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Zeta determinants on cones and products

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Abstract. We study the zeta determinant of the Laplace operator on two classes of (compact) geometries: cones and product spaces. The result is obtained by applying a new technique introduced in Spreafico (arXiv:math/0607816, 2008) to deal with some class of regularized products.

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1. Introduction

Zeta regularization is a fundamental technique in mathematical and theoretical physics and has been deeply investigated (see for example [9] or [12] and references therein). An interesting problem in this context is to study the analytic properties of the zeta function of the Laplace operator, and in particular the zeta determinant, on specific geometries. The case of a bounded cone was deeply investigated [1–3, 7, 10, 11] because the analytic geometric setting is well known by seminal works of Cheeger [5, 6]. The case of a product space is less studied, but see [13]. In all these works, the approach is based on the so called heat kernel methods (see for example [14]). These methods allow to determine many useful properties of the zeta function, in particular analytic extension, localization of poles and evaluation of residues, but in general they are not particularly efficient to dealing with the derivative of the zeta function at zero, namely with the determinant. In this note we announce some results on the zeta determinant of the Laplace operator on two classes of (compact) geometries: cones and product spaces. These results are obtained by applying some techniques in zeta determinants and regularized products introduced and developed in a series of recent works [19–22]. In particular, a detailed account and complete proofs can be found in [22]. We show that using this method more effective results on the zeta determinant are obtained. We postpone further comments to the end of the paper. This note is organized as follows. In Sect. 2, we briefly present our technique. In Sect. 3, we recall some general facts on the zeta function of the Laplacian on a compact connected Riemannian manifold, that will be used in the following sections. In Sects. 4 and 5, we give the zeta determinant for the Laplace operator on a cone and on a product space, respectively.

2. Zeta determinants for double sequences of spectral type

We recall in this section the method we use to evaluate the zeta determinants. This is based on [19–22]. Let $S = \{a_n\}_{n=1}^{\infty}$ be a sequence of non vanishing complex numbers, ordered by increasing modules, with unique point of accumulation at infinite. Denote by $\mathsf{e}(S)$ the exponent of convergence of S, and assume $\mathsf{e}(S) < \infty$. Denote by $\mathsf{g}(S)$ the genus of S. We define the zeta function associated to S by the uniformly convergent series

$$\zeta(s, S) = \sum_{n=1}^{\infty} a_n^{-s},$$

when $\mathrm{Re}(s) > \mathsf{e}(S)$, and by analytic continuation otherwise. We call the open subset $\rho(S) = \mathbb{C} - S$ of the complex plane the resolvent set of S. For all $\lambda \in \rho(S)$, we define the Gamma function associated to S by the canonical product

$$\frac{1}{\Gamma(-\lambda, S)} = \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{a_n}\right) \mathrm{e}^{\sum_{j=1}^{\mathsf{g}(S)} \frac{(-1)^j}{j} \frac{(-\lambda)^j}{a_n^j}}.$$

When necessary in order to define the meromorphic branch of an analytic function, the domain for λ will be the open subset $\mathbb{C} - [0, \infty)$ of the complex plane. We use the notation $\Sigma_{\theta,c} = \left\{z \in \mathbb{C} \mid |\arg(z - c)| \leq \frac{\theta}{2}\right\}$, with $c \geq \delta > 0$, $0 < \theta < \pi$. We use $D_{\theta,c} = \mathbb{C} - \Sigma_{\theta,c}$, for the complementary (open) domain and $\Lambda_{\theta,c} = \partial\Sigma_{\theta,c} = \left\{z \in \mathbb{C} \mid |\arg(z - c)| = \frac{\theta}{2}\right\}$, oriented counter clockwise, for the boundary. For simplicity, we assume that each of our sequences S is contained in the interior of some sector $\Sigma_{\theta,c}$, and we call the complementary domain $D_{\theta,c}$ the *asymptotic domain* of S. We define now a particular subclass of sequences. Let S be as above, and assume that $\mathsf{e}(S) < \infty$, and that the logarithm of the associated Gamma function has a uniform asymptotic expansion for large $\lambda \in D_{\theta,c}$ of the following form

$$\log \Gamma(-\lambda, S) \sim \sum_{j=0}^{\infty} a_{\alpha_j,0} (-\lambda)^{\alpha_j} + \sum_{k=0}^{\mathsf{g}(S)} a_{k,1} (-\lambda)^k \log(-\lambda),$$

where $\{\alpha_j\}$ is a decreasing sequence of real numbers. Then, we say that S is a *totally regular sequence of spectral type with infinite order*. Next, let $S = \{\lambda_{n,k}\}_{n,k=1}^{\infty}$ be a double sequence of non vanishing complex numbers with unique accumulation point at the infinity, finite exponent $s_0 = \mathsf{e}(S)$ and genus $p = \mathsf{g}(S)$. Assume if necessary that the elements of S are ordered as $0 < |\lambda_{1,1}| \leq |\lambda_{1,2}| \leq |\lambda_{2,1}| \leq \cdots$. We use the notation $S_n(S_k)$ to denote the simple sequence with fixed $n(k)$. We call the exponents of S_n and S_k the relative exponents of S, and we use the notation $(s_0 = \mathsf{e}(S), s_1 = \mathsf{e}(S_k), s_2 = \mathsf{e}(S_n))$. We define relative genus accordingly.

