Effective Interactions

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Projection operators in QM

In quantum mechanics (QM) one describes a physical state by an abstract vector Ψ in the Hilbert space. Various representations are possible, for example:

- coordinate space representation: $\Psi(\vec{x})$
- momentum space representation: $\tilde{\Psi}(\vec{p})$
- A representation: A is a hermitean operator ("an observable") with eigenfunctions $\Phi_\mu(\vec{x})$, i.e.

  $$A\Phi_\mu(\vec{x}) = a_\mu \Phi_\mu(\vec{x}) \,,$$

  where a_μ are the real eigenvalues, e.g.,

  $$A = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2 \,.$$

Hamiltonian of a particle in a three-dimensional isotropic harmonic oscillator potential with energy $E_n = (n + \frac{3}{2})\hbar\omega$ and the usual harmonic oscillator wave functions: Φ_μ^{HO}

Definition of Projection Operators:

Assume that P is some subspace of total Hilbert space H and that $\{\phi_\nu\}$ is a set of ortonormal vectors which span subspace P. Then one defines a projection operator for space P by

$$P = \sum_\nu |\phi_\nu\rangle\langle\phi_\nu| = \sum_\nu |\nu\rangle\langle\nu|$$

Thus for any $f \in$ H one obtains: $P \cdot f \in$ P. Note operator depends only on subspace P and not on set of orthonormal functions. Take two different orthormal sets, $\{|\nu\rangle\}$ and $\{|\mu\rangle\}$:

$$|\nu\rangle = \sum_\mu |\mu\rangle\langle\mu|\nu\rangle\,, \quad |\mu\rangle = \sum_\nu |\nu\rangle\langle\nu|\mu\rangle$$

$$\langle\mu|\mu'\rangle = \sum_\nu \langle\mu|\nu\rangle\langle\nu|\mu'\rangle = \delta_{\mu\mu'}$$

$$\begin{aligned}
\sum_\nu |\nu\rangle\langle\nu| &= \sum_{\nu\mu\mu'} |\mu'\rangle\langle\mu'|\nu\rangle\langle\nu|\mu\rangle\langle\mu| \\
&= \sum_{\mu\mu'} |\mu'\rangle\delta_{\mu\mu'}\langle\mu| \\
&= \sum_\mu |\mu\rangle\langle\mu|
\end{aligned}$$

Properties of Projection Operators:

$$
\begin{aligned}
P^\dagger &= \Big(\sum_\nu |\nu\rangle\langle\nu|\Big)^\dagger \\
&= \sum_\nu (\langle\nu|)^\dagger (|\nu\rangle)^\dagger \\
&= \sum_\nu |\nu\rangle\langle\nu| \\
&= P
\end{aligned}
$$

$$
\begin{aligned}
P^2 &= P\sum_\nu |\nu\rangle\langle\nu| \\
&= \sum_{\nu'\nu} |\nu'\rangle\langle\nu'|\nu\rangle\langle\nu| \\
&= \sum_{\nu'\nu} |\nu'\rangle\delta_{\nu\nu'}\langle\nu| \\
&= \sum_\nu |\nu\rangle\langle\nu| \\
&= P
\end{aligned}
$$

Note P is defined in whole Hilbert space H

Separation of Hilbert space

The Hilbert space H can be divided in two orthogonal subspaces P and Q with projection operators P and Q with $P + Q$=1. One thus has

$$f = (P + Q) \cdot f = P \cdot f + Q \cdot f = f_P + f_Q$$

with $f_P \in$ P and $f_Q \in$ Q

Aim:

Microscopic calculation of wavefunction of A-body nuclear system, i.e. exact solution of Schrödinger equation

$$H\Psi_A(1, 2, ..., A) = E\Psi_A(1, 2, ..., A) \qquad (1)$$

in a Hilbert space associated with the A-body system spanned by ortonormal set $\{|\nu\rangle\}$ with

$$H = \sum_{i=1}^{A} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i<j} V_{ij} \;\; (+ \sum_{i<j<k} V_{ijk})$$

