Analytical and numerical analysis of the complete Lipkin-Meshkov-Glick Hamiltonian

The Lipkin-Meshkov-Glick is a simple, but not trivial, model of a quantum many-body system which allows us to solve the many-body Schrödinger equation without making any approximation. The model, which in its unperturbed case is composed only by two energy levels, includes two interacting terms. A first one, the V interaction, which promotes or degrade pairs of particles, and a second one, the W interaction, which scatters one particle in the upper and another in the lower energy level. In comparing this model with other approximation methods, the W term interaction is often set to zero. In this paper, we show how the presence of this interaction changes the global structure of the system, generates degeneracies between the various eigenstates and modifies the energy eigenvalues structure. We present analytical solutions for systems of two and three particles and, for some specific cases, also for four, six and eight particles. The solutions for systems with more than eight particles are only numerical but their behaviour can be well understood by considering the extrapolations of the analytical results. Of particular interest it is the study of how the W interaction affects the energy gap between the ground state and the first-excited state.

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I. Introduction

The goal of the non relativistic many-body physics is to solve a Schrödinger equation which describes a system composed by many particles [1, 2]. Since two and, eventually, three-particles systems are used to define the effective interaction between the particles, in this context, "many" means more than three. The techniques to solve the many-body Schrödinger equation without approximations are rather involved and, usually, limited in the number of particles composing the systems. For this reason, the methodologies most commonly used to solve the quantum many-body problem are based on some simplifying approximations. One of the key problems of this branch of physics is the possibility of testing the validity of these approximations. With this aim, in the middle '60s of the last century, Lipkin, Meshkov, and Glick (LMG) [3–5] built a many-fermion system for which the Schrödinger equation can be solved without approximations. In this model, the non interacting, one-body, part of the hamiltonian has only two energy levels, which are occupied by fermions. The exact description of this system has been compared with those obtained by using effective theories [6–10]. This model is a beautiful

example of application of basic quantum mechanics, and one of the few examples of many-body systems which can be studied in relatively simple manner, without making approximations.

In this article, we consider the two level model containing in its hamiltonian both the V and the W interaction terms. After showing how to obtain the solutions of the one-body part of the hamiltonian, we present a simple derivation of the exact solutions for the complete LGM hamiltonian. We show analytical expressions for system of two and three particles and numerical results for those with more particles. We discuss the role of the various terms of the hamiltonian, and some general features of the solutions, mainly those concerning the degeneracies of some eigenstates.

II. The Lipkin-Meshkov-Glick model

The original LMG model [3] considers a hamiltonian of the type

$$H_{\mathrm{LMG}} = E_0\,J_0 + \frac{V}{2}\left(\frac{J_+^2 + J_-^2}{2}\right) + \frac{W}{2}\left(\frac{J_+J_- + J_-J_+}{2}\right) \,. \tag{1}$$

Nevertheless, since the model was often used to test the treatment of ground-state correlation in the random phase approximation, the LMG hamiltonian is usually simplified by setting $W = 0$. Another possible choice for the W interaction parameter is given by $W = V$. In this case the hamiltonian can be easily solved and its eigenvalues and eigenvectors are used to study the instability of the Hartree-Fock state against collective oscillations.

The aim of this paper is to present a detailed discussion of the hamiltonian (1) without any assumption or constraint on the W interation term.

We start our study by considering a system composed by N non-interacting particles. The configuration space available to each particle is composed by only two states whose energy values are separated by an amount E_0,

$$\left\{-\frac{E_0}{2}, \frac{E_0}{2}\right\} \,. \tag{2}$$

When the particles do not interact with each other only the one-body part of the hamiltonian is present, E_0J_0. In this case, the ground state of the system is that where all the N particles are lying on the lowest state. The excited states are obtained by promoting particles from the lower to the higher of the two energy levels.

A convenient representation of the many-body states of this system is obtained by defining the quantum numbers $j = N/2$ and m, this latter one can assume the values

$$m = -j, -j+1, ..., j-1, j \,.$$

The eigenstates of the unperturbed hamiltonian ($V = W = 0$) then satisfy

$$J_0\,|j,m\rangle = m\,|j,m\rangle \,. \tag{3}$$

The global number of possible eigenstates of this system is $2j + 1$, and the energies of these eigenstates are $m\,E_0$. A pictorial representation of the eigenstates of systems with $N = 2$, 3, and 4 particles is given in Table 1.

The J_+ operator removes a particle from the lower level and put it on the higher one. The lowering operator, J_-, acts in the opposite direction. The action of these operators on the eigenstates of J_0, Eq. (3), is

$$J_\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j,m\pm 1\rangle \,. \tag{4}$$

The J_+^2 term promotes particle pairs from the lower to the upper state, while the J_-^2 term operates in the opposite direction. The J_+J_- operator and its hermitian conjugate promotes one particle and lowers another one. The constants V and W are the strengths of the interactions between the two particles involved in the processes described above.

We prefer to work with dimensionless quantities and with attractive interactions

$$(V \, , \, W \,) = - \, (\, |V| \, , \, |W| \,) \; ,$$

therefore we divide the LMG hamiltonian (1) by a constant factor related to the the two-level model energy eigenvalue

$$\mathcal{H} \; = \; \frac{H_{\mathrm{LMG}}}{E_0} \; = \; J_0 - \frac{\nu}{2} \left(J_+^2 + J_-^2 \right) - \frac{\omega}{2} \left(J_+ J_- + J_- J_+ \right), \tag{5}$$

where we have defined

$$\nu = |V| \, / \, E_0 \quad \text{and} \quad \omega = |W| \, / \, E_0 \; .$$

Many of the of matrix elements of $\mathcal{H}$ between the eigenstates of J_0 are zero, those different from zero are

$$\begin{aligned}
\langle j, m | \mathcal{H} | j, m \rangle \; &= \; m - \omega [\, j(j+1) - m^2 \,], \\
\langle j, m | \mathcal{H} | j, m+2 \rangle \; &= \; -\frac{\nu}{2} \sqrt{[\, j \, (j+1) - (m+1) \, m \,] \, [\, j \, (j+1) - (m+2) \, (m+1) \,]}, \\
\langle j, m+2 | \mathcal{H}_{\mathrm{LMG}} | j, m \rangle \; &= \; \langle j, m | \mathcal{H} | j, m+2 \rangle.
\end{aligned} \tag{6}$$

We define a new quantum number r, related to N and m by the expression

$$m = \frac{N}{2} - r + 1 \quad (r = 1, 2, ..., N+1), \tag{7}$$

and we express the above matrix elements as a function of the particle number N and of the interaction parameters ν and ω,

$$\mathcal{H}_{rs}^{[N]} = \begin{cases}
\dfrac{N}{2} - r + 1 - \left(N \, r - \dfrac{N}{2} - r^2 + 2 \, r - 1 \right) \omega & \text{for } s = r \; ; \; (r = 1, 2, ..., N+1), \\[4mm]
-\dfrac{\sqrt{(N-r)(N-r+1)(r+1) \, r}}{2} \, \nu & \text{for } s = r + 2 \; ; \; (r = 1, 2, ..., N-1), \\[4mm]
-\dfrac{\sqrt{(N-s)(N-s+1)(s+1) \, s}}{2} \, \nu & \text{for } r = s + 2 \; ; \; (s = 1, 2, ..., N-1), \\[4mm]
0 & \text{otherwise.}
\end{cases} \tag{8}$$

The solution of the Schrödinger equation consists in diagonalizing the above matrix. In this manner, we obtain the eigenvalues of the hamiltonian (5) in terms of the N, r quantum numbers and of the ν and ω stregths. The corresponding eigenvectors are expressed in the basis formed by the eigenstates of J_0.

