergences cancel. Therefore, reinserting the expressions of A_{41} and A_{37} extracted from Eq. (A.66) into the expression of $\Gamma^{\mu\nu\alpha\beta}(p,q)$ one finds another mapping between the form factors A_i and F_i, as previously done in Eqs. (2.72-2.84)

$$
F_1 = \frac{c}{3k\cdot k}, \tag{A.67}
$$

$$
F_2 = 0, \tag{A.68}
$$

$$
F_3 = \frac{A_4}{4} - \frac{c}{12\,k\cdot k}, \tag{A.69}
$$

$$
F_4 = \frac{A_7}{4\,p\cdot p}, \tag{A.70}
$$

$$
F_5 = \frac{A_{16}}{4} - \frac{c}{12k\cdot k}, \tag{A.71}
$$

$$
F_6 = \frac{A_{12}}{4\,q\cdot q}, \tag{A.72}
$$

$$
F_7 = -\frac{c}{6k\cdot k} + \frac{A_{11}}{2} + \frac{(A_9\,p\cdot p + A_{14}\,p\cdot q)\,q\cdot q}{2\,p\cdot q^2} + \frac{p\cdot p(A_6\,p\cdot q + A_9\,q\cdot q)}{2\,p\cdot q^2}, \tag{A.73}
$$

$$
F_8 = -\frac{A_9}{2\,p\cdot q}, \tag{A.74}
$$

$$
F_9 = \frac{A_6}{p\cdot q} + A_9\frac{q\cdot q}{p\cdot q^2}, \tag{A.75}
$$

$$
F_{10} = A_9\frac{p\cdot p}{p\cdot q^2} + \frac{A_{14}}{p\cdot q}, \tag{A.76}
$$

$$
F_{11} = \frac{A_{12}}{2\,q\cdot q} - \frac{A_2}{2\,p\cdot p}, \tag{A.77}
$$

$$
F_{12} = \frac{A_3}{2\,p\cdot q} + \frac{A_7}{2\,p\cdot p}, \tag{A.78}
$$

$$F_{13} = \frac{1}{4\,p\cdot q}\,\big[2A_{11}\,p\cdot q^2 + c\,p\cdot q + 4\,A_{42}\,p\cdot q + A_4\,p\cdot p\,p\cdot q + 2\,A_6\,p\cdot p\,p\cdot q$$
$$+ 2\,A_{14}\,q\cdot q\,p\cdot q + A_{16}\,q\cdot q\,p\cdot q + 4\,A_9\,p\cdot p\,q\cdot q\big]\,; \tag{A.79}$$

This new mapping leaves the invariant amplitudes from F_9 to F_{12} the same as before, so the condition c), i.e. the WI derived from Eq. (2.36) and c') are perfectly equivalent in determining these 4 form factors.

A.4 Comparison with the parametric approach and numerical checks

The parametric approach of [51] allows, by combining the denominators of the various tensor amplitudes, to give parametric expressions for the form factors F_i starting from a set scalar parametric integrals. Our results correspond to an explicit computation of these integrals. We will not give each integral separately, since they are rather lengthy. The mapping between the F_i's in the parametric form and our expressions allow to perform numerical checks of our result. We have perfect agreement between the parametric forms derived in [51], computed numerically, and our explicit expressions in all the euclidean regions of the external momenta. We briefly clarify this point.

Explicit formulae for all twelve finite coefficient functions may be given in the Feynman parameterized form,

$$C_j(k^2;p^2,q^2) = \frac{e^2}{4\pi^2}\int_0^1 dx \int_0^{1-x} dy\; \frac{c_j(x,y)}{p^2\,x(1-x) + q^2\,y(1-y) + 2\,xy\,p\cdot q + m^2}\,, \tag{A.80}$$

where the polynomials $c_i(x,y)$ for $i = 1,\ldots,12$ are listed in Table A.1.

$$F_1 = \frac{C_7+C_8+C_9}{3} + \frac{p^2}{3k^2}\,(-C_1+C_3+C_8-C_9) + \frac{q^2}{3k^2}\,(-C_7+C_8+C_{10}-C_{12})\,, \tag{A.81}$$
$$F_3 = \frac{2C_7-C_8-C_9}{12} + \frac{p^2}{12k^2}\,(C_1-C_3-C_8+C_9) + \frac{q^2}{12k^2}\,(C_7-C_8-C_{10}+C_{12})\,, \tag{A.82}$$
$$F_5 = \frac{-C_7-C_8+2C_9}{12} + \frac{p^2}{12k^2}\,(C_1-C_3-C_8+C_9) + \frac{q^2}{12k^2}\,(C_7-C_8-C_{10}+C_{12})\,, \tag{A.83}$$
$$F_7 = \frac{-C_7+2C_8-C_9}{6} + \frac{p^2}{6k^2}\,(C_1-C_3-C_8+C_9) + \frac{q^2}{6k^2}\,(C_7-C_8-C_{10}+C_{12})$$
$$+\frac{p^2\,q^2}{(p\cdot q)^2}\,C_5 + \frac{p^2C_2+q^2C_{11}}{2\,(p\cdot q)}\,. \tag{A.84}$$

j	C_j = coefficient of	$c_j(x,y)$
1	$p^\mu p^\nu p^\alpha p^\beta$	$-4x^2(1-x)(1-2x)$
2	$(p^\mu q^\nu + q^\mu p^\nu)p^\alpha p^\beta$	$-x(1-x)(1-4x+8xy)+xy$
3	$q^\mu q^\nu p^\alpha p^\beta$	$2x(1-2y)(1-x-y+2xy)$
4	$p^\mu p^\nu p^\alpha q^\beta$	$-2x(1-x)(1-2x)(1-2y)$
5	$(p^\mu q^\nu + q^\mu p^\nu)p^\alpha q^\beta$	$x(1-x)(1-2y)^2+y(1-y)(1-2x)^2$
6	$q^\mu q^\nu p^\alpha q^\beta$	$-2y(1-y)(1-2x)(1-2y)$
7	$p^\mu p^\nu q^\alpha p^\beta$	$2xy(1-2x)^2$
8	$(p^\mu q^\nu + q^\mu p^\nu)q^\alpha p^\beta$	$-2xy(1-2x)(1-2y)$
9	$q^\mu q^\nu q^\alpha p^\beta$	$2xy(1-2y)^2$
10	$p^\mu p^\nu q^\alpha q^\beta$	$2y(1-2x)(1-x-y+2xy)$
11	$(p^\mu q^\nu + q^\mu p^\nu)q^\alpha q^\beta$	$-y(1-y)(1-4y+8xy)+xy$
12	$q^\mu q^\nu q^\alpha q^\beta$	$-4y^2\,(1-2y)(1-y)$

Table A.1: The twelve tensors with four free indices ($\mu\nu\alpha\beta$) on p, q used in ref. [51] for the construction of the form factors F_i. At each coefficient functions $C_j(k^2;p^2,q^2)$ correspond a polynomial c_j in the Feynman parameterized form, as given in Eq. (A.80).

$$F_2 = \frac{C_1}{3q^2} + \frac{C_{12}}{3p^2} + \frac{-C_1 + 2C_2 - 2C_5 + 2C_{11} - C_{12}}{3k^2}\,, \tag{A.85}$$

$$F_4 = -\frac{C_1}{12q^2} + \frac{3C_{10} - C_{12}}{12p^2} + \frac{C_1 - 2C_2 + 2C_5 - 2C_{11} + C_{12}}{12k^2}\,, \tag{A.86}$$

$$F_6 = \frac{-C_1 + 3C_3}{12q^2} - \frac{C_{12}}{12p^2} + \frac{C_1 - 2C_2 + 2C_5 - 2C_{11} + C_{12}}{12k^2}\,, \tag{A.87}$$

$$F_8 = -\frac{C_5}{2p\cdot q} - \frac{C_1}{6q^2} - \frac{C_{12}}{6p^2} + \frac{C_1 - 2C_2 + 2C_5 - 2C_{11} + C_{12}}{6k^2}\,. \tag{A.88}$$

$$F_9 = \frac{C_2}{p\cdot q} + \frac{q^2\,C_5}{(p\cdot q)^2}\,, \tag{A.89}$$

$$F_{10} = \frac{p^2\,C_5}{(p\cdot q)^2} + \frac{C_{11}}{p\cdot q}\,, \tag{A.90}$$

$$F_{11} = \frac{C_3}{2q^2} - \frac{C_{12}}{2p^2}\,, \tag{A.91}$$

$$F_{12} = \frac{C_{10}}{2p^2} - \frac{C_1}{2q^2}. \tag{A.92}$$

Finally, numerical checks on F_{13} are performed on the UV convergent amplitude

$$
\begin{aligned}
F_{13} &= \frac{\Pi_R(p^2)+\Pi_R(q^2)}{2} + \frac{p^2\, q^2}{p\cdot q}\, C_5 + \frac{p^4 C_4 + q^4 C_6}{4 p\cdot q} + \frac{p\cdot q}{4}\,(2C_2 + C_3 + C_{10} + 2C_{11}) \\
&+ \frac{p^2}{4}\,(2C_2 + C_4 + 2C_5 + C_{10}) + \frac{q^2}{4}\,(C_3 + 2C_5 + C_6 + 2C_{11}) \qquad \text{(A.93)}
\end{aligned}
$$

where the scalar two-point functions have been renormalized by the subtraction of the UV $1/\epsilon$ pole.

A.5 The massive invariant amplitudes

The off-shell massive form factors F_i, with

- $s \neq 0 \qquad s_1 \neq 0 \qquad s_2 \neq 0 \qquad m \neq 0$

and with $\gamma \equiv s - s_1 - s_2$, $\sigma \equiv s^2 - 2(s_1+s_2)\,s + (s_1-s_2)^2$ are given by [1]

$$
\begin{aligned}
\mathbf{F_1(s;\, s_1,\, s_2,\, m^2)} &= \frac{e^2 \gamma m^2}{3\pi^2 s\sigma} + \frac{e^2\,\mathcal{D}_2(s,s_2,m^2)\, s_2\left[s^2 + 4s_1 s - 2s_2 s - 5s_1^2 + s_2^2 + 4s_1 s_2\right] m^2}{3\pi^2 s\sigma^2} \\
&- \frac{e^2}{18\pi^2 s} - \frac{e^2\,\mathcal{D}_1(s,s_1,m^2)\, s_1\left[-(s-s_1)^2 + 5s_2^2 - 4(s+s_1)\, s_2\right] m^2}{3\pi^2 s\sigma^2} \\
&- e^2\, C_0(s,s_1,s_2,m^2)\left[\frac{m^2\gamma\left[(s-s_1)^3 - s_2^3 + (3s+s_1)\, s_2^2 + \left(-3s^2 - 10 s_1 s + s_1^2\right) s_2\right]}{6\pi^2 s\sigma^2} - \frac{2m^4\gamma}{3\pi^2 s\sigma}\right], \qquad \text{(A.94)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F_2(s;\, s_1,\, s_2,\, m^2)} &= -\frac{2e^2 m^2}{3\pi^2 s\sigma} - \frac{2\,e^2\,\mathcal{D}_2(s,s_2,m^2)\,\left[(s-s_1)^2 - 2s_2^2 + (s+s_1)\, s_2\right] m^2}{3\pi^2\, s\,\sigma^2} \\
&- \frac{2e^2\,\mathcal{D}_1(s,s_1,m^2)\, m^2}{3\pi^2 s\sigma^2}\left[s^2 + (s_1 - 2s_2)\, s - 2s_1^2 + s_2^2 + s_1 s_2\right] \\
&- e^2\,\mathcal{C}_0(s,s_1,s_2,m^2)\left[\frac{4m^4}{3\pi^2 s\sigma} + \frac{m^2}{3\pi^2 s\sigma^2}\left[s^3 - (s_1+s_2)\, s^2 - \left(s_1^2 - 6 s_2 s_1 + s_2^2\right) s \right.\right. \\
&\left.\left. + (s_1 - s_2)^2\,(s_1+s_2)\right]\right], \qquad \text{(A.95)}
\end{aligned}
$$