V_{ij}: nucleon-nucleon (NN) potential
V_{ijk}: NNN potential

Expansion of wavefunction:

$$\Psi_A = \sum_{\nu=1}^{N} c_\nu |\nu\rangle$$

One obtains the following Schrödinger equation

$$(H - E) \sum_{\nu=1}^{N} c_\nu |\nu\rangle = 0$$

Projection from left with $\langle \nu' |$:

$$\langle \nu' | (H - E) \sum_{\nu=1}^{N} c_\nu |\nu\rangle = 0$$

One has N equations ($\nu' = 1, ..., N$) and can determine the N coefficients c_ν

Convergence: Check convergence of expansion by increasing N

Generally one encounters convergence problems for $A > 2$

Method of Effective Interaction

Introducing an effective interaction one can in principle overcome the convergence problem. One divides the total Hilbert space in P and Q subspaces (projection operators P and Q with $P+Q$=1). Multiplying Schrödinger equation (1) on the left with P yields

$$PH\Psi_A = PE\Psi_A = EP\Psi_A$$

$$P+Q=1 \;\Rightarrow PH(P+Q)\Psi_A = EP\Psi_A$$

$$P^2 = P \text{ and } Q^2 = Q \;\Rightarrow$$

$$PHP(P\Psi_A) + PHQ(Q\Psi_A) = E(P\Psi_A) \quad (2)$$

Similarly one finds

$$QHQ(Q\Psi_A) + QHP(P\Psi_A) = E(Q\Psi_A)$$

or

$$(QHQ - E)(Q\Psi_A) = -QHP(P\Psi_A)$$

$$\Rightarrow \quad (Q\Psi_A) = \frac{1}{E - QHQ} QHP(P\Psi_A)$$

Substitution of $Q\Psi_A$ in eq. (2) leads to

$$PHP(P\Psi_A)+PHQ\frac{1}{E-QHQ}QHP(P\Psi_A) = EP\Psi_A$$

or

$$H_{eff}(P\Psi_A) = EP\Psi_A$$

with

$$H_{eff} = PHP + PHQ\frac{1}{E-QHQ}QHP$$

For a given model space P exists an effective Hamiltonian H_{eff} which solves the full problem in the subspace P

In fact this is true for any operator

$$\langle\Psi_A|O|\Psi_A\rangle = \langle P\Psi_A|O_{eff}|P\Psi_A\rangle$$

Problem:

How to determine H_{eff} ?

Later we come back to this question!

For a better understanding of the effective interaction let us have a look at the two-body system.

Question: Is it sufficient to consider the NN interaction in the following way?

Answer: No!

Nucleon has excitation spectrum: lowest resonance $\Delta(1232)$ with spin und isospin equal to 3/2. The following process with a virtual Δ is possible

Therefore one should consider the Δ in the wavefunction

$$\Psi(1,2) = \Psi_{NN}(1,2) + \Psi_{N\Delta}(1,2)$$

leading to the following Schrödinger equation

$$(H - E)\Big(\Psi_{NN}(1,2) + \Psi_{N\Delta}(1,2)\Big) = 0$$

where

$$H = H_N + H_\Delta + H_{N\Delta} + H_{\Delta N}$$

with

$$H_N = \frac{\vec{p}^2}{2\mu_{NN}} + V_{NN}$$

$$H_\Delta = \frac{\vec{p}^2}{2\mu_{N\Delta}} + (M_\Delta - M_N) + V_{N\Delta}$$

$$H_{N\Delta} = V_{NN,N\Delta}$$

$$H_{\Delta N} = V_{N\Delta,NN} = \Big(V_{NN,N\Delta}\Big)^\dagger$$

($\vec{p}$: relative momentum of particles 1 and 2, μ_{NN} and $\mu_{N\Delta}$ reduced masses of NN and $N\Delta$)

leads to a coupled channel Schrödinger equation:

$$\begin{aligned}(H_N - E)\Psi_{NN} &= -V_{NN,N\Delta}\Psi_{N\Delta} \\ (H_\Delta - E)\Psi_{N\Delta} &= -V_{N\Delta,NN}\Psi_{NN}\end{aligned}$$