III. Solutions for $N = 2$ and $N = 3$.

For the systems composed by 2 or 3 particles we found analytical expressions of eiegenvalues and eigenvectors of the full hamiltonian (5). We obtained analytical expressions of eiegenvalues and eigenvectors also for systems composed by 4, 6 and 8 particles, but only when in the hamiltonian we set $\omega = 0$.

We exploit the analytical form of solutions to investigate the effects of the terms proportional to ν and ω as they evolve with the increasing number of particles and to suggest how to analyse the numerical solutions.

The explicit expression of the hamiltonian matrix for the system composed by two particles is

$$\mathcal{H}^{[2]} = \begin{pmatrix} 1 - \omega & 0 & -\nu \\ 0 & -2\,\omega & 0 \\ -\nu & 0 & -(1+\omega) \end{pmatrix} \,, \tag{9}$$

where we have indicated in the upperindex between square brackets the number of particles composing the system. We write the eigenvalues, $\mathcal{E}^{[2]}(\nu, \omega)$ and the eigenvectors, $\mathcal{V}^{[2]}(\nu, \omega)$ of this system as

$$\left\{ \mathcal{E}^{[2]}(\nu, \omega) \,;\, \mathcal{V}^{[2]}(\nu, \omega) \right\} = \left\{ \begin{array}{lcl} +\sqrt{1+\nu^2} - \omega & ; & \left(\dfrac{1 + \sqrt{1+\nu^2}}{2}, 0, -\dfrac{\nu}{2} \right)^\top \\[2ex] -2\,\omega & ; & (0, 1, 0)^t \\[2ex] -\sqrt{1+\nu^2} - \omega & ; & \left(\dfrac{\sqrt{1+\nu^2} - 1}{\nu}, 0, 1 \right)^\top \end{array} \right\}, \tag{10}$$

where $\top$ indicates the transpose which, in this case, transform the rows into columns. We have selected a global normalisation of the eigenvectors which allows us to recover, in the limit where both ν and ω are zero, the simple form $(1, 0, 0)^\top$, $(0, 1, 0)^\top$ and $(0, 0, 1)^\top$.

For the case of three particles, we found the following expression of the hamiltonian matrix

$$H^{[3]} = \begin{pmatrix} \frac{3}{2}(1-\omega) & 0 & -\sqrt{3}\,\nu & 0 \\ 0 & \frac{1}{2}(1 - 7\,\omega) & 0 & -\sqrt{3}\,\nu \\ -\sqrt{3}\,\nu & 0 & -\frac{1}{2}(1 + 7\,\omega) & 0 \\ 0 & -\sqrt{3}\,\nu & 0 & -\frac{3}{2}(1+\omega) \end{pmatrix}. \tag{11}$$

The energy eigenvalues are

$$\mathcal{E}^{[3]}(\nu, \omega) = \begin{bmatrix} \mathcal{E}_4^{[3]}(\nu, \omega) \\ \mathcal{E}_3^{[3]}(\nu, \omega) \\ \mathcal{E}_2^{[3]}(\nu, \omega) \\ \mathcal{E}_1^{[3]}(\nu, \omega) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} +1 + 2\sqrt{(1+\omega)^2 + 3\,\nu^2} - 5\,\omega \\ -1 + 2\sqrt{(1-\omega)^2 + 3\,\nu^2} - 5\,\omega \\ +1 - 2\sqrt{(1+\omega)^2 + 3\,\nu^2} - 5\,\omega \\ -1 - 2\sqrt{(1-\omega)^2 + 3\,\nu^2} - 5\,\omega \end{bmatrix}, \tag{12}$$

where they are ordered with increasing values for $\nu = 0$ and $\omega = 0$.

We show in Fig. 1 the evolution of these four eigenvalues as a function of ν for 6 selected values of ω. Since we are interest in studying when degeneracy appear, we write the energy differences between the r-th and s-th levels as

$$\Delta_{r,s}^{[N]}(\nu, \omega) = \mathcal{E}_r^{[N]}(\nu, \omega) - \mathcal{E}_s^{[N]}(\nu, \omega) \,.$$

In the case of $\omega = 0$ we recover the traditional LMG model [3–5]. For this case we have

$$\Delta^{[3]}(\nu, 0) = \begin{bmatrix} \Delta_{4,3}^{[3]}(\nu, 0) \\ \Delta_{3,2}^{[3]}(\nu, 0) \\ \Delta_{2,1}^{[3]}(\nu, 0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 + 2\sqrt{1 + 3\,\nu^2} \\ 1 \end{bmatrix}. \tag{13}$$

We observe that the energy gap between the first/second and third/fourth energy levels remains unchanged by increasing the value of ν. The energy gap between the second/third energy level increases when we increase the ν value and, consequently, the energy levels in the traditional LMG model never cross with each other, see the panel (a) of Fig. 1.

Values of ω different from zero break this symmetry. We show in Fig. 2 the evolution of the four egeinenergies as a function of ω for 6 values of ν. When $\nu = 0$, see the panel (a) of Fig. 2, we find

$$\Delta^{[3]}(0,\omega) = \begin{bmatrix} \Delta^{[3]}_{4,3}(0,\omega) \\ \Delta^{[3]}_{3,2}(0,\omega) \\ \Delta^{[3]}_{2,1}(0,\omega) \end{bmatrix} = \begin{bmatrix} 1+2\,\omega \\ 1 \\ 1-2\,\omega \end{bmatrix}. \tag{14}$$

In this case, the values of the first and second energy eigenvalues cross at $\omega = 1/2$. In general, the crossing point, for a given value of ν, satisfies the equation

$$\Delta^{[3]}_{2,1}(\nu,\omega) = \mathcal{E}^{[3]}_2(\nu,\omega) - \mathcal{E}^{[3]}_1(\nu,\omega) = 0, \tag{15}$$

which implies that the ω and ν terms are related by the expression

$$\omega^{[3]}_{2,1} = \sqrt{\nu^2 + \frac{1}{4}} \ . \tag{16}$$

The eigenvectors of the 3 particle system related to the four eigenvalues can be expressed as

$$\mathcal{V}^{[3]}(\nu,\omega) = \mathcal{A}(\nu,\omega) \begin{cases} \left(\dfrac{1+\sqrt{1+a_+^2(\nu,\omega)}}{2} , 0 , -\dfrac{a_+(\nu,\omega)}{2} , 0 \right)^\top , \\[2.2em] \left(0 , \dfrac{1+\sqrt{1+a_-^2(\nu,\omega)}}{2} , 0 , -\dfrac{a_-(\nu,\omega)}{2} \right)^\top , \\[2.2em] \left(\dfrac{\sqrt{1+a_+^2(\nu,\omega)}-1}{a_+(\nu,\omega)} , 0 , 1 , 0 \right)^\top , \\[2.2em] \left(0 , \dfrac{\sqrt{1+a_-^2(\nu,\omega)}-1}{a_-(\nu,\omega)} , 0 , 1 \right)^\top . \end{cases} \tag{17}$$

where we have defined the quantity

$$a_\pm(\nu,\mu) = \frac{\sqrt{3}\,\nu}{1\pm\omega}. \tag{18}$$

In the limit of $\nu = \omega = 0$ the eigenvectors assume the simple form $(0,0,0,1)^\top$, etc. and the normalisation constant

$$\mathcal{A}(\nu,\omega) = \frac{a_-(\nu,\omega)}{\sqrt{2\sqrt{1+a_-^2(\nu,\omega)}\left(\sqrt{1+a_-^2(\nu,\omega)}-1\right)}} \tag{19}$$

is chosen to guarantee that each eigenvector is normalised to one.