[1] We use boldfaced notation to facilitate their identification in the lengthier expressions

$$
\begin{aligned}
\underline{\underline{\mathbf{F_3(s;\, s_1,\, s_2,\, m^2)}}} \;&=\; \underline{\underline{\mathbf{F_5(s;\, s_2,\, s_1,\, m^2)}}} = -\frac{e^2}{144\pi^2 s\sigma^3}\Big[s^6 - 3\,(s_1-4s_2)\,s^5 + 6\,(3s_1-7s_2)\,s_2 s^4 \\
&+2\left(5s_1^3-69s_2s_1^2+117s_2^2s_1+23s_2^3\right)s^3 - 3\left(5s_1^4-62s_2s_1^3+72s_2^2s_1^2+50s_2^3s_1+7s_2^4\right)s^2 \\
&+3\,(s_1-s_2)^2\left(3s_1^3-24s_2s_1^2-33s_2^2s_1+2s_2^3\right)s - 2\,(s_1-s_2)^6\Big] \\
&-\frac{e^2\gamma m^2}{6\pi^2 s\sigma^2}\Big[s^2-2\,(s_1-3s_2)\,s+(s_1-s_2)^2\Big] \\
&-\frac{e^2\,\gamma}{12\pi^2 s\sigma^2}\Big[s^2+(5s_2-2s_1)\,s+(s_1-s_2)^2\Big]\left[\mathcal{B}_0(s,m^2)m^2-\mathcal{A}_0(m^2)\right] \\
&-\frac{e^2\,m^2}{12\pi^2\,s\,\sigma^3}\,\mathcal{D}_1(s,s_1,m^2)\Big[(2s+s_1)\,(s-s_1)^4 - 12\,(s+s_1)\,s_2^2\,(s-s_1)^2 \\
&\qquad +s_1\,(41s+2s_1)\,s_2\,(s-s_1)^2 - (6s+5s_1)\,s_2^4 + \left(16s^2-41s_1s+14s_1^2\right)s_2^3\Big] \\
-\frac{e^2\,s_1}{48\pi^2\,\sigma^4}\,\mathcal{D}_1(s,s_1,m^2)&\Big[\,(s-s_1)^6 + 2\,(14s+11s_1)\,s_2\,(s-s_1)^4 \\
&\qquad -\left(23s^2-214s_1s+19s_1^2\right)s_2^2\,(s-s_1)^2 + 2 - 21s_2^6 + (5s_1-2s)\,s_2^5 \\
&\qquad +\left(107s^2-318s_1s+71s_1^2\right)s_2^4 + 8\left(-11s^3+18s_1s^2+17s_1^2s-8s_1^3\right)s_2^3\Big] \\
-\frac{e^2\,s_2\,m^2}{12\pi^2\,s\,\sigma^3}\,\mathcal{D}_2(s,s_2,m^2)&\Big[s_2^4+(19s+2s_1)\,s_2^3 - 2\left(12s^2-23s_1s+6s_1^2\right)s_2^2 \\
&-(s-s_1)\left(13s^2-49s_1s+14s_1^2\right)s_2 + (s-s_1)^3\,(17s+5s_1)\Big] \\
-\frac{e^2\,s_2}{48\pi^2\,\sigma^4}\,\mathcal{D}_2(s,s_2,m^2)&\Big[s_2^6 - 2\,(s-14s_1)\,s_2^5 + \left(s^2+120s_1s-37s_1^2\right)s_2^4 \\
&-4\left(s^3+49s_1s^2-69s_1^2s+13s_1^3\right)s_2^3 + (s-s_1)\left(11s^3-69s_1s^2+309s_1^2s-83s_1^3\right)s_2^2 \\
&-2\,(s-s_1)^3\left(5s^2-49s_1s-4s_1^2\right)s_2 + 3\,(s-s_1)^5\,(s+5s_1)\Big] \\
&-\,e^2\,\mathcal{C}_0(s,s_1,s_2,m^2)\left[\frac{\gamma\,m^4}{3\pi^2\,s\,\sigma^2}\left[s^2+(7s_2-2s_1)\,s+(s_1-s_2)^2\right]\right. \\
&+\frac{m^2}{24\pi^2\,s\,\sigma^3}\Big[-s_2^6+(2s_1-9s)\,s_2^5+\left(12s^2-65s_1s+s_1^2\right)s_2^4 \\
&+2\left(13s^3-54s_1s^2+55s_1^2s-2s_1^3\right)s_2^3 - (s-s_1)\left(45s^3-133s_1s^2+15s_1^2s+s_1^3\right)s_2^2 \\
&+(s-s_1)^3\left(15s^2+47s_1s-2s_1^2\right)s_2 + (s-s_1)^5\,(2s+s_1)\Big] \\
&+\frac{s_1s_2}{8\pi^2\sigma^4}\Big[2s^6+3\,(s_2-3s_1)\,s^5+\left(15s_1^2+6s_2s_1-13s_2^2\right)s^4 \\
&\qquad +2\left(-5s_1^3-19s_2s_1^2+29s_2^2s_1+s_2^3\right)s^3 + 12s_2\left(4s_1^3-4s_2s_1^2-3s_2^2s_1+s_2^3\right)s^2 \\
&\qquad \left.+\,(s_1-s_2)^2\left(3s_1^3-15s_2s_1^2-31s_2^2s_1-5s_2^3\right)s - (s_1-s_2)^4\,(s_1+s_2)^2\Big]\right],
\end{aligned}
\tag{A.96}
$$

$$\underline{\underline{\mathbf{F_4(s;\,s_1,\,s_2,\,m^2)}}} \;=\; \underline{\underline{\mathbf{F_6(s;\,s_2,\,s_1,\,m^2)}}} =$$

$$\frac{e^2m^2}{6\pi^2 s\sigma^2 s_1}\Big[3s^3-2\left(2s_1+3s_2\right)s^2+\left(-s_1^2+6s_2s_1+3s_2^2\right)s+2s_1\left(s_1-s_2\right)^2\Big]$$
$$+\frac{e^2}{12\pi^2 s\sigma^2 s_1}\left[\mathcal{B}_0(s,m^2)\,m^2-\mathcal{A}_0(m^2)\right]\Big[3s^3-2\left(2s_1+3s_2\right)s^2+$$
$$\left(-s_1^2+4s_2s_1+3s_2^2\right)s+2s_1\left(s_1-s_2\right)^2\Big]$$
$$+\frac{e^2}{24\pi^2\sigma^3 s_1}\Big[-s_2^5+\left(6s+11s_1\right)s_2^4-\left(14s^2+s_1s+5s_1^2\right)s_2^3+\left(16s^3-35s_1s^2+46s_1^2s-15s_1^3\right)s_2^2$$
$$-\left(s-s_1\right)^2\left(9s^2-11s_1s-6s_1^2\right)s_2+2\left(s-s_1\right)^4\left(s+2s_1\right)\Big]$$
$$-e^2\mathcal{D}_2(s,s_2,m^2)\Bigg[\frac{m^2}{12\pi^2 s\sigma^3 s_1}\Big(-2\left(2s+s_1\right)\left(s-s_1\right)^4+\left(3s^2-43s_1s+2s_1^2\right)s_2\left(s-s_1\right)^2$$
$$+\left(9s+4s_1\right)s_2^4+\left(-23s^2+29s_1s-10s_1^2\right)s_2^3+\left(15s^3+2s_1s^2+5s_1^2s+6s_1^3\right)s_2^2\Big)$$
$$-\frac{1}{48\pi^2\sigma^4 s_1}\Big(3\left(s+s_1\right)\left(s-s_1\right)^6-4\left(4s^2-14s_1s-5s_1^2\right)s_2\left(s-s_1\right)^4$$
$$+\left(35s^3-119s_1s^2+169s_1^2s-13s_1^3\right)s_2^2\left(s-s_1\right)^2+\left(s-3s_1\right)s_2^6-8\left(s^2+9s_1s+7s_1^2\right)s_2^5$$
$$+\left(25s^3+159s_1s^2-197s_1^2s+157s_1^3\right)s_2^4+4\left(-10s^4+21s_1^2s^2+28s_1^3s-27s_1^4\right)s_2^3\Big)\Bigg]$$
$$+e^2\mathcal{D}_1(s,s_1,m^2)\Bigg[\frac{m^2}{12\pi^2 s\sigma^3 s_1}\Big(2s^5+\left(15s_1-8s_2\right)s^4+\left(-53s_1^2-5s_2s_1+12s_2^2\right)s^3$$
$$+\left(49s_1^3+46s_2s_1^2-33s_2^2s_1-8s_2^3\right)s^2-\left(s_1-s_2\right)\left(9s_1^3+52s_2s_1^2+23s_2^2s_1+2s_2^3\right)s$$
$$-2s_1\left(s_1-s_2\right)^3\left(2s_1+s_2\right)\Big)+\frac{1}{48\pi^2\sigma^4}\Big(s_2^6+4\left(6s+11s_1\right)s_2^5+\left(-87s^2+106s_1s-91s_1^2\right)s_2^4$$
$$+4\left(22s^3-69s_1s^2+40s_1^2s+s_1^3\right)s_2^3+\left(s-s_1\right)\left(3s^3-29s_1s^2+209s_1^2s-79s_1^3\right)s_2^2$$
$$-8\left(s-s_1\right)^3\left(6s^2-13s_1s-4s_1^2\right)s_2+\left(s-s_1\right)^5\left(19s+5s_1\right)\Big)\Bigg]$$
$$+e^2\mathcal{C}_0(s,s_1,s_2,m^2)\Bigg\{\frac{m^4}{6\pi^2\sigma^2}\left[\frac{\sigma\left(3s+2s_1\right)}{s\,s_1}+18s_2\right]-\frac{m^2}{24\pi^2\sigma^3}\Big[-\frac{\sigma^2}{ss_1}\left(9s^2+\left(59s_1+3s_2\right)s\right.$$
$$\left.+2s_1\left(s_1+s_2\right)\right)+12\left(3s^2-3\left(22s_1+7s_2\right)s+s_1\left(3s_1-17s_2\right)\right)\sigma$$
$$+720\,s\,s_1\left(\left(s-s_1\right)^2-2\left(s+s_1\right)s_2\right)\Big]-\frac{1}{16\pi^2\sigma^4}\Big[-2s\left(s-s_1\right)^6$$
$$-2\left(s^2+7s_1s+2s_1^2\right)s_2\left(s-s_1\right)^4+2\left(7s^3+6s_1s^2+11s_1^2s-4s_1^3\right)s_2^4$$
$$+12\left(2s^3-3s_1s^2-2s_1^2s+s_1^3\right)s_2^2\left(s-s_1\right)^2-4\left(s+s_1\right)s_2^6+6\left(s^2-5s_1s+2s_1^2\right)s_2^5$$
$$-4\left(9s^4-25s_1s^3+33s_1^2s^2-15s_1^3s+2s_1^4\right)s_2^3\Big]\Bigg\},\tag{A.97}$$

