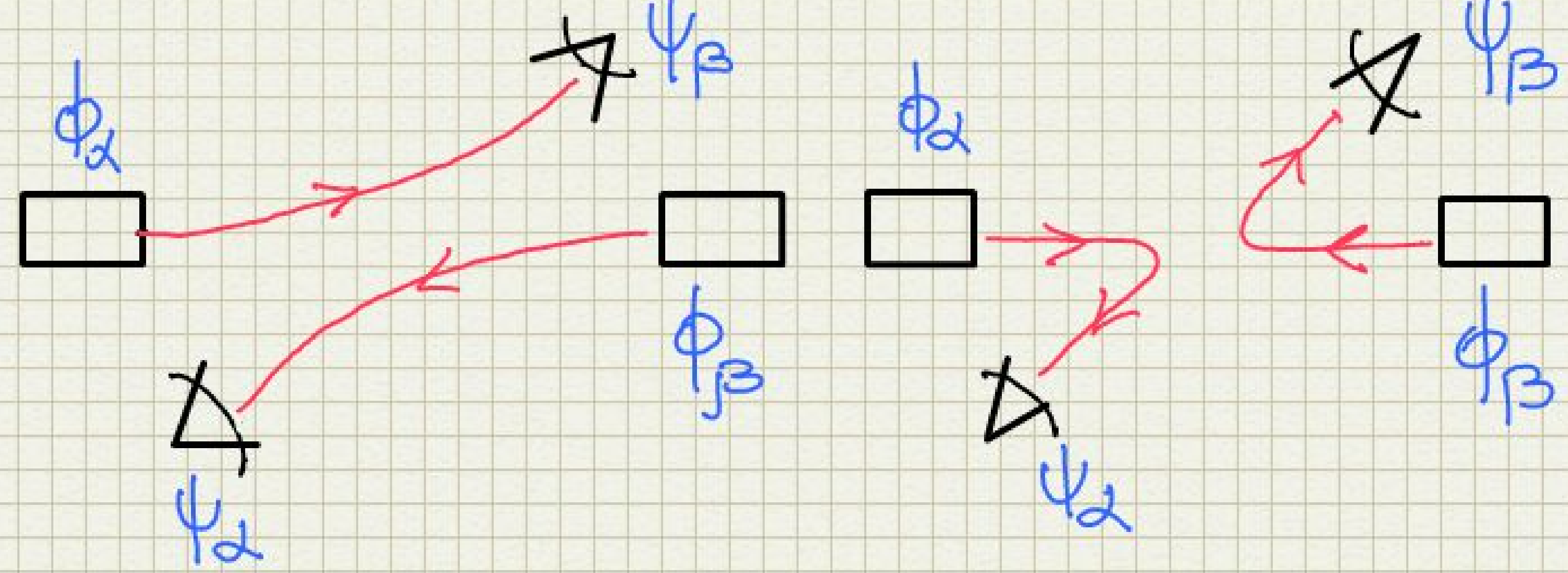
Particelle identiche



$$\langle \phi_\alpha(1)\,\phi_\beta(2)\,|\,\psi_\beta(1)\,\psi_\alpha(2)\rangle \qquad \langle \phi_\alpha(1)\,\phi_\alpha(2)\,|\,\psi_\alpha(1)\,\psi_\beta(2)\rangle$$

Operatori di permutazione

$$P(2,1)\,|\psi_\alpha(1)\,\psi_\beta(2)\rangle = |\psi_\alpha(2)\,\psi_\beta(1)\rangle = |\psi_\beta(1)\,\psi_\alpha(2)\rangle$$

In termini di prodotto tensoriale di spazi vettoriali: $\mathcal{E}(1)\otimes\mathcal{E}(2) = \mathcal{E}(2)\otimes\mathcal{E}(1)$

$P^2(2,1) = \underline{1}$

$$P(2,1)\,P(2,1)\,|\psi_\alpha(1)\,\psi_\beta(2)\rangle = P(2,1)\,|\psi_\alpha(2)\,\psi_\beta(1)\rangle$$
$$= |\psi_\alpha(1)\,\psi_\beta(2)\rangle$$