We studied how the eigenvectors changes as a function of the interaction strengths ν and ω. As example of these changes, we show in Fig. 3, for the lowest energy state, the behaviour of the squares of the two non zero components. We remark that for $\nu \ll |1-\omega|$, i.e. in the limit $a_-^2(\nu,\omega) \to 0$, the lowest energy eigenvector assumes the values

$$(0 , 0 , 0 , 1)^\top.$$

For $\nu = |1-\omega|$, i.e. when $a_-^2(\nu,\omega) \to 3$, we find

$$(0 , 0.25 , 0 , 0.75)^\top .$$

Finally, in the limit for $a_-^2(\nu,\omega) \to \infty$ ($\nu \gg |1-\omega|$), we obtain for the energy eigenvector the representation

$$(0 , 0.5 , 0 , 0.5)^\top .$$

In all the panels of Fig. 3 and 4 the full black line and the red dashed line represent, respectively, the second and fourth component of the lowest energy eigenvector. The difference between the two components start from 1 and tends to zero when the value of ν inceases. This can be see in all the panels of Fig. 3 and 4 . The interaction depending on ω potential accelerate this behaviour when the values of ω are between 0 and 1, see Fig. 3. At $\omega = 1$, the trend is inverted, as it is shown in Fig. 4, and the difference between these two components of the lowest energy eigenstate tends to 1. These two components are equal, and assume the value of 0.5, for $\omega = 1 \mp \nu$, see Fig. 4.

IV. Solutions for $N \geq 4$

We obtain analytical expressions for the solutions of the systems with $N = 4, 6, 8$ when we set $\omega = 0$,

$$
\begin{aligned}
\mathcal{E}^{[4]}(\nu, 0) &= 0, \quad \pm\sqrt{1 + 9\nu^2}, \quad \pm 2\sqrt{1 + 3\nu^2}, \\
\mathcal{E}^{[6]}(\nu, 0) &= 0, \quad \pm 2\sqrt{1 + 15\nu^2}, \quad \pm\sqrt{5 + 33\nu^2 \pm 4\sqrt{1 + 6\nu^2 + 54\nu^4}}, \\
\mathcal{E}^{[8]}(\nu, 0) &= 0, \quad \pm\sqrt{5 + 113\nu^2 \pm 4\sqrt{1 + 38\nu^2 + 550\nu^4}}, \quad \pm\sqrt{10 + 118\nu^2 \pm 6\sqrt{1 - 2\nu^2 + 225\nu^4}}.
\end{aligned}
\tag{20}
$$

The above expressions correct those given in the original paper of Lipkin et al. [3] where a factor 4 in $E^{[6]}$, and a factor 6 in $E^{[8]}$, are missing.

The energy differences between the first excited state and the ground state in each of these systems are

$$
\begin{aligned}
\Delta_{2,1}^{[4]}(\nu, 0) &= 2\sqrt{1 + 3\nu^2} - \sqrt{1 + 9\nu^2}, \\
\Delta_{2,1}^{[6]}(\nu, 0) &= \sqrt{5 + 33\nu^2 + 4\sqrt{1 + 6\nu^2 + 54\nu^4}} - 2\sqrt{1 + 15\nu^2}, \\
\Delta_{2,1}^{[8]}(\nu, 0) &= \sqrt{10 + 118\nu^2 + 6\sqrt{1 - 2\nu^2 + 225\nu^4}} - \sqrt{5 + 113\nu^2 + 4\sqrt{1 + 38\nu^2 + 550\nu^4}}.
\end{aligned}
$$

As expected, for a free hamiltonian we have $\Delta_{2,1}^{[4]}(0, 0) = \Delta_{2,1}^{[6]}(0, 0) = \Delta_{2,1}^{[8]}(0, 0) = 1$.

The above results indicate that, for a given ν value, these quantities become smaller with the increase of the particle number N. For example, in the case $\nu = 1$, we obtain

$$
\begin{aligned}
\Delta_{2,1}^{[4]}(1, 0) &= 4 - \sqrt{10} \approx 0.838, \\
\Delta_{2,1}^{[6]}(1, 0) &= \sqrt{38 + 4\sqrt{61}} - 8 \approx 0.321, \\
\Delta_{2,1}^{[8]}(1, 0) &= 2\sqrt{43 + 6\sqrt{14}} - \sqrt{118 + 4\sqrt{589}} \approx 0.093.
\end{aligned}
$$

The energy eigenvalues for the cases investigated are shown in Fig. 5 for $0 \leq \nu \leq 1$ when $\omega = 0$. The various lines never cross with each other and show a symmetry with respect to the zero value. The differences from this value increase with the increase of the value of ν. In the figure it is evident the lowering of the $\Delta_{2,1}^{[N]}(\nu, 0)$ values with increasing of N and of ν.

We investigate the role of ω in the hamiltonian (1) by considering the case of $\nu = 0$. We found that, for a system composed by N particles, the eigenvalues of the m state can be expressed as

$$
\mathcal{E}_m^{[N]}(0, \omega) = m - \omega\left[j(j+1) - m^2\right], \tag{21}
$$

where m can assume the integer values $(-j, -j+1, ..., j-1, j)$ with $j = N/2$.

The energy difference between the level $m + p$ and the level m is

$$
\Delta_{m+p,m}^{[N]} = p\left[1 + \omega(2m + p)\right] \tag{22}
$$

which is a constant, i.e. independent of ω, when $p = -2\,m$.

The energy level r $(m = -N/2 + r - 1)$ intercepts the level s $(p = s - r)$ when ω acquires the value

$$\omega_{r,s}^{[N]} = \frac{1}{N - r - s + 2} \; . \tag{23}$$

We show in Fig. 6 the behaviour of the energy eigenvalues for the $N = 4, 6, 8$ systems as a function of ω, when $\nu = 0$. The vertical lines highlight the position of the crossing points obtained in (23). This expression implies degeneracies for equal values of $r + s$. For example, for $N = 8$, see Fig. 6(c), we have a triple degeneracy when

$$\omega_{1,6}^{[8]} = \omega_{2,5}^{[8]} = \omega_{3,4}^{[8]} = \frac{1}{3} \quad \text{and} \quad \omega_{1,7}^{[8]} = \omega_{2,6}^{[8]} = \omega_{3,5}^{[8]} = \frac{1}{2} \; ,$$

and a fourfold degeneracy for

$$\omega_{18}^{[8]} = \omega_{27}^{[8]} = \omega_{36}^{[8]} = \omega_{45}^{[8]} = 1 \; .$$

We have numerically studied the solution for $N > 8$. We show in Fig. 7 the energy difference between the two lowest states of a system composed by 10 interacting particles as a function of the renormalised interaction strength ν. The various lines indicate the results obtained with different values of ω. We observe that by increasing the ω values, i.e. the strength of the last term of the hamiltonian (1), this difference decreases, see panel (a) of Fig. 7.