Definition 2.1. Let $S = \{\lambda_{n,k}\}_{n,k=1}^{\infty}$ be a double sequence with finite exponents (s_0, s_1, s_2), and genus (p_0, p_1, p_2). Let $U = \{u_n\}_{n=1}^{\infty}$ be a totally regular sequence of spectral type of infinite order with exponent r_0, and genus q. We say that S is spectrally decomposable over U with power κ, length ℓ and asymptotic domain $D_{\theta,c}$, if there exist positive real numbers κ, and ℓ (integer), such that: (1) the sequence $u_n^{-\kappa} S_n = \left\{\frac{\lambda_{n,k}}{u_n^{\kappa}}\right\}_{k=1}^{\infty}$ is a totally regular sequence of spectral type of infinite order for each n; (2) the logarithmic Γ-function associated to S_n/u_n^{κ} has an asymptotic expansion for large n uniformly in λ for λ in $D_{\theta,c}$, of the following form

$$\log \Gamma(-\lambda, u_n^{-\kappa} S_n) = \sum_{h=0}^{\ell} \phi_{\sigma_h}(\lambda) u_n^{-\sigma_h} + \sum_{l=0}^{L} P_{\rho_l}(\lambda) u_n^{-\rho_l} \log u_n + o(u_n^{-r_0}),$$

where σ_h and ρ_l are real numbers with $\sigma_0 < \cdots < \sigma_\ell$, $\rho_0 < \cdots < \rho_L$, the $P_{\rho_l}(\lambda)$ are polynomials in λ satisfying the condition $P_{\rho_l}(0) = 0$, ℓ and L are the larger integers such that $\sigma_\ell \leq r_0$ and $\rho_L \leq r_0$.

In order to state our main result, we need some more notation. First, we define

$$\Phi_{\sigma_h}(s) = \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} \phi_{\sigma_h}(\lambda) d\lambda dt.$$

Second, for all n, we have the expansions:

$$\log \Gamma(-\lambda, S_n/u_n^{\kappa}) \sim \sum_{j=0}^{\infty} a_{\alpha_j,0,n} (-\lambda)^{\alpha_j} + \sum_{k=0}^{p_2} a_{k,1,n} (-\lambda)^k \log(-\lambda),$$

$$\phi_{\sigma_h}(\lambda) \sim \sum_{j=0}^{\infty} b_{\sigma_h,\alpha_j,0} (-\lambda)^{\alpha_j} + \sum_{k=0}^{p_2} b_{\sigma_h,k,1} (-\lambda)^k \log(-\lambda),$$

for large λ in $D_{\theta,c}$. Then, we set (see Lemma 3.5 of [22]), for $0 \leq k \leq p_2$,

$$A_{\alpha_j,0}(s) = \sum_{n=1}^{\infty} \left(a_{\alpha_j,0,n} - \sum_{h=0}^{\ell} b_{\sigma_h,\alpha_j,0} u_n^{-\sigma_h} \right) u_n^{-\kappa s}, \quad \alpha_j \neq 0, 1, \ldots, p_2,$$

$$A_{0,0}(s) = \sum_{n=1}^{\infty} \left(a_{0,0,n} - \sum_{h=0}^{\ell} b_{\sigma_h,0,0} u_n^{-\sigma_h} \right) u_n^{-\kappa s},$$

$$A_{k,0}(s) = \sum_{n=1}^{\infty} \left(a_{k,0,n} - \sum_{h=0}^{\ell} b_{\sigma_h,k,0} u_n^{-\sigma_h} - \sum_{l=0}^{L} p_{\rho_l,k} u_n^{-\rho_l} \log u_n \right) u_n^{-\kappa s}, \quad k \neq 0,$$

$$A_{k,1}(s) = \sum_{n=1}^{\infty} \left(a_{k,1,n} - \sum_{h=0}^{\ell} b_{\sigma_h,k,1} u_n^{-\sigma_h} \right) u_n^{-\kappa s}.$$

With this notation, we have the following theorem (see [22, Theorem 3.9], and [20, Proposition 1]).

Theorem 2.2 (Spectral decomposition lemma). *Let the double sequence S be spectrally decomposable over U with power κ and length ℓ, then*