It corresponds to a subdivision of the Hilbert space in NN and $N\Delta$ subspaces. Identifying the NN subspace with the P-space and the $N\Delta$ subspace with the Q-space one finds

$$H_N = PHP\,, \quad H_\Delta = QHQ$$

$$H_{\Delta N} = QHP\,, \quad H_{N\Delta} = PHQ$$

We derived before

$$H_{eff} = PHP + PHQ\frac{1}{E - QHQ}QHP$$

Thus the usual NN potential $\tilde{V}_{NN}$ is in reality an effective potential

$$\begin{aligned}\tilde{V}_{NN} &= V_{NN,eff} \\ &= V_{NN} + V_{NN,N\Delta}\frac{1}{E - H_\Delta}V_{N\Delta,NN}\end{aligned}$$

Conclusion:

In NN scattering one does not determine the bare NN interaction PVP, but an effective interaction $P\tilde{V}P$! For example in meson theoretical NN potential models Δ are partially simulated by the exchange of a hypothetical scalar σ meson with mass of about 500 MeV:

Because of term $(M_\Delta - M_N)$ in H_Δ the energy denominator $(E - H_\Delta)^{-1}$ is negative, thus one gets attraction (note $V_{NN,N\Delta}$ and $V_{N\Delta,NN}$ are real in meson theoretical models, therefore $V_{NN,N\Delta} = V_{N\Delta,NN}$)

Consequence for CC calculation:

In a coupled channel (CC) calculation one needs the bare NN potential V_{NN} and thus one cannot use one of the usual realistic NN potential models. One has to fit all potential pieces:
$(V_{NN},\ V_{\Delta\Delta},\ V_{NN,N\Delta},\ V_{N\Delta,NN})$.

In the past there are only a few examples: Argonne V28, CC-Bonn.

Also done:
determination of bare V_{NN} from $\tilde{V}_{NN}$ via

$$V_{NN} = \tilde{V}_{NN} - V_{NN,N\Delta}\frac{1}{E - H_{\Delta}}V_{N\Delta,NN}$$

with a meson theoretical model of $V_{NN,N\Delta}$ (π- plus ρ-exchange) and approximation of $(E - H_{N\Delta})^{-1}$ by a fixed energy denominator.

Conclusion for A $\geq$2:

- Normally one works with effective NN potential models (all modern realistic NN potentials are effective potentials (e.g. AV18, CD-Bonn, Nijmegen models). NN wavefunctions are "effective" wavefunctions relying on the fact that the omission of Δ-degrees of freedom is parametrized in a proper way. In particular at higher energies the NN wavefunction at short and intermediate distances could be somewhat different than usually reflected in realistic NN potential models

  On the other hand wavefunctions cannot be measured directly only observables, which are given in terms of $|\langle\Psi_f|O|\Psi_i\rangle|^2$

- For intrinsic energies E $> m_\pi$: Δ can become a real particle decaying eventually in $N + \pi$ and one has to take into account Δ degrees of freedom explicitly

Conclusion for A>2:

- In a CC calculation one can use the same potential model as for the two-nucleon case

- If one renounces to explicit Δ degrees of freedom, i.e. using the effective interaction $\tilde{V}_{NN}$, one has to correct for the reduction to model space P by taking into account additional NNN-forces

Back to the A-body problem

$$H\Psi_A(1,2,...,A) = E\Psi_A(1,2,...,A)$$

which we reduced to

$$H(P\Psi_A(1,2,...A)) = E(P\Psi_A(1,2,...,A))$$

with

$$H_{eff} = PHP + PHQ\frac{1}{E-QHQ}QHP$$

How can we determine H_{eff}?

