Gravitational Wave Emission from Compact Stars
Lecture 2

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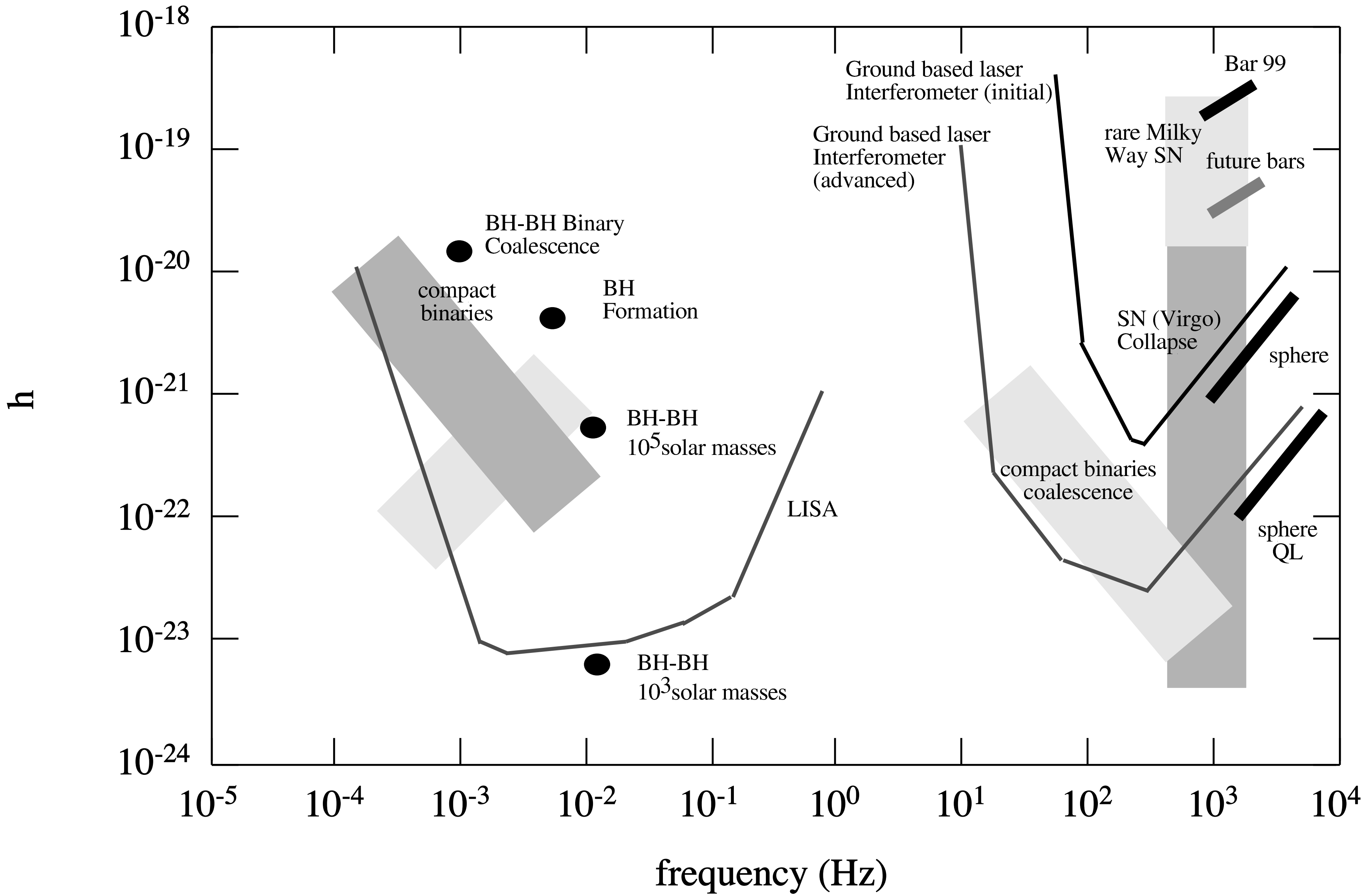
Summary of Lecture 1: Detectors vs sources



Outline

2.1 Quasi-normal modes of neutron (compact) stars and gravitational wave emission

2.2 Dependence of the gravitational wave emission from stellar structure

2.3 Towards gravitational wave asteroseismology

- Bottom line: a star emits gravitational waves at the frequencies of its non radial quasi-normal (complex eigenfrequencies) modes
- The complex frequencies of the quasi-normal modes carry information on the internal structure of the emitting source.
  - ▷ For black holes, it has been shown that they only depend on the parameters that identify spacetime geometry: mass, charge and angular momentum.
  - ▷ For stars, the situation is far less simple, since the quasi-normal mode eigenfrequencies depend on the equation of state (EOS) prevailing in the interior, on which not much is known.
- It is interesting to study these frequencies for different EOS's proposed to describe matter at supernuclear densities, in view of the possibility of extracting information on the internal structure of the star

Preliminaries

- Consider a star characterized by a static and spherically symmetric distribution of matter in chemical, hydrostatic and thermodynamic equilibrium
- The metric of the gravitational field generated by the star can be written ($x^0 = t, x^1 = \varphi, x^2 = r, x^3 = \theta$)

$$ds^2 = g^0_{\mu\nu} dx^\mu dx^\nu = \mathrm{e}^\nu dt^2 - \mathrm{e}^\lambda dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right)$$