$$
\begin{aligned}
&\underline{\underline{\mathbf{F_7(s;\, s_1,\, s_2,\, m^2)}}} \;=\; \frac{e^2m^2}{3\pi^2 s\sigma^2}\Big[\left(s^2+12s_2s-s_2^2\right)s_1+s_1^3-(2s+s_2)\,s_1^2+(s-s_2)^{\,2}s_2\Big]\\
&+\frac{e^2}{72\pi^2}\Bigg[\frac{840s\left(2\,(s+s_1)\,s_2-(s-s_1)^{\,2}\right)s_1^2}{\sigma^3}+\frac{6\left(-13s^2+166s_1s-13s_1^2+39\,(s+s_1)\,s_2\right)s_1}{\sigma^2}\\
&\qquad+\frac{3\,(-s+27s_1+s_2)}{\sigma}+\frac{2}{s}+\frac{9s}{\gamma^2}-\frac{6}{\gamma}\Bigg]\\
&+\frac{e^2}{6\pi^2}\left[\mathcal{B}_0(s,m^2)\,m^2-\mathcal{A}_0(m^2)\right]\left[\frac{14s_1s_2}{\sigma^2}+\frac{s+s_1+s_2}{s\sigma}-\frac{3}{\gamma^2}\right]\\
&-\frac{e^2}{16\pi^2}\,\mathcal{C}_0(s,s_1,s_2,m^2)\Bigg\{m^4\left[\frac{16\gamma}{3s\sigma}-\frac{96s_1s_2}{\sigma^2}-\frac{16}{\gamma^2}\right]+m^2\Bigg[\frac{960s\left((s-s_1)^{\,2}-2\,(s+s_1)\,s_2\right)s_1^2}{\sigma^3}-\frac{4}{3s}\\
&\qquad+\frac{4}{\gamma}+\frac{16\left(7s_1^2-(74s+21s_2)\,s_1+s\,(7s-19s_2)\right)s_1}{\sigma^2}-\frac{4\,(3s\,(2s+s_2)+s_1\,(87s+4s_2))}{3s\sigma}\Bigg]\\
&\qquad-\frac{4s_1s_2}{\gamma^2\sigma^4}\Big[\left(-9s^2+22s_2s-4s_2^2\right)s_1^6+2s\left(20s^2-39s_2s+21s_2^2\right)s_1^5+(s-s_2)^{\,6}s_2\,(4s+s_2)\\
&\qquad+2s\,(s-s_2)^{\,4}\left(2s^2+5s_2s+11s_2^2\right)s_1+\left(-65s^4+96s_2s^3+33s_2^2s^2-62s_2^3s+6s_2^4\right)s_1^4\\
&\qquad+2s\left(27s^4-22s_2s^3-108s_2^2s^2+102s_2^3s-31s_2^4\right)s_1^3-(s-s_2)^{\,2}\big(23s^4+40s_2s^3-105s_2^2s^2\\
&\qquad-34s_2^3s+4s_2^4\big)s_1^2+s_1^8-2ss_1^7\Big]\Bigg\}\\
&-\frac{e^2}{16\pi^2}\,\mathcal{D}_1(s,s_1,m^2)\Bigg[\frac{2}{3}m^2\Bigg(\frac{3}{\gamma^2}(1-\frac{3s_2}{s-s_1})+\frac{20s_1^2-37ss_1+s\,(9s_2-19s)}{\sigma s\,(s-s_1)}\\
&\qquad+\frac{8s_1\left(3s_1^2-(61s+3s_2)\,s_1+s\,(3s-19s_2)\right)}{\sigma^2 s}+\frac{440s_1^2\left((s-s_1)^{\,2}-(3s+s_1)\,s_2\right)}{\sigma^3}\Bigg)\\
&\qquad+\frac{2s_1}{3\gamma^2\sigma^4}\Big(-\left(317s^2+227s_1s+64s_1^2\right)s_2^6+(s_1-s)^{\,5}\left(-7s^2+39s_1s+32s_1^2\right)s_2\\
&\qquad+\left(397s^3+846s_1s^2-539s_1^2s+312s_1^3\right)s_2^5-(s-s_1)^{\,3}\left(23s^3+114s_1s^2+463s_1^2s-16s_1^3\right)s_2^2\\
&\qquad-\left(275s^4+s_1\left(1181s^3+s_1\left(3s_1\,(93s+94s_1)-1441s^2\right)\right)\right)s_2^4+(s-s_1)\big(103s^4+767s_1s^3\\
&\qquad-79s_1^2s^2-563s_1^3s-36s_1^4\big)s_2^3-23s_2^8+(133s+4s_1)\,s_2^7+(s_1-s)^{\,7}\,(2s+s_1)\Big)\Bigg]\\
&-e^2\mathcal{D}_2(s,s_2,m^2)\Bigg[\frac{m^2}{6\pi^2 s\gamma^2\sigma^3}\Big(\left(-6s^2+49s_1s-7s_1^2\right)s_2^5+(s-s_1)^{\,4}\left(16s^2-5s_1\,(s+s_1)\right)s_2\\
&\qquad+s_2^7-(s-4s_1)\,s_2^6-4s\,(s-s_1)^{\,6}+2\left(5s^3-87s_1s^2+56s_1^2s-4s_1^3\right)s_2^4\\
&\quad-(s-s_1)^{\,2}\big(21s^3+40s_1s^2+147s_1^2s-4s_1^3\big)s_2^2+\left(5s^4+164s_1s^3-68s_1^2s^2-16s_1^3s+11s_1^4\right)s_2^3\Big)\\
&\qquad+\frac{s_2}{24\pi^2\gamma^2\sigma^4}\Big(-23s_1^8+(133s+4s_2)\,s_1^7-\left(317s^2+227s_2s+64s_2^2\right)s_1^6\\
&\qquad+(s-s_2)^{\,5}\left(7s^2-39s_2s-32s_2^2\right)s_1-(s-s_2)^{\,7}\,(2s+s_2)\\
&\qquad+\left(397s^3+846s_2s^2-539s_2^2s+312s_2^3\right)s_1^5-(s-s_2)^{\,3}\left(23s^3+114s_2s^2+463s_2^2s-16s_2^3\right)s_1^2\\
&\qquad-\left(275s^4+s_2\left(1181s^3+s_2\left(3s_2\,(93s+94s_2)-1441s^2\right)\right)\right)s_1^4+(s-s_2)\big(103s^4+767s_2s^3\\
&\qquad-79s_2^2s^2-563s_2^3s-36s_2^4\big)s_1^3\Big)\Bigg],
\end{aligned}
\tag{A.98}
$$

$$
\begin{aligned}
&\underline{\underline{\mathbf{F_8(s;\, s_1,\, s_2,\, m^2)}}} \;=\; -\frac{e^2 m^2}{6\pi^2 s\sigma^2}\Big[3s^2-2\left(s_1+s_2\right)s-\left(s_1-s_2\right)^2\Big]\\
&-\frac{e^2}{3\pi^2 s\,\gamma\sigma^2}\Big(\mathcal{B}_0(s,m^2)\,m^2-\mathcal{A}_0(m^2)\Big)\Big[4s^3-7\left(s_1+s_2\right)s^2+2\left(s_1^2+s_2s_1+s_2^2\right)s\\
&\quad+\left(s_1-s_2\right)^2\left(s_1+s_2\right)\Big]-\frac{e^2}{12\pi^2\gamma\sigma^3}\Big[4s_2^5+\left(14s_1-11s\right)s_2^4+2\left(s-s_1\right)\left(2s+9s_1\right)s_2^3\\
&\quad+2\left(7s^3-43s_1s^2+33s_1^2s-9s_1^3\right)s_2^2-2\left(s-s_1\right)^2\left(8s^2-21s_1s-7s_1^2\right)s_2\\
&\quad+\left(s-s_1\right)^4\left(5s+4s_1\right)\Big]-e^2\mathcal{D}_2(s,s_2,m^2)\left[\frac{m^2}{3\pi^2 s\gamma\sigma^3}\Big(-2s_2^5+\left(3s_1-10s\right)s_2^4\right.\\
&\quad+\left(39s^2-33s_1s+2s_1^2\right)s_2^3+7s\left(s-s_1\right)^2\left(s+5s_1\right)s_2\\
&\quad+\left(-37s^3+20s_1s^2+9s_1^2s-4s_1^3\right)s_2^2+\left(s-s_1\right)^4\left(3s+s_1\right)\Big)+\frac{1}{24\pi^2\gamma\sigma^4}\Big(-5s_2^7\\
&\quad+3\left(s-13s_1\right)s_2^6+\left(57s^2-128s_1s+43s_1^2\right)s_2^5+\left(-155s^3+567s_1s^2-341s_1^2s+121s_1^3\right)s_2^4\\
&\quad+3\left(55s^4-176s_1s^3+86s_1^2s^2+56s_1^3s-53s_1^4\right)s_2^3+3\left(s-s_1\right)^6\left(s+s_1\right)\\
&\quad\left.-\left(s-s_1\right)^2\left(75s^3+103s_1s^2-311s_1^2s-11s_1^3\right)s_2^2+\left(s-s_1\right)^4\left(7s^2+124s_1s+25s_1^2\right)s_2\Big)\right]\\
&-e^2\mathcal{D}_1(s,s_1,m^2)\left[\frac{m^2}{3\pi^2 s\gamma\sigma^3}\Big(s_2^5-ss_2^4+\left(-6s^2+35s_1s-4s_1^2\right)s_2^3+\left(14s^3-63s_1s^2+9s_1^2s+2s_1^3\right)s_2^2\right.\\
&\quad+\left(-11s^4+21s_1s^3+20s_1^2s^2-33s_1^3s+3s_1^4\right)s_2+\left(s-s_1\right)^3\left(3s^2+16s_1s+2s_1^2\right)\Big)\\
&\quad+\frac{1}{24\pi^2\gamma\sigma^4}\Big(3s_2^7+5\left(5s_1-3s\right)s_2^6+\left(27s^2+24s_1s+11s_1^2\right)s_2^5-\left(15s^3+339s_1s^2\right.\\
&\quad\left.-289s_1^2s+159s_1^3\right)s_2^4+\left(-15s^4+616s_1s^3-714s_1^2s^2+168s_1^3s+121s_1^4\right)s_2^3\\
&\quad+\left(s-s_1\right)\left(27s^4-402s_1s^3+40s_1^2s^2+298s_1^3s-43s_1^4\right)s_2^2-\left(s-s_1\right)^3\left(15s^3-51s_1s^2\right.\\
&\quad\left.\left.-245s_1^2s-39s_1^3\right)s_2+\left(s-s_1\right)^5\left(3s^2+22s_1s+5s_1^2\right)\Big)\right]\\
&-e^2\mathcal{C}_0(s,s_1,s_2,m^2)\left[\frac{2m^4}{3\pi^2 s\gamma\sigma^2}\Big(2s^3-3\left(s_1+s_2\right)s^2+10s_1s_2s+\left(s_1-s_2\right)^2\left(s_1+s_2\right)\Big)\right.\\
&\quad+\frac{m^2}{6\pi^2 s\sigma^3}\Big(11s^5-18\left(s_1+s_2\right)s^4+\left(-11s_1^2+94s_2s_1-11s_2^2\right)s^3\\
&\quad+\left(s_1+s_2\right)\left(31s_1^2-90s_2s_1+31s_2^2\right)s^2-4\left(s_1-s_2\right)^2\left(3s_1^2+11s_2s_1+3s_2^2\right)s\\
&\quad-\left(s_1-s_2\right)^4\left(s_1+s_2\right)\Big)+\frac{1}{4\pi^2\gamma\sigma^4}\Big(\left(s_1+s_2\right)s^7-6\left(s_1^2-s_2s_1+s_2^2\right)s^6\\
&\quad+3\left(s_1+s_2\right)\left(5s_1^2-12s_2s_1+5s_2^2\right)s^5+2\left(-10s_1^4+3s_2s_1^3\right.\\
&\quad\left.+54s_2^2s_1^2+3s_2^3s_1-10s_2^4\right)s^4+\left(s_1+s_2\right)\left(15s_1^4+16s_2s_1^3-126s_2^2s_1^2+16s_2^3s_1+15s_2^4\right)s^3\\
&\quad-6\left(s_1^6+5s_2s_1^5-s_2^2s_1^4-18s_2^3s_1^3-s_2^4s_1^2+5s_2^5s_1+s_2^6\right)s^2+\left(s_1-s_2\right)^2\left(s_1+s_2\right)\\
&\quad\left.\left(s_1^4+6s_2s_1^3+34s_2^2s_1^2+6s_2^3s_1+s_2^4\right)s+2s_1\left(s_1-s_2\right)^4s_2\left(s_1+s_2\right)^2\Big)\right],
\end{aligned}
\tag{A.99}
$$