To answer this question a short historical review is given

(for more details see B.R. Barrett, Czech. J. Phys. 49, 1 (1999) and references therein)

Short Review of Effective Interaction

Historically the problem of effective interaction in many-body nuclear physics started with shell model calculations. As set $\{\phi_v\}$ which spans the A-body Hilbert space harmonic oscillator wavefunctions are used

Typical problem:

Calculation of wavefunction for a nucleus with only a few nucleons in the valence shell. One makes a separation assumption in a core (c) and a valence (v) part

$$\Psi_A(1, ..., A) = \Psi_c(1, ..., A_c)\Psi_v(A_c + 1, ..., A)$$

with

$$\begin{aligned} H_c\Psi_c &= E_c\Psi_c \\ H_v\Psi_v &= E_v\Psi_v \end{aligned}$$

assuming that Ψ_c is already solved

Example: nucleus ^{18}O with $A_c = 16$:

$$\Psi_v(17,18) = \sum_{i=1}^{\infty} \phi_i^{2\ particle}$$

with two-particle states ϕ_i above core

Infinite number of such states: truncate wave-function and work in a reduced model space

$$\Psi_v^{trun} = \sum_{i}^{sd\ shell} \phi_i^{2\ particle}$$

But:

$$H_v \Psi_v^{trun} \neq E_v \Psi_v^{trun}$$

Introduce effective interaction:

$$H_{v,eff} = H_{v,0} + V_{eff}$$

Methods for determining V_{eff}

- parametrization method

  Take for example $V_{eff} = V_0 exp(-\mu r)$ and fit parameters to sd shell data.

  How well are data described not included in the fit?

- least square method

  Take the two-body matrix elements of $H_{v,eff}$ in the basis space $(\phi_i\phi_j)^{JT}$:

  $$\langle(\phi_i\phi_j)^{JT}|H_{v,eff}|\phi_k\phi_l)^{JT}\rangle$$

  Determine matrix elements by fit to "good" data ("good" data: state is core+valence state and not intruder state)

  Not enough good data to fit all matrix elements → include also sd shell data from A=19, 20, ...

Interesting empirical fact: method leads to extremely good extrapolations with great success in predicting energy levels and other nuclear properties not included in the fit. Seems to work also for nuclei within the same shell with higher values of A not included in the fit

From a more fundamental point of view parametrization and least square methods are not satisfactory.

One would like to have a microscopic theory, i.e. calculate

$$H_{v,eff} = PH_vP + PH_vQ\frac{1}{E - QH_vQ}QH_vP$$

with

$$\begin{aligned} H_v &= T + V \\ &= T + U + V - U = H_{v,0} + \bar{V} \end{aligned}$$

where $H_{v,0} = T + U$ is the unperturbed Hamiltonian and $\bar{V} = V - U$ is the residual interaction

$H_{v,0}$ is diagonal in the single particle basis:

$$H_{v,0}\phi_i = \epsilon_i \phi_i \quad \Rightarrow \text{ for model space P}$$

$$H_{v,eff} = H_{v,0} + \bar{V} + \bar{V}Q\frac{1}{E_v - QH_vQ}Q\bar{V}$$

Expansion of denominator:

$$\bar{V}Q\frac{1}{E_v - H_v}Q\bar{V} = \bar{V}Q\frac{1}{E_v - H_{v,0} - \bar{V}}Q\bar{V}$$

$$= \bar{V}Q\frac{1}{(E_v - H_{v,0})(1 - \frac{\bar{V}}{E_v - H_{v,0}})}Q\bar{V}$$

$$= \bar{V}\frac{Q}{E_v - H_{v,0}}\bar{V} + \bar{V}\frac{Q}{E_v - H_{v,0}}\bar{V}\frac{Q}{E_v - H_{v,0}}\bar{V} + ...$$

note: (i) $(1 - x)^{-1} = 1 + x + x^2 + ...$ and

(ii) $\bar{V}\frac{Q}{E_v - H_{v,0}}\bar{V} = \bar{V}Q\frac{1}{E_v - H_{v,0}}Q\bar{V}$

$$\Rightarrow \quad H_{v,eff} = H_{v,0} + V_{eff}$$

with $V_{eff} = \bar{V} + \bar{V}\frac{Q}{E_v - H_{v,0}}V_{eff} = \bar{V} + \bar{V}\frac{Q}{E_v - H_v}\bar{V}$