$P^\dagger(2,1) = P(2,1)$ Hermitiano

$$\langle \psi_{\alpha'}(1)\psi_{\beta'}(2)|P(2,1)|\psi_\alpha(1)\psi_\beta(2)\rangle = \langle\psi_{\alpha'}(1)\psi_{\beta'}(2)|\psi_\beta(1)\psi_\alpha(2)\rangle = \delta_{\alpha'\beta}\delta_{\beta'\alpha}$$

$$\langle \psi_{\alpha'}(1)\psi_{\beta'}(2)|P^\dagger(2,1)|\psi_\alpha(1)\psi_\beta(2)\rangle = \left(\langle\psi_\alpha(1)\psi_\beta(2)|P(2,1)|\psi_{\alpha'}(1)\psi_{\beta'}(2)\rangle\right)^*$$

$$= \left(\langle\psi_\alpha(1)\psi_\beta(2)|\psi_{\beta'}(1)\psi_{\alpha'}(2)\rangle\right)^* = \delta_{\alpha'\beta}\delta_{\alpha\beta'}$$

$P^\dagger(2,1)P(2,1) = P(2,1)P^\dagger(2,1) = 1$ Unitario

P hermitiano $\rightarrow$ autovalori reali

Poiché $|P|^2 = \mathbb{I} \rightarrow$ autovalori ± 1

$P(2,1)|\psi_S\rangle = |\psi_S\rangle$ simmetrico

$P(2,1)|\psi_A\rangle = -|\psi_A\rangle$ asimmetrico

$$S \equiv \frac{1}{2}\left(1+P(2,1)\right) \qquad A \equiv \frac{1}{2}\left(1-P(2,1)\right)$$

$$S^2 = \frac{1}{2}(1+P)\frac{1}{2}(1+P) = \frac{1}{4}\left(1+2P+P^2\right) = \frac{1}{4}(2+2P) = S$$

$$A^2 = \frac{1}{2}(1-P)\frac{1}{2}(1-P) = \frac{1}{4}\left(1-2P+P^2\right) = \frac{1}{4}(2-2P) = A$$

$$S^\dagger = S \qquad A^\dagger = A$$

$$SA = \frac{1}{2}(1+P)\frac{1}{2}(1-P) = \frac{1}{4}\left(1+P-P-P^2\right) = \frac{1}{4}(1-1) = 0$$

$$S + A = \underline{1}$$

$$P S|\psi\rangle = P\frac{1}{2}(1+P)|\psi\rangle = \frac{1}{2}(P+P^2)|\psi\rangle$$
$$= \frac{1}{2}(P+1)|\psi\rangle = S|\psi\rangle$$

$$P A|\psi\rangle = P\frac{1}{2}(1-P)|\psi\rangle = \frac{1}{2}(P-P^2)|\psi\rangle = \frac{1}{2}(P-1)|\psi\rangle$$
$$= -\frac{1}{2}(1-P)|\psi\rangle = -A|\psi\rangle$$

Operatori di proiezione

$\boxed{\text{Osservabili}}$ $B|\psi_\alpha\rangle = b_\alpha|\psi_\alpha\rangle$

$$P(2,1)B(1)P^\dagger(2,1)|\psi_\alpha(1)\psi_\beta(2)\rangle = P(2,1)B(1)|\psi_\beta(1)\psi_\alpha(2)\rangle$$
$$= P(2,1)b_\beta|\psi_\beta(1)\psi_\alpha(2)\rangle = b_\beta P(2,1)|\psi_\beta(1)\psi_\alpha(2)\rangle$$
$$= b_\beta|\psi_\alpha(1)\psi_\beta(2)\rangle$$

Quindi $P(2,1)B(1)P^\dagger(2,1) = B(2)$ e $P(2,1)B(2)P^\dagger(2,1) = B(1)$

$$P(2,1)\left[B(1)+C(2)\right]P^\dagger(2,1) = B(2)+C(1)$$

$$P(2,1)\left[B(1)C(2)\right]P^\dagger(2,1) = P(2,1)B(1)P^\dagger(2,1)P(2,1)C(2)P^\dagger(2,1)$$
$$= B(2)C(1)$$

$$P(2,1)O(1,2)P^\dagger(2,1) = O(2,1)$$

Operatore simmetrico $O_S(1,2) = O_S(2,1)$

$$P(2,1)O_S(1,2)P^\dagger(2,1) = O_S(2,1) = O_S(1,2)$$

$$P O_S P^\dagger P = O_S P = P O_S = O_S P \rightarrow [O_S, P] = 0$$

Generalizzazione a N particelle

Esempio con 3 particelle

$|\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3)\rangle$

$P(n,p,q)\,|\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3)\rangle = |\psi_\alpha(n)\psi_\beta(p)\psi_\gamma(q)\rangle$

