

Lezione 13

Equazione di Dirac

Equazione di Schrödinger col principio di equivalenza

$$E = \frac{p^2}{2m} \quad ; \quad E \rightarrow i\hbar \frac{\partial}{\partial t} \quad ; \quad \mathbf{p} \rightarrow -i\hbar \nabla$$

$$i\hbar \frac{\partial}{\partial t} \psi = \frac{(-i\hbar \nabla)^2}{2m} \psi = \frac{-\hbar^2}{2m} \nabla^2 \psi$$

Conservazione della corrente

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{J} = 0$$

Moltiplichiamo l'equazione di Schrödinger per ψ^* e poi ne sottraiamo il complesso coniugato.

$$\psi^* \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] \psi = 0 \quad ; \quad \psi \left[-i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] \psi^* = 0$$

$$i\hbar \left[\psi^* \frac{\partial}{\partial t} \psi + \psi \frac{\partial}{\partial t} \psi^* \right] + \frac{\hbar^2}{2m} [\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*] = 0$$

Dividendo per $i\hbar$

$$\frac{\partial}{\partial t} (\psi^* \psi) - \frac{i\hbar}{2m} \nabla \cdot [\psi^* \nabla \psi - \psi \nabla \psi^*] = 0$$

$$\rho = |\psi|^2 \quad ; \quad \mathbf{J} = -\frac{i\hbar}{2m} [\psi^* \nabla \psi - \psi \nabla \psi^*]$$

Soluzione

$$\psi(\mathbf{r}, t) = N e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

$$\rho = |\psi|^2 = |N|^2 \quad ; \quad \mathbf{J} = -\frac{i\hbar}{2m} [i\mathbf{k} - (-i\mathbf{k})] |N|^2 = \frac{\hbar \mathbf{k}}{m} |N|^2$$

Equazione di Klein-Gordon

$$E^2 = p^2 c^2 + m^2 c^4$$

$$\left(i\hbar \frac{\partial}{\partial t}\right)^2 \psi - (-i\hbar \nabla)^2 c^2 \psi = m^2 c^4 \psi$$

$$\left[-\hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 \nabla^2 c^2\right] \psi = m^2 c^4 \psi ; \quad \left[-\frac{\partial^2}{\partial t^2} + \nabla^2 c^2\right] \psi = \frac{m^2}{\hbar^2} c^4 \psi$$

Moltiplichiamo per $-i\psi^*$ e consideriamo la complessa coniugata

$$\begin{aligned} -i\psi^* \left[-\frac{\partial^2}{\partial t^2} + \nabla^2 c^2\right] \psi &= \frac{m^2}{\hbar^2} c^4 \psi \psi^* (-i) \\ i\psi \left[-\frac{\partial^2}{\partial t^2} + \nabla^2 c^2\right] \psi^* &= \frac{m^2}{\hbar^2} c^4 \psi \psi^* (i) \end{aligned}$$

Sommiamo

$$i \left[\psi^* \frac{\partial^2}{\partial t^2} \psi - \psi \frac{\partial^2}{\partial t^2} \psi^* \right] - i\psi^* \nabla^2 c^2 \psi + i\psi \nabla^2 c^2 \psi^* = 0$$

$$\frac{\partial}{\partial t} \left[i\psi^* \frac{\partial}{\partial t} \psi - i\psi \frac{\partial}{\partial t} \psi^* \right] + \nabla \cdot \left[-i \left(\psi^* \nabla c^2 \psi - \psi \nabla c^2 \psi^* \right) \right] = 0$$

$$\rho = i \left[\psi^* \frac{\partial}{\partial t} \psi - \psi \frac{\partial}{\partial t} \psi^* \right] \quad ; \quad \mathbf{J} = \left[-i \left(\psi^* \nabla c^2 \psi - \psi \nabla c^2 \psi^* \right) \right]$$