We found a relation between the ν and ω strengths indicating when the two levels are degenerated. We generalise the results obtained for the 3 particle system, Eq. (16), and we found the following relation for N particles:

$$\omega_{2,1}^{[N]} = \sqrt{\nu^2 + \frac{1}{(N - 1)^2}} \; . \tag{24}$$

In the specific case presented in Fig. 7, $N = 10$, we have that for a strength $\omega = 1/9$ the two levels become degenerated for $\nu = 0$, see the red line in the panel (b) of Fig. 7. For stronger ω strengths the degeneracy appear at positive values of ω as it is shown in the panel (b) of Fig. 7 by the black line.

In Fig. 8 we compare some result of the $N = 10$ system with those obtained for $N = 20$ and $N = 50$. The blue lines indicate the $\Delta_{2,1}^{[N]}$ values when $\omega = 0$, this is the result of the usual LMG model. We observe that the degeneracy of the two levels appear at smaller values of the interaction strength ω.

The red lines show the results obtained by including also the second interaction term of the Hamiltonian. We have chosen the values $\omega = 1/(N - 1)$ which satisfy Eq. (24) when $\nu = 0$. We observe that the degeneracies for $\nu > 0$ appear for values always smaller with the increase of N. It is worth to remark in the figure, the differences in the ν scale between the different panels and the amplification factors of the red lines.

V. Conclusions

In this article, we have investigated the two-level LMG model when the hamiltonian includes a two-body interaction term which allows particle-hole type of excitations. We have studied the effects of the interaction terms of the LMG hamiltonian (1) by changing ω and ν, the strengths of the two interaction terms. The $\omega = 0$ case reproduce the traditional LMG model [3–5].

For $N = 2, 3$ we obtain fully analytical solutions for both eigenvalues and eigenvectors. The presence of the new interaction term implies the degeneracy of some of the solutions. By changing the values of the strengths ν and ω the energies of the lowest states is modified and the higher states becomes a new ground state.

This feature is emphasised when the number of the particle composing the system increases. We have pointed out cases when the energy differences become always smaller with increasing the ν values but they are never zero. In other cases, instead, it is evident the crossing between the different eigenvalues resulting in a change of the ground state eigenvector.

We give in Eq. (24) a semi-empirical expression which relates the zeros of the energy differences between the two lowest states of a system with N particles with the strength ν of the interaction.

References

[1]   A. L. Fetter and J. D. Walecka, Quantum Theory of Many-Particle Systems (McGraw-Hill, 1971).

[2]   W. Nolting, Fundamentals of Many-Body Physics (Springer-Verlag, 2009).

[3]   H. J. Lipkin, N. Meshkov, and A. J. Glick, *Validity of many-body approximation methods for a solvable model [(I) Exact solutions and perturbation theory]*, Nucl. Phys. **62**, 188-198 (1965).

[4]   N. Meshkov, A. J. Glick, and H. J. Lipkin, *Validity of many-body approximation methods for a solvable model [(II) Linearization procedures]*, Nucl. Phys. **62**, 199-210 (1965).

[5]   A. J. Glick, H. J. Lipkin, and N. Meshkov, *Validity of many-body approximation methods for a solvable model [(III) Diagram summations]*, Nucl. Phys. **62**, 211-224 (1965).

[6]   K. Hagino and G. mF. Bertsch, *Random-phase approximation approach to rotational symmetry restoration in a three-level Lipkin model*, Phys. Rev. C **61**, 024307-5 (2000).

[7]   O. Castanos, R. Lopez-Pena, and J. G. Hirsch, *Classical and quantum phase transitions in the Lipkin-Meshkov-Glick model*, Phys. Rev. B **74**, 104118-14 (2008).

[8]   R. Puebla and A. Relano, *Irreversible processes without energy dissipation in an isolated Lipkin-Meshkov-Glick model*, Phys. Rev. E **92**, 012101-9 (2015).

[9]   G. Co' and S. De Leo, *Hartree-Fock and random phase approximation theories in a many-fermion solvable model*, Mod. Phys. Lett. A **30**, 1550196-15 (2015).

[10]   S. Campbell, G. De Chiara, M. Paternostro, G. M. Palma, and R. Fazio, *Shortcut to Adiabaticity in the Lipkin-Meshkov-Glick Model*, Phys. Rev. Lett. **114**, 177206-6 (2015).