$P(1,2,3)\; P(3,1,2)\; P(2,3,1) \qquad P(1,3,2)\; P(2,1,3)\; P(3,2,1)$

Esempio

$P(2,3,1)\,|\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3)\rangle = |\psi_\alpha(2)\psi_\beta(3)\psi_\gamma(1)\rangle$

Trasposizioni - si invertono solo due particelle le altre rimangono intatte

Ad esempio $P(1,3,2)$ è una trasposizione

Ogni permutazione può essere costruita come prodotto di trasposizioni.

$P(3,1,2) = P(3,2,1)P(2,1,3) = P(1,3,2)P(3,2,1)$

Verifica

$P(3,1,2)\,|\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3)\rangle = |\psi_\alpha(3)\psi_\beta(1)\psi_\gamma(2)\rangle$

$P(1,3,2)P(3,2,1)\,|\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3)\rangle = P(1,3,2)\,|\psi_\alpha(3)\psi_\beta(2)\psi_\gamma(1)\rangle$

$= |\psi_\alpha(3)\psi_\beta(1)\psi_\gamma(2)\rangle$

$P(3,2,1)P(2,1,3)\,|\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3)\rangle = P(3,2,1)\,|\psi_\alpha(2)\psi_\beta(1)\psi_\gamma(3)\rangle$

$= |\psi_\alpha(3)\psi_\beta(1)\psi_\gamma(2)\rangle$

Gli operatori di permutazione

NON COMMUTANO

$$P(3,2,1)P(1,3,2)|\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3)\rangle = P(3,2,1)|\psi_\alpha(1)\psi_\beta(3)\psi_\gamma(2)\rangle$$
$$= |\psi_\alpha(2)\psi_\beta(3)\psi_\gamma(1)\rangle$$

Quindi

$$P(1,3,2)P(3,2,1) = P(3,1,2) \neq P(3,2,1)P(1,3,2) = P(2,3,1)$$

Per ogni permutazione rimane fisso il numero di trasposizioni.

$P(1,2,3)$ $P(3,1,2)$ $P(2,3,1)$ ha un numero <u>pari</u> di trasposizioni

$P(1,3,2)$ $P(2,1,3)$ $P(3,2,1)$ numero <u>dispari</u>

Generalizzazione a N particelle

$P(\alpha)$ operatore di permutazione che agisce su uno stato di N particelle

Definiamo stati simmetrici S o antisim. A

$P(\alpha)|\psi_S\rangle = |\psi_S\rangle$ simmetrico

$P(\alpha)|\psi_A\rangle = \varepsilon_\alpha |\psi_A\rangle$ antisimmetrico

$$\varepsilon_\alpha \begin{cases} +1 & \text{se } P(\alpha) \text{ è pari} \\ -1 & \text{se } P(\alpha) \text{ è dispari} \end{cases}$$

I due operatori

$S = \frac{1}{N!}\sum_\alpha P(\alpha)$ $A = \frac{1}{N!}\sum_\alpha \varepsilon_\alpha P(\alpha)$ <u>sono Proiettori</u>

Se A sono hermitiani

$S^+ = S \qquad A^+ = A$

si dimostra in maniera analoga a come è stato fatto per 3 particelle.

$P(\alpha) S = S P(\alpha) = S \qquad P(\alpha) A = A P(\alpha) = \varepsilon_\alpha A$

Questo è dovuto al fatto che

$P(\alpha) P(\beta) = P(\gamma) \quad e \quad \varepsilon_\alpha \varepsilon_\beta = \varepsilon_\gamma$

$$P(\alpha) S = \frac{1}{N!} \sum_\beta P(\alpha) P(\beta) = \frac{1}{N!} \sum_\gamma P(\gamma) = S$$

$$P(\alpha) A = \frac{1}{N!} \sum_\beta \varepsilon_\beta P(\alpha) P(\beta) = \frac{1}{N!} \sum_\gamma \varepsilon_\alpha \varepsilon_\gamma P(\gamma) = \varepsilon_\alpha A$$