- ν and λ are functions of r, to be determined solving Einstein's equations (in geometric units)

$$G_{\mu\nu} = 2\,T_{\mu\nu} \quad \text{inside the star}$$

$$G_{\mu\nu} = 0 \quad \text{outside the star}$$

- Assuming that matter inside the star can be described as a perfect fluid characterized by the four-velocity field u^μ

$$T_{\mu\nu} = (\epsilon + P)\, u_\mu u_\nu - P g_{\mu\nu}$$

  - ▷ ϵ : energy density , P : pressure

- From the tt component of Einstein's equations it follows that

$$\mathrm{e}^{-\lambda(r)} = 1 - \frac{2}{r}\, M(r)$$

$$M(r) = 4\pi \int_0^r \epsilon(r') r'^2 dr'$$

- The rr component yields

$$\frac{d\nu(r)}{dr} = -2 \left(4\pi P(r) r + \frac{M(r)}{r^2} \right) \left(1 - 2\, \frac{M(r)}{r} \right)^{-1}$$

- From energy-momentum conservation

$$\frac{d\nu(r)}{dr} = -\frac{2}{\epsilon(r) + P(r)}\frac{dP}{dr}$$

- Combining together the above relations leads to the equation describing hydrostatic equilibrium of a spherically symmetric star in general relativity: the Tolman-Oppenheimer-Volkoff equation

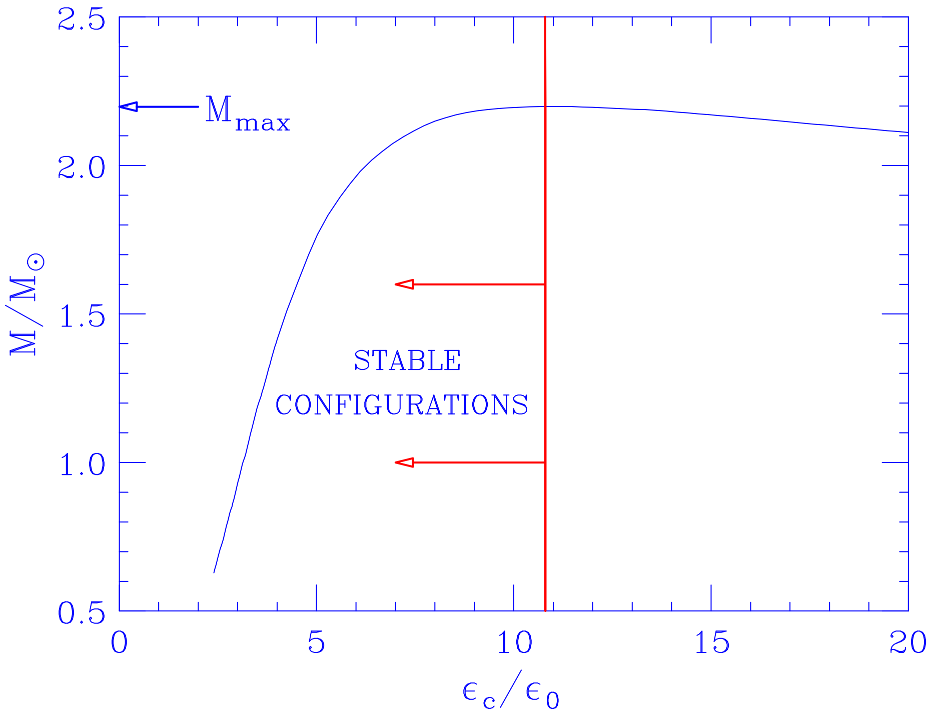
$$\frac{dP}{dr} = -\frac{[\epsilon(r) + P(r)]\left[M(r) + 4\pi r^3 P(r)\right]}{r^2\left[1 - 2M(r)/r\right]} \underrightarrow{c \to \infty} \frac{\epsilon(r)M(r)}{r^2}$$

- In vacuum $\epsilon = P = 0$ and we find the Schwarzschild solution

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\vartheta d\varphi^2\right)$$

Equilibrium configurations of non rotating stars

- typical mass-central energy-density curve



- maximum mass given by

$$M_{max} = M(\overline{\epsilon}_c) \quad , \quad \left(\frac{dM}{d\epsilon_c}\right)_{\epsilon_c=\overline{\epsilon}_c} = 0$$

Non radial oscillations of a star in general relativity

- Consider a perturbation inducing a small amplitude motion described by the displacement 3-vector $\xi^i(x)$
- Due to the fluid motion the geometry of spacetime will no longer be described by the metric tensor $g^0_{\mu\nu}$.
- In the familiar notation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(g^0 + h_{\mu\nu}\right) dx^\mu dx^\nu = ds_0^2 + h_{\mu\nu} dx^\mu dx^\nu$$

- g^0 corresponds to the unperturbed state and can be determined from TOV equation
- At first order, Einstein's equations become a set of *linear* differential equations linking the thirteen functions $\xi^i(x)$ and $h_{\mu\nu}(x)$
- Studying non radial oscillations amounts to determining the solutions of these equations

Properties of the solutions of perturbed equations

- We can assume that the time dependence of the solutions be of the standard form $\exp i\sigma t$ (linearity + stationary background metric)
- It is convenient to single out the radial and angular dependence of the solutions
  - ▷ Scalar functions can be expanded in ordinary spherical harmonics

$$f(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}(r) Y_{\ell m}(\theta,\phi)$$