$$\underline{\underline{\mathbf{F_9(s;\,s_1,\,s_2,\,m^2)}}} \;=\; \underline{\underline{\mathbf{F_{10}(s;\,s_2,\,s_1,\,m^2)}}} = -\frac{2e^2m^2}{3\pi^2\sigma s_1} - \frac{e^2}{6\pi^2 s_1}\Big[\mathcal{B}_0(s,m^2)\,m^2 - \mathcal{A}_0(m^2)\Big]\left(\frac{3}{\gamma^2}+\frac{1}{\sigma}\right)$$
$$+\frac{e^2}{12\pi^2\gamma^2\sigma^2}\Big[(s-s_1)^4 - 4\,(4s+s_1)\,s_2\,(s-s_1)^2 - 3s_2^4 + 4\,(s_1-2s)\,s_2^3 + 2\left(13s^2-2s_1s+s_1^2\right)s_2^2\Big]$$
$$-\,e^2\mathcal{C}_0(s,s_1,s_2,m^2)\left[\frac{4s_2m^4}{\pi^2\gamma^2\sigma} + \frac{m^2}{2\pi^2\gamma\sigma^2}\Big((s-s_1)^3 + (7s+s_1)\,s_2\,(s-s_1) - 3s_2^3 + 5\,(s_1-s)\,s_2^2\Big)\right.$$
$$+\frac{8ss_2}{\gamma^2\sigma^3}\Big(s_2^5 + (2s_1-3s)\,s_2^4 + 2\,(s-s_1)\,(s+2s_1)\,s_2^3 + 2\left(s^3-7s_1s^2+3s_1^2s-s_1^3\right)s_2^2$$
$$\left.-\,(s-3s_1)\,(s-s_1)^2\,(3s+s_1)\,s_2 + s\,(s-s_1)^4\Big)\right]$$
$$-\,e^2\mathcal{D}_2(s,s_2,m^2)\left[\frac{2s_2m^2}{3\pi^2\gamma^2\sigma^2}\Big(8\,(s-s_1)^2 - 5s_2^2 - 3\,(s+s_1)\,s_2\Big)\right.$$
$$+\frac{s_2}{12\pi^2\gamma^2\sigma^3}\Big(s_2^5 - (35s+11s_1)\,s_2^4 + 30\left(3s^2+s_1^2\right)s_2^3 + 2\left(-35s^3+17s_1s^2\right.$$
$$\left.\left.+11s_1^2s-17s_1^3\right)s_2^2 + (s-s_1)^2\left(5s^2+26s_1s+17s_1^2\right)s_2 + 3\,(s-s_1)^4\,(3s-s_1)\Big)\right]$$
$$-\,e^2\mathcal{D}_1(s,s_1,m^2)\left[\frac{2m^2}{3\pi^2\gamma^2\sigma^2s_1}\Big(-s_2^4 + 2\,(2s+3s_1)\,s_2^3 + \left(-6s^2-6s_1s+s_1^2\right)s_2^2\right.$$
$$+\,(s-s_1)\left(4s^2-2s_1s+3s_1^2\right)s_2 - (s-3s_1)\,(s-s_1)^3\Big) + \frac{1}{12\pi^2\gamma^2\sigma^3}\Big(-s_2^6 + (18s+11s_1)\,s_2^5$$
$$-3\left(21s^2-3s_1s+10s_1^2\right)s_2^4 + 2\left(46s^3-37s_1s^2+2s_1^2s+17s_1^3\right)s_2^3$$
$$\left.-\left(63s^4-82s_1s^3+2s_1^3s+17s_1^4\right)s_2^2 + 3\,(s-s_1)^3\left(6s^2+7s_1s-s_1^2\right)s_2 - s\,(s-s_1)^5\Big)\right], \tag{A.100}$$

$$\underline{\underline{\mathbf{F_{11}(s;\,s_1,\,s_2,\,m^2)}}} \;=\; \underline{\underline{\mathbf{F_{12}(s;\,s_2,\,s_1,\,m^2)}}} = \frac{2e^2m^2}{3\pi^2\sigma s_2} + \frac{e^2}{6\pi^2\sigma s_2}[\mathcal{B}_0(s,m^2)\,m^2 - \mathcal{A}_0(m^2)]$$
$$+\frac{e^2}{12\pi^2\sigma^2s_2}\Big[2s^3 - (5s_1+2s_2)\,s^2 + \left(4s_1^2+4s_2s_1-2s_2^2\right)s - (s_1-2s_2)\,(s_1-s_2)^2\Big]$$
$$-\,e^2\mathcal{C}_0(s,s_1,s_2,m^2)\;\left[\frac{m^4}{\pi^2\sigma s_2} + \frac{m^2}{4\pi^2\sigma^2s_2}\Big(3s^3 - (5s_1+3s_2)\,s^2 + \left(s_1^2+10s_2s_1-3s_2^2\right)s\right.$$
$$+\,(s_1-s_2)^2\,(s_1+3s_2)\Big) + \frac{s}{4\pi^2\sigma^3}\Big(s^4 + (s_1-4s_2)\,s^3 - (s_1-s_2)\,(5s_1+6s_2)\,s^2$$
$$\left.+\,(s_1+s_2)\left(3s_1^2+3s_2s_1-4s_2^2\right)s + (s_1-s_2)^2s_2\,(3s_1+s_2)\Big)\right]$$
$$-\,e^2\mathcal{D}_2(s,s_2,m^2)\;\left[\frac{m^2}{6\pi^2\sigma^2s_2}\left(-4\,(s-s_1)^2 + 9s_2^2 - 5\,(s+s_1)\,s_2\right) + \frac{1}{24\pi^2\sigma^3}\Big(-17s^4 + (26s_1+48s_2)\,s^3\right.$$
$$\left.-42s_2\,(s_1+s_2)\,s^2 - 2\,(s_1-s_2)\left(5s_1^2+17s_2s_1+4s_2^2\right)s + (s_1-3s_2)\,(s_1-s_2)^3\Big)\right]$$
$$+\,e^2\mathcal{D}_1(s,s_1,m^2)\;\left[\frac{m^2}{6\pi^2\sigma^2s_2}\left(4s^2+5s_1s-8s_2s-9s_1^2+4s_2^2+5s_1s_2\right) - \frac{2}{3\sigma^3s_2}\Big(3s^5\right.$$
$$-\,(10s_1+9s_2)\,s^4 + \;2\left(6s_1^2+26s_2s_1+3s_2^2\right)s^3 - 6\left(s_1^3+4s_2s_1^2+14s_2^2s_1-s_2^3\right)s^2$$
$$\left.+\,(s_1-s_2)\left(s_1^3-19s_2s_1^2-43s_2^2s_1+9s_2^3\right)s + (s_1-3s_2)\,(s_1-s_2)^3s_2\Big)\right], \tag{A.101}$$

$$
\begin{aligned}
\underline{\underline{\mathbf{F_{13,R}(s;\, s_1,\, s_2,\, m^2)}}} \quad &= \quad -\frac{e^2 m^2\,(s_1+s_2)}{12\pi^2 s_1 s_2} + \frac{e^2}{48\pi^2}\left(\frac{s\gamma}{\sigma} + \frac{3s}{\gamma} + 1\right) \\
&-\frac{1}{2}\left[\Pi_R(s_1,m^2) + \Pi_R(s_2,m^2)\right] - \frac{e^2}{12\pi^2}\left[\mathcal{B}_0(s,m^2)\,m^2 - \mathcal{A}_0(m^2)\right]\left(\frac{1}{s_1} + \frac{1}{s_2} + \frac{3}{\gamma}\right) \\
&+ e^2\mathcal{C}_0(s,s_1,s_2,m^2)\left[\frac{m^4}{2\pi^2\gamma} + \frac{m^2 s\gamma}{4\pi^2\sigma} + \frac{s^2 s_1 s_2\left(s^2 - 2\,(s_1+s_2)\,s + s_1^2 + s_2^2\right)}{4\pi^2\gamma\,\sigma^2}\right] \\
&- e^2\mathcal{D}_1(s,s_1,m^2)\left[\frac{1}{24\pi^2}m^2\left(-\frac{5\,(s+s_1-s_2)}{\sigma} - \frac{2}{s_1} - \frac{3}{\gamma}\right)\right. \\
&\quad + \frac{1}{24\pi^2\gamma\sigma^2}\left((s-s_1)\left(5s^3 + s_1^2 s - 4s_1^3\right)s_2 + \left(10s^2 + 5s_1 s + 7s_1^2\right)s_2^3\right. \\
&\quad \left.\left. - (s-s_1)^3\left(s^2 + 2s_1 s - s_1^2\right) + \left(-10s^3 + 3s_1 s^2 - 7s_1^3\right)s_2^2 + s_2^5 - (5s+4s_1)\,s_2^4\right)\right] \\
&-e^2\,\mathcal{D}_2(s,s_2,m^2)\left[\frac{1}{24\pi^2}m^2\left(-\frac{5\,(s-s_1+s_2)}{\sigma} - \frac{2}{s_2} - \frac{3}{\gamma}\right) + \frac{1}{24\pi^2\gamma\sigma^2}\left(\left(4s^3 + s_1 s^2 + 7s_1^3\right)s_2^2\right.\right. \\
&\quad \left.\left. - \left(8s^2 + 5s_1 s + 7s_1^2\right)s_2^3 - (s-s_1)^5 + (s-4s_1)(s+s_1)\,s_2\,(s-s_1)^2 - s_2^5 + (5s+4s_1)\,s_2^4\right)\right],
\end{aligned}
\tag{A.102}
$$

where as previously done the master integrals are collected in Appendix A.2. These expressions have been analyzed in the text in various kinematical limits to show the appearance of anomaly poles and of all the other poles in the off-shell formulation.