Soluzione di onda piana

$$\psi(\mathbf{r}, t) = N e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

$$\left[-\frac{\partial^2}{\partial t^2} + \nabla^2 c^2 \right] N e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = \frac{m^2}{\hbar^2} c^4 N e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

$$(\omega^2 - k^2 c^2) = \frac{m^2 c^4}{\hbar^2} \quad ; \quad \hbar^2 \omega^2 - \hbar^2 k^2 c^2 = m^2 c^4$$

$$E = \hbar \omega \quad ; \quad \mathbf{p} = \hbar \mathbf{k}$$

$$\begin{aligned}\rho &= iN^*e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(-i\omega)Ne^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - iNe^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(i\omega)N^*e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ &= 2\omega|N|^2 = 2\frac{E}{\hbar}|N|^2\end{aligned}$$

$$\mathbf{J} = 2\mathbf{k}|N|^2 = 2\frac{\mathbf{p}}{\hbar}|N|^2$$

$$E = \pm\sqrt{p^2c^2 + m^2c^4}$$

Densità di probabilità negative

Equazione di Dirac

$$E\psi = i\hbar \frac{\partial}{\partial t} \psi = H\psi \quad ; \quad H = \boldsymbol{\alpha} \cdot (-i\hbar \boldsymbol{\nabla})c + \beta mc^2$$

$\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$ e β costanti da determinare.

Per la particella libera

$$\begin{aligned} E^2 &= p^2 c^2 + m^2 c^4 \\ &= [\boldsymbol{\alpha} \cdot \mathbf{p}c + \beta mc^2] [\boldsymbol{\alpha} \cdot \mathbf{p}c + \beta mc^2] \\ &= (\boldsymbol{\alpha} \cdot \mathbf{p})(\boldsymbol{\alpha} \cdot \mathbf{p})c^2 + \beta mc^2 \boldsymbol{\alpha} \cdot \mathbf{p}c + \boldsymbol{\alpha} \cdot \mathbf{p}c \beta mc^2 + \beta^2 m^2 c^4 \\ &= \left(\sum_{i=1}^3 \alpha_i p_i \right) \left(\sum_{j=1}^3 \alpha_j p_j \right) c^2 + mc^3 \beta \sum_{i=1}^3 \alpha_i p_i + mc^3 \sum_{j=1}^3 \alpha_j p_j \beta + \beta^2 m^2 c^4 \\ &= \sum_i \alpha_i p_i \alpha_i p_i c^2 + \sum_{i \neq j} (\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j c^2 \\ &\quad + mc^3 \sum_i (\beta \alpha_i + \alpha_i \beta) p_i + \beta^2 m^2 c^4 \\ &= \sum_i p_i^2 c^2 + m^2 c^4 \end{aligned}$$

Proprietà di α e β

$$\alpha_i^2 = 1 \quad ; \quad \alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad ; \quad \alpha_i \beta + \beta \alpha_i = 0 \quad ; \quad \beta^2 = 1$$

- 1) Non commutano, sono matrici.
- 2) Sono hermitiane perché H è hermitiano
 $\alpha_i = \alpha_i^\dagger$, $\beta = \beta^\dagger$
- 3) Dato che $\alpha_i^2 = \alpha_i \alpha_i^\dagger = \mathbf{I}$ e $\beta \beta^\dagger = \mathbf{I}$,
sono unitarie, hanno autovalori ± 1 .
- 4) Dato che le tracce sono nulle hanno dimensioni pari.

$$Tr[AB] = Tr[BA]$$

$$\begin{aligned} Tr[\alpha_i] &= Tr[(\alpha_i \beta) \beta] = Tr[\beta(\alpha_i \beta)] = Tr[\beta(-\beta \alpha_i)] = \\ Tr[-\beta^2 \alpha_i] &= Tr[-\alpha_i] = 0 \end{aligned}$$

- 5) Per matrici 2×2 non c'è nessuna β che anticommuti con $\alpha_{1,2,3}$.
Si deve passare alle dimensioni successive 4×4 .