Si ha che $S^2 = S \quad A^2 = A \qquad AS = SA = 0$

I due proiettori proiettano su spazi composti da stati totalmente S o A

$P(\alpha) S|\psi\rangle = S|\psi\rangle \equiv |\psi_S\rangle \longrightarrow \varepsilon_S$

$P(\alpha) A|\psi\rangle = \varepsilon_\alpha A|\psi\rangle \equiv |\psi_A\rangle \longrightarrow \varepsilon_A$

Per $N > 2$ S e A non proiettano su spazi supplementari

Per 3 particelle

$$S + A = \frac{1}{6}\Big[\underline{P(1,2,3)+P(3,1,2)+P(2,3,1)+P(1,3,2)+P(2,1,3)+P(3,2,1)}$$
$$\underline{+P(1,2,3)+P(3,1,2)+P(2,3,1)-P(1,3,2)-P(2,1,3)-P(3,2,1)}\Big]$$
$$= \frac{1}{3}\Big[P(1,2,3)+P(3,1,2)+P(2,3,1)\Big] \neq 1$$

Postulato di simmetrizzazione

Un sistema di particelle identiche è descritto da stati che godono di un solo tipo di simmetria rispetto alle permutazioni di queste particelle.

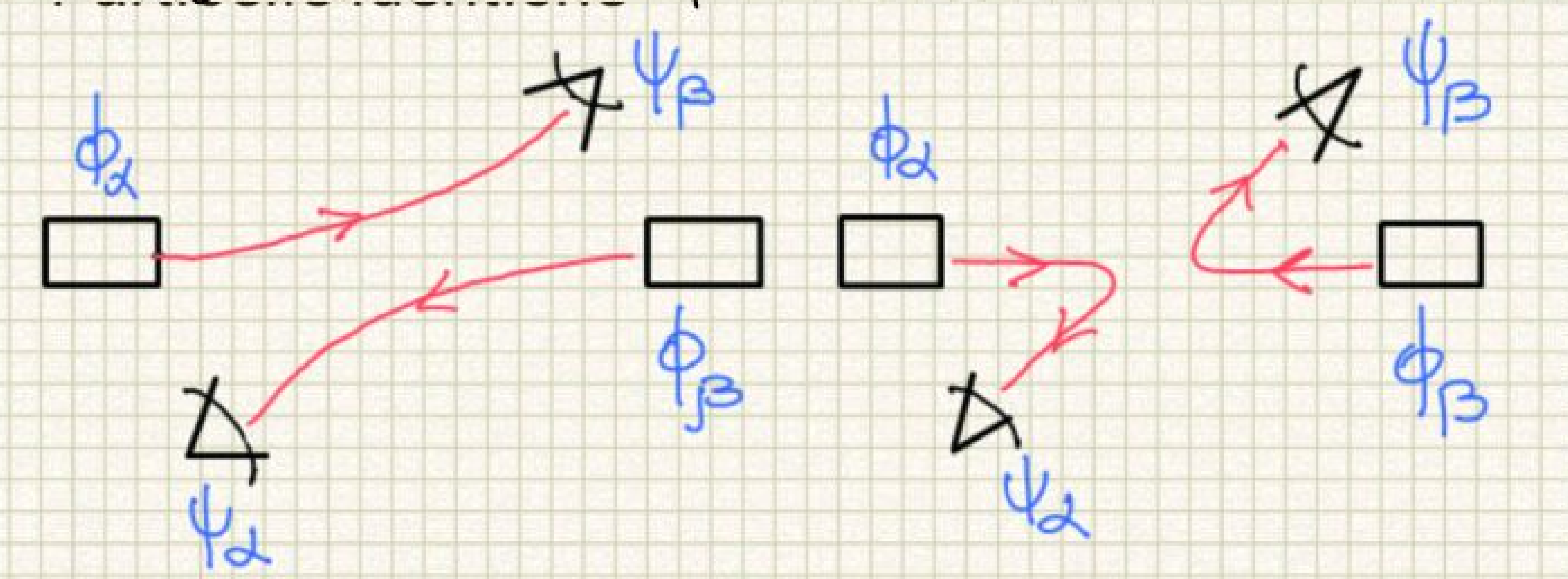
Per particelle a spin intero, o nullo, dette Bosoni, gli stati sono simmetrici.

Per particelle a spin semi-intero, dette Fermioni, gli stati sono antisimmetrici.