The last relation for V_{eff} is known as the energy dependent expression for the effective interaction in open shell nuclei

Calculation of V_{eff} in self-consistent way starting from an NN potential within a perturbation theory

- Bethe-Goldstone-Equation

- Brückner-Reaction-Matrix
  (Brückner K-Matrix: today G-Matrix)

Enormous wealth of calculations

However there is a problem: Study of Schucan and Weidenmüller for the convergence properties of the expansion of V_{eff}: Only convergence if P- and Q-spaces do not overlap!

Problem Intruder States
i.e. shell model assumption in core plus valence nucleons

In all known spectra of nuclei exist states (intruder states) of different structure intruding into the spectrum (e.g. ^{18}O a 4p-2h state)

B. Barrett (Czech. J. Phys. 49, 1 (1999)):

"Theorist cannot hope to compute the structure of finite nuclei from a microscopic point of view without experimental information about the nuclei being investigated, e.g., to calculate spectra in a truncated model space. The theorist must know in advance, which experimental states are intruder states and which ones are good model space states"

Barrett's summary in J. Phys. G 31, 1 (2005):

"Thus, after 40 years of Many-Body-Perturbation-Theory (MBPT) to calculate $V_{NN,eff}$, we have learned that

- it is extremely difficult to calculate $V_{NN,eff}$ beyond third-order in G

- there is a problem with the convergence of individual terms ...

- there is a spurious centre-of-mass motion problem ...

- one needs non-perturbative resumming techniques ...

- ...no clear sign of convergence ... due to the intruder state problem"

Barrett's Conclusion: "G-Matrix is severely challenged ... new and better methods exist now such as the effective interactions generated using the unitary transformation approach"

Unitary Transformation Approach
(Lee-Suzuki)

Once again our problem:

$$H\Psi_A = E\Psi_A\,, \quad H = H_0 + V$$

where H_0 is unperturbed Hamiltonian and V residual interaction. P- and Q-spaces are defined with projection operators P and Q

Consider similarity transformation

$$\mathcal{H} = X^{-1}HX$$

where X is a transformation operator (defined in entire Hilbert space) and has its inverse X^{-1}

Decompose $\mathcal{H}$:

$$\begin{aligned}\mathcal{H} &= 1 \cdot \mathcal{H} \cdot 1 \\ &= (P+Q)\mathcal{H}(P+Q) \\ &= P\mathcal{H}P + P\mathcal{H}Q + Q\mathcal{H}P + Q\mathcal{H}Q\end{aligned}$$

The condition that $P\mathcal{H}P = P(X^{-1}HX)P$ is the P-space effective Hamiltonian:

$$Q\mathcal{H}P = Q(X^{-1}HX)P = 0$$

In fact if $|\phi\rangle$ is a state in the P-space it follows

$$H_{eff}|\phi\rangle = \mathcal{H}|\phi\rangle$$

Many choices possible for X, Lee and Suzuki consider

$$X = \exp(\omega)$$

Properties:

$$\begin{aligned}1 \cdot \omega \cdot 1 &= (P+Q)\omega(P+Q) \\ &= P\omega P + P\omega Q + Q\omega P + Q\omega Q\end{aligned}$$

with

$P\omega P = 0$, $P\omega Q = 0$, $Q\omega P = \omega$, $Q\omega Q = 0$

The ω operator is introduced as an operator which acts as a transformation of a P-space state to a Q-space state. From its properties follows:

$$\omega^2 = Q\omega P Q\omega P = \omega^3 = \ldots = 0 \text{ note } PQ = 0$$

$$\Rightarrow \quad X = \exp(\omega) = 1 + \omega$$

Effective P-space Hamiltonian:

$$\begin{aligned}
P\mathcal{H}P &= P(X^{-1}HX) \\
&= P(1 - Q\omega P)(H_0 + V)(1 + Q\omega P)P \\
&= P\Big[(H_0 + V) - Q\omega P(H_0 + V) \\
&\quad + (H_0 + V)Q\omega P - Q\omega P(H_0 + V)Q\omega\Big]P \\
&= PHP + PVQ\omega P
\end{aligned}$$