Equazione di Dirac

$$i\hbar \frac{\partial}{\partial t} \psi = -i\hbar c \alpha \cdot \nabla \psi + \beta m c^2 \psi \quad ; \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

Continuità dell'equazione di Dirac

$$\begin{aligned} \psi^\dagger \left[i\hbar \frac{\partial}{\partial t} \psi \right] &= \psi^\dagger \left[-i\hbar c \alpha \cdot \nabla \psi + \beta m c^2 \psi \right] \\ \left[-i\hbar \frac{\partial}{\partial t} \psi^\dagger \right] \psi &= \left[i\hbar c \nabla \psi^\dagger \cdot \alpha + m c^2 \psi^\dagger \beta \right] \psi \end{aligned}$$

Sottraendo

$$\begin{aligned} i\hbar \left[\psi^\dagger \frac{\partial}{\partial t} \psi + \left(\frac{\partial}{\partial t} \psi^\dagger \right) \psi \right] &= -i\hbar c \left[\psi^\dagger \alpha \cdot (\nabla \psi) + (\nabla \psi^\dagger) \cdot \alpha \psi \right] \\ &\quad + m c^2 \left[(\psi^\dagger \beta \psi) - (\psi^\dagger \beta) \psi \right] \end{aligned}$$

$$i\hbar \frac{\partial}{\partial t} (\psi^\dagger \psi) = -i\hbar c \nabla \cdot [\psi^\dagger \alpha \psi]$$

$$\rho = |\psi|^2 \quad ; \quad \mathbf{J} = c \psi^\dagger \alpha \psi$$

Unità naturali $\hbar = c = 1$

$$\begin{aligned} E\psi &= (\alpha \cdot \mathbf{p} + \beta m) \psi \\ i\frac{\partial}{\partial t}\psi &= (-i\alpha \cdot \nabla + \beta m) \psi \\ \left[i\frac{\partial}{\partial t} + i\alpha \cdot \nabla - \beta m \right] \psi &= 0 \end{aligned}$$

α e β non hanno espressione univoca. Scelta comune.

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \quad \alpha_i^2 = \beta^2 = 1$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Moltiplico a sinistra per β .

$$\left[i\beta \frac{\partial}{\partial t} + i\beta \alpha \cdot \nabla - \beta^2 m \right] \psi = 0$$

Definiamo le 4 componenti di un quadrivettore

$$\gamma \equiv [\beta, \beta \alpha] \equiv [\gamma^0, \gamma]$$

Forma covariante dell'equazione di Dirac

$$[i\gamma^0 \frac{\partial}{\partial t} + i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - m] \psi = 0$$

$$\gamma^i = \beta \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}$$

Indici $i, j, k = 1, 2, 3$ $\mu, \nu = 0, 1, 2, 3$

Proprietà delle matrici γ

$$\gamma^\nu \gamma^\mu = -\gamma^\mu \gamma^\nu \quad (\nu \neq \mu) \quad (\gamma^i)^2 = -\mathbf{I} \quad (\gamma^0)^2 = \mathbf{I}$$

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \quad (\gamma^i)^\dagger = -\gamma^i \quad (\gamma^0)^\dagger = \gamma^0$$

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \quad (\gamma^5)^2 = \mathbf{I} \quad (\gamma^5)^\dagger = \gamma^5 \quad \gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5$$

$$\gamma^5 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}$$

Prodotto scalare

$$A^\mu \equiv (A_0, \mathbf{A}) \quad A_\mu \equiv (A_0, -\mathbf{A})$$

$$A_\mu B^\mu \equiv \sum_{\mu=0,1,2,3} A_\mu B^\mu = A_0 B^0 - \mathbf{A} \cdot \mathbf{B}$$

$$x^\mu \equiv (t, \mathbf{r}) \quad x_\mu x^\mu = t^2 - \mathbf{r}^2$$

$$p^\mu \equiv (E, \mathbf{p}) \quad p_\mu p^\mu = E^2 - \mathbf{p}^2 = m^2$$

Invarianti relativistici

$$\frac{\partial}{\partial x_\mu} \equiv \left(\frac{\partial}{\partial t}, -\nabla \right) \quad p_\mu \equiv \left(i \frac{\partial}{\partial t}, i \nabla \right)$$