Table 1: Representation of the eigenstates of the one-body hamiltonian for systems with $N = 2$, 3, and 4 particles. The number of green and red dots represents the number of particles lying on the upper and on the lower energy level, respectively.
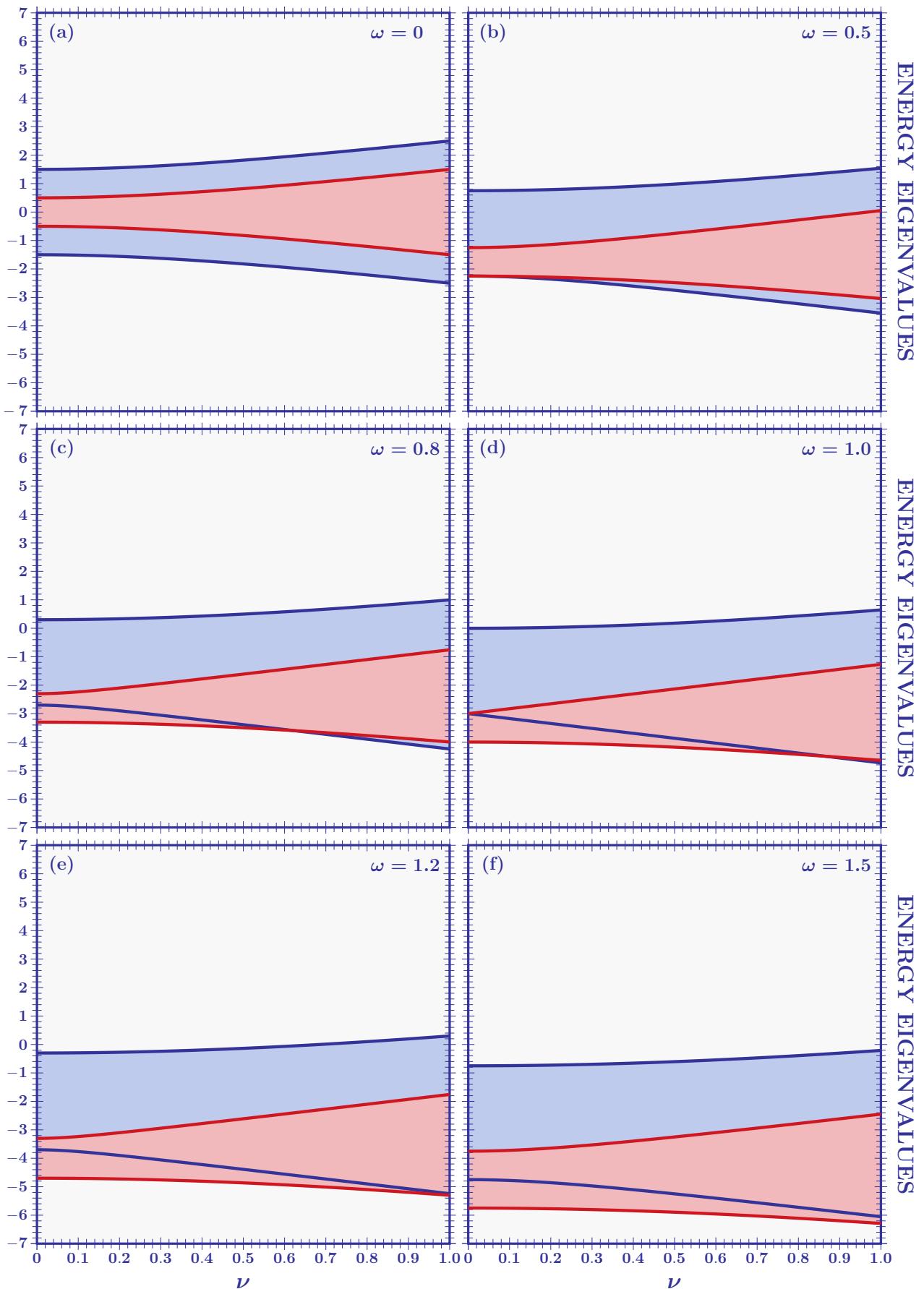
Figure 1: Eigenenergies of a system of 3 particles, Eq. (12), as a function of the strength ν for some fixed values of the strength ω. The blues and red lines identify the the specific eigenvalues. The coloured areas are drawn to highlight the differences between the eigenvalues.