  - ▷ Under parity transformations spherical harmonics transform according to

$$Y_{\ell m}(\pi-\theta,\pi+\phi) = (-1)^{\ell} Y_{\ell m}(\theta,\phi)$$

- Similarly, any quantity transforming as a tensor under rotations , e.g. h^{ij}, can be expanded in *tensorial* spherical harmonics

$$h^{ij} = \sum_{\ell m} \left[\sum_{k=1}^{3} h^{k,ax}_{\ell m}(r) \left\{ A^{ij}_{\ell m} \right\}^{k} (\theta,\phi) + \sum_{h=1}^{7} h^{h,pol}_{\ell m}(r) \left\{ P^{ij}_{\ell m} \right\}^{h} (\theta,\phi) \right]$$

$$= h^{ij}_{ax} + h^{ij}_{pol}$$

where the three tensorial harmonics called *axial* are *odd* and the seven called *polar* are *even*, as under parity transformations they tranform according to

$$\left\{ P^{ij}_{\ell m} \right\}^{h} (\pi-\theta,\pi+\phi) = (-1)^{\ell} \left\{ P^{ij}_{\ell m} \right\}^{h} (\theta,\phi)$$

$$\left\{ A^{ij}_{\ell m} \right\}^{k} (\pi-\theta,\pi+\phi) = (-1)^{\ell+1} \left\{ A^{ij}_{\ell m} \right\}^{k} (\theta,\phi)$$

- Quasi-normal modes are labelled *polar* or *axial* according to the parity of the corresponding perturbation

Further classification of quasi-normal modes

- Consider the equation describing in Newtonian theory the displacement associated with the perturbation

$$\rho = \rho_0 + \delta\rho \;, \; P = P_0 + \delta P \;, \; \Phi = \Phi_0 + \delta\Phi$$

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\rho_0 \boldsymbol{\nabla} \delta\Phi - \delta\rho \boldsymbol{\nabla} \Phi_0 - \boldsymbol{\nabla} \delta P$$

- In the rhs of the above equation, the *restoring force* consists of three contributions
  - ▷ $\rho_0 \boldsymbol{\nabla} \delta\Phi$: change of the gravitational field
  - ▷ $\delta\rho \boldsymbol{\nabla} \Phi_0$: change of density (buoyancy)
  - ▷ $\boldsymbol{\nabla} \delta P$: gradient of pressure
- Quasi-normal modes are classified according to the prevailing restoring force

Gravitational waves from neutron stars

- a neutron star emits GW at the (complex) frequencies of its quasi-normal modes
  - g-modes: main restoring force is the buoyancy force
  - p-modes: main restoring force is pressure
  - f-modes: intermediate between g- and p-modes
  - w-modes: pure space-time modes
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- in newtonian theory the frequency of the f-mode is proportional to the average density of the star
- g-modes appear in presence of thermal or composition gradients

Gravitational wave emission & equation of state (EOS)

- How do neutron star oscillation modes associated with GW emission depend upon the EOS (i.e. the relation linking pressure and energy-density) describing the properties of matter inside the star ?

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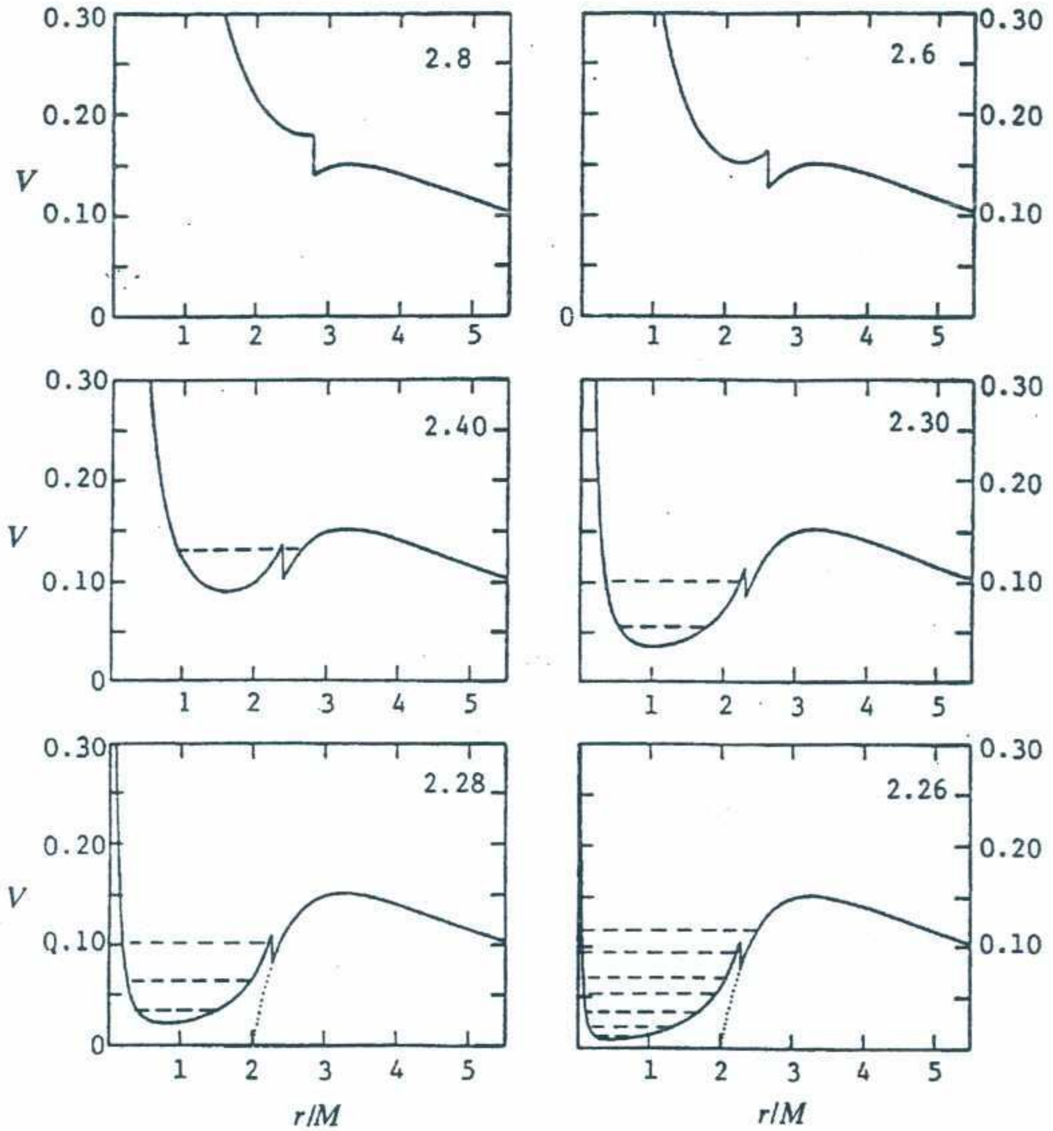
- Consider, for example, the axial (i.e. odd parity) w-modes. Their frequencies are (complex) eigenvalues of a Schrödinger-like equation, whose "potential" $V_\ell(r)$ explicitly depends upon the EOS

