$$\sum_\eta \langle \lambda\ N-\eta\ 1\eta | L N\rangle\langle \lambda-1\ N\ 1-\eta | \lambda\ N-\eta\rangle (-)^\eta =$$

$$= (-)^{\lambda+L+\eta+\eta} \frac{[\lambda]}{[\lambda-1]} \sum_\eta \langle \lambda\ N-\eta\ 1-\eta | \lambda-1\ N\rangle\langle \lambda\ N-\eta\ 1-\eta | L N\rangle =$$

$$= (-)^{\lambda+L} \frac{[\lambda]}{[\lambda-1]} \delta_{L,\lambda-1} = (-)^{2L+1}\left(\frac{2(L+1)+1}{2(L+1-1)+1}\right)^{\frac{1}{2}} = -\left(\frac{2L+3}{2L+1}\right)^{\frac{1}{2}}$$

$$\sum_\eta \langle \lambda\ N-\eta\ 1\eta | L N\rangle\langle \lambda+1\ N\ 1-\eta | \lambda\ N-\eta\rangle (-)^\eta$$

$$= (-)^{\lambda+L} \frac{[\lambda]}{[\lambda+1]} \sum_\eta (-)^{2\eta} \langle \lambda\ N-\eta\ 1-\eta | L N\rangle\langle \lambda\ N-\eta\ 1-\eta | \lambda+1\ N\rangle$$

$$= (-)^{\lambda+L} \frac{[\lambda]}{[\lambda+1]} \delta_{L,\lambda+1} = (-)^{L-1+L}\left(\frac{2(L-1)+1}{2L+1}\right)^{\frac{1}{2}} = -\left(\frac{2L-1}{2L+1}\right)^{\frac{1}{2}}$$

quindi

$$\vec\nabla\left(j_\lambda(kr)\vec{\mathcal{Y}}^N_{L,\lambda,1}(\hat r)\right) = -k\, j_L(kr)\mathcal{Y}_{LN}(\hat r)\left[\left(\frac{L+1}{\cancel{2L+3}}\cdot\frac{\cancel{2L+3}}{2L+1}\right)^{\frac{1}{2}}\delta_{L,\lambda-1} + \left(\frac{L-1+1}{\cancel{2L-1}}\,\frac{\cancel{2L-1}}{2L+1}\right)^{\frac{1}{2}}\delta_{L,\lambda+1}\right]$$

per quanto riguarda la 51 a) $\lambda=L$ quindi $\vec\nabla\cdot A_{LM}(\vec r,m)=0$

Per la 51b) questo significa che:

$$\vec\nabla\vec A_{LM}(\vec r,e) = c_{L-1}\,\vec\nabla\cdot\left(j_{L-1}\,\vec{\mathcal{Y}}^N_{L,L-1,1}\right) + c_{L+1}\,\vec\nabla\cdot\left(j_{L+1}\,\vec{\mathcal{Y}}^N_{L,L+1,1}\right) =$$

$$= c_{L-1}\left\{-k\, j_L\mathcal{Y}_{LM}\left[\left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}\delta_{L,L-2} + \left(\frac{L}{2L+1}\right)^{\frac{1}{2}}\delta_{L,L}\right]\right\}$$
$$+ c_{L+1}\left\{-k\, j_L\mathcal{Y}_{LM}\left[\left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}\delta_{L,L} + \left(\frac{L}{2L+1}\right)^{\frac{1}{2}}\delta_{L,L+2}\right]\right\} =$$

dato che L si accoppia con $\vec e$ che ha spin 1 $\delta_{L,L\pm2}=0$ sempre, quindi

$$= -k\, j_L(kr)\mathcal{Y}_{LM}(\hat r)\left[\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}c_{L-1} - \left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}c_{L+1}\right] = 0$$

per $\quad c_{L-1} = \left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}} \qquad c_{L+1} = -\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}$.