Particelle composite

Teoria dei campi

Diffusione di 2 particelle identiche



Stato iniziale $\frac{1}{\sqrt{2}}\left(1+\varepsilon P(1,2)\right)|\phi_\alpha(1)\phi_\beta(2)\rangle$

Stato finale $\frac{1}{\sqrt{2}}\left(1+\varepsilon P(1,2)\right)|\psi_\alpha(1)\psi_\beta(2)\rangle$

$\varepsilon = 1$ bosoni $\qquad \varepsilon = -1$ fermioni

Normalizzazione $= 1$

$$\langle\phi_\alpha(1)\phi_\beta(2)|\frac{1}{\sqrt{2}}\left(1+\varepsilon P^\dagger(1,2)\right)\frac{1}{\sqrt{2}}\left(1+\varepsilon P(1,2)\right)|\phi_\alpha(1)\phi_\beta(2)\rangle$$

$$= \frac{1}{2}\langle\phi_\alpha(1)\phi_\beta(2)|\left(1+\varepsilon P^\dagger(1,2)+\varepsilon P(1,2)+\varepsilon^2 P^\dagger(1,2)P(1,2)\right)|\phi_\alpha(1)\phi_\beta(2)\rangle$$

$$= \frac{1}{2}\langle\phi_\alpha(1)\phi_\beta(2)|\left(1+2\varepsilon P(1,2)+1\right)|\phi_\alpha(1)\phi_\beta(2)\rangle$$

$$= \langle\phi_\alpha(1)\phi_\beta(2)|\phi_\alpha(1)\phi_\beta(2)\rangle \to 1 \text{ normaliz.}$$

$$+\,\varepsilon\langle\phi_\alpha(1)\phi_\beta(2)|\phi_\alpha(2)\phi_\beta(1)\rangle \to 0 \text{ ortogonalità}$$

$$\langle\phi_\alpha|\phi_\beta\rangle = \delta_{\alpha\beta}$$

Ampiezza di transizione.

$$<\phi_\alpha(1)\phi_\beta(2)|\frac{1}{\sqrt{2}}(1+\varepsilon P^+(1,2))\frac{1}{\sqrt{2}}(1+\varepsilon P(1,2))|\psi_\alpha(1)\psi_\beta(2)>$$

$$= <\phi_\alpha(1)\phi_\beta(2)|\psi_\alpha(1)\psi_\beta(2)> + \varepsilon<\phi_\alpha(1)\phi_\beta(2)|\psi_\alpha(2)\psi_\beta(1)>$$

Probabilità è $|\quad|^2$ dell'ampiezza. Interferenza.

Nel caso in cui $\alpha=\beta$ il processo è inibito per fermioni.

Principio di esclusione di Pauli.

Diffusione di due nucleoni identici.

$$\psi(1,2)=\sum_L R_L(r)\mathcal{Y}_{L,M}(\Omega)\chi^S_M \qquad \begin{matrix} r \text{ - coordinata relativa} \\ \Omega \end{matrix}$$

$$\chi^0_0=\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow) \quad \text{singoletto - antisim.}$$

$$\left.\begin{aligned} \chi^1_1 &= \uparrow\uparrow \\ \chi^1_0 &= \frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow) \\ \chi^1_{-1} &= \downarrow\downarrow \end{aligned}\right\} \text{tripletto - simmetrico}$$

$$\mathcal{Y}_{L,M}(-\Omega)=\mathcal{Y}_{L,M}(\pi-\vartheta,\varphi+\pi)=(-1)^L\mathcal{Y}_{L,M}(\vartheta,\varphi)=(-1)^L\mathcal{Y}_{L,M}(\Omega)$$