$$V_\ell(r) = \frac{\mathrm{e}^{2\nu(r)}}{r^3} \left\{ \ell(\ell+1)r + r^3 \left[\epsilon(r) - P(r)\right] - 6M(r) \right\}$$

wiht (recall)

$$\frac{d\nu}{dr} = -\frac{1}{\left[\epsilon(r) + P(r)\right]} \frac{dP}{dr}$$



2.8
2.6
2.40
2.30
2.28
2.26
V
r/M
0.30
0.20
0.10
0
1
2
3
4
5

Digression: why do we want to learn about the EOS ?

- The EOS is a nontrivial relationship linking the termodynamic variables specifying the state of a system (e.g. Boyle's law of ideal gas)

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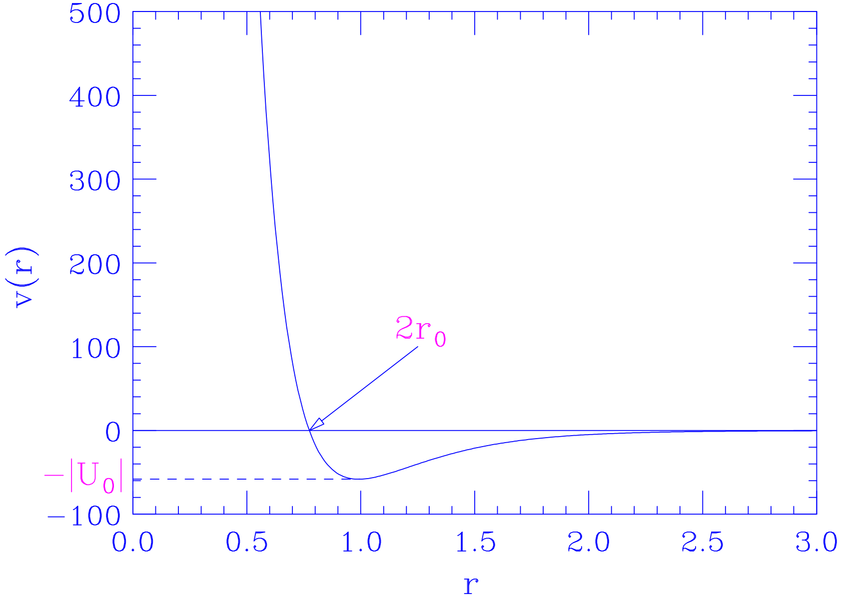

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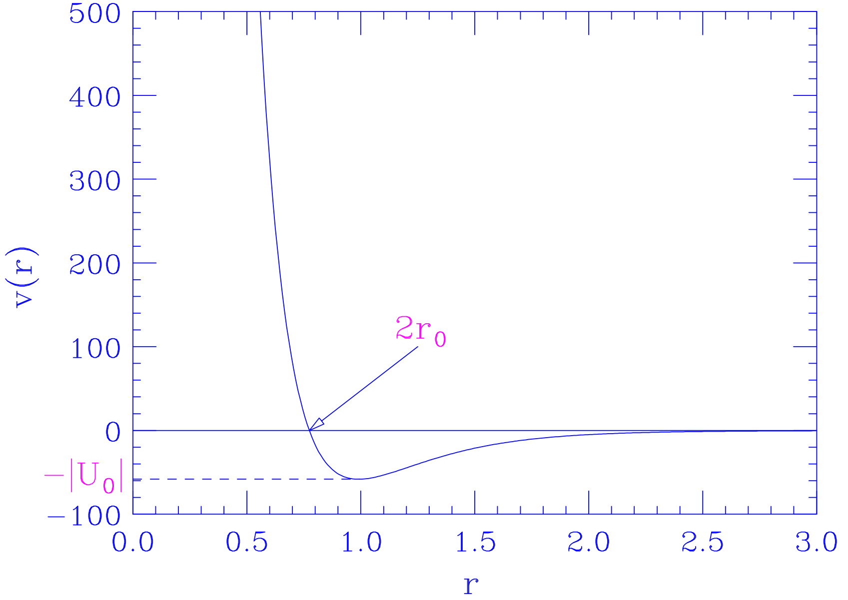

- EOS @ $|U_0|/T \ll 1$

$$P = \frac{\rho T}{1 - b\rho} - a\rho^2$$

$$b = \frac{16}{3}\pi r_0^3 \ , \quad a = \pi \int_{2r_0}^{\infty} |v(r)| r^2 dr$$