Equazione di Dirac

$$\left[i \gamma^\mu \frac{\partial}{\partial x_\mu} - m \right] \psi = 0$$

Hamiltoniana NON commuta con \mathbf{L}^2 e L_z ma commuta con \mathbf{J}^2 e J_z dove
 $\mathbf{J} = \mathbf{L} + \boldsymbol{\sigma}/2$

Soluzione di onde piane

Sistema di riferimento a riposo $\mathbf{p} = 0$.

$$i\frac{\partial}{\partial t}\psi = \beta m\psi \quad ; \quad \beta = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \quad ; \quad \psi = \phi(\mathbf{r}, t)u$$

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\beta u_{1,2} = u_{1,2} \quad ; \quad \beta u_{3,4} = -u_{3,4}$$

$$i\frac{\partial}{\partial t}\phi(\mathbf{r}, t)u_{1,2} = m\phi(\mathbf{r}, t)u_{1,2} \quad ; \quad \phi(\mathbf{r}, t) = e^{-imt} \quad ; \quad \psi_{1,2} = e^{-imt}u_{1,2}$$

$$i\frac{\partial}{\partial t}\phi(\mathbf{r}, t)u_{3,4} = -m\phi(\mathbf{r}, t)u_{3,4} \quad ; \quad \phi(\mathbf{r}, t) = e^{imt} \quad ; \quad \psi_{3,4} = e^{imt}u_{3,4}$$

Autovalori dell'hamiltoniana sono $\pm m$.

Energie negative !!!

Invariante relativistico

$$mt = p_\nu x^\nu = E't' - \mathbf{p}' \cdot \mathbf{r}'$$

Le soluzioni dell'equazione si modificano in un sistema di riferimento in moto

$$\psi_{1,2} = e^{i(\mathbf{p} \cdot \mathbf{r} - Et)} u_{1,2} \quad ; \quad \psi_{3,4} = e^{-i(\mathbf{p} \cdot \mathbf{r} - Et)} u_{3,4}$$

$$i \frac{\partial}{\partial t} \psi + i\alpha \cdot \nabla \psi - \beta m \psi = 0$$

$$\psi = e^{i(\mathbf{p} \cdot \mathbf{r} - Et)} u = e^{i(\mathbf{p} \cdot \mathbf{r} - Et)} \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$(E - \alpha \cdot \mathbf{p} - \beta m)u = \left[\begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix} E - \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \cdot \mathbf{p} - \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} m \right] \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0$$

$$E\phi - \boldsymbol{\sigma} \cdot \mathbf{p}\chi - m\phi = 0 \quad (E - m)\phi = \boldsymbol{\sigma} \cdot \mathbf{p}\chi$$

$$E\chi - \boldsymbol{\sigma} \cdot \mathbf{p}\phi + m\chi = 0 \quad (E + m)\chi = \boldsymbol{\sigma} \cdot \mathbf{p}\phi$$

$$\phi = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E - m} \chi \quad \chi = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \phi$$

$$\det \begin{vmatrix} E - m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ -\boldsymbol{\sigma} \cdot \mathbf{p} & E + m \end{vmatrix} = (E - m)(E + m) - (\boldsymbol{\sigma} \cdot \mathbf{p})^2 = E^2 - m^2 - \mathbf{p}^2 = 0$$

$$E = \pm \sqrt{\mathbf{p}^2 + m^2}$$

Energie positive $E = |E|$

$$\phi = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E - m} \chi$$

Regime non relativistico $|E| \simeq m$ quindi $\phi \gg \chi$

Scegliamo

$$\phi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \phi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Soluzioni ad energia positiva

$$\psi_+ = N e^{i(\mathbf{p} \cdot \mathbf{r} - Et)} u_+ = N e^{i(\mathbf{p} \cdot \mathbf{r} - Et)} \begin{bmatrix} \phi \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \phi \end{bmatrix}$$