Notice that F_{13} contains two vacuum polarization diagrams with different momenta on the external lines and has been renormalized by a subtraction at zero momentum

$$
\Pi_R(s,m^2) \equiv \Pi(s,m^2) - \Pi(0,m^2) = \frac{e^2}{36\,\pi^2}\left[\left(3 + \frac{6\,m^2}{s}\right)a_3\,\log\frac{a_3+1}{a_3-1} - \frac{12\,m^2}{s} - 5\right],
\tag{A.103}
$$

where $\Pi(s,m^2)$ is defined in Eq. (2.41), $a_3 = \sqrt{1-4m^2/s}$ and

$$
\Pi(0,m^2) = -\frac{e^2}{12\,\pi^2}\,\mathcal{B}_0(0,m^2) = -\frac{e^2}{12\,\pi^2}\left[\frac{1}{\bar\epsilon} - \log\left(\frac{m^2}{\mu^2}\right)\right]
\tag{A.104}
$$

with $1/\bar\epsilon$ defined in (A.52).

A.6 The massless invariant amplitudes

We present here the expressions of the invariant amplitudes in the massless limit. We obtain

$$
\underline{\underline{\mathbf{F_1(s;\, s_1,\, s_2,\, 0)}}} \quad = \quad -\frac{e^2}{18\pi^2 s},
\tag{A.105}
$$

$$\underline{\underline{\mathbf{F_2(s;\, s_1,\, s_2,\, 0)}}} \;=\; 0, \tag{A.106}$$

$$\begin{aligned}
\underline{\underline{\mathbf{F_3(s;\, s_1,\, s_2,\, 0)}}} \;&=\; \underline{\underline{\mathbf{F_5(s;\, s_2,\, s_1,\, 0)}}} = -\frac{e^2}{144\pi^2 s\sigma^3}\Big[s^6 - 3\,(s_1 - 4s_2)\,s^5 + 6\,(3s_1 - 7s_2)\,s_2 s^4 \\
&+2\left(5s_1^3 - 69s_2 s_1^2 + 117s_2^2 s_1 + 23s_2^3\right)s^3 - 3\left(5s_1^4 - 62s_2 s_1^3 + 72s_2^2 s_1^2 + 50s_2^3 s_1 + 7s_2^4\right)s^2 \\
&+3\,(s_1 - s_2)^2\left(3s_1^3 - 24s_2 s_1^2 - 33s_2^2 s_1 + 2s_2^3\right)s - 2\,(s_1 - s_2)^6\Big] \\
-\frac{e^2\, s_1}{48\pi^2\,\sigma^4}\,\mathcal{D}_1(s, s_1, 0)&\Big[\,(s - s_1)^6 + 2\,(14s + 11s_1)\,s_2\,(s - s_1)^4 - \left(23s^2 - 214s_1 s + 19s_1^2\right)s_2^2\,(s - s_1)^2 \\
&- 21s_2^6 + 2\,(5s_1 - 2s)\,s_2^5 + \left(107s^2 - 318s_1 s + 71s_1^2\right)s_2^4 + 8\left(-11s^3 + 18s_1 s^2 + 17s_1^2 s - 8s_1^3\right)s_2^3\Big] \\
-\frac{e^2\, s_2}{48\pi^2\sigma^4}\,\mathcal{D}_2(s, s_2, 0)&\Big[s_2^6 - 2\,(s - 14s_1)\,s_2^5 + \left(s^2 + 120s_1 s - 37s_1^2\right)s_2^4 \\
&-4\left(s^3 + 49s_1 s^2 - 69s_1^2 s + 13s_1^3\right)s_2^3 + (s - s_1)\left(11s^3 - 69s_1 s^2 + 309s_1^2 s - 83s_1^3\right)s_2^2 \\
&-2\,(s - s_1)^3\left(5s^2 - 49s_1 s - 4s_1^2\right)s_2 + 3\,(s - s_1)^5\,(s + 5s_1)\Big] \\
-\frac{e^2}{16\pi^2}\mathcal{C}_0(s, s_1, s_2, 0)&\Bigg[\frac{2s_1 s_2}{\sigma^4}\left[2s^6 + 3\,(s_2 - 3s_1)\,s^5 + \left(15s_1^2 + 6s_2 s_1 - 13s_2^2\right)s^4\right. \\
&+2\left(-5s_1^3 - 19s_2 s_1^2 + 29s_2^2 s_1 + s_2^3\right)s^3 + 12s_2\left(4s_1^3 - 4s_2 s_1^2 - 3s_2^2 s_1 + s_2^3\right)s^2 \\
&\left.+\,(s_1 - s_2)^2\left(3s_1^3 - 15s_2 s_1^2 - 31s_2^2 s_1 - 5s_2^3\right)s - (s_1 - s_2)^4\,(s_1 + s_2)^2\right]\Bigg],
\end{aligned} \tag{A.107}$$

$$\begin{aligned}
&\underline{\underline{\mathbf{F_4(s;\, s_1,\, s_2,\, 0)}}} \;=\; \underline{\underline{\mathbf{F_6(s;\, s_2,\, s_1,\, 0)}}} = \\
&\frac{e^2}{24\pi^2\sigma^3 s_1}\Big[- s_2^5 + (6s + 11s_1)\,s_2^4 - \left(14s^2 + s_1 s + 5s_1^2\right)s_2^3 + \left(16s^3 - 35s_1 s^2 + 46s_1^2 s - 15s_1^3\right)s_2^2 \\
&\qquad - (s - s_1)^2\left(9s^2 - 11s_1 s - 6s_1^2\right)s_2 + 2\,(s - s_1)^4\,(s + 2s_1)\Big] \\
&-\frac{e^2}{16\pi^2}\mathcal{D}_2(s, s_2, 0)\Bigg[- \frac{1}{3\sigma^4 s_1}\Bigg(3\,(s + s_1)\,(s - s_1)^6 - 4\left(4s^2 - 14s_1 s - 5s_1^2\right)s_2\,(s - s_1)^4 \\
&\qquad + \left(35s^3 - 119s_1 s^2 + 169s_1^2 s - 13s_1^3\right)s_2^2\,(s - s_1)^2 + (s - 3s_1)\,s_2^6 - 8\left(s^2 + 9s_1 s + 7s_1^2\right)s_2^5 \\
&\qquad + \left(25s^3 + 159s_1 s^2 - 197s_1^2 s + 157s_1^3\right)s_2^4 + 4\left(-10s^4 + 21s_1^2 s^2 + 28s_1^3 s - 27s_1^4\right)s_2^3\Bigg)\Bigg] \\
&-\frac{e^2}{16\pi^2}\mathcal{D}_1(s, s_1, 0)\Bigg[- \frac{1}{3\sigma^4}\Bigg(s_2^6 + 4\,(6s + 11s_1)\,s_2^5 + \left(-87s^2 + 106s_1 s - 91s_1^2\right)s_2^4 \\
&\qquad +4\left(22s^3 - 69s_1 s^2 + 40s_1^2 s + s_1^3\right)s_2^3 + (s - s_1)\left(3s^3 - 29s_1 s^2 + 209s_1^2 s - 79s_1^3\right)s_2^2 \\
&\qquad -8\,(s - s_1)^3\left(6s^2 - 13s_1 s - 4s_1^2\right)s_2 + (s - s_1)^5\,(19s + 5s_1)\Bigg)\Bigg] \\
&-\frac{e^2}{16\pi^2}\mathcal{C}_0(s, s_1, s_2, 0)\Bigg[\frac{1}{\sigma^4}\Bigg(- 2s\,(s - s_1)^6 - 2\left(s^2 + 7s_1 s + 2s_1^2\right)s_2\,(s - s_1)^4 \\
&\qquad +2\left(7s^3 + 6s_1 s^2 + 11s_1^2 s - 4s_1^3\right)s_2^4 + 12\left(2s^3 - 3s_1 s^2 - 2s_1^2 s + s_1^3\right)s_2^2\,(s - s_1)^2 \\
&\qquad -4\,(s + s_1)\,s_2^6 + 6\left(s^2 - 5s_1 s + 2s_1^2\right)s_2^5 - 4\left(9s^4 - 25s_1 s^3 + 33s_1^2 s^2 - 15s_1^3 s + 2s_1^4\right)s_2^3\Bigg)\Bigg],
\end{aligned} \tag{A.108}$$

$$
\begin{aligned}
\underline{\underline{\mathbf{F_7(s;\, s_1,\, s_2,\, 0)}}} \;=\; & \frac{e^2}{72\pi^2}\Bigg[\frac{840 s\, s_1^2}{\sigma^3}\left(2\,(s+s_1)\,s_2-(s-s_1)^2\right) \\
& + \frac{6s_1}{\sigma^2}\left(-13s^2+166 s_1 s-13s_1^2+39\,(s+s_1)\,s_2\right)+\frac{3\,(-s+27s_1+s_2)}{\sigma}+\frac{2}{s}+\frac{9s}{\gamma^2}-\frac{6}{\gamma}\Bigg] \\
& -\frac{e^2}{16\pi^2}\mathcal{C}_0(s,s_1,s_2,0)\Bigg\{-\frac{4s_1s_2}{\gamma^2\sigma^4}\Bigg[\left(-9s^2+22s_2s-4s_2^2\right)s_1^6+2s\left(20s^2-39s_2s+21s_2^2\right)s_1^5 \\
& + (s-s_2)^6 s_2\,(4s+s_2)+2s\,(s-s_2)^4\left(2s^2+5s_2s+11s_2^2\right)s_1 \\
& + \left(-65s^4+96s_2s^3+33s_2^2s^2-62s_2^3s+6s_2^4\right)s_1^4 \\
& + 2s\left(27s^4-22s_2s^3-108s_2^2s^2+102s_2^3s-31s_2^4\right)s_1^3 \\
& - (s-s_2)^2\left(23s^4+40s_2s^3-105s_2^2s^2-34s_2^3s+4s_2^4\right)s_1^2+s_1^8-2ss_1^7\Bigg]\Bigg\} \\
& -\frac{e^2}{16\pi^2}\mathcal{D}_1(s,s_1,0)\Bigg\{\frac{2s_1}{3\gamma^2\sigma^4}\Bigg[-\left(317s^2+227s_1s+64s_1^2\right)s_2^6+(s_1-s)^5\left(-7s^2+39s_1s+32s_1^2\right)s_2 \\
& + \left(397s^3+846s_1s^2-539s_1^2s+312s_1^3\right)s_2^5-(s-s_1)^3\left(23s^3+114s_1s^2+463s_1^2s-16s_1^3\right)s_2^2 \\
& - \left(275s^4+s_1\left(1181s^3+s_1\left(3s_1\,(93s+94s_1)-1441s^2\right)\right)\right)s_2^4+(s-s_1)\left(103s^4+767s_1s^3\right. \\
& \left. -79s_1^2s^2-563s_1^3s-36s_1^4\right)s_2^3-23s_2^8+(133s+4s_1)\,s_2^7+(s_1-s)^7\,(2s+s_1)\Bigg]\Bigg\} \\
& -\frac{e^2}{16\pi^2}\mathcal{D}_2(s,s_2,0)\Bigg\{\frac{2s_2}{3\gamma^2\sigma^4}\Bigg[-23s_1^8+(133s+4s_2)\,s_1^7-\left(317s^2+227s_2s+64s_2^2\right)s_1^6 \\
& + (s-s_2)^5\left(7s^2-39s_2s-32s_2^2\right)s_1-(s-s_2)^7\,(2s+s_2) \\
& + \left(397s^3+846s_2s^2-539s_2^2s+312s_2^3\right)s_1^5-(s-s_2)^3\left(23s^3+114s_2s^2+463s_2^2s-16s_2^3\right)s_1^2 \\
& - \left(275s^4+s_2\left(1181s^3+s_2\left(3s_2\,(93s+94s_2)-1441s^2\right)\right)\right)s_1^4+(s-s_2)\left(103s^4+767s_2s^3\right. \\
& \left. -79s_2^2s^2-563s_2^3s-36s_2^4\right)s_1^3\Bigg]\Bigg\},
\end{aligned}
\tag{A.109}
$$