Towards gravitational-wave asteroseismology

- Bottom line: want to exploit the (hopefully upcoming) detection of gravitational waves to extract information on the properties of the emitting star (e.g. its radius, or the composition of matter in its interior)

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- Two possible strategies:
  1. Find a set of empirical relations allowing to express the mode frequencies in terms of appropriate "scaling" variables independent of the choice of EOS $\longrightarrow$ use the detected signal to obtain the star radius knowing its mass
  2. Study the dependence of the pattern of emitted gravitational waves for different stellar models, corresponding to different EOS (i.e. different dynamics and composition) $\longrightarrow$ use the detected signal to discriminate between models

Early attempts within strategy 1

- Andersson & Kokkotas, 1998
  - ▷ compute the real and imaginary part of the frequencies of the (polar and axial) f-mode and the first p- and w- modes for a variety of EOS
  - ▷ identify the “scaling” variables

$$\frac{\bar{M}}{\bar{R}^3} \;,\quad \frac{\bar{M}}{\bar{R}}$$

$$\bar{M} = \frac{M}{1.4\ M_\theta} \;,\quad R = \frac{R}{10\ \mathrm{km}}$$

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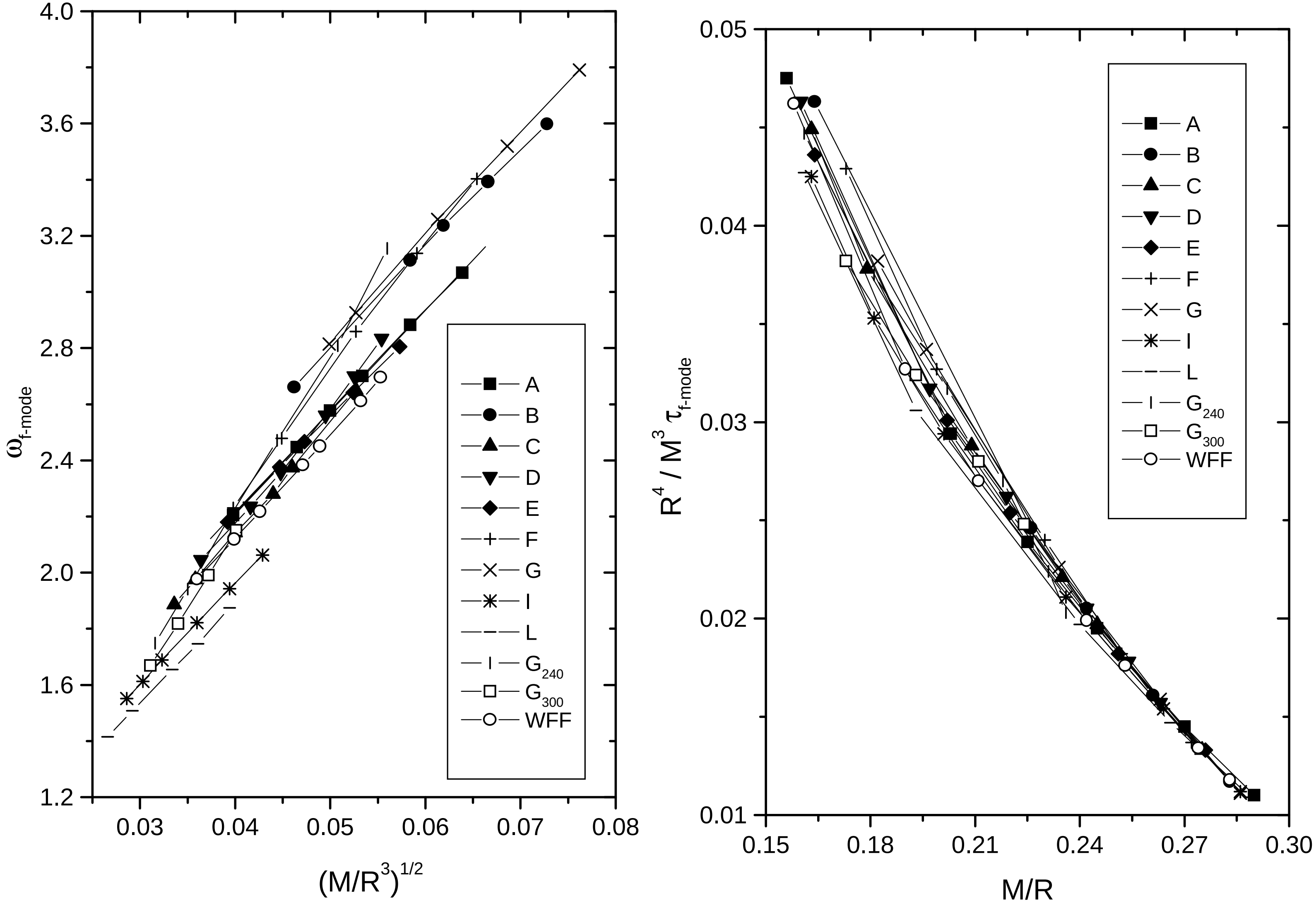
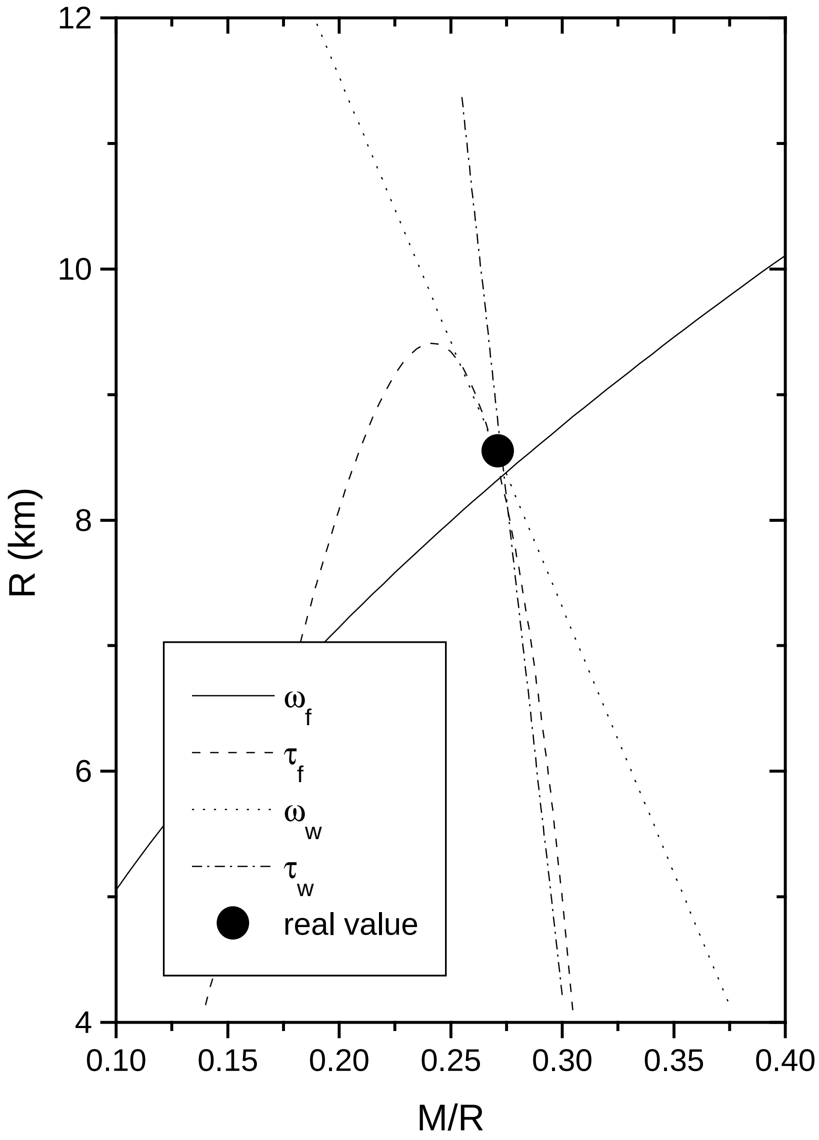
$$\bar{M} = \frac{M}{1.4\ M_\theta}\ ,\ \ R = \frac{R}{10\ \mathrm{km}}$$