L		Spin		Totale
0	S	0	A	A
0	S	1	S	~~S~~
1	A	0	A	~~S~~
1	A	1	S	A

Diffusione elastica n-n

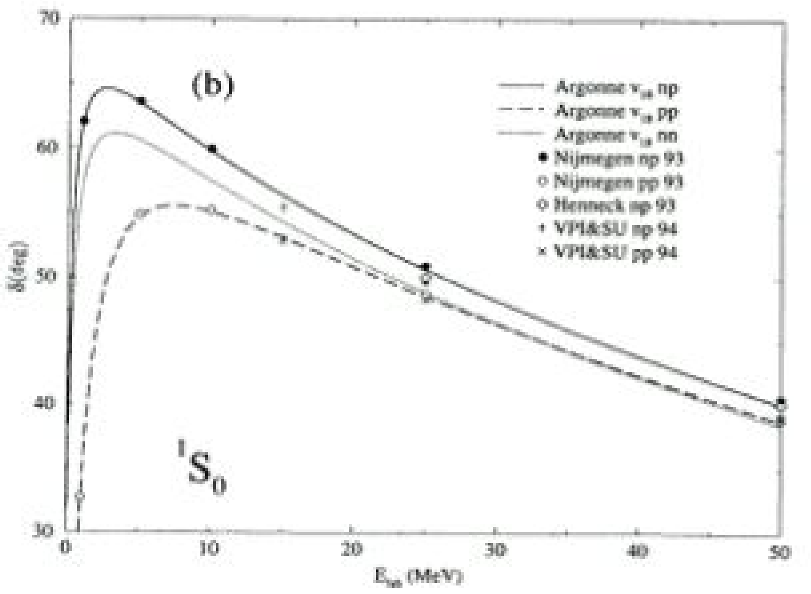


FIG. 2. Phase shifts in the $^1S_0$ channel for np, nn, and pp scattering, compared to various partial-wave phase-shift analyses.

FIG. 4. The ϵ_1 mixing parameter compared to various partial-wave phase-shift analyses.

S=0 L=1

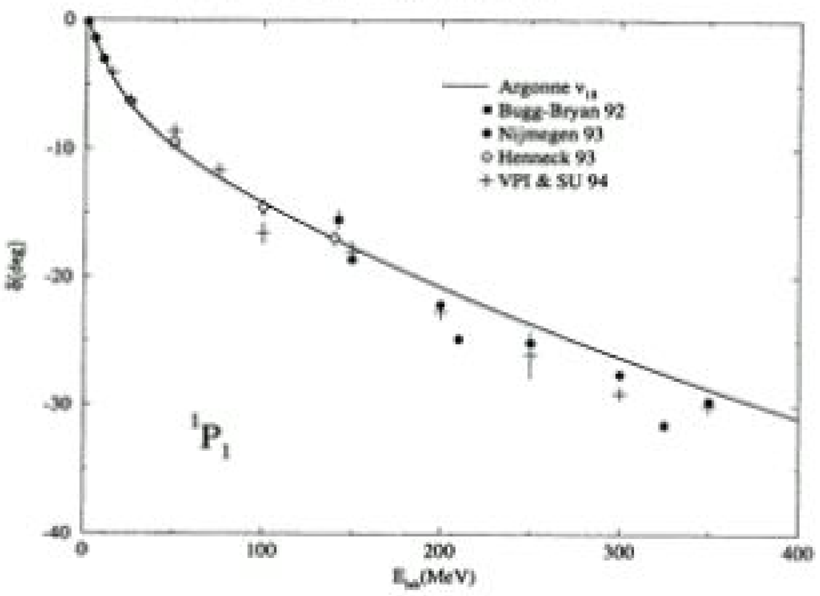


FIG. 5. Phase shifts in the $^1P_1$ channel, compared to various partial-wave phase-shift analyses.

S=1 L=1

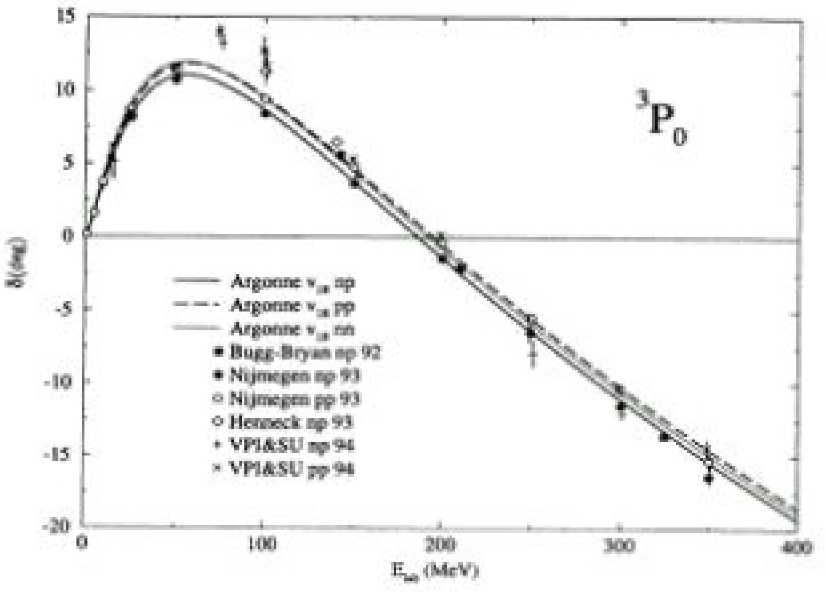


FIG. 3. Phase shifts in the $^3P_0$ channel for np, nn, and pp scattering, compared to various partial-wave phase-shift analyses.