$$\underline{\underline{\mathbf{F_8(s;\, s_1,\, s_2,\, 0)}}} = -\frac{e^2}{12\pi^2\gamma\sigma^3}\Big[4s_2^5+(14s_1-11s)\,s_2^4+2\,(s-s_1)\,(2s+9s_1)\,s_2^3$$
$$+2\left(7s^3-43s_1s^2+33s_1^2s-9s_1^3\right)s_2^2-2\,(s-s_1)^{\,2}\left(8s^2-21s_1s-7s_1^2\right)s_2$$
$$+(s-s_1)^{\,4}\,(5s+4s_1)\Big]$$
$$-\frac{e^2}{16\pi^2}\mathcal{D}_2(s,s_2,0)\left\{\frac{2}{3\gamma\sigma^4}\Big[-5s_2^7+3\,(s-13s_1)\,s_2^6+\left(57s^2-128s_1s+43s_1^2\right)s_2^5\right.$$
$$+\left(-155s^3+567s_1s^2-341s_1^2s+121s_1^3\right)s_2^4$$
$$+3\left(55s^4-176s_1s^3+86s_1^2s^2+56s_1^3s-53s_1^4\right)s_2^3+3\,(s-s_1)^{\,6}\,(s+s_1)$$
$$\left.-(s-s_1)^{\,2}\left(75s^3+103s_1s^2-311s_1^2s-11s_1^3\right)s_2^2+(s-s_1)^{\,4}\left(7s^2+124s_1s+25s_1^2\right)s_2\Big]\right\}$$
$$-\frac{e^2}{16\pi^2}\mathcal{D}_1(s,s_1,0)\left\{\frac{2}{3\gamma\sigma^4}\Big[3s_2^7+5\,(5s_1-3s)\,s_2^6+\left(27s^2+24s_1s+11s_1^2\right)s_2^5-\left(15s^3+339s_1s^2\right.\right.$$
$$\left.-289s_1^2s+159s_1^3\right)s_2^4+\left(-15s^4+616s_1s^3-714s_1^2s^2+168s_1^3s+121s_1^4\right)s_2^3$$
$$+(s-s_1)\left(27s^4-402s_1s^3+40s_1^2s^2+298s_1^3s-43s_1^4\right)s_2^2$$
$$\left.-(s-s_1)^{\,3}\left(15s^3-51s_1s^2-245s_1^2s-39s_1^3\right)s_2+(s-s_1)^{\,5}\left(3s^2+22s_1s+5s_1^2\right)\Big]\right\}$$
$$-\frac{e^2}{16\pi^2}\mathcal{C}_0(s,s_1,s_2,0)\left\{\frac{4}{\gamma\sigma^4}\Big[(s_1+s_2)\,s^7-6\left(s_1^2-s_2s_1+s_2^2\right)s^6\right.$$
$$+3\,(s_1+s_2)\left(5s_1^2-12s_2s_1+5s_2^2\right)s^5+2\left(-10s_1^4+3s_2s_1^3+54s_2^2s_1^2+3s_2^3s_1-10s_2^4\right)s^4$$
$$+(s_1+s_2)\left(15s_1^4+16s_2s_1^3-126s_2^2s_1^2+16s_2^3s_1+15s_2^4\right)s^3$$
$$-6\left(s_1^6+5s_2s_1^5-s_2^2s_1^4-18s_2^3s_1^3-s_2^4s_1^2+5s_2^5s_1+s_2^6\right)s^2$$
$$+(s_1-s_2)^{\,2}\,(s_1+s_2)\left(s_1^4+6s_2s_1^3+34s_2^2s_1^2+6s_2^3s_1+s_2^4\right)s$$
$$\left.+2s_1\,(s_1-s_2)^{\,4}s_2\,(s_1+s_2)^{\,2}\Big]\right\}, \tag{A.110}$$

$$\underline{\underline{\mathbf{F_9(s;\, s_1,\, s_2,\, 0)}}} = \underline{\underline{\mathbf{F_{10}(s;\, s_2,\, s_1,\, 0)}}} = \frac{e^2}{12\pi^2\gamma^2\sigma^2}\Big[(s-s_1)^{\,4}-4\,(4s+s_1)\,s_2\,(s-s_1)^{\,2}$$
$$-3s_2^4+4\,(s_1-2s)\,s_2^3+2\left(13s^2-2s_1s+s_1^2\right)s_2^2\Big]$$
$$-\frac{e^2}{16\pi^2}\mathcal{C}_0(s,s_1,s_2,0)\left[\frac{8ss_2}{\gamma^2\sigma^3}\Big(s_2^5+(2s_1-3s)\,s_2^4+2\,(s-s_1)\,(s+2s_1)\,s_2^3\right.$$
$$\left.+2\left(s^3-7s_1s^2+3s_1^2s-s_1^3\right)s_2^2-(s-3s_1)\,(s-s_1)^{\,2}\,(3s+s_1)\,s_2+s\,(s-s_1)^{\,4}\Big)\right]$$
$$-\frac{e^2}{16\pi^2}\mathcal{D}_2(s,s_2,0)\left[\frac{4s_2}{3\gamma^2\sigma^3}\Big(s_2^5-(35s+11s_1)\,s_2^4+30\left(3s^2+s_1^2\right)s_2^3+2\left(-35s^3+17s_1s^2\right.\right.$$
$$\left.\left.+11s_1^2s-17s_1^3\right)s_2^2+(s-s_1)^{\,2}\left(5s^2+26s_1s+17s_1^2\right)s_2+3\,(s-s_1)^{\,4}\,(3s-s_1)\Big)\right]$$
$$-\frac{e^2}{16\pi^2}\mathcal{D}_1(s,s_1,0)\left[\frac{4}{3\gamma^2\sigma^3}\Big(-s_2^6+(18s+11s_1)\,s_2^5-3\left(21s^2-3s_1s+10s_1^2\right)s_2^4\right.$$
$$+2\left(46s^3-37s_1s^2+2s_1^2s+17s_1^3\right)s_2^3-\left(63s^4-82s_1s^3+2s_1^3s+17s_1^4\right)s_2^2$$
$$\left.+3\,(s-s_1)^{\,3}\left(6s^2+7s_1s-s_1^2\right)s_2-s\,(s-s_1)^{\,5}\Big)\right], \tag{A.111}$$

$$
\begin{aligned}
\underline{\underline{\mathbf{F_{11}(s;\, s_1,\, s_2,\, 0)}}} \;&=\; \underline{\underline{\mathbf{F_{12}(s;\, s_2,\, s_1,\, 0)}}} = \\
&\frac{e^2}{12\pi^2\sigma^2 s_2}\Big[2s^3-(5s_1+2s_2)\,s^2+\left(4s_1^2+4s_2s_1-2s_2^2\right)\,s-(s_1-2s_2)\,(s_1-s_2)^{\,2}\Big] \\
&-\frac{e^2}{16\pi^2}\,\mathcal{C}_0(s,s_1,s_2,0)\,\Big[-\frac{4s}{\sigma^3}\Big(s^4+(s_1-4s_2)\,s^3-(s_1-s_2)\,(5s_1+6s_2)\,s^2 \\
&\qquad+(s_1+s_2)\left(3s_1^2+3s_2s_1-4s_2^2\right)s+(s_1-s_2)^{\,2}s_2\,(3s_1+s_2)\Big)\Big] \\
&-\frac{e^2}{16\pi^2}\,\mathcal{D}_2(s,s_2,0)\,\Big[\frac{2}{3\sigma^3}\Big(-17s^4+(26s_1+48s_2)\,s^3-42s_2\,(s_1+s_2)\,s^2 \\
&\qquad-2\,(s_1-s_2)\left(5s_1^2+17s_2s_1+4s_2^2\right)s+(s_1-3s_2)\,(s_1-s_2)^{\,3}\Big)\Big] \\
&-\frac{e^2}{16\pi^2}\,\mathcal{D}_1(s,s_1,0)\,\Big[-\frac{2}{3\sigma^3 s_2}\Big(3s^5-(10s_1+9s_2)\,s^4 \\
&\qquad+\,2\left(6s_1^2+26s_2s_1+3s_2^2\right)s^3-6\left(s_1^3+4s_2s_1^2+14s_2^2s_1-s_2^3\right)s^2 \\
&\qquad+(s_1-s_2)\left(s_1^3-19s_2s_1^2-43s_2^2s_1+9s_2^3\right)s+(s_1-3s_2)\,(s_1-s_2)^{\,3}s_2\Big)\Big]\,,
\end{aligned}
\tag{A.112}
$$

$$
\begin{aligned}
\underline{\underline{\mathbf{F_{13,R}(s;\, s_1,\, s_2,\, 0)}}} \;&=\; -\frac{1}{2}\left[\Pi_R(s_1,0)+\Pi_R(s_2,0)\right]+\frac{e^2}{48\pi^2}\left(\frac{s\gamma}{\sigma}+\frac{3s}{\gamma}+1\right) \\
&+\frac{e^2}{16\pi^2}\,\mathcal{C}_0(s,s_1,s_2,0)\left[\frac{4s^2s_1s_2\left(s^2-2\,(s_1+s_2)\,s+s_1^2+s_2^2\right)}{\gamma\sigma^2}\right] \\
&-\frac{e^2}{16\pi^2}\,\mathcal{D}_1(s,s_1,0)\,\Big[\frac{2}{3\gamma\sigma^2}\Big(\,(s-s_1)\left(5s^3+s_1^2s-4s_1^3\right)s_2+\left(10s^2+5s_1s+7s_1^2\right)s_2^3 \\
&\qquad-(s-s_1)^{\,3}\left(s^2+2s_1s-s_1^2\right)+\left(-10s^3+3s_1s^2-7s_1^3\right)s_2^2+s_2^5-(5s+4s_1)\,s_2^4\Big)\Big] \\
&-\frac{e^2}{16\pi^2}\,\mathcal{D}_2(s,s_2,0)\,\Big[\frac{2}{3\gamma\sigma^2}\Big(\,\left(4s^3+s_1s^2+7s_1^3\right)s_2^2-\left(8s^2+5s_1s+7s_1^2\right)s_2^3 \\
&\qquad-(s-s_1)^{\,5}+(s-4s_1)\,(s+s_1)\,s_2\,(s-s_1)^{\,2}-s_2^5+(5s+4s_1)\,s_2^4\Big)\Big];
\end{aligned}
\tag{A.113}
$$

as already noticed above for the case of the massive form factors the last one, i.e. $F_{13,R}(s;\, s_1,\, s_2,\, 0)$, has been affected by the renormalization procedure for which the one-loop transverse photon propagator with a virtual pair of massless fermions is given by

$$
\Pi_R(s,0)=-\frac{e^2}{12\,\pi^2}\,\left[\frac{5}{3}-\log\left(-\frac{s}{\mu^2}\right)\right], \tag{A.114}
$$

where the dependence on the renormalization scale μ remains explicit.