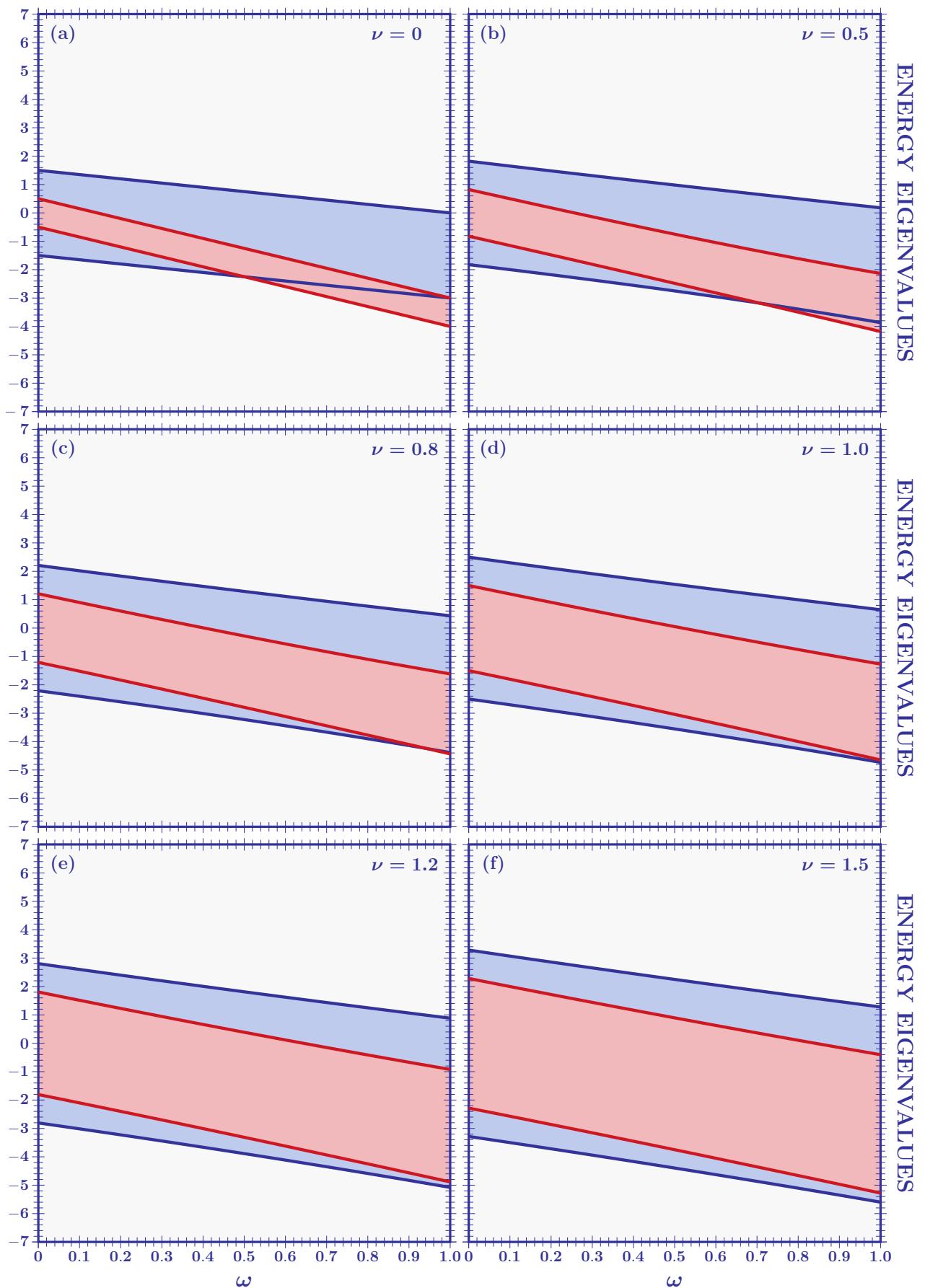
Figure 2: The same as Fig. 1, but as a function of the strength ω for six values of ν.
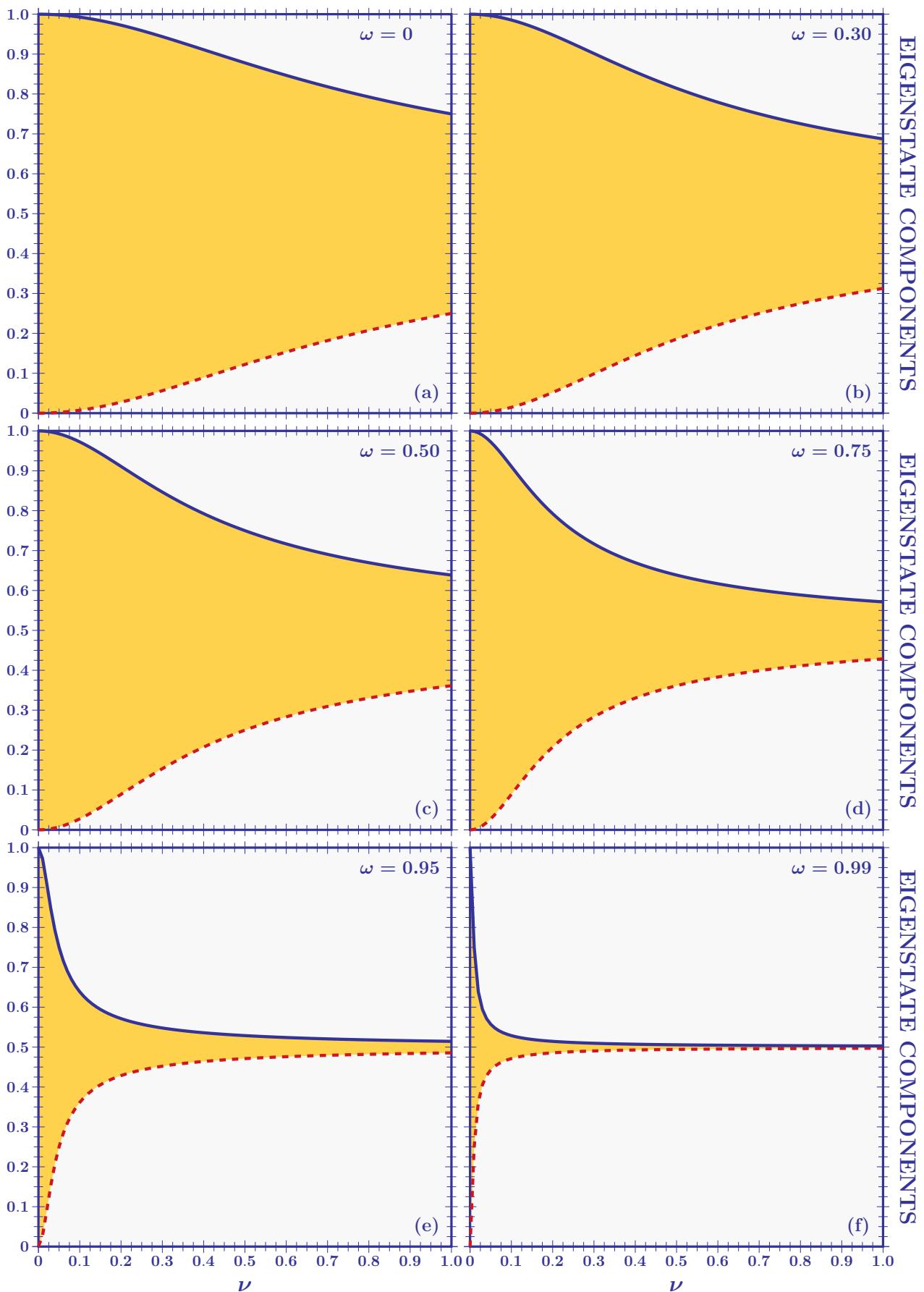
Figure 3: The square of the two non-zero components of the lowest energy eigenstate, Eq. (17), as a function of ν for six values of ω. The system is composed by 3 particles. The yellow areas emphasise the differences between the two components.
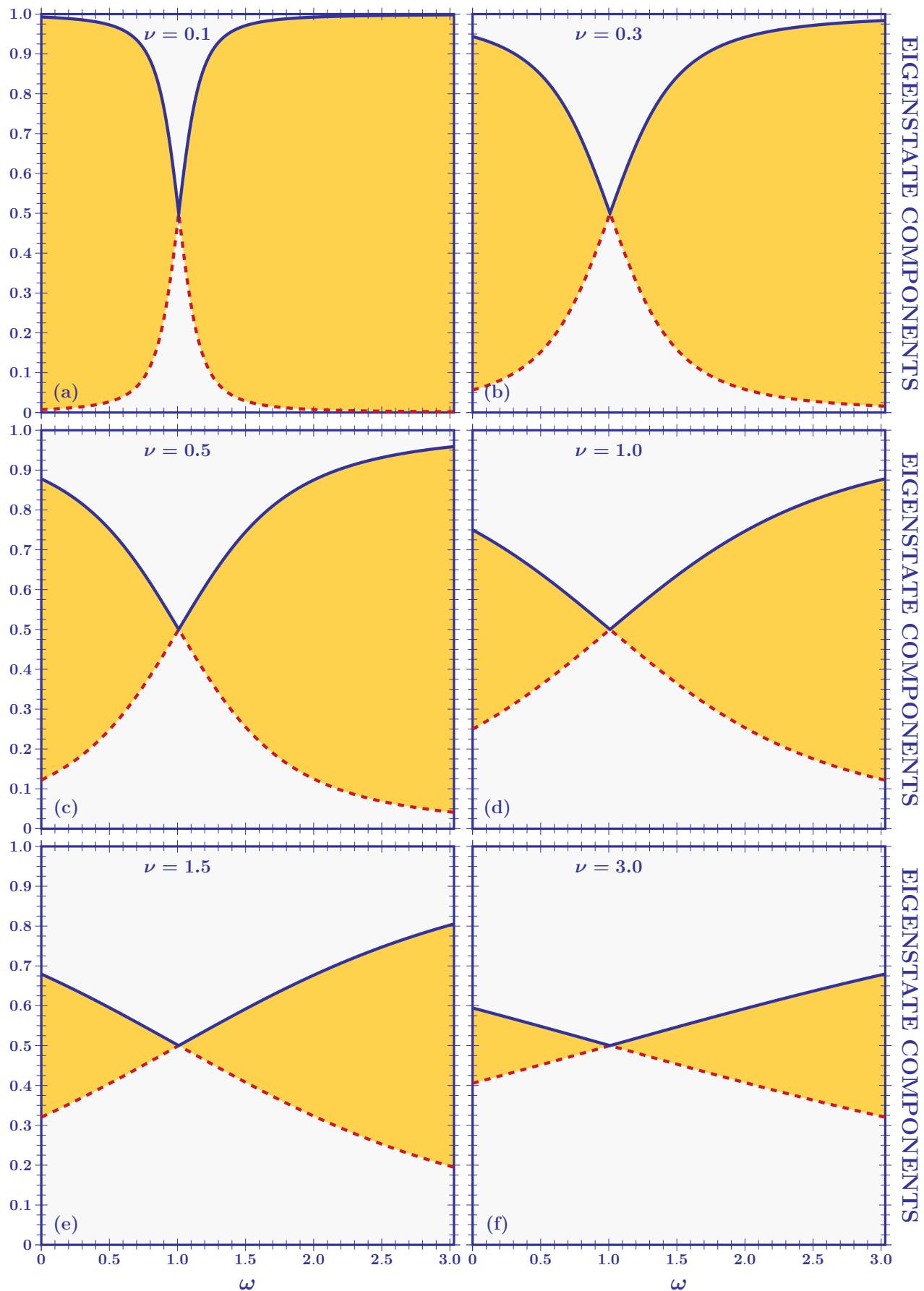