$$\begin{aligned}
\operatorname*{Res_1}_{s=0} \zeta(s,S) &= \frac{1}{\kappa}\sum_{h=0}^{\ell} \operatorname*{Res_2}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_1}_{s=\sigma_h} \zeta(s,U),\\
\operatorname*{Res_0}_{s=0} \zeta(s,S) &= \sum_{h=0}^{\ell} \operatorname*{Res_2}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_0}_{s=\sigma_h} \zeta(s,U) - A_{0,1}(0)\\
&\quad + \frac{1}{\kappa}\sum_{h=0}^{\ell} \operatorname*{Res_1}_{s=\sigma_h} \zeta(s,U) \left(\operatorname*{Res_1}_{s=0} \Phi_{\sigma_h}(s) + \gamma \operatorname*{Res_2}_{s=0} \Phi_{\sigma_h}(s)\right),\\
\operatorname*{Res_0}_{s=0} \zeta'(s,S) &= \frac{1}{\kappa}\left(\frac{\gamma^2}{2} - \frac{\pi^2}{12}\right) \sum_{h=0}^{\ell} \operatorname*{Res_2}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_1}_{s=\sigma_h} \zeta(s,U)\\
&\quad + \frac{\gamma}{\kappa}\sum_{h=0}^{\ell} \operatorname*{Res_1}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_1}_{s=\sigma_h} \zeta(s,U) + \gamma \sum_{h=0}^{\ell} \operatorname*{Res_2}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_0}_{s=\sigma_h} \zeta(s,U)\\
&\quad + \frac{1}{\kappa}\sum_{h=0}^{\ell} \operatorname*{Res_0}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_1}_{s=\sigma_h} \zeta(s,U) + \kappa \sum_{h=0}^{\ell} \operatorname*{Res_2}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_0}_{s=\sigma_h} \zeta'(s,U)\\
&\quad + \sum_{h=0}^{\ell} \operatorname*{Res_1}_{s=0} \Phi_{\sigma_h}(s) \operatorname*{Res_0}_{s=\sigma_h} \zeta(s,U) - A_{0,0}(0) - A'_{0,1}(0).
\end{aligned}$$

Next, we consider the particular case where S is the sum of two sequences. More precisely, let $S_{(i)} = \{\lambda_{(i),n_i}\}_{n_i\in\mathbb{N}_0}$, $i = 1, 2$, be two totally regular sequences of spectral type with finite exponents $s_{(i)}$, genus $p_{(i)}$, and orders $\alpha_{(i),N_{(i)}} \leq 0$. Assume $\lambda_{(1),n_1} + \lambda_{(2),n_2} \neq 0$ for all (n_1, n_2). Then the sum sequence $S_{(0)} = S_{(1)} + S_{(2)}$ is a totally regular sequence of spectral type with exponent $s_{(0)} = s_{(1)} + s_{(2)}$, genus $p_{(0)} = [s_{(0)}]$, and order $\alpha_{(0),N_{(0)}} = \min(\alpha_{(i),N(i)}) \leq 0$, and we have:

Theorem 2.3. *Suppose that $\alpha_{(1),N_{(1)}} < -p_{(2)} - 1$, and that $-\alpha_{(2),N_{(2)}} \geq s_{(1)}$. Then, the sequence $S_{(0)} = \{\lambda_{(1),n_1} + \lambda_{(2),n_2}\}_{n_i\in\mathbb{N}_0}$ is spectrally decomposable over $S_{(1)}$ with power 1, and finite length $\ell \leq N_{(2)}$. The length ℓ is the larger integer such that $-\alpha_{(2),\ell} \leq s_{(1)}$, where $\alpha_{(2),h}$ are the powers of the terms of the expansion of the Γ-function $\log\Gamma(-\lambda, S_{(2)})$. The zeta function associated to the sum sequence $S_{(0)}$ is regular at $s = 0$, and*

$$\begin{aligned}
\zeta(0, S_{(0)}) &= \sum_{h=0}^{\ell} c_{(1),-\alpha_{(2),h},0}c_{(2),\alpha_{(2),h},0} = a_{(1),0,1}a_{(2),0,1}\\
&\quad + \sum_{j=1}^{p_{(2)}} (-1)^{j+1} j a_{(1),-j,0}a_{(2),j,1} + \sum_{h=0}^{N} \frac{a_{(1),-\alpha_{(2),h},0}a_{(2),\alpha_{(2),h},0}}{\Gamma(-\alpha_{(2),h})\Gamma(\alpha_{(2),h})},\\
\zeta'(0, S_{(0)}) &= -\sum_{l=0}^{p_{(2)}} a_{(2),l,1}\left(\zeta'(-l, S_{(1)}) + (\gamma + \psi(l+1))\zeta(-l, S_{(1)})\right)
\end{aligned}$$

$$+\sum_{\substack{h=0,\\ \alpha_{(2),h}\notin\mathbb{N}}}^{\ell} a_{(2),\alpha_{(2),h},0}\left(\operatorname*{Res_0}_{s=-\alpha_{(2),h}}\zeta(s,S_{(1)})+(\gamma+\psi(-\alpha_{(2),h}))\operatorname*{Res_1}_{s=-\alpha_{(2),h}}\zeta(s,S_{(1)})\right)$$
$$+\log\prod_{n_1=1}^{\infty}\mathrm{e}^{-\sum_{j=0}^{p_{(2)}}a_{(2),j,1}\lambda_{(1),n_1}^{j}\log\lambda_{(1),n_1}-\sum_{h=0}^{\ell}a_{(2),\alpha_{(2),h},0}\lambda_{(1),n_1}^{\alpha_{(2),h}}}\Gamma(\lambda_{(1),n_1},S_{(2)}).$$

3. Zeta invariants of Riemannian manifolds

Let (M,g_M) be a compact connected Riemannian manifold of dimension m, with metric g_M. Let Δ_M denote the (negative of the) metric Laplacian, and $\mathrm{Sp}\Delta_M=\{\lambda_n\}_{n=0}^{\infty}$ ($\lambda_0=0$) the spectrum of Δ_M. It is well known that there exists a full asymptotic expansion for the trace of the heat kernel of the Laplacian for small t,

$$\mathrm{Tr}_{L^2}\mathrm{e}^{-t\Delta_M}=t^{-\frac{m}{2}}\sum_{j=0}^{\infty}e_j t^{\frac{j}{2}}, \tag{3.1}$$

where the coefficients depend only on local invariants constructed from the metric tensor, and are in principle calculable from it (and all the coefficients of odd index vanish if the manifold has no boundary).

