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MULTIPLE POISSON KERNELS

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Recently, some results appeared [2] [1] on generalizations of the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)d\theta = \frac{1}{1-r^2}, \tag{1}$$

where $P_r(\theta) = \frac{P(\theta,r)}{1-r^2}$, and

$$P(\theta,r) = \frac{1-r^2}{(1-r\mathrm{e}^{i\theta})(1-r\mathrm{e}^{-i\theta})},$$

is the Poisson kernel in two dimensions, with $|r|<1$. Moving on along this line of investigation, we prove in this note the following result, that is the natural generalization of equation (1), and contains the results of [2] and [1].

Proposition 0.1. *Let $a=(a_n)$ and $b=(b_n)$ be two vectors in the complex open unit N-ball, and define*

$$P_{a,b}(\theta) = \prod_{n=1}^{N} \frac{1}{(1-a_n\mathrm{e}^{i\theta})(1-b_n\mathrm{e}^{-i\theta})}.$$

Then,

$$\frac{1}{2\pi}\int_0^{2\pi} P_{a,b}(\theta)d\theta = \sum_{k=1}^{N}\frac{b_k^{N-1}}{1-a_kb_k}\prod_{n=1,n\neq k}^{N}\frac{1}{(1-a_nb_k)(b_k-b_n)}.$$

Proof. The proof is a straightforward application of the residue theorem. In fact, assuming first that the components of b are all different, namely that $b_j\neq b_k$, for all $j\neq k$, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} P_{a,b}(\theta)d\theta =& \frac{1}{2\pi i}\int_{|z|=1}\prod_{n=1}^{N}\frac{1}{(1-a_nz)(1-b_n\bar{z})}\frac{dz}{z}\\ =&\frac{1}{2\pi i}\int_{|z|=1} z^{N-1}\prod_{n=1}^{N}\frac{1}{(1-a_nz)(z-b_n)}dz\\ =&\sum_{k=1}^{N}\operatorname*{Res}_{z=b_k} z^{N-1}\prod_{n=1}^{N}\frac{1}{(1-a_nz)(z-b_n)}.\end{aligned}$$

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This gives the thesis in this case. To finish the proof, it is easy to verify that the formula given in the thesis extends analytically when some components of b are equal.

□

References

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