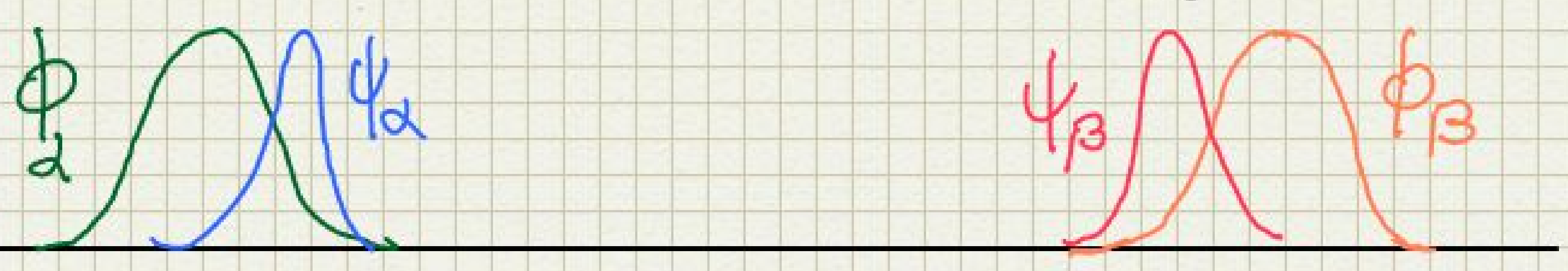
TABLE VIII. Scattering lengths and effective ranges in fm.

	Experiment	Argonne v_{18}	w/o v^{EM}
$^1a_{pp}$	-7.8063 ± 0.0026[a]	-7.8064	-17.164
$^1r_{pp}$	2.794 ± 0.014[a]	2.788	2.865
$^1a_{nn}$	-18.5 ± 0.4[b]	-18.487	-18.818
$^1r_{nn}$	2.80 ± 0.11[b]	2.840	2.834
$^1a_{np}$	-23.749 ± 0.008[c]	-23.732	-23.084
$^1r_{np}$	2.81 ± 0.05[c]	2.697	2.703
$^3a_{np}$	5.424 ± 0.003[c]	5.419	5.402
$^3r_{np}$	1.760 ± 0.005[c]	1.753	1.752

[a]Reference [32].
[b]Reference [28].
[c]Reference [35].

Quando si trascura lo scambio?

$\phi_\alpha(\vec{r})$ e $\phi_\beta(\vec{r})$ localizzate in regioni dello spazio molto distanti e NON interagenti



$$\langle \phi_\alpha(1)\phi_\beta(2) | \psi_\alpha(2)\psi_\beta(1)\rangle =$$

$$\underbrace{\int d^3r_1\, \phi^*_\alpha(\vec{r}_1)\psi_\beta(\vec{r}_1)}_{\downarrow\, 0} \underbrace{\int d^3r_2\, \phi^*_\beta(\vec{r}_2)\psi_\alpha(\vec{r}_2)}_{\downarrow\, 0}$$

Modello a particelle indipendenti

$$H(1 \ldots N) = \sum_{i=1}^{N} h(i) \qquad h(i)|\phi_i\rangle = \varepsilon_i |\phi_i\rangle$$

$|\Phi\rangle$ autostato di H è prodotto di $|\phi_i\rangle$

Se le particelle sono fermioni $P(\alpha,\beta)|\Phi\rangle = -|\Phi\rangle$

$$|\Phi\rangle = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_1(1) & \phi_2(1) & \cdots & \phi_1(N) \\ \phi_1(2) & \phi_2(2) & & \\ & & & \\ \phi_1(N) & \phi_2(N) & & \phi_N(N) \end{vmatrix}$$

Principio di esclusione di Pauli