Proposition 3.1. *The sequence* $\mathrm{Sp}_+\Delta_M$ *of the positive eigenvalues of the metric Laplacian on a compact connected Riemannian manifold of dimension* m*, is a totally regular sequence of spectral type, with finite exponent* $\mathsf{e}=\frac{m}{2}$*, genus* $\mathsf{g}=[\mathsf{e}]$*, spectral sector* $\Sigma_{\epsilon,c}$ *with shift* $0<c<\lambda_1$*, asymptotic domain* $D_{\epsilon,c}$*, and infinite order.*

We have the following formulas for the coefficients in the expansion of the logarithmic Γ-function: $\alpha_h=\frac{m-h}{2}$, and

$$\begin{aligned}\log\Gamma(-\lambda,\mathrm{Sp}_+\Delta_M)&=(\dim\ker\Delta_M-e_m)\log(-\lambda)\\&\quad+\sum_{j=1}^{[m/2]}\frac{(-1)^{j+1}}{j!}e_{m-2j}(-\lambda)^j\log(-\lambda)\\&\quad+\sum_{h=0}^{\infty}a_{\frac{m-h}{2},0}(-\lambda)^{\frac{m-h}{2}},\end{aligned}$$

where ((n) denotes the parity of n)

$$a_{\frac{m-h}{2},0}=\begin{cases}\Gamma\left(\frac{h-m}{2}\right)e_h, & (h)\neq(m)\text{ or }(h)=(m)\text{ and }h>m,\\ (-1)^{\frac{m-h}{2}}\dfrac{\frac{e_h}{\frac{m-h}{2}!}-\operatorname*{Res_0}_{s=\frac{m-h}{2}}\zeta(s,\mathrm{Sp}_+\Delta_M)}{\frac{m-h}{2}}, & (h)=(m)\text{ and }h<m,\\ -\zeta'(0,\mathrm{Sp}_+\Delta_M), & h=m.\end{cases}$$

This allows to resume all the information on the zeta function as follows.

Proposition 3.2. *The zeta function $\zeta(s, \mathrm{Sp}_+\Delta_M)$ has a meromorphic continuation to the whole complex plane up to simple poles at the values of $s = \frac{m-h}{2}$, $h = 0, 1, 2, \ldots$, that are not negative integers nor zero, with residues*

$$\mathrm{Res}_1{}_{s=\frac{m-h}{2}}\,\zeta(s, \mathrm{Sp}_+\Delta_M) = \frac{e_h}{\Gamma\left(\frac{m-h}{2}\right)} = \begin{cases} \dfrac{a_{\frac{h-m}{2},0}}{\Gamma\left(\frac{h-m}{2}\right)\Gamma\left(\frac{m-h}{2}\right)}, & (h) \neq (m), \\ (-1)^{\frac{h-m}{2}+1}\dfrac{h-m}{2}a_{\frac{h-m}{2},1} & (h) = (m), \end{cases}$$

$$\mathrm{Res}_0{}_{s=\frac{m-h}{2}}\,\zeta(s, \mathrm{Sp}_+\Delta_M) = (-1)^{\frac{h-m}{2}+1}\frac{h-m}{2}a_{\frac{h-m}{2},1} + \frac{e_h}{\frac{m-h}{2}!}, \qquad (h) = (m);$$

the point $s = -k = 0, -1, -2, \ldots$ are regular points and

$$\begin{aligned} \zeta(0, \mathrm{Sp}_+\Delta_M) &= a_{0,1} = e_m - \mathrm{dimker}\Delta_M, \\ \zeta'(0, \mathrm{Sp}_+\Delta_M) &= -a_{0,0}, \\ \zeta(-k, \mathrm{Sp}_+\Delta_M) &= (-1)^{k+1}k a_{k,0} = (-1)^k k! e_{m+2k}. \end{aligned}$$

4. The zeta determinant of a cone

Assume in this section that (M, g_M) is a compact connected Riemannian manifold of dimension m without boundary. Let $C_\nu M$ be the metric cone over M, namely the space $[0, 1] \times M$ with metric

$$g = (dx)^2 + \frac{x^2}{\nu^2} g_M,$$

on $(0, 1] \times M$, and where ν is a positive constant [5]. Particular instances of this setting have been studied in [1,2,10] (m-ball), (cone over a circle) [19], and (deformed spheres) [23]. The zeta function on the cone $C_\nu M$, is defined by the series

$$\zeta(s, \mathrm{Sp}_+\Delta_{C_\nu M}) = \sum_{\lambda \in \mathrm{Sp}_+\Delta_{C_\nu M}} \lambda^{-s},$$