- The mode frequencies and damping times, when plotted as a function of the above variables, show very little dependence on the choice of EOS

Frequency and damping time of the f-mode

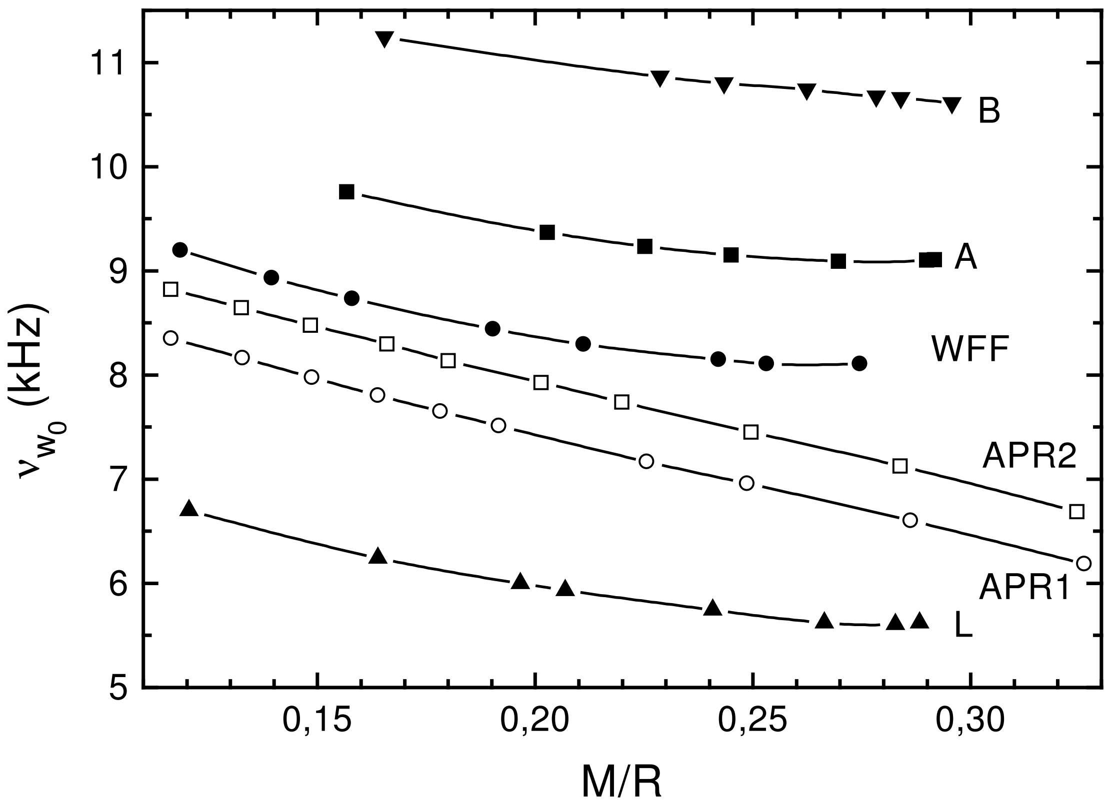


A numerical experiment



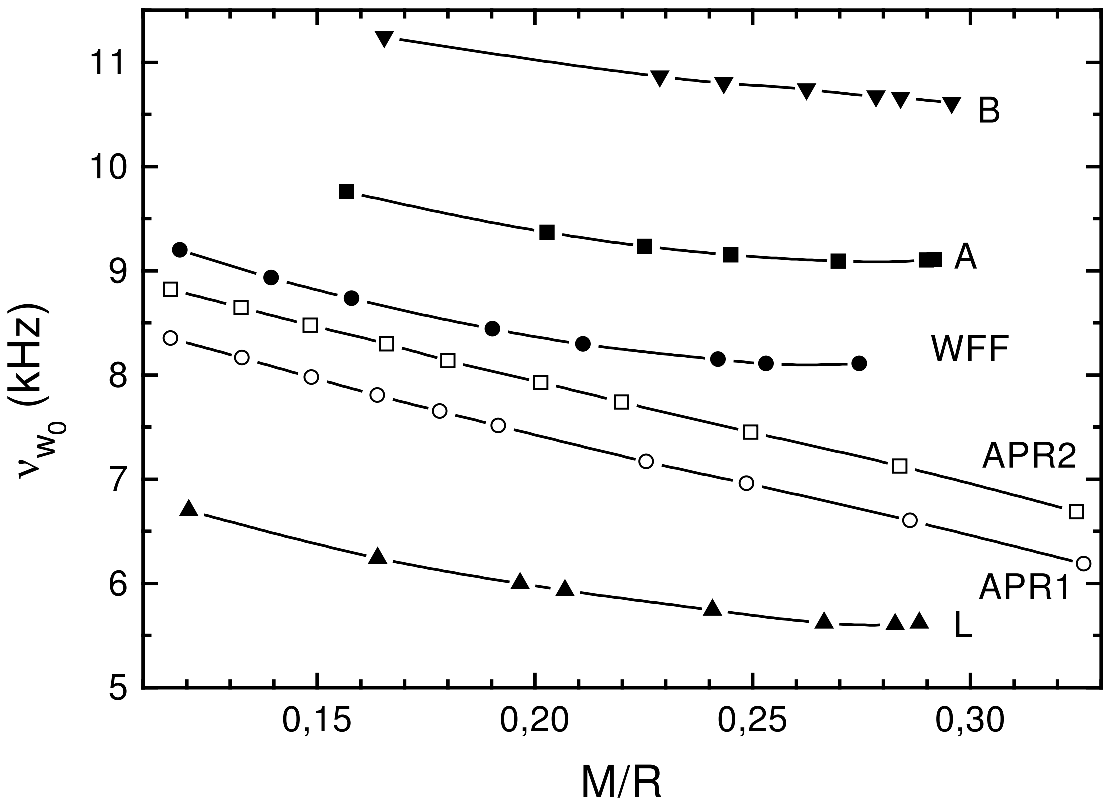
- select a model polytropic star ($P \propto \epsilon^{\Gamma}$ EOS, easily solvable) and compute M and R
- compute frequency and damping time of the f-mode and the 1st w-mode
- plot the four lines corresponding to the empirical relations