Energie negative $E = -|E|$

$$\chi = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \phi = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{-|E| + m} \phi$$

Regime non relativistico $|E| \simeq m$ quindi $\chi \gg \phi$

Scegliamo

$$\chi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Soluzioni ad energia negativa

$$\psi_- = N e^{i(\mathbf{p} \cdot \mathbf{r} + |E|t)} u_- = N e^{i(\mathbf{p} \cdot \mathbf{r} + |E|t)} \begin{bmatrix} -\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|E| + m} \chi \\ \chi \end{bmatrix}$$

N costante di normalizzazione. Per $\psi^\dagger \psi = 1$

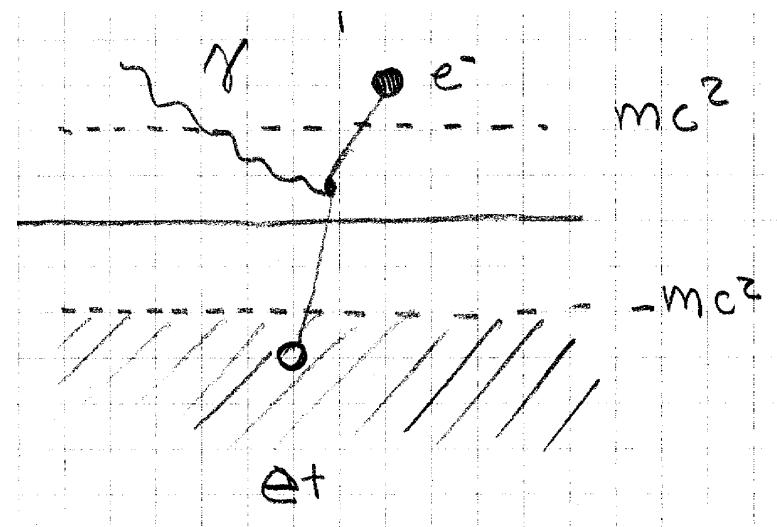
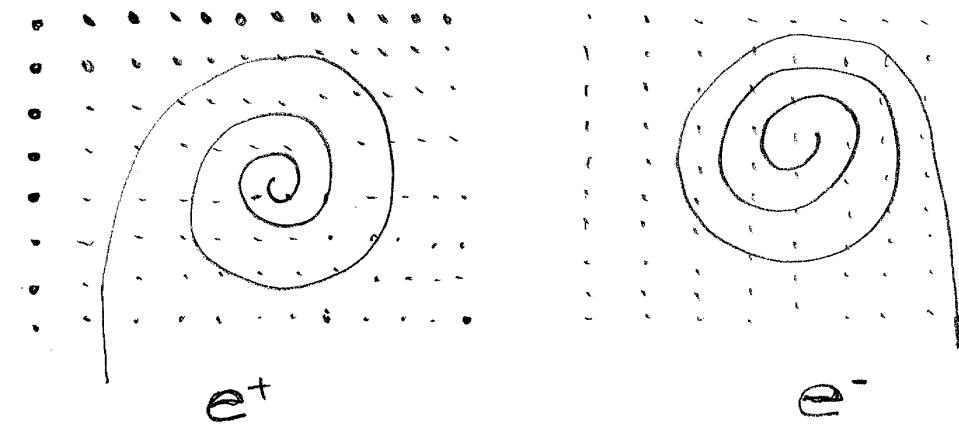
$$N = \sqrt{\frac{m + |E|}{2|E|}}$$