when $\mathrm{Re}(s) > \frac{m+1}{2}$, and by analytic extension elsewhere, and we are interested in the expansion of this function at $s = 0$. More precisely, the main purpose in this context is to provide formulas that relate the zeta invariants of the cone to the zeta invariants of the section, or, more generally, to some spectral invariants of the section, namely invariants that can be constructed using only the spectral information of the section. Therefore, our aim is to relate the coefficients of the expansion at $s = 0$ of $\zeta(s, \mathrm{Sp}_+\Delta_{C_\nu M})$ to some spectral invariants of M. For we decompose the induced metric Laplacian on $C_\nu M$

$$\Delta_{C_\nu M} = -d_x^2 + \frac{1}{x^2}\left(\nu^2\Delta_M - \frac{m}{4}\right),$$

on the eigenspaces of Δ_M, as in [4] or [19], and we obtain a family of singular Sturm operators that can be solved in term of Bessel functions (see [19]). With the opportune boundary conditions, that generalize standard Dirichlet conditions (see [4,5,19]), the positive spectrum of the metric Laplacian on the cone is

$$S = \mathrm{Sp}_+\Delta_{C_\nu M} = \left\{j^2_{\mu_n,k}\right\}_{n=0,k=1}^{\infty}, \quad \mu_n = \sqrt{\nu^2\lambda_n + \frac{(m-1)^2}{4}}.$$

where the $j_{\nu,k}$ are the positive zeros of the Bessel function J_ν. This means that the relevant zeta functions are:

$$\zeta(s,S)=\zeta(s,\mathrm{Sp}_+\Delta_{C\nu M})=\sum_{n=0,k=1}^{\infty} j_{\mu_n,k}^{-2s},$$
$$\zeta(s,U)=\zeta\left(s,\mathrm{Sp}\left(\nu^2\Delta_M+\frac{(m-1)^2}{4}\right)\right)=\sum_{n=0}^{\infty}\left(\nu^2\lambda_n+\frac{(m-1)^2}{4}\right)^{-s}.$$

where the second zeta function is the zeta function on the section of the cone, twisted by the parameter ν and shifted by the constant $\frac{(m-1)^2}{4}$. Note that we must omit the zero mode when $m=1$ in order to have a proper definition. The relevant sequences are $U=Sp\left(\nu^2\Delta_M+\frac{(m-1)^2}{4}\right)$, i.e. $u_n=\mu_n^2$, and $S=\mathrm{Sp}_+\Delta_{C\nu M}$. By Proposition 3.1, and using classical estimates for the zeros of Bessel functions, U and S are sequences of spectral type of genus $\left[\frac{m}{2}\right]$ and $\left[\frac{m+1}{2}\right]$, respectively. Next, we claim that the sequence S is spectrally decomposable over the sequence U. For it is easy to see that S has relative genus $(p_0,p_1,p_2)=\left(\left[\frac{m+1}{2}\right],\left[\frac{m}{2}\right],0\right)$, and that U is a totally regular sequence of spectral type by Proposition 3.1 has genus $\left[\frac{m}{2}\right]$ and infinite order. The key point, in order to prove decomposability of S over U, is to show that the Fredholm determinant $\log\Gamma(-\lambda,\tilde S_n)$ associated to the sequence $\tilde S_n=\{S_n/u_n\}$, has a uniform asymptotic expansion for large μ_n. This is of course the key point in all development of spectral analysis on spaces with conical singularities, as can be seen reading the works of Cheeger. Such an expansion is known from asymptotic theory of special functions (see for example [16, 10.7]). We obtain (see also [19])

$$\begin{aligned}\log\Gamma(-\lambda,\tilde S_n)&=-\sum_{k=1}^{\infty}\log\left(1+\frac{\mu_n^2(-\lambda)}{j_{\mu_n,k}^2}\right)\\&=\left(1-\log 2+\log(1+\sqrt{1-\lambda})-\sqrt{1-\lambda}\right)\mu_n+\frac14\log(1-\lambda)\\&\quad+\sum_{j=1}^{\infty}\left(\frac{(-1)^j}{j}\left(\sum_{k=1}^{\infty}\frac{U_k(1/\sqrt{1-\lambda})}{\mu_n^k}\right)^j-\frac{B_{2j}}{2j(2j-1)}\mu_n^{1-2j}\right),\end{aligned}$$

where the B_k are the Bernoulli numbers, and the $U_k(z)$ are polynomials in z of order $3k$. The first polynomials are given in [16], where we can also find a recursive formula. Writing

$$\sum_{h=1}^{\infty}D_h(1/\sqrt{1-\lambda})\mu_n^{-h}=\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\left(\sum_{k=1}^{\infty}\frac{U_k(1/\sqrt{1-\lambda})}{\mu_n^k}\right)^j,$$

we have

$$\log\Gamma(-\lambda,\tilde S_n)=\textstyle\sum_{h=-1}^{m}\phi_{\frac{h}{2}}(\lambda)u_n^{-\frac{h}{2}}+O(u_n^{-m-1}),$$

with

$$\begin{aligned}
\phi_{-\frac{1}{2}}(\lambda) &= 1 - \log 2 + \log(1+\sqrt{1-\lambda}) - \sqrt{1-\lambda},\\
\phi_0(\lambda) &= \frac{1}{4}\log(1-\lambda),\\
\phi_{\frac{h}{2}}(\lambda) &= D_h(1/\sqrt{1-\lambda}) - \frac{B_{h+1}}{h(h+1)}, \quad 1\leq h\leq m.
\end{aligned}$$