**Figure 4:** Same as Fig. 3 as a function of the ω strength, for different values of ν.
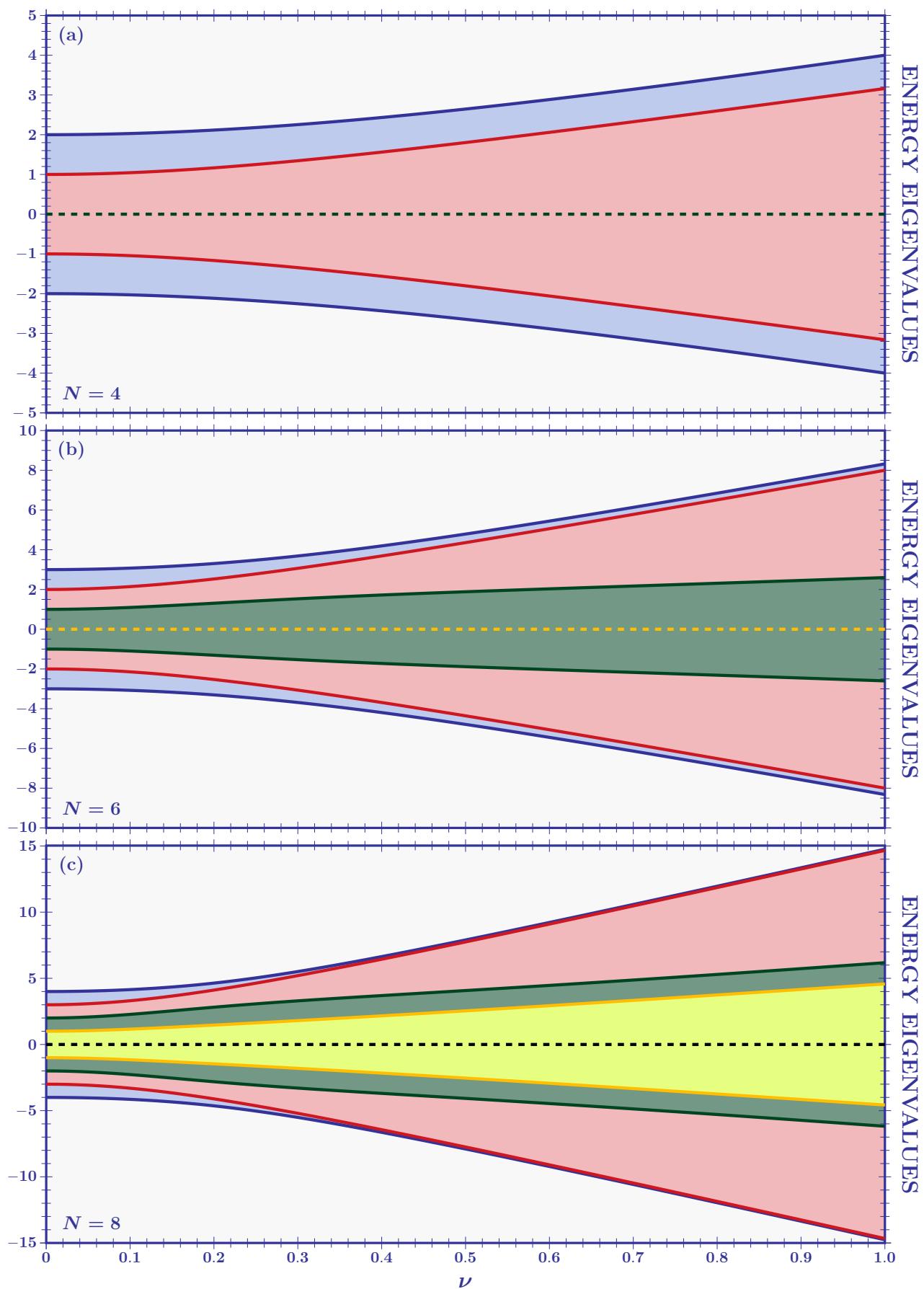
Figure 5: Energy eigenvalues of the $N = 4, 6, 8$ particle systems as a function of ν, when $\omega = 0$.
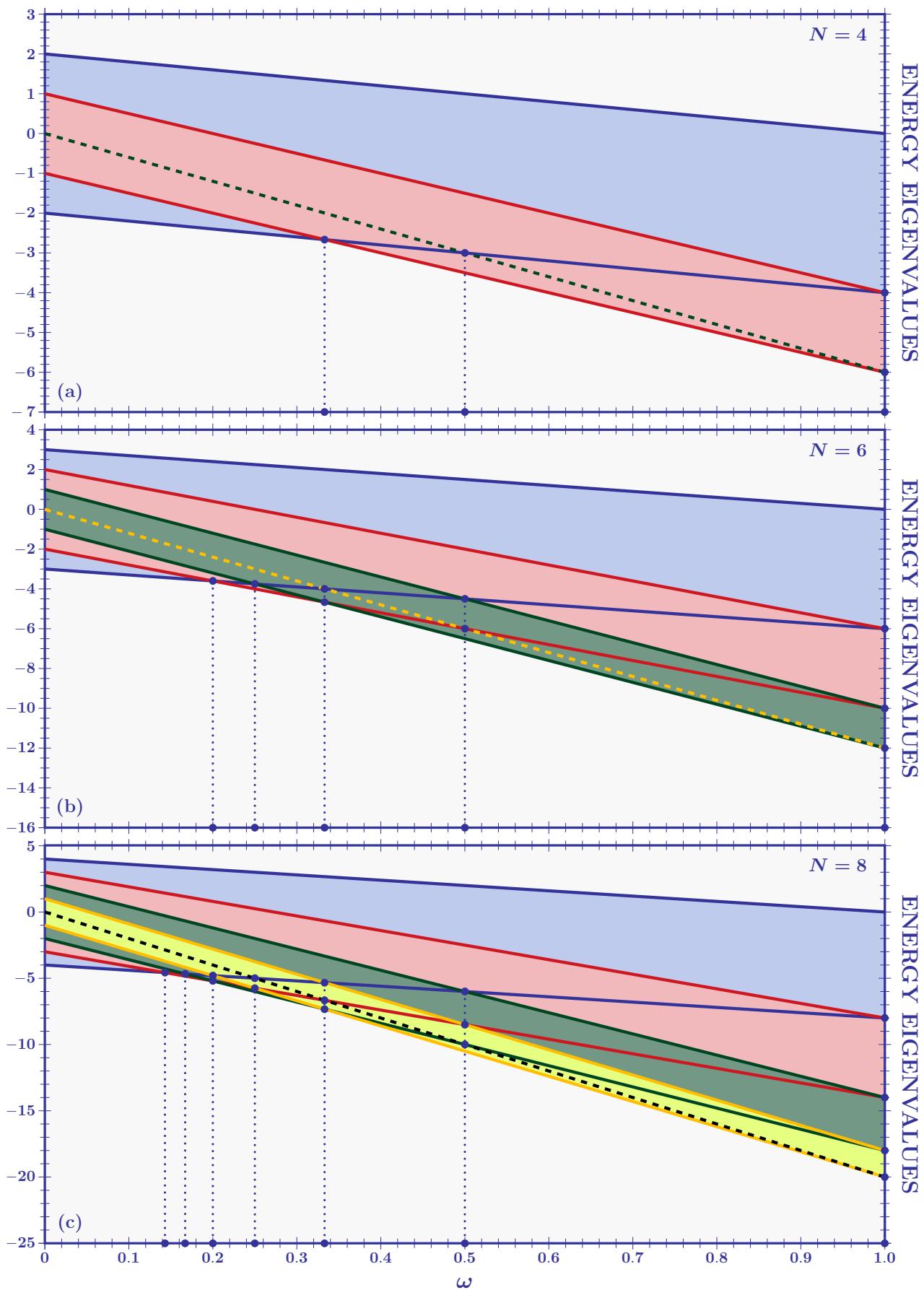
Figure 6: The same as Fig. 5 as a function of ω, when $\nu = 0$. The vertical lines indicate the degeneracy points where the energies of different eigenstates becomes equal.
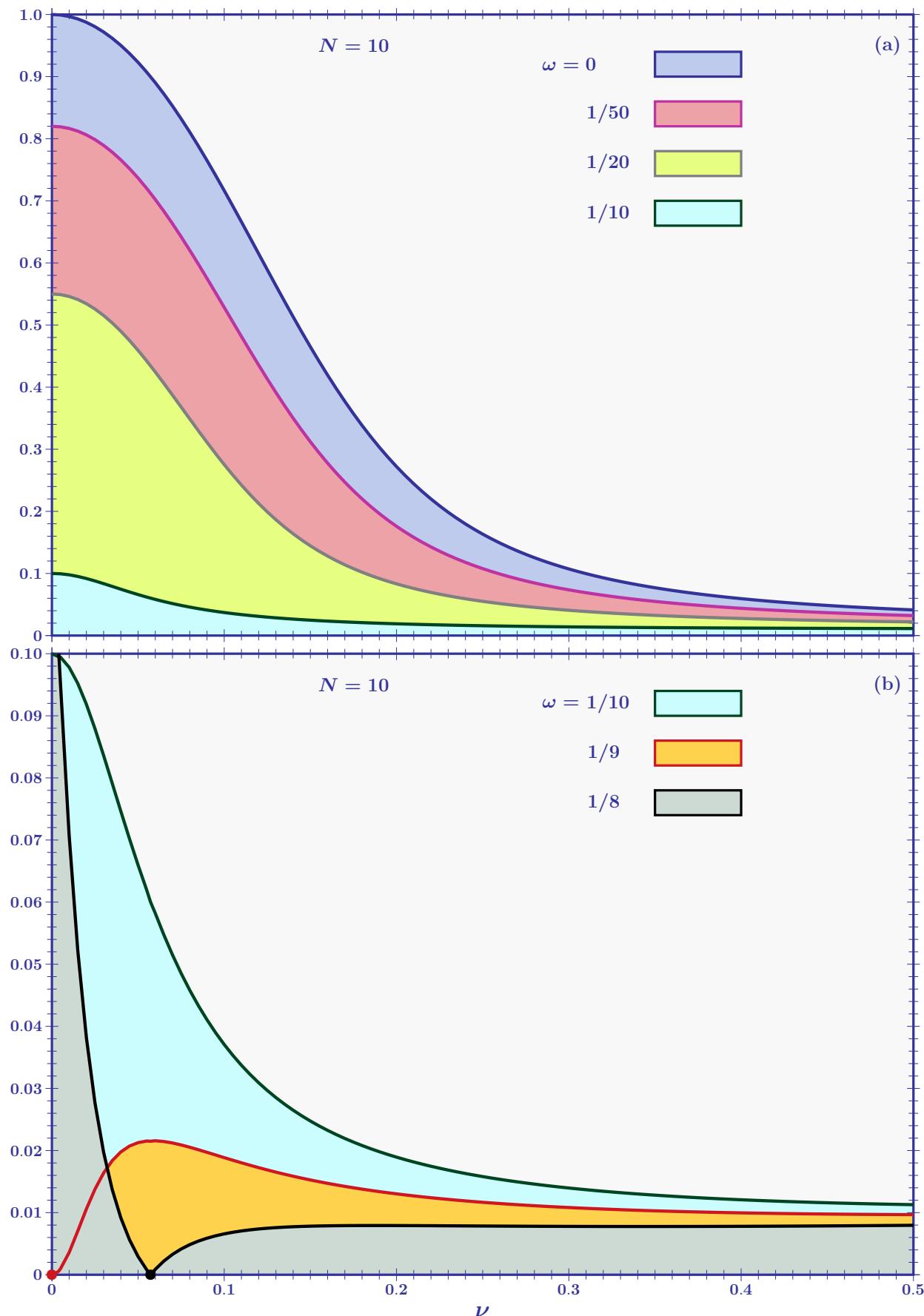