Spinori in onde piane

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{|E|+m} \\ \frac{p_x+ip_y}{|E|+m} \end{bmatrix} ; \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{|E|+m} \\ \frac{-p_z}{|E|+m} \end{bmatrix}$$

$$u_3 = \begin{bmatrix} -\frac{p_z}{|E|+m} \\ -\frac{p_x+ip_y}{|E|+m} \\ 1 \\ 0 \end{bmatrix} ; \quad u_4 = \begin{bmatrix} -\frac{p_x-ip_y}{|E|+m} \\ \frac{p_z}{|E|+m} \\ 0 \\ 1 \end{bmatrix}$$

Antimateria



Antiparticelle

Proprietà che **NON** variano
massa, spin, isospin, vita media

Proprietà uguali in valore assoluto e segno opposto
carica elettrica, 3^a componente isospin, momento magnetico, n^o leptónico, n^o barionico, stranezza, charm, beauty

Fermioni identificati con una barra \bar{p} .
Tranne leptoni e^\pm, μ^\pm, τ^\pm .

Equazione di Klein-Gordon per densità di carica

$$-e \left[-\frac{\partial^2}{\partial t^2} + \nabla^2 c^2 - m^2 \right] \psi = 0 \quad \psi = N e^{-i(\mathbf{p} \cdot \mathbf{r} - Et)}$$

Carica unitaria con segno -.

$$\rho(e^-) = -e2E|N|^2 \quad \mathbf{J}(e^-) = -e2\mathbf{p}|N|^2$$

Cambio carica

$$\rho(e^+) = e2E|N|^2 \quad \mathbf{J}(e^+) = e2\mathbf{p}|N|^2$$

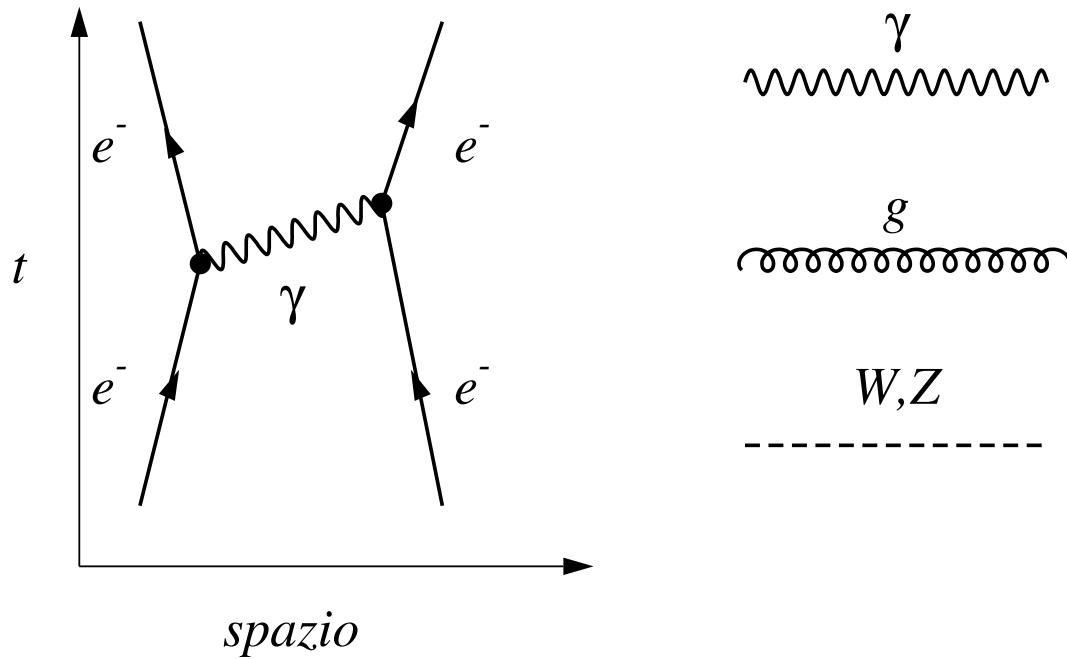
Posso ottenere il cambio carica anche cambiando $E \rightarrow -E$ e $\mathbf{p} \rightarrow -\mathbf{p}$. Nella soluzione ottengo lo stesso risultato cambiando $t \rightarrow -t$, che cambia $\mathbf{p} \rightarrow -\mathbf{p}$.