Note that the polynomial U_h or D_h represent a set of invariants that completely characterize the geometry of the cone, at least for what is concerned with the heat kernel and the zeta function. This emerges clearly from the results of [3] and [7], as well as from the original results of Cheeger. In fact, all the formulas related to the analytic properties of the zeta function are given using information on the zeta function on the section and information contained in the above polynomials. Also note that $D_h(1) = -\frac{\zeta_R(-h)}{h}$. Applying Theorem 2.2, and writing $D_h(z) = \sum_{j=0}^h c_j(h) z^{h+2j}$, we obtain:

Theorem 4.1. *The zeta function on the cone $\zeta(s,S)$ has an analytic extension near $s=0$, with at most a simple pole at $s=0$ and*:

$$\operatorname*{Res}_{s=0}{}_1\,\zeta(s,S) = -\frac{1}{2}\operatorname*{Res}_{s=-\frac{1}{2}}{}_1\,\zeta(s,U), \tag{4.1}$$

$$\begin{aligned}
\operatorname*{Res}_{s=0}{}_0\,\zeta(s,S) &= -\frac{1}{2}\operatorname*{Res}_{s=-\frac{1}{2}}{}_0\,\zeta(s,U) - \frac{1}{4}\operatorname*{Res}_{s=0}{}_0\,\zeta(s,U)\\
&\quad + (\log 2 - 1)\operatorname*{Res}_{s=-\frac{1}{2}}{}_1\,\zeta(s,U) - \sum_{h=1}^m \frac{\zeta_R(-h)}{h}\operatorname*{Res}_{s=\frac{h}{2}}{}_1\,\zeta(s,U),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\operatorname*{Res}_{s=0}{}_0\,\zeta'(s,S) &= -\left(\frac{\pi^2}{12} + (\log 2 - 1)^2 + 1\right)\operatorname*{Res}_{s=-\frac{1}{2}}{}_1\,\zeta(s,U)\\
&\quad +(\log 2 - 1)\operatorname*{Res}_{s=-\frac{1}{2}}{}_0\,\zeta(s,U) - \frac{1}{2}\operatorname*{Res}_{s=-\frac{1}{2}}{}_0\,\zeta'(s,U)\\
&\quad -\frac{1}{4}\zeta'(0,U) - \sum_{h=1}^m \frac{\zeta_R(-h)}{h}\operatorname*{Res}_{s=\frac{h}{2}}{}_0\,\zeta(s,U)\\
&\quad +\sum_{h=1}^m\sum_{j=0}^h c_j(h)\left(\gamma + \psi\left(\frac{h}{2}+j\right)\right)\operatorname*{Res}_{s=\frac{h}{2}}{}_1\,\zeta(s,U)\\
&\quad +\sum_{n=0}^\infty\left(\sum_{j=1}^\infty \frac{j}{2(j+1)(j+2)}\sum_{k=1}^\infty \frac{1}{(\mu_n+k)^{j+1}} - \sum_{h=1}^m \frac{B_{h+1}}{h(h+1)\mu_n^h}\right).
\end{aligned} \tag{4.3}$$

Some remarks on this result are in order.

(1) An other interesting way of writing the last term in Eq. (4.3) is

$$-\log\sqrt{2\pi}\prod_{n=1}^\infty \mu_n^{\left(\frac{1}{2}+\mu_n\right)} \mathrm{e}^{(\gamma+1)\mu_n + \sum_{h=1}^m \frac{B_{h+1}}{h(h+1)\mu_n^h}} \prod_{k=1}^\infty \left(1 - \frac{\mu_n}{k}\right)\mathrm{e}^{-\frac{\mu_n}{k}}.$$

(2) Equation (4.1) was given in [7, equation (12)], and Eq. (4.2) in incomplete form in [7] (equations (12) and (19)), and in complete form in [3] (equation (4.5)). The main result, Eq. (4.3), is new in this form. However, we note that a formula for the derivative of the zeta function at zero was also given in [3] (equations (9.8) or equations (3.8) plus (9.1) and (9.2)). The approach of [3] is based on purely heat kernel methods, and as a result the final formula for the derivative at zero contains a term (coming from equation (9.2)) given by an integral of some complicate function, and should be compared with the last term in Eq. (4.3).

(3) If m is even, $\zeta(s, \mathrm{Sp}_+\Delta_{C_\nu M})$ is regular at $s=0$. In fact, we can apply for example Proposition 1 of [17] to write

$$\operatorname*{Res_1}_{s=-\frac{1}{2}} \zeta(s,U) = -\frac{\nu}{2\sqrt{\pi}} \sum_{j,k\geq 0, j+2k=m+1} \frac{(-1)^k}{k!} e_j \left(\frac{m-1}{2\nu}\right)^k,$$

where the e_j are the coefficients in the heat kernel expansion of Δ_M (see Eq. (3.1). If m is odd, all the indices j are odd, and hence the coefficients vanish.