Figure 7: Energy difference between the lowest two states $\Delta_{1,2}$ of a system of 10 particles for different values of ω, as a function of ν. The different coloured areas emphasize the differences.
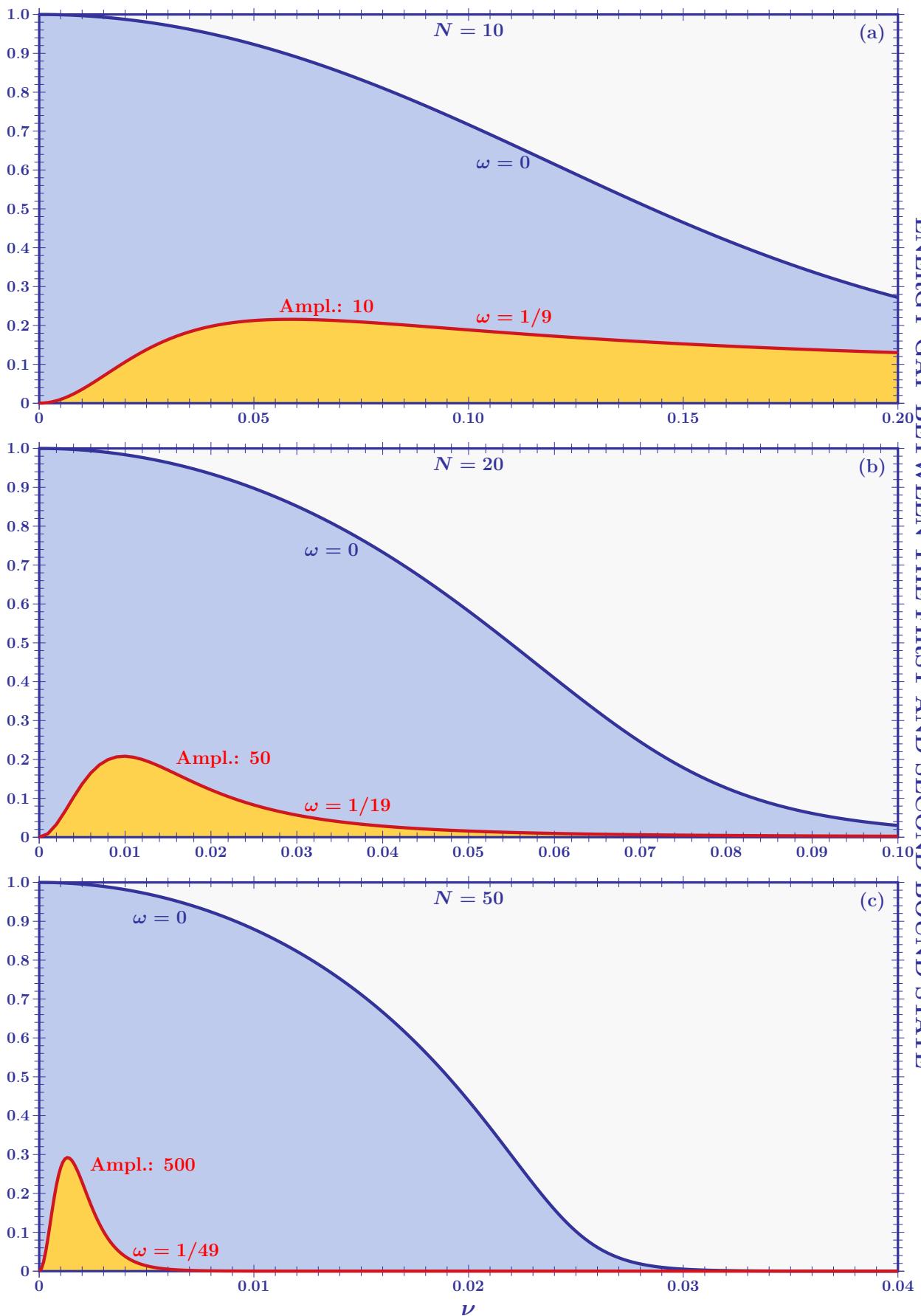
Figure 8: Energy difference $\Delta_{12}^{[N]}$ between the two lowest states of systems with $N = 10, 20, 50$ particles. The blue lines show results obtained with $\omega = 0$ while the red lines for $\omega = 1/(N - 1)$. We point out the different ν scales in the various panels, and the amplification factors of the red lines.