A.7 The asymptotic behavior of the off-shell massless $\langle TJJ \rangle$ correlator

We present here the asymptotic expression of the form factor in the high energy limit. The leading contributions to the expansion in each expression come from the pole singularities (conformal or anomalous) except for F_{13} which has a constant asymptotic term.

$$F_1(s,s_1,s_2,0) = -\frac{e^2}{18\pi^2 s}, \tag{A.115}$$

$$F_2(s,s_1,s_2,0) = 0, \tag{A.116}$$

$$F_3(s,s_1,s_2,0) = -\frac{e^2}{144\pi^2 s} - \frac{e^2}{48\pi^2 s^2}\left[s_1 + 6s_2 + s_1\log\left(\frac{s_1}{s}\right) + 3s_2\log\left(\frac{s_2}{s}\right)\right] + O\left(\frac{1}{s^3}\right),$$

$$\begin{aligned} F_4(s,s_1,s_2,0) &= \frac{e^2}{48\pi^2 s_1 s}\left[3\log\left(\frac{s_2}{s}\right)+4\right] + \frac{e^2}{48\pi^2 s_1 s^2}\Bigg[2\pi^2 s_1 + 16 s_1 + 6s_2 + 19 s_1\log\left(\frac{s_1}{s}\right) \\ &\quad + \log\left(\frac{s_2}{s}\right)\left(9s_1 + 8s_2 + 6s_1\log\left(\frac{s_1}{s}\right)\right)\Bigg] + O\left(\frac{1}{s^3}\right), \end{aligned} \tag{A.117}$$

$$F_7(s,s_1,s_2,0) = \frac{e^2}{36\pi^2 s} + \frac{e^2}{24\pi^2 s^2}\left[3\left(s_1+s_2\right) + 2s_1\log\left(\frac{s_1}{s}\right) + 2s_2\log\left(\frac{s_2}{s}\right)\right] + O\left(\frac{1}{s^3}\right), \tag{A.118}$$

$$F_8(s,s_1,s_2,0) = -\frac{e^2}{24\pi^2 s^2}\left[3\log\left(\frac{s_1}{s}\right) + 3\log\left(\frac{s_2}{s}\right) + 10\right] + O\left(\frac{1}{s^3}\right), \tag{A.119}$$

$$F_9(s,s_1,s_2,0) = \frac{e^2}{12\pi^2 s^2}\left[\log\left(\frac{s_1}{s}\right) + 1\right] + O\left(\frac{1}{s^3}\right), \tag{A.120}$$

$$\begin{aligned} F_{11}(s,s_1,s_2,0) &= \frac{e^2}{24\pi^2 s_2 s}\left[3\log\left(\frac{s_1}{s}\right)+4\right] + \frac{e^2}{24\pi^2 s_2 s^2}\Bigg[6s_1 + 2\pi^2 s_2 + 12 s_2 + 17 s_2\log\left(\frac{s_2}{s}\right) \\ &\quad + \log\left(\frac{s_1}{s}\right)\left(8s_1 + 9s_2 + 6s_2\log\left(\frac{s_2}{s}\right)\right)\Bigg] + O\left(\frac{1}{s^3}\right), \end{aligned} \tag{A.121}$$

$$\begin{aligned} F_{13}(s,s_1,s_2,0) &= -\frac{1}{2}\left[\Pi_R(s_1,0) + \Pi_R(s_2,0)\right] + \frac{e^2}{24\pi^2}\left[\log\left(\frac{s_1}{s}\right) + \log\left(\frac{s_2}{s}\right) + \frac{5}{2}\right] \\ &\quad + \frac{e^2}{12\pi^2 s}\left[s_1 + s_2 + 2s_1\log\left(\frac{s_1}{s}\right) + 2s_2\log\left(\frac{s_2}{s}\right)\right] \\ &\quad + \frac{e^2}{24\pi^2 s^2}\Bigg[2\left(s_1^2 + \left(3+\pi^2\right)s_2 s_1 + s_2^2\right) + s_2\left(13 s_1 + 6 s_2\right)\log\left(\frac{s_2}{s}\right) \\ &\quad + s_1\log\left(\frac{s_1}{s}\right)\left(6s_1 + 13 s_2 + 6s_2\log\left(\frac{s_2}{s}\right)\right)\Bigg] + O\left(\frac{1}{s^3}\right). \end{aligned} \tag{A.122}$$

A.8 The asymptotic behavior of the on-shell massive $\langle TJJ\rangle$ correlator

This appendix contains the asymptotic expansion of the relevant on-shell massive form factors, that is their dominant contributions as $s \to \infty$ with $s > 0$ after taking into account the suitable analytic continuation. They result

$$F_1(s,0,0,m^2) = -\frac{e^2}{18\pi^2 s} + \frac{e^2 m^2}{12\pi^2 s^2}\left[4 - \log^2\left(\frac{m^2}{s}\right) - 2\,i\pi\log\left(\frac{m^2}{s}\right) + \pi^2\right] + O\left(\frac{1}{s^3}\right), \tag{A.123}$$

$$\begin{aligned} F_3(s,0,0,m^2) &= F_5(s,0,0,m^2) = -\frac{e^2}{144\pi^2 s} - \frac{e^2 m^2}{24\pi^2 s^2}\Bigg[-\log^2\left(\frac{m^2}{s}\right) \\ &\quad -(6+2i\pi)\log\left(\frac{m^2}{s}\right) + \pi^2 - 6i\pi - 14\Bigg] + O\left(\frac{1}{s^3}\right), \end{aligned} \tag{A.124}$$

$$F_7(s,0,0,m^2) = -4F_3(s,0,0,m^2), \tag{A.125}$$

$$\begin{aligned} F_{13,R}(s,0,0,m^2) &= \frac{e^2}{144\,\pi^2}\left[12\log\left(\frac{m^2}{s}\right) + 35 + 12\,i\,\pi\right] + \frac{e^2\,m^2}{8\,\pi^2\,s}\Bigg[\log^2\left(\frac{m^2}{s}\right) \\ &\quad +10 - \pi^2 + 2\,i\,\pi + (\,2 + 2i\,\pi\,)\log\left(\frac{m^2}{s}\right)\Bigg] - \frac{e^2 m^4}{4\,\pi^2\,s^2}\Bigg[-\log^2\left(\frac{m^2}{s}\right) \\ &\quad +(2-2i\pi)\log\left(\frac{m^2}{s}\right) + \pi^2 + 2i\pi - 3\Bigg] + O\left(\frac{1}{s^3}\right) \end{aligned} \tag{A.126}$$

A.9 Form factors for the off-shell $\langle TJ_AJ_A\rangle$ correlator

This appendix contains the form factors involved in the decomposition of the $\langle TJ_AJ_A\rangle$ correlator, as in Eq. (3.103), expressed in terms of scalar integrals after the tensorial reduction

$$\begin{aligned} R_1(s,s_1,s_2,m^2) &= \frac{g^2\,m^2}{6\pi^2\,s}\Big[\mathcal{D}_1(s,s_1,m^2) + \mathcal{D}_2(s,s_2,m^2) - 2\mathcal{B}_0(s^2,m^2) - 2 \\ &\quad + (s-4m^2)\mathcal{C}_0(s,s_1,s_2,m^2)\Big] \end{aligned} \tag{A.127}$$

$$\begin{aligned} R_2(s,s_1,s_2,m^2) &= \frac{g^2\,m^2}{4\pi^2\,\sigma}\Big[2(s-s_1-s_2)\mathcal{D}_1(s,s_1,m^2) + 4s_2\mathcal{D}_2(s,s_2,m^2) \\ &\quad + ((s-s_1)^2 - s_2^2)\mathcal{C}_0(s,s_1,s_2,m^2)\Big] \end{aligned} \tag{A.128}$$

$$R_3(s,s_1,s_2,m^2) = \frac{g^2\,m^2}{4\pi^2\,\sigma}\Big[4s_1\mathcal{D}_1(s,s_1,m^2) + 2(s-s_1-s_2)\mathcal{D}_2(s,s_2,m^2) + ((s-s_2)^2 - s_1^2)\mathcal{C}_0(s,s_1,s_2,m^2)\Big] \tag{A.129}$$

$$R_4(s,s_1,s_2,m^2) = R_2(s,s_1,s_2,m^2) \tag{A.130}$$

$$R_5(s,s_1,s_2,m^2) = R_3(s,s_1,s_2,m^2) \tag{A.131}$$

$$R_6(s,s_1,s_2,m^2) = \frac{g^2 m^2}{2\pi^2(s-s_1-s_2)}\Big[2\mathcal{B}_0(s,m^2) - \mathcal{D}_1(s,s_1,m^2) - \mathcal{D}_2(s,s_2,m^2)\Big] \tag{A.132}$$

$$\begin{aligned}R_7(s,s_1,s_2,m^2) = \frac{g^2 m^2}{24\pi^2 s}\Big[&2\mathcal{B}_0(s,m^2) + \frac{2}{\sigma^2}\mathcal{C}_0(s,s_1,s_2,m^2)\left((s-s_1)^2 + s_2^2 + 4ss_2 - 2s_1s_2\right)\times\\
&\left(2m^2\left(s^2 - 2(s_1+s_2)s + (s_1-s_2)^2\right) + s\left(s^2 - 2(s_1+s_2)s + s_1^2 + s_2^2 + 4s_1s_2\right)\right)\\
&+ \frac{\mathcal{D}_1(s,s_1,m^2)}{\sigma^2}\Big(5s^4 - 2(7s_1+s_2)s^3 + 4\left(3s_1^2 + 5s_2s_1 - 3s_2^2\right)s^2\\
&- 2(s_1-s_2)\left(s_1^2 + 12s_2s_1 + 5s_2^2\right)s - (s_1-s_2)^4\Big)\\
&+ \frac{\mathcal{D}_2(s,s_2,m^2)}{\sigma^2}\Big(-2\left(9s^2 + 8s_1s + 3s_1^2\right)s_2^2 - (s-s_1)^4 + 4(7s+s_1)s_2(s-s_1)^2\\
&- s_2^4 + 4(s_1-2s)s_2^3\Big) + \frac{2\left((s-s_1)^2 + s_2^2 + 4ss_2 - 2s_1s_2\right)}{\sigma}\Big]\end{aligned} \tag{A.133}$$

$$\begin{aligned}R_8(s,s_1,s_2,m^2) = \frac{g^2 m^2}{24\pi^2 s}\Big[&2\mathcal{B}_0(s,m^2) + \frac{2\mathcal{C}_0(s,s_1,s_2,m^2)}{\sigma^2}\left(s^2 + 4s_1s + s_1^2 + s_2^2 - 2(s+s_1)s_2\right)\times\\
&\left(2m^2\left(s^2 - 2(s_1+s_2)s + (s_1-s_2)^2\right) + s\left(s^2 - 2(s_1+s_2)s + s_1^2 + s_2^2 + 4s_1s_2\right)\right)\\
&- \frac{\mathcal{D}_1(s,s_1,m^2)}{\sigma^2}\Big(s^4 - 4(7s_1+s_2)s^3 + 2\left(9s_1^2 + 26s_2s_1 + 3s_2^2\right)s^2\\
&+ 4(s_1-s_2)\left(2s_1^2 + 6s_2s_1 + s_2^2\right)s + (s_1-s_2)^4\Big)\\
&+ \frac{\mathcal{D}_2(s,s_2,m^2)}{\sigma^2}\Big(5s^4 - 2(s_1+7s_2)s^3 + 4\left(-3s_1^2 + 5s_2s_1 + 3s_2^2\right)s^2 - (s_1-s_2)^4\\
&+ 2(s_1-s_2)\left(5s_1^2 + 12s_2s_1 + s_2^2\right)s\Big) + \frac{2\left(s^2 + 4s_1s + s_1^2 + s_2^2 - 2(s+s_1)s_2\right)}{\sigma}\Big]\end{aligned} \tag{A.134}$$