(4) If $M = S^m$, and $\nu = 1$, then $C_1 S^m = B^{m+1}$, the disc of dimension $m+1$, and $\zeta(s, \mathrm{Sp}_+\Delta_{C_1 S^m})$ is regular at $s=0$ (this was studied in [1,2,10]). For it is known that the spectrum of the metric Laplacian on the sphere is $\left\{n(n+m-1)\right\}_{n=1}^{\infty}$, and hence $\mathrm{Sp}_+\left(\Delta_M + \frac{(m-1)^2}{4}\right) = \left(n+\frac{m-1}{2}\right)^2$. By theoretical argument in zeta function theory (see for example [20]), the zeta function associated to these series is regular at $s = -\frac{1}{2}$. This is an expected result, since B^{m+1} is a smooth manifold.

Example. The particular case when the manifold M is the circle S^1_l of radius l was studied in [19]. The relevant sequences are $S_\nu = \{j_{\nu|n|,k}\}$, and $U = \{\nu n\}$, and

$$\log\Gamma(\lambda, (\nu n)^{-2} S_\nu) = -\log I_{\nu n}(\nu n\sqrt{-\lambda}) - \log \frac{(2^{\nu n}\Gamma(\nu n+1)}{(\nu n\sqrt{-\lambda})^{\nu n}}.$$

Applying Theorem 2.2, we obtain ([19, Theorem 1], see also [4, Section 11])

$$\begin{aligned}
\zeta(0,S_\nu) &= \frac{1}{12}\left(\nu+\frac{1}{\nu}\right),\\
\zeta'(0,S_\nu) &= \frac{1}{6}\left(\nu+\frac{1}{\nu}\right)\log l + \frac{\nu}{6} - \frac{\nu}{6}\log 2\nu - 2\nu\zeta_R'(-1) - \frac{1}{2}\log\nu\\
&\quad + \frac{1}{6}\sum_{n=1}^{\infty}\left(\zeta_H(2,\nu n+1) - \frac{1}{\nu n+1}\right)\\
&\quad - \frac{1}{6\nu}\left(\log 2\nu - \frac{5}{2} + \psi\left(1+\frac{1}{\nu}\right)\right).\\
&\quad + \sum_{m=2}^{\infty}\frac{m}{(m+1)(m+2)}\sum_{n,k=1}^{\infty}\frac{1}{(\nu n+k)^{m+1}}.
\end{aligned}$$

In particular, when $\nu=1$ (compare with [1]):

$$\zeta(0,S_1)=\frac{1}{6},\quad \zeta'(0,S_1)=2\zeta_R'(-1)+\frac{5}{12}-\frac{1}{3}\log 2+\frac{1}{2}\log 2\pi.$$

5. The zeta determinant of a product space

Let $M_{(0)}=M_{(1)}\times M_{(2)}$ be the product of two compact connected Riemannian manifolds of dimension $m_{(i)}$ without boundary. The metric Laplacian $\Delta_{M_{(1)}\times M_{(2)}}$ has real spectrum with positive part $\mathrm{Sp}_+\Delta_{M_{(0)}}=\{\lambda_{n_1,n_2}\}'_{n_i\in\mathbb{N}}$, $\lambda_{n_1,n_2}=\lambda_{(1),n_1}+\lambda_{(2),n_2}$. By Proposition 3.1, both $S_{(i)}$ are totally regular sequences of spectral type. They have exponents $s_{(i)}=\frac{m_{(i)}}{2}$, genus $p_{(i)}=\left[\frac{m_{(i)}}{2}\right]$, and infinite orders. This implies that the hypothesis of Definition 2.1 are satisfied, and consequently $S_{(0)}$ is spectrally decomposable over, say, $S_{(1)}$, with power $\kappa=1$. It also follows from the characterization of the length given in Theorem 2.3 and the formulas for the coefficients $\alpha_h=\frac{m-h}{2}$, just after Proposition 3.1, that $\ell=m_{(0)}=m_{(1)}+m_{(2)}$. We are precisely in the situation described at the end of Sect. 3, and therefore we obtain the analytic properties of the zeta function $\zeta(s,S_{(0)})$ near $s=0$ applying Theorem 2.3. This gives the following result.

Theorem 5.1. *The meromorphic extension of the zeta function $\zeta(s,S_{(0)})$ is regular at $s=0$ and (writing $S_{(j)}=\mathrm{Sp}_+\Delta_{M_{(j)}}$)*