$$\psi = N e^{-i((-p) \cdot r - E(-t))}$$

Interpetazione di Feynmann-Stückelbeg

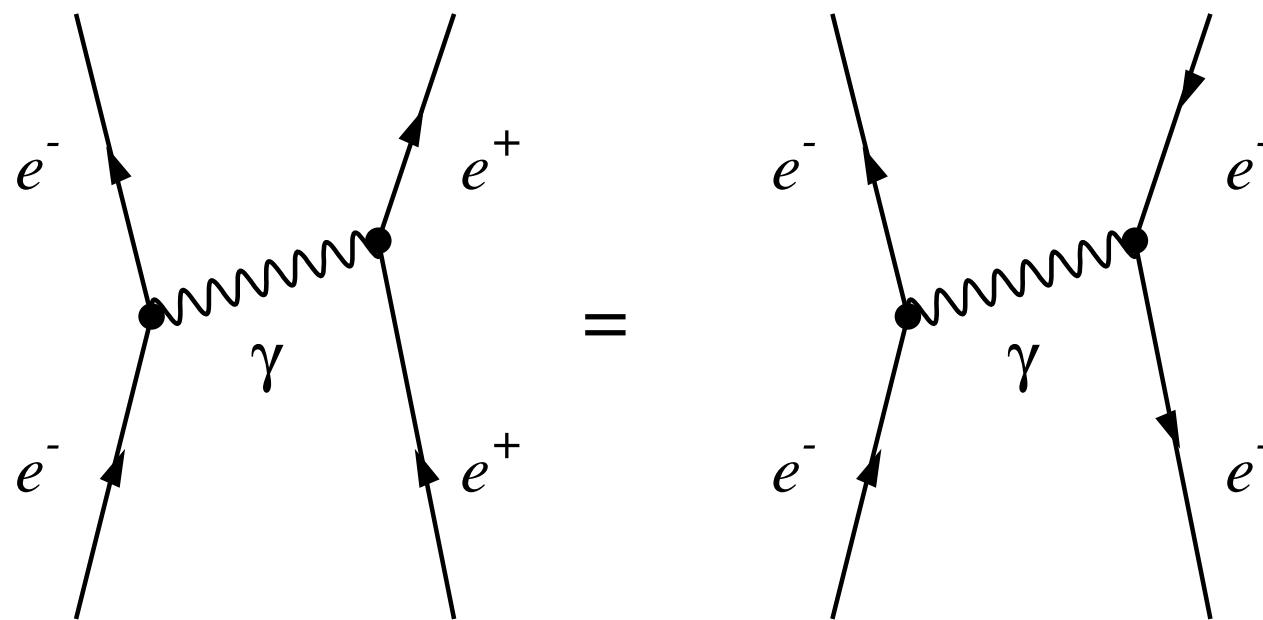
Le antiparticelle sono particelle che simuovono all'indietro nel tempo.

Diagrammi di Feynmann



Particelle scambiate virtuali. Non soddisfano $E^2 = \mathbf{p}^2 c^2 + m^2 c^4$.
I punti che uniscono 3 o più linee sono vertici.
Costante di accoppiamento dell'interazione.
Propagatore $1/(\mathbf{q}^2 - \omega^2 + M^2 c^2)$.

Mie convenzioni



Bibliografia

Approfondimenti degli argomenti presentati possono essere trovati nei cap. 1, 2 e 3 del libro

J. D. Bjorken and S. D. Drell
Relativistic Quantum Mechanics
Mc Graw - Hill, (New York) 1964