- the intersection of the four lines gives the correct M and R with a few percent accuracy

Early attempts within strategy 2



▷ frequency of the 1st axial w-mode vs star compactness (OB, Berti & Ferrari, 1999)

Early attempts within strategy 2



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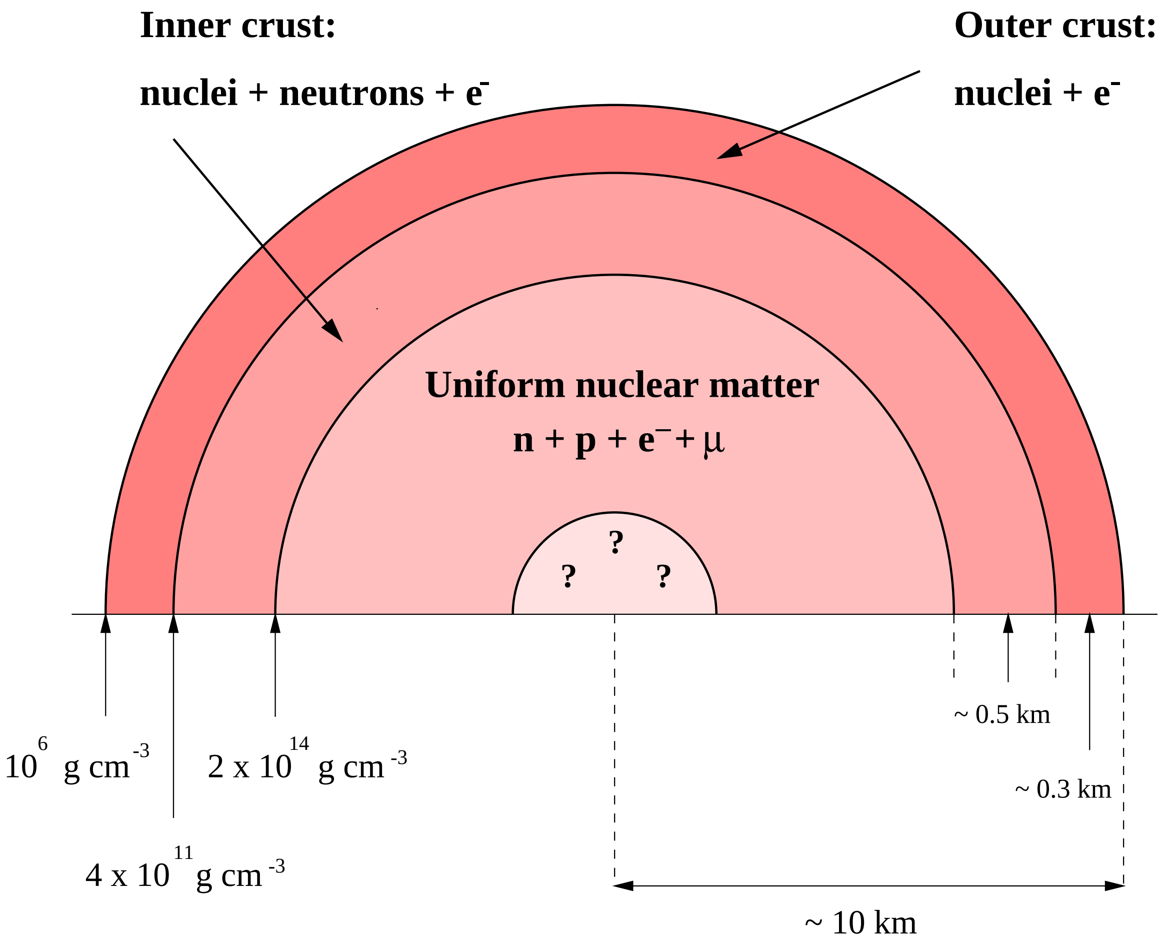
- ▷ the pattern of frequencies strictly reflects the (local) stiffness of the EOS ($\Gamma = d \log P / d \log \rho$). Softer (i.e. lower Γ) EOS correspond to higher frequencies
- ▷ for a given EOS, the frequency depends weakly upon M/R