$$\begin{aligned}\zeta(0,S_{(0)})=&\sum_{h=0}^{m_{(1)}+m_{(2)}} e_{(1),\frac{m_{(2)}-h}{2}}e_{(2),\frac{h-m_{(1)}}{2}}\\&-e_{(1),0}\mathrm{dimker}\Delta_{M_{(2)}}-e_{(2),0}\mathrm{dimker}\Delta_{M_{(1)}},\\ \zeta'(0,S_{(0)})=&\,\zeta(0,S_{(2)})\zeta'(0,S_{(1)})+\sum_{h=0}^{m_{(1)}+m_{(2)}}\Gamma\left(\frac{h-m_{(2)}}{2}\right)e_{(2),h}\\&\times\left(\mathop{\mathrm{Res}_0}_{s=\frac{h-m_{(2)}}{2}}\zeta(s,S_{(1)})+\left(\gamma+\psi\left(\frac{h-m_{(2)}}{2}\right)\right)\mathop{\mathrm{Res}_1}_{s=\frac{h-m_{(2)}}{2}}\zeta(s,S_{(1)})\right)\\&+\sum_{l=1}^{\left[\frac{m_{(1)}}{2}\right]}(l-1)!e_{(2),m_{(2)}+2l}\left(\mathop{\mathrm{Res}_0}_{s=l}\zeta(s,S_{(1)})+(\gamma+\psi(l))\mathop{\mathrm{Res}_1}_{s=l}\zeta(s,S_{(1)})\right)\\&+\sum_{l=1}^{\left[\frac{m_{(2)}}{2}\right]}\frac{(-1)^l e_{(2),m_{(2)}-2l}}{l!}\left(\zeta'(-l,S_{(1)})+(\gamma+\psi(l+1))\zeta(-l,S_{(1)})\right)\\&-\log\prod_{n_1=1}^{\infty}\left(\mathrm{e}^{-\sum_{l=1}^{\left[\frac{m_{(2)}}{2}\right]}\frac{(-1)^l e_{(2),m_{(2)}-2l}}{l!}\lambda_{(1),n_1}^l\log\lambda_{(1),n_1}}\right.\\&\times\mathrm{e}^{\sum_{h=0}^{m_{(1)}+m_{(2)}}\Gamma\left(\frac{h-m_{(2)}}{2}\right)e_{(2),h}\lambda_{(1),n_1}^{\frac{m_{(2)}-h}{2}}+\sum_{l=1}^{\left[\frac{m_{(1)}}{2}\right]}(l-1)!e_{(2),m_{(2)}+2l}\lambda_{(1),n_1}^{-l}}\end{aligned}$$

$$\times \mathrm{e}^{-\sum_{l=1}^{\left[\frac{m_{(2)}}{2}\right]} \frac{(-1)^l}{l}\left(\operatorname{Res}_{0\atop s=l}\zeta(s,S_{(2)})-\frac{1}{l}\operatorname{Res}_{1\atop s=l}\zeta(s,S_{(2)})\right)\lambda_{(1),n_1}^l}$$

$$\times\, \mathrm{e}^{-\zeta(0,S_{(2)})\log\lambda_{(1),n_1}-\zeta'(0,S_{(2)})}\prod_{n_2=1}^{\infty}\left(1+\frac{\lambda_{(1),n_1}}{\lambda_{(2),n_2}}\right)\mathrm{e}^{\sum_{j=1}^{\left[\frac{m_{(2)}}{2}\right]}\frac{(-1)^j}{j}\frac{\lambda_{(1),n_1}^j}{\lambda_{(2),n_2}^j}}\Bigg).$$

In particular, we have the following formula for the determinant:

$$\det{}_\zeta\Delta_{M_{(1)}\times M_{(2)}}=\det{}_\zeta\Delta_{M_{(1)}}\det{}_\zeta\Delta_{M_{(2)}}\mathrm{e}^{-\zeta'(0,\mathrm{Sp}_+\Delta_{M_{(1)}}+\mathrm{Sp}_+\Delta_{M_{(2)}})}.$$

Also in this case a result for the zeta determinant using pure heat kernel methods is possible. For in the case of a product manifold one can write the regular term in the Mellin transform of the heat function by adding and subtracting the singular part of the integrand (see for example [15, Section 3]), since this singular part is known (it corresponds to the product of the expansions of the singular parts of the heat kernels of the factors). This approach provides a formula for the regularized determinant involving, in the regular part, a finite integral of some complicate function, and was used in a somehow formal way in [13]. Since some derivative of the logarithmic Gamma function $\Gamma(-\lambda,S)$ is the Mellin Laplace transform of the heat function (see the proof of Proposition 2.7 of [20] for details), the result for $\zeta'(0,S_{(0)})$ given in Theorem 2.3 is an evaluation of the finite integrals appearing in the formulas given in Section 3 of [13].

Example. Consider the product $S^1_{1/y}\times M$, where $S^1_{1/y}$ is the circle of radius $\frac{1}{y}$ and M is a compact connected Riemannian manifold without boundary of dimension m, with $\mathrm{Sp}_+\Delta_M=\{\lambda_k\}_{k=1}^\infty$. We obtain

$$\det{}_\zeta\Delta_{S^1_{\frac{1}{2}}\times M}$$
$$=\frac{4\pi^2}{y^2}\mathrm{e}^{\frac{2\pi}{y}\left(\operatorname{Res}_{0\atop s=-\frac{1}{2}}\zeta(s,\mathrm{Sp}_+\Delta_M)+2(1-\log 2)\operatorname{Res}_{1\atop s=-\frac{1}{2}}\zeta(s,\mathrm{Sp}_+\Delta_M)\right)}\prod_{k=1}^{\infty}\left(1-\mathrm{e}^{-\frac{2\pi}{y}\sqrt{\lambda_k}}\right)^2.$$

This equation is particularly important in theoretical physics, since it gives the quantistic partition function at finite temperature $T=\frac{y}{2\pi}$, for a scalar field in the Euclidean product space time $S^1_{1/2\pi T}\times M$ (see also [17]).

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