$$
\begin{aligned}
R_9(s,s_1,s_2,m^2) = \frac{g^2 m^2}{12\pi^2 s}\Bigg[& 2\mathcal{B}_0(s,m^2)\left[1+\frac{3s}{\gamma}\right] - \frac{2\mathcal{C}_0(s,s_1,s_2,m^2)}{\sigma^2}\left(2s^2-(s_1+s_2)\,s-(s_1-s_2)^2\right)\cdot \\
& \left(2m^2\left(s^2-2(s_1+s_2)\,s+(s_1-s_2)^2\right)+s\left(s^2-2(s_1+s_2)\,s+s_1^2+s_2^2+4s_1s_2\right)\right) \\
+ \frac{\mathcal{D}_1(s,s_1,m^2)}{\gamma\,\sigma^2}\Bigg(& -10s^5+(23s_1+41s_2)\,s^4-2\left(5s_1^2+27s_2s_1+32s_2^2\right)s^3 \\
& -2(s_1+s_2)\left(4s_1^2+5s_2s_1-23s_2^2\right)s^2+2(s_1-s_2)\left(2s_1^3+19s_2s_1^2+8s_2^2s_1+7s_2^3\right)s \\
& +(s_1-s_2)^4(s_1+s_2)\Bigg) \\
+ \frac{\mathcal{D}_2(s,s_2,m^2)}{\gamma\,\sigma^2}\Bigg(& 2\left(-4s^2+17s_1s+s_1^2\right)s_2^3+(s-s_1)^2\left(23s^2-8s_1s-3s_1^2\right)s_2 \\
& -2\left(5s^3+9s_1s^2+11s_1^2s-s_1^3\right)s_2^2+s_2^5+(4s-3s_1)\,s_2^4-(s-s_1)^4(10s-s_1)\Bigg) \\
& + \frac{2\left(-2s^2+(s_1+s_2)\,s+(s_1-s_2)^2\right)}{\sigma}\Bigg], \qquad \text{(A.135)}
\end{aligned}
$$

where $s=k^2=(p+q)^2$, $s_1=p^2$, $s_2=q^2$, $\gamma\equiv s-s_1-s_2$, $\sigma\equiv s^2-2(s_1+s_2)\,s+(s_1-s_2)^2$ and the scalar integrals $\mathcal{B}_0(s^2,m^2)$, $\mathcal{D}_1(s,s_1,m^2)$, $\mathcal{D}_2(s,s_1,m^2)$, $\mathcal{C}_0(s,s_1,s_2,m^2)$ for generic virtualities and masses are defined in Appendix A.2.

A.10 Form factors for the $\Lambda_{VV}^{\alpha\beta}$ amplitude

We write in this appendix the form factors G_1 and G_2 appearing in eq. 3.127 as contributions to the classical trace obtained for the $\langle T J_V J_V\rangle$ correlator

$$
\begin{aligned}
G_1(s,s_1,s_2,m^2) \;=\; & \frac{g^2\gamma m^2}{\pi^2\sigma}+\frac{g^2\,\mathcal{D}_2(s,s_2,m^2)\,s_2m^2}{\pi^2\sigma^2}\left[s^2+4s_1s-2s_2s-5s_1^2+s_2^2+4s_1s_2\right] \\
-\frac{g^2\,\mathcal{D}_1(s,s_1,m^2)\,s_1m^2}{\pi^2\sigma^2} & \left[-(s-s_1)^2+5s_2^2-4(s+s_1)\,s_2\right] \\
-g^2\,\mathcal{C}_0(s,s_1,s_2,m^2) & \left[\frac{m^2\gamma}{2\pi^2\sigma^2}\left[(s-s_1)^3-s_2^3+(3s+s_1)\,s_2^2+\left(-3s^2-10s_1s+s_1^2\right)s_2\right]-\frac{2m^4\gamma}{\pi^2\sigma}\right], \qquad \text{(A.136)}
\end{aligned}
$$

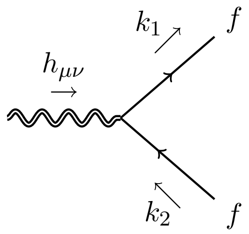
$$
\begin{aligned}
G_2(s,s_1,s_2,m^2) \;=\; & -\frac{2g^2m^2}{\pi^2\sigma}-\frac{2g^2\,\mathcal{D}_2(s,s_2,m^2)m^2}{\pi^2\sigma^2}\left[(s-s_1)^2-2s_2^2+(s+s_1)\,s_2\right] \\
-\frac{2\,g^2\mathcal{D}_1(s,s_1,m^2)\,m^2}{\pi^2\sigma^2} & \left[s^2+(s_1-2s_2)\,s-2s_1^2+s_2^2+s_1s_2\right] \\
-g^2\mathcal{C}_0(s,s_1,s_2,m^2) & \left[\frac{4m^4}{\pi^2\sigma}+\frac{m^2}{\pi^2\sigma^2}\left[s^3-(s_1+s_2)\,s^2-\left(s_1^2-6s_2s_1+s_2^2\right)s\right.\right. \\
& \left.\left.+(s_1-s_2)^2(s_1+s_2)\right]\right], \qquad \text{(A.137)}
\end{aligned}
$$

where $\gamma\equiv s-s_1-s_2$, $\sigma\equiv s^2-2(s_1+s_2)\,s+(s_1-s_2)^2$ and the scalar integrals $\mathcal{D}_1(s,s_1,m^2)$, $\mathcal{D}_2(s,s_1,m^2)$ and $\mathcal{C}_0(s,s_1,s_2,m^2)$ have been already defined in Appendix A.2.

A.11 Feynman rules

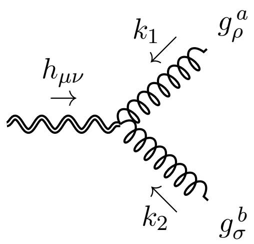
The Feynman rules used throughout the paper are collected here

- Graviton - fermion - fermion vertex



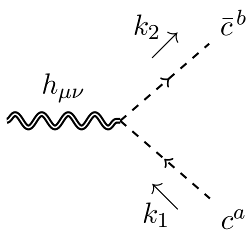
$$\begin{aligned} &= -i\,\frac{\kappa}{2}\,V'_{\mu\nu}(k_1,k_2) \\ &= -i\,\frac{\kappa}{2}\,\left\{\frac{1}{4}\left[\gamma_\mu(k_1+k_2)_\nu+\gamma_\nu(k_1+k_2)_\mu\right]-\frac{1}{2}g_{\mu\nu}[\gamma^\lambda(k_1+k_2)_\lambda-2m]\right\} \end{aligned} \tag{A.138}$$

- Graviton - gluon - gluon vertex



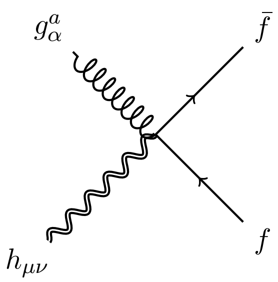
$$\begin{aligned} &= -i\,\frac{\kappa}{2}\,\delta_{ab}\,V^{Ggg}_{\mu\nu\rho\sigma}(k_1,k_2) \\ &= -i\,\frac{\kappa}{2}\,\delta_{ab}\left\{k_1\cdot k_2\,C_{\mu\nu\rho\sigma}+D_{\mu\nu\rho\sigma}(k_1,k_2)+\frac{1}{\xi}\,E_{\mu\nu\rho\sigma}(k_1,k_2)\right\} \end{aligned} \tag{A.139}$$

- Graviton - ghost - ghost vertex



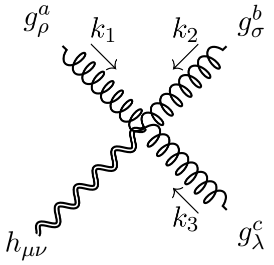
$$= -i\,\frac{\kappa}{2}\,\delta^{ab}\,C_{\mu\nu\rho\sigma}\,k_{1\,\rho}\,k_{2\,\sigma} \tag{A.140}$$

- Graviton - fermion - fermion - gauge boson vertex



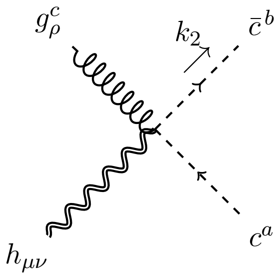
$$= ig\,\frac{\kappa}{2}\,T^a\,W'_{\mu\nu\alpha} = ig\,\frac{\kappa}{2}\,T^a\left\{-\frac{1}{2}(\gamma_\mu\,g_{\nu\alpha}+\gamma_\nu\,g_{\mu\alpha})+g_{\mu\nu}\,\gamma_\alpha\right\} \tag{A.141}$$

- Graviton - gluon - gluon - gluon vertex



$$
\begin{aligned}
&= -g\frac{\kappa}{2} f^{abc} V^{Gggg}_{\mu\nu\rho\sigma\lambda}(k_1,k_2,k_3) \\
&= -g\frac{\kappa}{2} f^{abc} \{C_{\mu\nu\rho\sigma}(k_1-k_2)_\lambda + C_{\mu\nu\rho\lambda}(k_3-k_1)_\sigma \\
&\qquad + C_{\mu\nu\sigma\lambda}(k_2-k_3)_\rho + F_{\mu\nu\rho\sigma\lambda}(k_1,k_2,k_3)\}
\end{aligned}
\tag{A.142}
$$

- Graviton - ghost - ghost - gauge boson vertex



$$
= \frac{\kappa}{2}\, g\, f^{abc}\, C_{\mu\nu\rho\sigma}\, k_2^\sigma \tag{A.143}
$$

$$
C_{\mu\nu\rho\sigma} = g_{\mu\rho}\, g_{\nu\sigma} + g_{\mu\sigma}\, g_{\nu\rho} - g_{\mu\nu}\, g_{\rho\sigma} \tag{A.144}
$$

$$
D_{\mu\nu\rho\sigma}(k_1,k_2) = g_{\mu\nu}\, k_{1\,\sigma}\, k_{2\,\rho} - \Big[g^{\mu\sigma} k_1^\nu k_2^\rho + g_{\mu\rho}\, k_{1\,\sigma}\, k_{2\,\nu} - g_{\rho\sigma}\, k_{1\,\mu}\, k_{2\,\nu} + (\mu \leftrightarrow \nu) \Big] \tag{A.145}
$$

$$
E_{\mu\nu\rho\sigma}(k_1,k_2) = g_{\mu\nu}\,(k_{1\,\rho}\, k_{1\,\sigma} + k_{2\,\rho}\, k_{2\,\sigma} + k_{1\,\rho}\, k_{2\,\sigma}) - \Big[g_{\nu\sigma}\, k_{1\,\mu}\, k_{1\,\rho} + g_{\nu\rho}\, k_{2\,\mu}\, k_{2\,\sigma} + (\mu \leftrightarrow \nu) \Big], \tag{A.146}
$$

$$
F_{\mu\nu\rho\sigma\lambda}(k_1,k_2,k_3) = g_{\mu\rho}\, g_{\sigma\lambda}\,(k_2-k_3)_\nu + g_{\mu\sigma}\, g_{\rho\lambda}\,(k_3-k_1)_\nu + g_{\mu\lambda}\, g_{\rho\sigma}(k_1-k_2)_\nu + (\mu \leftrightarrow \nu) \tag{A.147}
$$

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