Overview of Neutron Stars' Structure

- recall: $\rho_0 \approx 0.16\ \mathrm{nucl/fm^3} = 2.67 \times 10^{14}\ \mathrm{g/cm^3}$



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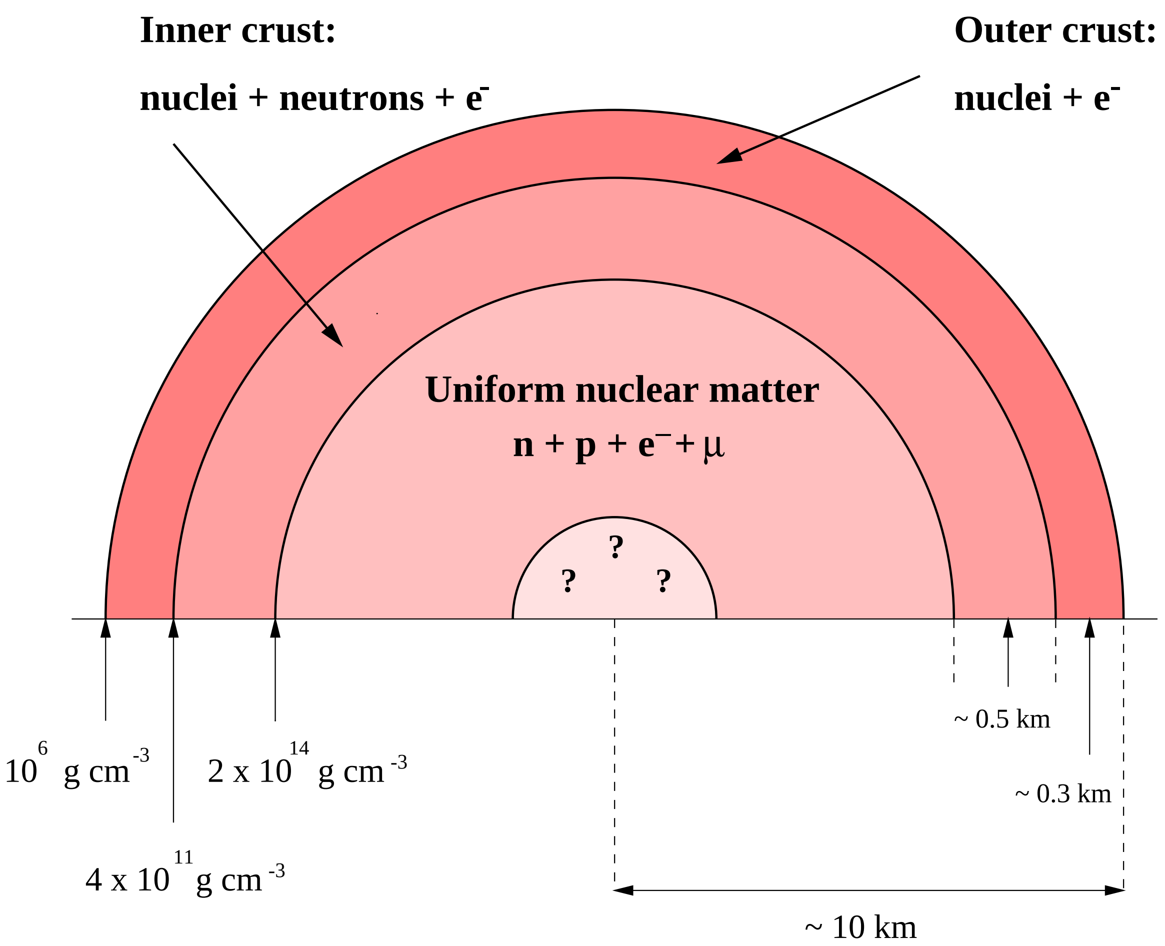

- ▷ ??? : hyperons, π-condensate, K-condensate, quark matter . . .
- ▷ note: most of the neutron star mass is in the region $\rho > \rho_0$