

Pur il caso specifico si ha:

$$\sum_M (\vec{\nabla} r^{\frac{\sigma}{2}} Y_{\sigma M}^*(\hat{r})) (\vec{\nabla} r^{\frac{\sigma}{2}} Y_{\sigma M}(\hat{r})) = \frac{2\sigma+1}{4\pi} \left[(\sigma r^{\sigma-1})^2 + \frac{\sigma(\sigma+1)}{r^2} r^{2\sigma} \right]$$

$$= \frac{\sigma(2\sigma+1)}{4\pi} r^{2\sigma-2}$$

Quindi:

$$\sum_{H_f} (\bar{E}_f - \bar{E}_i) |\langle f | Q_{\sigma M} | i \rangle|^2 = \frac{\hbar^2}{2m} \frac{\sigma(2\sigma+1)}{4\pi} \sum_a \langle i | r^{2\sigma-2} | i \rangle e_a^2$$

• Dimostrazione della 3.

$$\sum_M (\vec{\nabla} \phi(r) Y_{lM}^*(\hat{r})) (\vec{\nabla} \phi(r) Y_{lM}(\hat{r})) =$$

$$\sum_M \left\{ -\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}} \left(\frac{d}{dr} - \frac{l}{r}\right) \phi \frac{\vec{r}}{r} Y_{l,l+1}^* + \left(\frac{l}{2l+1}\right)^{\frac{1}{2}} \left(\frac{d}{dr} + \frac{l+1}{r}\right) \phi \frac{\vec{r}}{r} Y_{l,l-1}^* \right\}$$

$$\left\{ -\left(\frac{l+1}{2l+1}\right)^{\frac{1}{2}} \left(\frac{d}{dr} - \frac{l}{r}\right) \phi \frac{\vec{r}}{r} Y_{l,l+1} + \left(\frac{l}{2l+1}\right)^{\frac{1}{2}} \left(\frac{d}{dr} + \frac{l+1}{r}\right) \phi \frac{\vec{r}}{r} Y_{l,l-1} \right\}$$

$$\sum_M \frac{\vec{r}}{r} Y_{l,l+1}^* \frac{\vec{r}}{r} Y_{l',l'+1}^* = \sum_M \sum_{\mu, \nu} \langle l, \mu | \nu | j, m \rangle \langle l', \mu' | \nu' | j, m \rangle Y_{l\mu}^* Y_{l'\mu'}^* \vec{e}_\nu \vec{e}_{\nu'}$$

$$\vec{e}_\nu \vec{e}_{\nu'} = \delta_{\nu\nu'} \Rightarrow \delta_{\mu\mu'}$$

$$= \sum_\mu \left(\sum_{\nu} \langle l, \mu | \nu | j, m \rangle \langle l', \mu | \nu | j, m \rangle \right) Y_{l\mu}^* Y_{l'\mu}^* =$$

$$= \sum_\mu \left(\sum_{\nu} (-)^{l+\nu} \left(\frac{2j+1}{2l+1}\right)^{\frac{1}{2}} \langle j-m | \nu | l-\mu \rangle (-)^{l'+\nu} \left(\frac{2j+1}{2l'+1}\right)^{\frac{1}{2}} \langle j-m | \nu | l'-\mu \rangle \right) Y_{l\mu}^* Y_{l'\mu}^*$$

$$= \sum_\mu \left(\frac{2j+1}{\sqrt{(2l+1)(2l'+1)}} \sum_{\nu} \langle j-m | \nu | l-\mu \rangle \langle j-m | \nu | l'-\mu \rangle \right) Y_{l\mu}^* Y_{l'\mu}^* =$$

$$= \sum_\mu \frac{2j+1}{2l+1} \delta_{ll'} Y_{l\mu}^* Y_{l'\mu}^* = \frac{2j+1}{2l+1} \frac{2l+1}{4\pi} = \frac{2j+1}{4\pi} \delta_{ll'}$$