Hamiltonian is not hermitean, can be hermitized by

$$\bar{H}_{eff} = [P(1+\omega^\dagger\omega)P]^{\frac{1}{2}} H_{eff} [P(1+\omega^\dagger\omega)P]^{\frac{-1}{2}}$$

(note $V_{eff} = \bar{H}_{eff} - H_0$)

How to calculate ω:

Correct calculation of ω only possible if A-body problem is solved in entire Hilbert space!

No-Core-Shell-Model (NCSM) calculations show:

sufficient to calculate effective two-body interaction via a solution of the two-body problem in entire two-body Hilbert space

No-Core-Shell-Model

(P. Navrátil, J.P. Vary, B.R. Barrett, PRC 62, 054311 (2000) and references therein)

<u>no-core:</u>

Treat core and valence nucleons together in the A-body Hilbert space

$$\text{Hamiltonian}: \; H = \sum_{i=1}^{A} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i,j}^{A} V_{ij}$$

Add a centre-of-mass harmonic oscillator potential:

$$V_{cm} = \frac{1}{2} A m \Omega^2 \vec{R}^{\,2}\,, \quad \vec{R} = \frac{1}{A} \sum_{i=1}^{A} \vec{r}_i$$

Potential does not influence intrinsic properties of the many-body system. It provides however a mean field felt by each nucleon and allows to work with a convenient harmonic oscillator basis

Rewrite $\vec{R}^2$:

$$\begin{aligned}
\sum_{i<j}(\vec{r}_i - \vec{r}_j)^2 &= \frac{1}{2}\sum_{i,j}(\vec{r}_i - \vec{r}_j)^2 \\
&= \frac{1}{2}\sum_{i,j}(\vec{r}_i^{\,2} + \vec{r}_j^{\,2} - 2\vec{r}_i \cdot \vec{r}_j) \\
\Rightarrow \quad \sum_{i,j}\vec{r}_i \cdot \vec{r}_j &= \frac{1}{2}\sum_{i,j}(\vec{r}_i^{\,2} + \vec{r}_j^{\,2}) - \sum_{i<j}(\vec{r}_i - \vec{r}_j)^2 \\
&= A\sum_i \vec{r}_i^{\,2} - \sum_{i<j}(\vec{r}_i - \vec{r}_j)^2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \vec{R}^2 &= \frac{1}{A^2}\sum_{i,j}\vec{r}_i \cdot \vec{r}_j \\
&= \frac{1}{A}\sum_i \vec{r}_i^{\,2} - \frac{1}{A^2}\sum_{i<j}(\vec{r}_i - \vec{r}_j)^2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad V_{cm} &= \frac{1}{2}Am\Omega^2\vec{R}^2 \\
&= \frac{1}{2}m\Omega^2\sum_{i=1}^{A}\vec{r}_i^{\,2} - \frac{1}{2A}m\Omega^2\sum_{i<j}(\vec{r}_i - \vec{r}_j)^2
\end{aligned}$$

$$
\begin{aligned}
H^{\Omega} &= \sum_{i=1}^{A}\Big[\frac{\vec{p}_i^{\,2}}{2m} + \frac{1}{2}m\Omega^2 \sum_{i=1}^{A} \vec{r}_i^{\,2}\Big] \\
&+ \sum_{i<j}^{A}\Big[V_{ij} - \frac{1}{2A}m\Omega^2(\vec{r}_i - \vec{r}_j)^2\Big]
\end{aligned}
$$

Separation of cm and relative motion by use of Jacobi coordinates. Use for intrinsic motion harmonic oscillator basis of an intrinsic Hamiltonian H_0 that cancels the additional Ω^2-term in the residual interaction. A convergent solution does not depend on the choice of Ω, while one gets an Ω dependent result before convergence is reached.

Construct effective two-body interaction from two-body Hamiltonian

$$
H = \Big(\frac{\vec{p}^{\,2}}{2m} + \frac{m\Omega^2\vec{r}^{\,2}}{4} + V_{12} - \frac{m\Omega^2\vec{r}^{\,2}}{4}\Big) = H_0 + \bar{V}
$$

Expand two-body wavefunctions for the various JT-channels in harmonic oscillator wavefunctions.

For a convergent solution excitations up to about 300 $\hbar\Omega$ are needed

No problem for two-body case!

The solution consists in the diagonalization of H with a dimension of d eigenvalues and eigenvectors

Define V_{eff} in model space $N << 300$

Two-nucleon states spanning the model space P: $|\alpha_P\rangle$ (dimension d_P)

Two-nucleon states spanning the Q-space: $|\alpha_Q\rangle$ (dimension d_Q, $d_P + d_Q = d$)

Q-space component of eigenvector $|k_i\rangle$

$$\langle\alpha_Q|k_i\rangle = \sum_{\alpha_P}\langle\alpha_Q|\omega|\alpha_P\rangle\langle\alpha_P|k_i\rangle \qquad (3)$$

(d_Q equations)

One chooses a set $\mathcal{K}$ of d_P eigenvectors $|k_i\rangle$ $\Rightarrow$ $d_P \cdot d_Q$ equations. If the $d_P \times d_P$ matrix $\langle\alpha_P|k_i\rangle$ is invertible for $k_i \in \mathcal{K}$ the operator ω can be determined from eq. (3)

Construction of effective Hamiltonian $H_{2,eff}$

$$\begin{aligned}\langle\gamma_P|H_{2,eff}|\alpha_P\rangle &= \sum_{k\in\mathcal{K}}\Big[\langle\gamma_P|k\rangle E_k\langle k|\alpha_P\rangle \\ &+ \sum_{\alpha_Q}\langle\gamma_P|k\rangle E_k\langle k|\alpha_Q\rangle\langle\alpha_Q|\omega|\alpha_P\rangle\Big]\end{aligned} \qquad (4)$$

Note $P|k\rangle = \sum_{\alpha_P}|\alpha_P\rangle\langle\alpha_P|k\rangle$ for $k\in\mathcal{K}$

is right eigenvector of (4)

<u>Hermitean Hamiltonian</u>

$$\bar{H}_{2,eff} = \Big[P(1+\omega^\dagger\omega)P\Big]^{\frac{1}{2}}H_{2,eff}\Big[P(1+\omega^\dagger\omega)P\Big]^{\frac{-1}{2}}$$

<u>Two-body effective Interaction:</u>

$$V_{2,eff} = \bar{H}_{2,eff} - H_2$$

Important:

$V_{2,eff}$ depends on model space ($V_{2,eff}(P_N)$, excitation up to $N\hbar\Omega$ are included in P-space)

Search for convergence:

increase P-space $V_{2,eff}(P_N) \rightarrow V_{2,eff}(P_{N+2})$ (note $V_{2,eff} \rightarrow V_2$ for $N \rightarrow \infty$)

Next steps:

(i) Use obtained $V_{2,eff}(P_N)$ in many-body Schrödinger equation in the many-body P-space defined by $N_A = N$.

(ii) calculate searched observable up to convergence (note convergence is obtained for $N_A << 300$)

Three-body effective interaction

Start from a solution of the three-body system and construct in an analogous way three-body effective interaction $V_{3,eff}(P_N)$ and increase N up to convergence of the desired observable in the many-body calculation

NCSM collaboration is also working on four-body effective interaction $V_{4,eff}(P_N)$

Note:

- Increase of number of nucleons in effective interaction leads to faster convergence

- A-body interaction $V_{A,eff}(P_N)$ would give same resut for any N!